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Full state approximation by Galerkin projection reduced order models for stochastic and bilinear systems

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ABSTRACT

In this paper, the problem of full state approximation by model reduction is studied for stochastic and bilinear systems. Our proposed approach relies on identifying the dominant subspaces based on the reachability Gramian of a system. Once the desired subspace is computed, the reduced order model is then obtained by a Galerkin projection. We prove that, in the stochastic case, this approach either preserves mean square asymptotic stability or leads to reduced models whose minimal realization is mean square asymptotically stable. This stability preservation guarantees the existence of the reduced system reachability Gramian which is the basis for the full state error bounds that we derive. This error bound depends on the neglected eigenvalues of the reachability Gramian and hence shows that these values are a good indicator for the expected error in the dimension reduction procedure. Subsequently, we establish the stability preservation result and the error bound for a full state approximation to bilinear systems in a similar manner. These latter results are based on a recently proved link between stochastic and bilinear systems. We conclude the paper by numerical experiments using a benchmark problem. We compare this approach with balanced truncation and show that it performs well in reproducing the full state of the system.

1. Introduction

Galerkin approximation is an important methodology to obtain surrogate models for high fidelity systems. It relies on the fact that, in many applications, the state of the system is well approximated in a lower-dimensional subspace. In other words, for $x(t) \in \mathbb{R}^n$ with $n \gg 1$, there exist $V \in \mathbb{R}^{n\times r}$ such that $x(t) \approx V\hat{x}(t)$. The choice of the right basis V plays a crucial role in the approximation quality. Several approaches to construct the dominant subspaces have been proposed for deterministic linear systems, see, e.g., [6]. Among them, a commonly used approach is the proper orthogonal decomposition (POD) [10,18], which identifies dominant subspaces empirically by extracting them from snapshot matrices. These empirical methods have been successfully used in many applications. However, they are *input-dependent* in the setup of control systems, i.e., the quality of reduced order model (ROM) will depend on the choices of inputs used to generate the snapshots. In the setup of stochastic systems considered in this paper, a POD approach would require the simulation of an enormous

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amount of samples being numerically costly in practice. Moreover, a deeper theoretical analysis of such empirical methods is often not feasible.

Given stability in the original model, it is of interest to preserve this property in the reduced system. Galerkin methods have been shown to preserve stability for deterministic linear dissipative systems, see, e.g., [29]. Additionally, the authors in [22] propose a POD scheme combined with linear matrix inequalities to construct ROMs that are locally stable in a non-linear deterministic setting. In general, Krylov based methods for deterministic linear [15] and bilinear systems [11] do not guarantee stability in the ROM. However, in [23], the authors have proposed equivalent dissipative realizations to arbitrary Galerkin projected linear systems that are stable. To the authors' knowledge, stability preservation using Galerkin projections has not been studied in the literature of stochastic systems.

In this work, we focus on model order reduction of linear stochastic and bilinear systems aiming for a full state approximation. Therefore, we study a Galerkin approach based on the dominant reachability subspaces, which leads to input-independent projections, i.e., the corresponding ROMs are (pathwise) accurate for a large set of inputs. This approach relies on the computation of the reachability Gramian of the underlying dynamical system. These Gramians are encoded by generalized Lyapunov equations and, hence, can be computed in a numerically efficient way in large-scale settings (see, e.g., the review papers [8,30] for low-rank methods). Once the Lyapunov equation is solved, the dominant subspaces are then identified by the span of eigenvectors of the Gramian associated with the large eigenvalues. Hence, the ROM is obtained by projecting the dynamical system onto the identified subspace. We show that this procedure either preserves the underlying stability or at least ensures the existence of the reduced order Gramian, which is vital for the error analysis. As a consequence, the minimal realization of the reduced model is stable. Subsequently, we propose error bounds for the approximation, which show how the reduction error is related to the neglected eigenvalues of the Gramian. It is worth noticing that this approach has already been successfully applied in the literature of deterministic linear time-invariant systems, e.g., in the context of structured systems [31], and port-Hamiltonian systems [21]. However, to the authors knowledge, the stability analysis and error bounds for stochastic and bilinear systems considered in this paper have not even been established for deterministic linear systems so far.

It is worth mentioning that one way to address the problem of full state approximation is to use balanced truncation, see [19] for deterministic linear systems and [1,3,7,24] for stochastic and bilinear systems. This method is generally suitable when one wants to approximate a quantity of interest $y(t) = Cx(t)$, with $C \in \mathbb{R}^{p\times n}$ and $p \ll n$. For this method, one needs to compute the observability Gramian in addition to the reachability Gramian. Subsequently, a ROM is obtained by Petrov-Galerkin projection based on these two Gramians. One advantage of balanced truncation is that it is, under some mild conditions, stability preserving [4,20] and it guarantees error bounds [7,14,26]. However, whenever, $p \approx n$, it suffers from the issue that the computation of the observability Gramian is not feasible in practice since low-rank methods are no longer applicable in this context. This scenario is given if the full state shall be approximated, since $C = I$ in this case. For the deterministic linear case, the authors in [9] propose a scheme enabling the computation of the Petrov-Galerkin projection without the explicit computation of the observability Gramian. The approach is numerically feasible but costly since it relies on a quadrature scheme using the low-rank factors of the reachability Gramian pre-multiplied by shifted systems. Additionally, those results are no longer applicable for stochastic and bilinear systems.

The paper is organized as follows. In Section 2, we present the main setup for linear stochastic systems with zero initial states and the concept of mean square asymptotic stability along with some literature results. Then, in Section 3, we described the proposed Galerkin projection based procedure using the reachability Gramian. Additionally, an interpretation of the dominant subspaces is derived therein. Section 4 is dedicated to showing the properties of the ROM for zero initial data. First, we prove that this procedure either constructs a reduced model which is mean square asymptotically stable or a ROM which has a realization satisfying the desired stability property. It is worth noticing that modified versions of those results are also valid for the class of deterministic bilinear systems, which we also establish in this paper. In Subsection 4.2, we derive bounds for the approximation error and their relation to the neglected singular values of the reachability Gramian. In Section 5, the presented results are extended to the case of stochastic systems with non-zero initial conditions. This extension relies on splitting the general system into two subsystems. One being uncontrolled including the initial data x_0 and the other one being the previously studied control system with zero initial condition. In particular, the results of Section 4 are transferred to the subsystem involving x_0. In Section 6, similar results for the class of bilinear systems, including stability preservation and error bounds, are derived. In Section 7, some numerical experiments are conducted to illustrate the performance of the proposed approach and in order to compare it with balanced truncation.

2. Linear stochastic systems and mean square stability

2.1. Stochastic problem setup

We consider the following linear stochastic systems

$$dx(t) = [Ax(t) + Bu(t)]dt + \sum_{i=1}^{q} N_i x(t) dW_i(t), \quad t \geq 0, \tag{1}$$

where we assume that $A, N_i \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant matrices and its initial condition are assumed to be zero, i.e., $x(0) = 0$. Later, we generalize the results to the case where non-zero initial conditions are considered. The vectors x and u

are called state and control input, respectively. Moreover, let $W = (W_1, \ldots, W_q)^\top$ be an $\mathbb{R}^q$-valued standard Wiener process for simplicity of the notation. The results can be extended to square integrable Lévy processes with mean zero and general covariance matrix (see, e.g., [24]). All stochastic processes appearing in this paper, are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$[1] In addition, W is $(\mathcal{F}_t)_{t\geq 0}$-adapted and its increments $W(t+h) - W(t)$ are independent of $\mathcal{F}_t$ for $t, h \geq 0$. Throughout this paper, we assume that u is an $(\mathcal{F}_t)_{t\geq 0}$-adapted control that is square integrable, meaning that

$$\|u\|_{L^2_T}^2 := \mathbb{E}\int_0^T \|u(s)\|_2^2 ds < \infty$$

for all $T > 0$, where $\|\cdot\|_2$ denotes the Euclidean norm. Moreover, $\|\cdot\|_F$ will denote the Frobenius norm, whereas $\|\cdot\|$ represents an arbitrary matrix/vector norm. The aim is to identify a low-dimensional subspace $\mathcal{V}$ of $\mathbb{R}^n$ that approximates the manifold of the state x. Choosing a matrix $V \in \mathbb{R}^{n\times r}$ of orthonormal basis vectors of $\mathcal{V}$, an approximation of the form $V\hat{x}(t) \approx x(t)$ can be constructed. Inserting this approximation into the original system (1), we enforce a Petrov-Galerkin condition by multiplying the residual with $V^\top$ leading to a ROM

$$d\hat{x}(t) = [\hat{A}\hat{x}(t) + \hat{B}u(t)]dt + \sum_{i=1}^q \hat{N}_i\hat{x}(t)dW_i(t), \quad t \geq 0, \tag{2}$$

where $\hat{A} = V^\top AV$, $\hat{B} = V^\top B$, $\hat{N}_i = V^\top \hat{N}_i V$ and $\hat{x}(t) \in \mathbb{R}^r, \hat{x}(0) = 0$, with $r \ll n$. Our main goal is to construct the matrix V, such that, the approximation error is small for every input u considered.

2.2. Mean square asymptotic stability and generalized Lyapunov operators

We introduce the fundamental solution Φ to (1). It is defined as the $\mathbb{R}^{n\times n}$-valued solution to

$$\Phi(t,s) = I + \int_s^t A\Phi(\tau,s)d\tau + \sum_{i=1}^q \int_s^t N_i\Phi(\tau,s)dW_i(\tau), \quad t \geq s. \tag{3}$$

It is the operator that maps the initial condition x_0 to the solution of the homogeneous state equation, i.e., $u \equiv 0$, with initial time $s \geq 0$. We additionally define $\Phi(t) := \Phi(t,0)$. Moreover, notice that we have $\Phi(t,s) = \Phi(t)\Phi^{-1}(s)$.

Throughout this paper, we assume that the uncontrolled state Eq. (1) is mean square asymptotically stable, i.e, $\mathbb{E}\|\Phi(t)\|^2 \lesssim \mathrm{e}^{-ct}$ for some constant $c > 0$. With $\lambda(\cdot)$ denoting the spectrum of a matrix/operator, this is equivalent to

$$\lambda(K) \subset \mathbb{C}_-, \tag{4}$$

where $K := I\otimes A + A\otimes I + \sum_{i=1}^q N_i\otimes N_i$ and $\cdot\otimes\cdot$ is the Kronecker product of two matrices, see for instance [13,17]. Moreover, the system is called mean square stable if $\lambda(K) \subset \overline{\mathbb{C}_-}$. Notice that $\lambda(K) = \lambda(\mathcal{L}_A + \Pi_N)$, where the generalized Lyapunov operator $\mathcal{L}_A + \Pi_N$ is defined by $X \mapsto \mathcal{L}_A(X) = AX + XA^\top$ and $X \mapsto \Pi_N(X) = \sum_{i=1}^q N_iXN_i^\top$. In Section 3, we will introduce a reduced system which does not necessarily preserve (4) but it is always mean square stable, i.e., K can additionally have eigenvalues on the imaginary axis. Therefore, we need the following sufficient conditions for mean square stability.

Lemma 2.1. *Given a matrix $Y \geq 0$, let us assume that there exists $X > 0$ such that*

$$\mathcal{L}_A(X) + \Pi_N(X) \leq -Y.$$

Then, we have $\lambda(K) \subset \overline{\mathbb{C}_-}$.

Proof. An algebraic proof can be found in [4], Corollary 3.2. We refer to [24], Lemma 6.12 for a probabilistic approach. □

Let $\alpha(\mathcal{L}_A + \Pi_N) := \max\{\Re(\mu) : \mu \in \lambda(\mathcal{L}_A + \Pi_N)\}$ be the spectral abscissa of the operator $\mathcal{L}_A + \Pi_N$, with $\Re(\cdot)$ being the real part of a complex number. Since the stability of a stochastic system is related to the eigenvalues of $\mathcal{L}_A + \Pi_N$, we formulate the following result.

Lemma 2.2. *There exists $V_1 \geq 0$, $V_1 \neq 0$, such that $\mathcal{L}_A(V_1) + \Pi_N(V_1) = \alpha(\mathcal{L}_A + \Pi_N)V_1$.*

Proof. A proof in a more general framework can be found in [13], Section 3.2. We also refer to [4], Theorem 3.1 and the references therein. □

Finally, spectral properties of the Kronecker matrix involving both the reduced and the original model matrices are required.

Lemma 2.3. *Given that the full model is mean square asymptotically stable, whereas the reduced system is just mean square stable, i.e., we have*

$$\lambda\left(I\otimes A + A\otimes I + \sum_{i=1}^q N_i\otimes N_i\right) \subset \mathbb{C}_- \text{ and } \lambda\left(I\otimes \hat{A} + \hat{A}\otimes I + \sum_{i=1}^q \hat{N}_i\otimes \hat{N}_i\right) \subset \overline{\mathbb{C}_-}.$$

[1] $(\mathcal{F}_t)_{t\geq 0}$ is right continuous and complete.

Then, it holds that

$$\lambda\left(I\otimes A+\hat{A}\otimes I+\sum_{i=1}^{q}\hat{N}_i\otimes N_i\right)\subset\mathbb{C}_-.$$

Proof. A probabilistic version can be found in [24], Lemma 6.12 and an algebraic approach is given in [4], Proposition 3.4. □

3. Dominant subspaces and reduced order model

3.1. Dominant subspaces of (1)

We identify the redundant information in the system by using the reachability Gramian

$$P:=\mathbb{E}\int_0^\infty \Phi(s)BB^\top\Phi^\top(s)ds.$$

Notice that P exists due to the exponential decay of the fundamental solution Φ. Practically, one can compute P by solving a generalized Lyapunov equation. Using Lemma A.1 with $s=0$, $\hat{A}=A$, $\hat{B}=B$, $\hat{N}_i=N_i$ and $t\to\infty$, we obtain that the reachability Gramian is a solution of the following generalized Lyapunov equation

$$AP+PA^\top+\sum_{i=1}^{q}N_iPN_i^\top=-BB^\top. \tag{5}$$

Let $x(t,x_0,u)$, $t\geq 0$, denote the solution of (1) with initial value x_0 and control u. Then, for $z\in\mathbb{R}^n$, we have

$$\sup_{t\in[0,T]}\mathbb{E}|\langle x(t,0,u),z\rangle_2|\leq\left(z^\top Pz\right)^{\frac{1}{2}}\|u\|_{L^2_T} \tag{6}$$

using the results in [25]. Let $(p_k)_{k=1,\dots,n}$ be an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of P. Then, the state variable can be written as

$$x(t,0,u)=\sum_{k=1}^{n}\langle x(t,0,u),p_k\rangle_2 p_k.$$

Setting $z=p_k$ in (6), we obtain

$$\sup_{t\in[0,T]}\mathbb{E}|\langle x(t,0,u),p_k\rangle_2|\leq\lambda_k^{\frac{1}{2}}\|u\|_{L^2_T}, \tag{7}$$

where λ_k is the corresponding eigenvalue. Consequently, we see that the direction p_k is completely irrelevant if $\lambda_k=0$. On the other hand, if λ_k is not zero but small, then a large component in the direction of p_k requires a large amount of energy by (7). Therefore, the eigenspaces of P belonging to the small eigenvalues can also be neglected.

A ROM can now be obtained by removing the unimportant subspaces from (1). This is done by first diagonalizing P. If P is diagonal, we have that p_k is the kth unit vector and the diagonal entries of P indicate the relevance of the respective unit vector. A reduced system can then be easily derived by truncating the components of x associated to the small/zero entries λ_k of a Gramian of the form $P=\operatorname{diag}(\lambda_1,\dots,\lambda_n)$.

3.2. Reduced order model by Galerkin projection

We introduce the eigenvalue decomposition of the reachability Gramian as follows

$$P=S^\top\Lambda S,$$

where $S^{-1}=S^\top$ and $\Lambda=\begin{bmatrix}\Lambda_1 & 0\\ 0 & \Lambda_2\end{bmatrix}=\operatorname{diag}(\lambda_1,\dots,\lambda_n)$ is the matrix of eigenvalues of P. For simplicity, let us assume that the spectrum of P is ordered, i.e., $\lambda_1\geq\ldots\geq\lambda_n\geq 0$ so that Λ_2 contains the small eigenvalues. Let us do a state space transformation using the matrix S. The transformed state variable then is $x_b=Sx$. Plugging this into (1), we find

$$\begin{aligned} dx_b(t) &= [A_bx_b(t)+B_bu(t)]dt+\textstyle\sum_{i=1}^{q}N_{i,b}x_b(t)dW_i(t), \quad t\geq 0,\\ x(t) &= S^\top x_b(t), \end{aligned} \tag{8}$$

where the balanced matrices are given by

$$A_b:=SAS^\top=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix},\quad B_b:=SB=\begin{bmatrix}B_1\\ B_2\end{bmatrix},\quad N_{i,b}:=SN_iS^\top=\begin{bmatrix}N_{i,11} & N_{i,12}\\ N_{i,21} & N_{i,22}\end{bmatrix}. \tag{9}$$

We refer to (9) as the balanced realization of the linear stochastic system. The fundamental solution of the balanced realization is $\Phi_b = S\Phi S^\top$ which can be seen by multiplying (3) with S from the left and with $S^\top$ from the right. Therefore, the reachability Gramian of (8) is

$$P_b := \mathbb{E}\int_0^\infty \Phi_b(s)B_bB_b^\top\Phi_b^\top(s)ds = SPS^\top = \Lambda. \tag{10}$$

We partition $x_b = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where x_1 and x_2 are associated to Λ_1 and Λ_2, respectively. Now, exploiting the insights of Section 3.1, x_2 barely contributes to the system dynamics. We obtain the reduced system by truncating the equation related to x_2 in (8). Furthermore, we set the remaining x_2 components equal to zero. This yields a reduced system (2) with matrices

$$\hat{A} = A_{11} = V^\top AV, \quad \hat{B} = B_1 = V^\top B, \quad \hat{N}_i = N_{i,11} = V^\top N_i V, \tag{11}$$

where V are the first r columns of $S^\top = \begin{bmatrix} V & S_2 \end{bmatrix}$.

In large-scale settings, the reachability Gramian can be computed using low-rank methods (see [8,30]), i.e., we find a matrix $Z_P \in \mathbb{R}^{n\times l}$, with $l \ll n$, such that $P \approx Z_P Z_P^\top$. Consequently, in this setup, the Galerkin projection can be identified using the singular value decomposition of Z_P.

4. Properties of the reduced system

4.1. Mean square stability and reduced order Gramian

In this section, we study stability preservation and the existence of the Gramian for the reduced system in (11). The next result guarantees mean square stability.

Proposition 4.1. *Suppose that $\Lambda_1 = \operatorname{diag}(\lambda_1, \dots, \lambda_r) > 0$. Then, the reduced order system (2) with $\hat{A} = A_{11}$ and $\hat{N}_i = N_{i,11}$, introduced in (9), is mean square stable, i.e.,*

$$\lambda(I\otimes A_{11} + A_{11}\otimes I + \sum_{i=1}^q N_{i,11}\otimes N_{i,11}) \subset \overline{\mathbb{C}_-}.$$

Proof. According to (10), the balanced reachability Gramian is the diagonal matrix Λ of eigenvalues of P. Using the partition of the balanced matrices in (9), we therefore have

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} + \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}\begin{bmatrix} A_{11}^\top & A_{21}^\top \\ A_{12}^\top & A_{22}^\top \end{bmatrix} + \sum_{i=1}^q \begin{bmatrix} N_{i,11} & N_{i,12} \\ N_{i,21} & N_{i,22} \end{bmatrix}\begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}\begin{bmatrix} N_{i,11}^\top & N_{i,21}^\top \\ N_{i,12}^\top & N_{i,22}^\top \end{bmatrix}$$
$$= -\begin{bmatrix} B_1B_1^\top & B_1B_2^\top \\ B_2B_1^\top & B_2B_2^\top \end{bmatrix}.$$

The left upper block of the above equation is

$$A_{11}\Lambda_1 + \Lambda_1 A_{11}^\top + \sum_{i=1}^q N_{i,11}\Lambda_1 N_{i,11}^\top = -B_1B_1^\top - \sum_{i=1}^q N_{i,12}\Lambda_2 N_{i,12}^\top \le 0.$$

Since $\Lambda_1 > 0$ by assumption, Lemma 2.1 yields the claim. □

Using Proposition 4.1 and Lemma 2.2, the reduced order system is asymptotically mean square stable if and only if $0 \notin \lambda(I\otimes A_{11} + A_{11}\otimes I + \sum_{i=1}^q N_{i,11}\otimes N_{i,11})$. With the following example it is shown that the zero eigenvalue can indeed occur.

Example 4.2. Let $N_i = 0$, $A = \begin{bmatrix} 0 & -10 \\ 1 & -10 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$. Then, system (1) is already balanced since the reachability Gramian is given by

$$P = \int_0^\infty e^{As}BB^\top e^{A^\top s}ds = \begin{bmatrix} 50 & 0 \\ 0 & 5 \end{bmatrix}.$$

Moreover, we have $\operatorname{rank}(\begin{bmatrix} B & AB \end{bmatrix}) = 2$ which means that the system is reachable or locally reachable if N_i were non zero. According to Section 3.2 the reduced matrices are $A_{11} = 0$, $B_1 = 0$ and $N_{i,11} = 0$. Consequently, the reduced system is not asymptotically stable and the reachability of the system is also lost since the reduced system is uncontrolled.

Example 4.2 shows a difference to balanced truncation for stochastic systems, where mean square asymptotic stability is preserved under relatively general conditions [4,5]. Mean square asymptotic stability ensures the existence of the reachability Gramian $\hat{P} := \mathbb{E}\int_0^\infty \hat{\Phi}(s)B_1B_1^\top\hat{\Phi}^\top(s)ds$, where $\hat{\Phi}$ represents the fundamental solution of the reduced system. However, asymptotic stability is only a sufficient condition for $\hat{P}$ to exist. The next example illustrates such a scenario.

Example 4.3. Let $N_i = 0$, $A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then, we have

$$\Phi(t) = \mathrm{e}^{At} = \begin{bmatrix} \frac{\mathrm{e}^{-2t}+1}{2} & \frac{\mathrm{e}^{-2t}-1}{2} \\ \frac{\mathrm{e}^{-2t}-1}{2} & \frac{\mathrm{e}^{-2t}+1}{2} \end{bmatrix} \quad \text{and} \quad \Phi(t)B = \begin{bmatrix} \mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix}.$$

Clearly, Φ does not decay exponential to zero but ΦB does. Therefore, the reachability Gramian exist and is $P = \begin{bmatrix} 0.25 & 0.25 \\ 0.25 & 0.25 \end{bmatrix}$, whereas the set of solutions to (5) is given by $P + y\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $y \in \mathbb{R}$.

Example 4.3 emphasizes that it is important to distinguish between a Gramian (given by an integral representation) and a solution of a Lyapunov equation. The next theorem proves that even if the reduced system is not mean square asymptotically stable, the existence of the reduced order reachability Gramian can be guaranteed. This is one of the main results of the paper which is also vital for later considerations, where we prove an error bound in which $\hat{P}$ is involved.

Theorem 4.4. *Given the reduced system (2) with matrices $\hat{A} = A_{11}$, $\hat{B} = B_1$, $\hat{N}_i = N_{i,11}$ defined in (9) and $\Lambda_1 = \operatorname{diag}(\lambda_1, \dots, \lambda_r) > 0$. Moreover, let $\hat{\Phi}$ denote the fundamental solution to this reduced order system. Then, there is a constant $c > 0$ such that $\mathbb{E}\left\|\hat{\Phi}(t)B_1\right\|_F^2 \lesssim \mathrm{e}^{-ct}$. Hence, the reachability Gramian $\hat{P} := \mathbb{E}\int_0^\infty \hat{\Phi}(s)B_1B_1^\top\hat{\Phi}^\top(s)ds$ exists and satisfies*

$$A_{11}\hat{P} + \hat{P}A_{11}^\top + \sum_{i=1}^q N_{i,11}\hat{P}N_{i,11}^\top = -B_1B_1^\top. \tag{12}$$

Proof. We set $\hat{K} := I \otimes A_{11} + A_{11} \otimes I + \sum_{i=1}^q N_{i,11} \otimes N_{i,11}$ and consider the case that the reduced system is mean square asymptotically stable, i.e., $0 \notin \lambda(\hat{K})$. According to Section 2.2 this is equivalent to $\mathbb{E}\left\|\hat{\Phi}(t)\right\|_F^2 \lesssim \mathrm{e}^{-ct}$ implying $\mathbb{E}\left\|\hat{\Phi}(t)B_1\right\|_F^2 \lesssim \mathrm{e}^{-ct}$. Given this condition the infinite integral $\hat{P}$ exists. Moreover, using Lemma A.1 with $A = \hat{A} = A_{11}$, $B = \hat{B} = B_1$ and $N_i = \hat{N}_i = N_{i,11}$ and exploiting that the left hand side of (58) tends to zero if $t \to \infty$, we see that $\hat{P}$ solves (12).

Let us consider the case of $0 \in \lambda(\hat{K}) = \lambda(\hat{K}^\top)$. If further $B_1 = 0$, the result of this theorem is true. Therefore, we additionally assume that $B_1 \neq 0$. Then, by Lemma 2.2, there exists $\hat{V} \geq 0$ such that

$$\mathcal{L}_{A_{11}^\top}(\hat{V}) + \Pi_{N_{11}^\top}(\hat{V}) = A_{11}^\top\hat{V} + \hat{V}A_{11} + \sum_{i=1}^q N_{i,11}^\top\hat{V}N_{i,11} = 0. \tag{13}$$

Moreover, according to the proof of Proposition 4.1, we have

$$A_{11}\Lambda_1 + \Lambda_1A_{11}^\top + \sum_{i=1}^q N_{i,11}\Lambda_1N_{i,11}^\top = -B_1B_1^\top - \sum_{i=1}^q N_{i,12}\Lambda_2N_{i,12}^\top =: -R. \tag{14}$$

We observe that

$$-\langle R, \hat{V}\rangle_F = \langle \mathcal{L}_{A_{11}}(\Lambda_1) + \Pi_{N_{11}}(\Lambda_1), \hat{V}\rangle_F = \langle \Lambda_1, \mathcal{L}_{A_{11}^\top}(\hat{V}) + \Pi_{N_{11}^\top}(\hat{V})\rangle_F = 0.$$

Using the properties of the trace this yields

$$\begin{aligned}\left\|\hat{V}^{\frac{1}{2}}B_1\right\|_F^2 + \sum_{i=1}^q \left\|\hat{V}^{\frac{1}{2}}N_{i,12}\Lambda_2^{\frac{1}{2}}\right\|_F^2 &= \operatorname{tr}(\hat{V}^{\frac{1}{2}}B_1B_1^\top\hat{V}^{\frac{1}{2}}) + \sum_{i=1}^q \operatorname{tr}(\hat{V}^{\frac{1}{2}}N_{i,12}\Lambda_2N_{i,12}^\top\hat{V}^{\frac{1}{2}}) \\ &= \langle R, \hat{V}\rangle_F = 0.\end{aligned}$$

This implies that

$$\hat{V}B_1 = 0 \quad \text{and} \quad \hat{V}N_{i,12}\Lambda_2^{\frac{1}{2}} = 0. \tag{15}$$

The case $\hat{V} > 0$ is excluded since then it holds that $B_1 = 0$. Therefore, we consider the scenario in which $\hat{V}$ does not have full rank. We then assume that $\hat{V}$ is an eigenvector with maximal rank, i.e., for any other eigenvector $\tilde{V} \geq 0$ corresponding to the zero eigenvalue, we have $\operatorname{rank}(\tilde{V}) \leq \operatorname{rank}(\hat{V})$.

Introducing the eigenvalue decomposition of $\hat{V}$:

$$\hat{V} = \begin{bmatrix} \hat{V}_1 & \hat{V}_2 \end{bmatrix}\begin{bmatrix} \hat{D} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{V}_1^\top \\ \hat{V}_2^\top \end{bmatrix} = \hat{V}_1\hat{D}\hat{V}_1^\top, \tag{16}$$

$\hat{D} > 0$, we find a basis of the kernel by the columns of $\hat{V}_2$, i.e., $\ker(\hat{V}) = \operatorname{im}(\hat{V}_2)$. Inserting (16) into (15) yields

$$\hat{V}_1^\top B_1 = 0 \quad \text{and} \quad \hat{V}_1^\top N_{i,12}\Lambda_2^{\frac{1}{2}} = 0. \tag{17}$$

We use a state space transformation based on $\hat{S} = \begin{bmatrix} \hat{V}_1^\top \\ \hat{V}_2^\top \end{bmatrix}$ involving the following matrices

$$\begin{aligned} &\hat{S}A_{11}\hat{S}^\top =: \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \quad \hat{S}N_{i,11}\hat{S}^\top =: \begin{bmatrix} \hat{N}_{i,11} & \hat{N}_{i,12} \\ \hat{N}_{i,21} & \hat{N}_{i,22} \end{bmatrix}, \quad \hat{S}\Lambda_1\hat{S}^\top =: \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{12}^\top & \hat{P}_{22} \end{bmatrix}, \\ &\hat{S}B_1 = \begin{bmatrix} \hat{V}_1^\top B_1 \\ \hat{V}_2^\top B_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{V}_2^\top B_1 \end{bmatrix}, \quad \hat{S}N_{i,12}\Lambda_2^{\frac{1}{2}} = \begin{bmatrix} \hat{V}_1^\top N_{i,12}\Lambda_2^{\frac{1}{2}} \\ \hat{V}_2^\top N_{i,12}\Lambda_2^{\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{V}_2^\top N_{i,12}\Lambda_2^{\frac{1}{2}} \end{bmatrix}, \end{aligned} \tag{18}$$

where (17) was exploited. We multiply (14) with $\hat{S}$ from the left and with $\hat{S}^\top$ from the right and obtain

$$\begin{aligned} &\begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}\begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{12}^\top & \hat{P}_{22} \end{bmatrix} + \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{12}^\top & \hat{P}_{22} \end{bmatrix}\begin{bmatrix} \hat{A}_{11}^\top & \hat{A}_{21}^\top \\ \hat{A}_{12}^\top & \hat{A}_{22}^\top \end{bmatrix} + \textstyle\sum_{i=1}^q \begin{bmatrix} \hat{N}_{i,11} & \hat{N}_{i,12} \\ \hat{N}_{i,21} & \hat{N}_{i,22} \end{bmatrix}\begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{12}^\top & \hat{P}_{22} \end{bmatrix}\begin{bmatrix} \hat{N}_{i,11}^\top & \hat{N}_{i,21}^\top \\ \hat{N}_{i,12}^\top & \hat{N}_{i,22}^\top \end{bmatrix} \\ &= -\begin{bmatrix} 0 & 0 \\ 0 & \hat{R} \end{bmatrix} \end{aligned} \tag{19}$$

with $\hat{R} = \hat{V}_2^\top B_1 B_1^\top \hat{V}_2 + \sum_{i=1}^q \hat{V}_2^\top N_{i,12}\Lambda_2 N_{i,12}^\top \hat{V}_2 \geq 0$. Before we evaluate the blocks of (19), we show that

$$\hat{A}_{12} = \hat{V}_1^\top A_{11}\hat{V}_2 = 0 \quad \text{and} \quad \hat{N}_{i,12} = \hat{V}_1^\top N_{i,11}\hat{V}_2 = 0. \tag{20}$$

To do so, we show that the kernel of $\hat{V}$ is invariant under multiplication with A_{11} and $N_{i,11}$. Let $z \in \ker(\hat{V})$. Then, we obtain

$$0 = z^\top\left(A_{11}^\top\hat{V} + \hat{V}A_{11} + \sum_{i=1}^q N_{i,11}^\top\hat{V}N_{i,11}\right)z = \sum_{i=1}^q z^\top N_{i,11}^\top\hat{V}N_{i,11}z = \sum_{i=1}^q \left\|\hat{V}^{\frac{1}{2}}N_{i,11}z\right\|_2^2$$

implying that $\hat{V}N_{i,11}z = 0$. Using this fact provides that

$$0 = \left(A_{11}^\top\hat{V} + \hat{V}A_{11} + \sum_{i=1}^q N_{i,11}^\top\hat{V}N_{i,11}\right)z = \hat{V}A_{11}z.$$

Hence, we have $A_{11}\ker(\hat{V}), N_{i,11}\ker(\hat{V}) \subset \ker(\hat{V})$. Since the columns of $\hat{V}_2$ span $\ker(\hat{V})$ and due to the invariance, there exist suitable matrices $\tilde{A}_{11}$ and $\tilde{N}_{i,11}$ such that

$$A_{11}\hat{V}_2 = \hat{V}_2\tilde{A}_{11} \quad \text{and} \quad N_{i,11}\hat{V}_2 = \hat{V}_2\tilde{N}_{i,11}. \tag{21}$$

Exploiting that $\hat{V}_1^\top\hat{V}_2 = 0$ gives us (20). Moreover, we see that $\hat{A}_{22} = \hat{V}_2^\top A_{11}\hat{V}_2 = \hat{V}_2^\top\hat{V}_2\tilde{A}_{11} = \tilde{A}_{11}$ and similarly $\hat{N}_{i,22} = \tilde{N}_{i,11}$ using that $\hat{V}_2^\top\hat{V}_2 = I$. Taking (20) into account, the left upper block of (19) is

$$\begin{aligned} &\hat{A}_{11}\hat{P}_{11} + \hat{P}_{11}\hat{A}_{11}^\top + \textstyle\sum_{i=1}^q \hat{N}_{i,11}\hat{P}_{11}\hat{N}_{i,11}^\top = 0 \\ &\Leftrightarrow \hat{P}_{11}^{-1}\hat{A}_{11}^\top + \hat{A}_{11}\hat{P}_{11}^{-1} + \textstyle\sum_{i=1}^q \hat{P}_{11}^{-1}\hat{N}_{i,11}\hat{P}_{11}\hat{N}_{i,11}^\top\hat{P}_{11}^{-1} = 0. \end{aligned} \tag{22}$$

The evaluation of the right upper block yields

$$\begin{aligned} &\hat{A}_{11}\hat{P}_{12} + \hat{P}_{11}\hat{A}_{21}^\top + \hat{P}_{12}\hat{A}_{22}^\top + \textstyle\sum_{i=1}^q \begin{bmatrix} \hat{N}_{i,11} & 0 \end{bmatrix}\begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{12}^\top & \hat{P}_{22} \end{bmatrix}\begin{bmatrix} \hat{N}_{i,21}^\top \\ \hat{N}_{i,22}^\top \end{bmatrix} = 0 \\ &\Leftrightarrow \hat{A}_{21}^\top = -\left(\hat{P}_{11}^{-1}\hat{A}_{11}\hat{P}_{12} + \hat{P}_{11}^{-1}\hat{P}_{12}\hat{A}_{22}^\top + \textstyle\sum_{i=1}^q \begin{bmatrix} \hat{P}_{11}^{-1}\hat{N}_{i,11} & 0 \end{bmatrix}\begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{12}^\top & \hat{P}_{22} \end{bmatrix}\begin{bmatrix} \hat{N}_{i,21}^\top \\ \hat{N}_{i,22}^\top \end{bmatrix}\right). \end{aligned} \tag{23}$$

Finally, the right lower block is given by

$$\hat{A}_{21}\hat{P}_{12} + \hat{A}_{22}\hat{P}_{22} + \hat{P}_{12}^\top\hat{A}_{21}^\top + \hat{P}_{22}\hat{A}_{22}^\top + \sum_{i=1}^q \begin{bmatrix} \hat{N}_{i,21} & \hat{N}_{i,22} \end{bmatrix}\begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{12}^\top & \hat{P}_{22} \end{bmatrix}\begin{bmatrix} \hat{N}_{i,21}^\top \\ \hat{N}_{i,22}^\top \end{bmatrix} = -\hat{R}. \tag{24}$$

We set $\hat{\mathbf{P}}_{22} = \hat{P}_{22} - \hat{P}_{12}^\top\hat{P}_{11}^{-1}\hat{P}_{12}$, $\hat{\mathbf{N}}_{i,21} = \hat{N}_{i,21} - \hat{P}_{12}^\top\hat{P}_{11}^{-1}\hat{N}_{i,11}$ and insert (23) into (24) in order to obtain

$$\begin{aligned} &\hat{A}_{22}\hat{\mathbf{P}}_{22} + \hat{\mathbf{P}}_{22}\hat{A}_{22}^\top - \hat{P}_{12}^\top(\hat{A}_{11}^\top\hat{P}_{11}^{-1} + \hat{P}_{11}^{-1}\hat{A}_{11})\hat{P}_{12} + \sum_{i=1}^q \begin{bmatrix} \hat{\mathbf{N}}_{i,21} & \hat{N}_{i,22} \end{bmatrix}\begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{12}^\top & \hat{P}_{22} \end{bmatrix}\begin{bmatrix} \hat{N}_{i,21}^\top \\ \hat{N}_{i,22}^\top \end{bmatrix} \\ &+ \sum_{i=1}^q \begin{bmatrix} \hat{N}_{i,21} & \hat{N}_{i,22} \end{bmatrix}\begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{12}^\top & \hat{P}_{22} \end{bmatrix}\begin{bmatrix} -(\hat{P}_{12}^\top\hat{P}_{11}^{-1}\hat{N}_{i,11})^\top \\ 0 \end{bmatrix} = -\hat{R}. \end{aligned}$$

Using (22) for the above relation leads to

$$\hat{A}_{22}\hat{\mathbf{P}}_{22} + \hat{\mathbf{P}}_{22}\hat{A}_{22}^\top + \sum_{i=1}^q \begin{bmatrix} \hat{\mathbf{N}}_{i,21} & \hat{N}_{i,22} \end{bmatrix}\begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{12}^\top & \hat{P}_{22} \end{bmatrix}\begin{bmatrix} \hat{\mathbf{N}}_{i,21}^\top \\ \hat{N}_{i,22}^\top \end{bmatrix} = -\hat{R}.$$

We add and subtract $\sum_{i=1}^q \hat{N}_{i,22}\hat{\mathbf{P}}_{22}\hat{N}_{i,22}^\top$ resulting in

$$\hat{A}_{22}\hat{\mathbf{P}}_{22}+\hat{\mathbf{P}}_{22}\hat{A}_{22}^\top+\sum_{i=1}^q \hat{N}_{i,22}\hat{\mathbf{P}}_{22}\hat{N}_{i,22}^\top+\sum_{i=1}^q \begin{bmatrix}\hat{\mathbf{N}}_{i,21} & \hat{N}_{i,22}\end{bmatrix}\begin{bmatrix}\hat{P}_{11} & \hat{P}_{12}\\ \hat{P}_{12}^\top & \hat{P}_{12}^\top\hat{P}_{11}^{-1}\hat{P}_{12}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{N}}_{i,21}^\top\\ \hat{N}_{i,22}^\top\end{bmatrix}=-\hat{R}.$$

$\begin{bmatrix}\hat{P}_{11} & \hat{P}_{12}\\ \hat{P}_{12}^\top & \hat{P}_{12}^\top\hat{P}_{11}^{-1}\hat{P}_{12}\end{bmatrix}$ is positive semidefinite since it holds that

$$\begin{bmatrix}y^\top & z^\top\end{bmatrix}\begin{bmatrix}\hat{P}_{11} & \hat{P}_{12}\\ \hat{P}_{12}^\top & \hat{P}_{12}^\top\hat{P}_{11}^{-1}\hat{P}_{12}\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=y^\top\hat{P}_{11}y+2y^\top\hat{P}_{12}z+z^\top\hat{P}_{12}^\top\hat{P}_{11}^{-1}\hat{P}_{12}z=\left\|\hat{P}_{11}^{\frac{1}{2}}y+\hat{P}_{11}^{-\frac{1}{2}}\hat{P}_{12}z\right\|_2^2\geq 0,$$

where $\begin{bmatrix}y\\ z\end{bmatrix}$ is an arbitrary vector of suitable dimension. Therefore, we have

$$\begin{aligned}\hat{A}_{22}\hat{\mathbf{P}}_{22}+\hat{\mathbf{P}}_{22}\hat{A}_{22}^\top+\sum_{i=1}^q \hat{N}_{i,22}\hat{\mathbf{P}}_{22}\hat{N}_{i,22}^\top &= -\left(\hat{R}+\sum_{i=1}^q \begin{bmatrix}\hat{\mathbf{N}}_{i,21} & \hat{N}_{i,22}\end{bmatrix}\begin{bmatrix}\hat{P}_{11} & \hat{P}_{12}\\ \hat{P}_{12}^\top & \hat{P}_{12}^\top\hat{P}_{11}^{-1}\hat{P}_{12}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{N}}_{i,21}^\top\\ \hat{N}_{i,22}^\top\end{bmatrix}\right)\\ &\leq 0\end{aligned}\tag{25}$$

and $\hat{\mathbf{P}}_{22}>0$ since it is the inverse of the right lower block of $\begin{bmatrix}\hat{P}_{11} & \hat{P}_{12}\\ \hat{P}_{12}^\top & \hat{P}_{22}\end{bmatrix}^{-1}$. By Lemma 2.1, this implies $\lambda(I\otimes\hat{A}_{22}+\hat{A}_{22}\otimes I+\sum_{i=1}^q \hat{N}_{i,22}\otimes\hat{N}_{i,22})\subset\overline{\mathbb{C}_-}$. Let $\hat{\Phi}_2$ denote the fundamental solution of the system with matrices $(\hat{A}_{22},\hat{N}_{i,22})$. Moreover, we set $\hat{B}_2:=\hat{V}_2^\top B_1$. Then, we can express

$$\mathbb{E}\left\|\hat{\Phi}(t)B_1\right\|_F^2=\mathbb{E}\left\|\hat{S}^\top(\hat{S}\hat{\Phi}(t)\hat{S}^\top)\hat{S}B_1\right\|_F^2=\mathbb{E}\left\|(\hat{S}\hat{\Phi}(t)\hat{S}^\top)\begin{bmatrix}0\\ \hat{B}_2\end{bmatrix}\right\|_F^2.\tag{26}$$

We partition $\hat{S}\hat{\Phi}(t)\hat{S}^\top=\begin{bmatrix}\hat{\Phi}_{11}(t) & \hat{\Phi}_{12}(t)\\ \hat{\Phi}_{21}(t) & \hat{\Phi}_{22}(t)\end{bmatrix}$ and find the associated equation by multiplying the one for $\hat{\Phi}$ with $\hat{S}$ from the left and $\hat{S}^\top$ from the right resulting in

$$\begin{bmatrix}\hat{\Phi}_{11} & \hat{\Phi}_{12}\\ \hat{\Phi}_{21} & \hat{\Phi}_{22}\end{bmatrix}=\begin{bmatrix}I & 0\\ 0 & I\end{bmatrix}+\int_0^t\begin{bmatrix}\hat{A}_{11} & 0\\ \hat{A}_{21} & \hat{A}_{22}\end{bmatrix}\begin{bmatrix}\hat{\Phi}_{11} & \hat{\Phi}_{12}\\ \hat{\Phi}_{21} & \hat{\Phi}_{22}\end{bmatrix}ds+\sum_{i=1}^q\int_0^t\begin{bmatrix}\hat{N}_{i,11} & 0\\ \hat{N}_{i,21} & \hat{N}_{i,22}\end{bmatrix}\begin{bmatrix}\hat{\Phi}_{11} & \hat{\Phi}_{12}\\ \hat{\Phi}_{21} & \hat{\Phi}_{22}\end{bmatrix}dW_i(s).\tag{27}$$

Evaluating the right upper block and subsequently the right lower block of (27), we see that $\hat{\Phi}_{12}=0$ and $\hat{\Phi}_{22}=\hat{\Phi}_2$. Therefore, (26) becomes

$$\mathbb{E}\left\|\hat{\Phi}(t)B_1\right\|_F^2=\mathbb{E}\left\|\hat{\Phi}_2(t)\hat{B}_2\right\|_F^2.\tag{28}$$

In addition, we obtain the following rank relation

$$\begin{aligned}r_0 &:= \operatorname{rank}(\begin{bmatrix}B_1 & A_{11}B_1 & \dots & A_{11}^{r-1}B_1\end{bmatrix})=\operatorname{rank}(\begin{bmatrix}\hat{S}B_1 & (\hat{S}A_{11}\hat{S}^\top)\hat{S}B_1 & \dots & (\hat{S}A_{11}\hat{S}^\top)^{r-1}\hat{S}B\end{bmatrix})\\ &= \operatorname{rank}(\begin{bmatrix}\hat{B}_2 & \hat{A}_{22}\hat{B}_2 & \dots & \hat{A}_{22}^{r-1}\hat{B}_2\end{bmatrix})=\operatorname{rank}(\begin{bmatrix}\hat{B}_2 & \hat{A}_{22}\hat{B}_2 & \dots & \hat{A}_{22}^{r_2-1}\hat{B}_2\end{bmatrix}),\end{aligned}\tag{29}$$

where r_2 is the number of rows/columns of $\hat{A}_{22}$. If there is no zero eigenvalue of the Kronecker matrix associated to $(\hat{A}_{22},\hat{N}_{i,22})$, then $\hat{\Phi}_2$ decays exponentially and the claim of this theorem follows by (28). If the projected system still has a zero eigenvalue, then by Lemma 2.2, there is $\hat{V}_{22}\geq 0$, $\hat{V}_{22}\neq 0$, such that

$$\hat{A}_{22}^\top\hat{V}_{22}+\hat{V}_{22}\hat{A}_{22}+\sum_{i=1}^q\hat{N}_{i,22}^\top\hat{V}_{22}\hat{N}_{i,22}=0.$$

Now, one can further project down the reduced system with matrices $(\hat{A}_{22},\hat{B}_2,\hat{N}_{i,22})$ by the same type of state space transformation as in (18) based on the factor of the eigenvalue decomposition of $\hat{V}_{22}$ instead of $\hat{S}$ and based on (25) instead of (14). Notice that $\hat{V}_{22}$ cannot have full rank since else we have $\hat{B}_2=\hat{V}_2^\top B_1=0$ which, together with (17), implies $B_1=0$. One proceeds with this procedure until a mean square asymptotically stable subsystem is achieved. Such a subsystem exists since if one reaches a system of dimension r_0, then it holds that $r_2=r_0$ in (29). This local reachability condition combined with (25) is equivalent to mean square asymptotic stability, see [13], Theorem 3.6.1. Since (28) is then also obtained with the mean square asymptotically stable subsystem, the result follows which completes the proof. □

Remark 1. The only structure of the reduced system that was used in the proof of Theorem 4.4 is the existence of an equation of the form (14). Therefore, this theorem can be extended to any reduced system for which there exists a matrix $\hat{X}>0$ such that

$$\hat{A}\hat{X}+\hat{X}\hat{A}^\top+\sum_{i=1}^q\hat{N}_i\hat{X}\hat{N}_i^\top\leq-\hat{B}\hat{B}^\top.$$

The following implication of Theorem 4.4 shows the square mean asymptotic stability of the ROM (11) is preserved in some extended way.

Corollary 4.5. *Suppose that the reduced system (2) with matrices $\hat{A} = A_{11}$, $\hat{B} = B_1 \neq 0$ and $\hat{N}_i = N_{i,11}$ defined in (9), and associated to the fundamental solution $\hat{\Phi}$, is not mean square asymptotically stable. If we further have that $\Lambda_1 > 0$, then there exists $V_0 \in \mathbb{R}^{r\times r_0}$, $r_0 < r$, with $V_0^\top V_0 = I$ leading to a projected system $\hat{A}_0 = V_0^\top A_{11} V_0$, $\hat{B}_0 = V_0^\top B_1$ and $\hat{N}_{0,i} = V_0^\top N_{i,11} V_0$, associated to the mean square asymptotically stable fundamental solution $\hat{\Phi}_0$. Moreover, it holds that*

$$\hat{\Phi}(t)B_1 = V_0\hat{\Phi}_0(t)\hat{B}_0.$$

Proof. As in (26), we can write $\hat{\Phi}(t)B_1 = \hat{S}^\top(\hat{S}\hat{\Phi}(t)\hat{S}^\top)\hat{S}B_1$ with the orthogonal matrix $\hat{S}^\top = \begin{bmatrix}\hat{V}_1 & \hat{V}_2\end{bmatrix}$. Following the steps of the proof of Theorem 4.4, we see that $\hat{\Phi}(t)B_1 = \hat{V}_2\hat{\Phi}_2(t)\hat{B}_2$, where $\hat{B}_2 = \hat{V}_2^\top B_1$ and where $\hat{\Phi}_2$ is the fundamental solution for the system with matrices $\hat{V}_2^\top A_{11}\hat{V}_2$ and $\hat{V}_2^\top N_{i,11}\hat{V}_2$. If $\hat{\Phi}_2$ is mean square asymptotically stable, we have that $V_0 = \hat{V}_2$. Else, by the proof of Theorem 4.4, the projection procedure can be repeated until an mean square asymptotically stable subsystem is achieved. In this case, V_0 is the product of matrices like $\hat{V}_2$. □

Corollary 4.5 shows that the obtained ROM always has a mean square asymptotically stable realization. In other words, the procedure described in Section 3 produces a ROM that is either mean square asymptotically stable or that can be further reduced to a mean square asymptotically stable system without an additional approximation error given that $x_0 = 0$.

The following corollary will be useful for interpreting error bounds for the approximation error in Section 4.2.

Corollary 4.6. *Given the assumptions of Theorem 4.4, we have that*

$$\operatorname{tr}(\hat{P}) \leq \operatorname{tr}(\Lambda_1),$$

where $\hat{P}$ is the reachability Gramian of the reduced system with coefficients $\hat{A} = A_{11}$, $\hat{B} = B_1$ and $\hat{N}_i = N_{i,11}$.

Proof. As in the proof of Theorem 4.4, three cases need to be considered. Let us first assume that the reduced system is mean square asymptotically stable, i.e., $0 \notin \lambda(\hat{K})$. Subtracting (12) from (14) we see that $\Lambda_1 - \hat{P}$ satisfies

$$A_{11}(\Lambda_1 - \hat{P}) + (\Lambda_1 - \hat{P})A_{11}^\top + \sum_{i=1}^q N_{i,11}(\Lambda_1 - \hat{P})N_{i,11}^\top = -\sum_{i=1}^q N_{i,12}\Lambda_2 N_{i,12}^\top =: -R_2. \tag{30}$$

Now, Eq. (30) is uniquely solvable. According to Section 3.1, this solution is represented by $\mathbb{E}\int_0^\infty \hat{\Phi}(s)R_2\hat{\Phi}^\top(s)ds \geq 0$. Therefore, we have that $\Lambda_1 \geq \hat{P}$ implying the claim of this corollary. Now, let us study the case of $0 \in \lambda(\hat{K})$. $B_1 = 0$ implies that $\hat{P} = 0$ leading to $\Lambda_1 \geq \hat{P}$. It remains to consider the case of an unstable reduced system with $B_1 \neq 0$. We use the arguments of the proof of Theorem 4.4 and assume w.l.o.g. that the projected reduced system with matrices $(\hat{A}_{22}, \hat{B}_2, \hat{N}_{i,22})$ and fundamental solution $\hat{\Phi}_2$ is already mean square asymptotically stable. Else we could project down the reduced system further and the same arguments apply as the ones we use below. Integrating both sides of (28) over $[0, \infty)$ and using the definition of the Frobenius norm, we obtain

$$\operatorname{tr}\Bigg(\underbrace{\mathbb{E}\int_0^\infty \hat{\Phi}(t)B_1B_1^\top\hat{\Phi}^\top(t)dt}_{=\hat{P}}\Bigg) = \operatorname{tr}\Bigg(\underbrace{\mathbb{E}\int_0^\infty \hat{\Phi}_2(t)\hat{B}_2\hat{B}_2^\top\hat{\Phi}_2^\top(t)dt}_{=:\hat{P}_2}\Bigg).$$

Due to the mean square asymptotic stability, we know that $\hat{P}_2$ is the unique solution to

$$\hat{A}_{22}\hat{P}_2 + \hat{P}_2\hat{A}_{22}^\top + \sum_{i=1}^q \hat{N}_{i,22}\hat{P}_2\hat{N}_{i,22}^\top = -\hat{B}_2\hat{B}_2^\top.$$

Comparing this equation with (25), we find that $\hat{P}_2 \leq \mathbf{P}_{22} = \hat{P}_{22} - \hat{P}_{12}^\top\hat{P}_{11}^{-1}\hat{P}_{12} \leq \hat{P}_{22}$. We exploit (18) leading to $\operatorname{tr}(\hat{P}) \leq \operatorname{tr}(\hat{P}_{22}) \leq \operatorname{tr}(\hat{P}_{22}) + \operatorname{tr}(\hat{P}_{11}) = \operatorname{tr}(\hat{S}\Lambda_1\hat{S}^\top) = \operatorname{tr}(\Lambda_1)$. □

4.2. Error bounds

In this subsection, we derive error bounds for the model reduction procedure proposed in Section 3. We begin with an error bound that is general in the sense that it only requires the existence of the Gramians P and $\hat{P}$ and does not exploit any further structure of the reduced system. Once this general bound is established, an error estimate for the choice in (11) is given allowing to identify the scenarios in which this ROM leads to a good approximation. The next result characterizes the error in a full state approximation. Notice that we use similar techniques as in [7,28], where output errors were considered. However, we state the following proposition under milder assumptions.

Proposition 4.7. *Suppose that Φ denotes the fundamental solutions of (1), and $\hat{\Phi}$ denotes the fundamental solutions of (2) obtained by Galerkin projection using $V \in \mathbb{R}^{n\times r}$ with $V^\top V = I$. Moreover, let x and $\hat{x}$ represent the solutions to both systems. If there* *is a constant $c > 0$ such that $\mathbb{E}\|\Phi(t)B\|^2$, $\mathbb{E}\|\hat{\Phi}(t)\hat{B}\|^2 \lesssim \mathrm{e}^{-ct}$ and if $x_0 = 0$ and $\hat{x}_0 = 0$, we have*

$$\sup_{t\in[0,T]} \mathbb{E}\left\|x(t) - V\hat{x}(t)\right\|_2 \leq \left(\mathrm{tr}(P) + \mathrm{tr}(\hat{P}) - 2\mathrm{tr}(P_2 V^\top)\right)^{\frac{1}{2}} \|u\|_{L_T^2},$$

where the matrices $P := \mathbb{E}\int_0^\infty \Phi(s)BB^\top\Phi^\top(s)ds$, $\hat{P} := \mathbb{E}\int_0^\infty \hat{\Phi}(s)\hat{B}\hat{B}^\top\hat{\Phi}^\top(s)ds$, $P_2 := \mathbb{E}\int_0^\infty \Phi(s)B\hat{B}^\top\hat{\Phi}^\top(s)ds$ satisfy

$$AP + PA^\top + \sum_{i=1}^q N_i P N_i^\top = -BB^\top, \tag{31a}$$

$$\hat{A}\hat{P} + \hat{P}\hat{A}^\top + \sum_{i=1}^q \hat{N}_i \hat{P} \hat{N}_i^\top = -\hat{B}\hat{B}^\top, \tag{31b}$$

$$AP_2 + P_2\hat{A}^\top + \sum_{i=1}^q N_i P_2 \hat{N}_i^\top = -B\hat{B}^\top. \tag{31c}$$

Proof. It can be shown that the solution of (1) is given by

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t,s)Bu(s)ds,$$

see, e.g., [13]. Setting the initial states in (1) and (2) equal to zero and using the solution representations for both systems, we obtain by the triangle inequality that

$$\begin{aligned}\mathbb{E}\left\|x(t) - V\hat{x}(t)\right\|_2 &\leq \mathbb{E}\int_0^t \left\|\left(\Phi(t,s)B - V\hat{\Phi}(t,s)\hat{B}\right)u(s)\right\|_2 ds\\ &\leq \mathbb{E}\int_0^t \left\|\Phi(t,s)B - V\hat{\Phi}(t,s)\hat{B}\right\|_F \|u(s)\|_2 ds.\end{aligned}$$

We apply the inequality of Cauchy-Schwarz and obtain

$$\mathbb{E}\left\|x(t) - V\hat{x}(t)\right\|_2 \leq \left(\mathbb{E}\int_0^t \left\|\Phi(t,s)B - V\hat{\Phi}(t,s)\hat{B}\right\|_F^2 ds\right)^{\frac{1}{2}} \|u\|_{L_t^2}.$$

The definition of the Frobenius norm and properties of the trace operator yield

$$\begin{aligned}\mathbb{E}\left\|\Phi(t,s)B - V\hat{\Phi}(t,s)\hat{B}\right\|_F^2 &= \mathrm{tr}\left(\mathbb{E}\left[\Phi(t,s)BB^\top\Phi^\top(t,s)\right]\right)\\ &\quad + \mathrm{tr}\left(V\mathbb{E}\left[\hat{\Phi}(t,s)\hat{B}\hat{B}^\top\hat{\Phi}^\top(t,s)V^\top\right]\right)\\ &\quad - 2\mathrm{tr}\left(\mathbb{E}\left[\Phi(t,s)B\hat{B}^\top\hat{\Phi}^\top(t,s)\right]V^\top\right).\end{aligned}$$

Using Corollary A.2, $\Phi(t,s)$ and $\hat{\Phi}(t,s)$ can be replaced by $\Phi(t-s)$ and $\hat{\Phi}(t-s)$ above. Writing the resulting trace expressions by the Frobenius norm again, we obtain

$$\begin{aligned}\mathbb{E}\left\|x(t) - V\hat{x}(t)\right\|_2 &\leq \left(\mathbb{E}\int_0^t \left\|\Phi(t-s)B - V\hat{\Phi}(t-s)\hat{B}\right\|_F^2 ds\right)^{\frac{1}{2}} \|u\|_{L_t^2}\\ &= \left(\mathbb{E}\int_0^t \left\|\Phi(s)B - V\hat{\Phi}(s)\hat{B}\right\|_F^2 ds\right)^{\frac{1}{2}} \|u\|_{L_t^2}\\ &\leq \left(\mathbb{E}\int_0^\infty \left\|\Phi(s)B - V\hat{\Phi}(s)\hat{B}\right\|_F^2 ds\right)^{\frac{1}{2}} \|u\|_{L_t^2}.\end{aligned}$$

The infinite integral above exists due to the exponential decay of ΦB and $\hat{\Phi}\hat{B}$. Taking the supremum over $[0,T]$, inserting the definition of the Frobenius norm and exploiting that $V^\top V = I$, we obtain

$$\sup_{t\in[0,T]} \mathbb{E}\left\|x(t) - V\hat{x}(t)\right\|_2 \leq \left(\mathrm{tr}(P) + \mathrm{tr}(\hat{P}) - 2\mathrm{tr}(P_2 V^\top)\right)^{\frac{1}{2}} \|u\|_{L_T^2}.$$

The infinite integrals P, $\hat{P}$ and P_2 satisfy (31) due to Lemma A.1 using the exponential decay of ΦB and $\hat{\Phi}\hat{B}$. □

Remark 2. Under the assumptions of Proposition 4.7, the solutions of (31) are not necessarily unique as Example 4.3 shows. Uniqueness can be ensured if we further have that Φ and $\hat{\Phi}$ decay exponentially in the mean square sense.

Based on the result in Proposition 4.7, we now find an error bound for the reduced system introduced in Section 3.2. Output error bounds for balanced truncation in the same norm based on different choices of Gramians are proved in [7,27].

The error analysis for the scheme in Section 3.2 is more challenging since less structure than in the case of balanced truncation can be exploited which is a method where the reachability and observability Gramian are both diagonal and equal (after a balancing transformation). Moreover, in contrast to balanced truncation, we need to discuss the case in which mean square asymptotic stability is not preserved.

Theorem 4.8. *Let x be the solution to the mean square asymptotically stable system (1) and $\hat{x}$ the solution to (2) with zero initial states and with $\hat{A} = A_{11}$, $\hat{B} = B_1$, $\hat{N}_i = N_{i,11}$ being submatrices of the balanced partition in (9). Let $\Lambda = \operatorname{diag}(\Lambda_1, \Lambda_2)$ be the matrix of ordered eigenvalues of the reachability Gramian P with $\Lambda_1 = \operatorname{diag}(\lambda_1, \dots, \lambda_r) > 0$. Let $S^\top = \begin{bmatrix} V & S_2 \end{bmatrix}$ denote the factor of the associated eigenvalue decomposition of P. Then, it holds that*

$$\sup_{t\in[0,T]} \mathbb{E}\left\|x(t) - V\hat{x}(t)\right\|_2 \leq \left(\operatorname{tr}(\hat{P} - \Lambda_1) + \operatorname{tr}(\Lambda_2 \mathcal{W}_0)\right)^{\frac{1}{2}} \|u\|_{L^2_T}, \tag{32}$$

where $\hat{P}$ is the reduced reachability Gramian and

$$\mathcal{W}_0 = I + 2A_{12}^\top Y_2 + \sum_{i=1}^{q} N_{i,12}^\top \left(2Y \begin{bmatrix} N_{i,12} \\ N_{i,22} \end{bmatrix}\right).$$

The matrix $Y = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix}$ is defined as the unique solution to

$$A_{11}^\top Y + Y A_b + \sum_{i=1}^{q} N_{i,11}^\top Y N_{i,b} = -(SV)^\top = -\begin{bmatrix} I & 0 \end{bmatrix}. \tag{33}$$

If it moreover holds that $0 \notin \lambda(I \otimes A_{11} + A_{11} \otimes I + \sum_{i=1}^{q} N_{i,11} \otimes N_{i,11})$, then $\hat{Q}$ can be introduced as the positive semidefinite solution to

$$A_{11}^\top \hat{Q} + \hat{Q} A_{11} + \sum_{i=1}^{q} N_{i,11}^\top \hat{Q} N_{i,11} = -I. \tag{34}$$

Hence, the error bound becomes

$$\sup_{t\in[0,T]} \mathbb{E}\left\|x(t) - V\hat{x}(t)\right\|_2 \leq \left(\operatorname{tr}(\Lambda_2 \mathcal{W})\right)^{\frac{1}{2}} \|u\|_{L^2_T},$$

where the weight is

$$\mathcal{W} = I + 2A_{12}^\top Y_2 + \sum_{i=1}^{q} N_{i,12}^\top \left(2Y \begin{bmatrix} N_{i,12} \\ N_{i,22} \end{bmatrix} - \hat{Q} N_{i,12}\right).$$

Proof. Since the original model is asymptotically mean square stable and due to Theorem 4.4, the assumptions of Proposition 4.7 are met such that we have

$$\sup_{t\in[0,T]} \mathbb{E}\left\|x(t) - V\hat{x}(t)\right\|_2 \leq \left(\operatorname{tr}(P) + \operatorname{tr}(\hat{P}) - 2\operatorname{tr}(P_2 V^\top)\right)^{\frac{1}{2}} \|u\|_{L^2_T}.$$

Notice that P uniquely solves (31a). Since the ROM is mean square stable by Proposition 4.1 and due to Lemma 2.3 P_2 is also the unique solution to (31c). However, there can still be infinitely many other solutions to (31b) besides $\hat{P}$. Using the balanced realization in (9), the error bound then becomes

$$\sup_{t\in[0,T]} \mathbb{E}\left\|x(t) - V\hat{x}(t)\right\|_2 \leq \left(\operatorname{tr}(\Lambda) + \operatorname{tr}(\hat{P}) - 2\operatorname{tr}(S^\top X V^\top)\right)^{\frac{1}{2}} \|u\|_{L^2_T}, \tag{35}$$

where Λ and $X = SP_2$ uniquely solve

$$A_b \Lambda + \Lambda A_b^\top + \sum_{i=1}^{q} N_{i,b} \Lambda N_{i,b}^\top = -B_b B_b^\top, \tag{36}$$

$$A_b X + X A_{11}^\top + \sum_{i=1}^{q} N_{i,b} X N_{i,11}^\top = -B_b B_1^\top. \tag{37}$$

By Lemma 2.3, there is a unique solution to (33) which we can use to rewrite $\operatorname{tr}(S^\top X V^\top) = \operatorname{tr}(Y B_b B_1^\top)$. Based on the partition (9), we evaluate the first r columns of (36) and obtain

$$\begin{aligned} -B_b B_1^\top &= A_b \begin{bmatrix} \Lambda_1 \\ 0 \end{bmatrix} + \Lambda \begin{bmatrix} A_{11}^\top \\ A_{12}^\top \end{bmatrix} + \sum_{i=1}^{q} N_{i,b} \Lambda \begin{bmatrix} N_{i,11}^\top \\ N_{i,12}^\top \end{bmatrix} \\ &= \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} \Lambda_1 + \begin{bmatrix} \Lambda_1 A_{11}^\top \\ \Lambda_2 A_{12}^\top \end{bmatrix} + \sum_{i=1}^{q} \left(\begin{bmatrix} N_{i,11} \\ N_{i,21} \end{bmatrix} \Lambda_1 N_{i,11}^\top + \begin{bmatrix} N_{i,12} \\ N_{i,22} \end{bmatrix} \Lambda_2 N_{i,12}^\top \right). \end{aligned}$$

Inserting this into $\mathrm{tr}(YB_bB_1^\top)$ yields

$$-\mathrm{tr}(S^\top XV^\top) = \mathrm{tr}\left(Y\left[\begin{bmatrix}A_{11}\\A_{21}\end{bmatrix}\Lambda_1 + \begin{bmatrix}\Lambda_1A_{11}^\top\\\Lambda_2A_{12}^\top\end{bmatrix} + \sum_{i=1}^q\left(\begin{bmatrix}N_{i,11}\\N_{i,21}\end{bmatrix}\Lambda_1N_{i,11}^\top + \begin{bmatrix}N_{i,12}\\N_{i,22}\end{bmatrix}\Lambda_2N_{i,12}^\top\right)\right]\right)$$
$$= \mathrm{tr}\left(\Lambda_1\left[Y\begin{bmatrix}A_{11}\\A_{21}\end{bmatrix} + A_{11}^\top Y_1 + \sum_{i=1}^q N_{i,11}^\top Y\begin{bmatrix}N_{i,11}\\N_{i,21}\end{bmatrix}\right]\right)$$
$$+\,\mathrm{tr}\left(\Lambda_2\left[A_{12}^\top Y_2 + \sum_{i=1}^q N_{i,12}^\top Y\begin{bmatrix}N_{i,12}\\N_{i,22}\end{bmatrix}\right]\right).$$

The first r columns of (33) yield

$$-\mathrm{tr}(S^\top XV^\top) = -\mathrm{tr}(\Lambda_1) + \mathrm{tr}\left(\Lambda_2\left[A_{12}^\top Y_2 + \sum_{i=1}^q N_{i,12}^\top Y\begin{bmatrix}N_{i,12}\\N_{i,22}\end{bmatrix}\right]\right).$$

Inserting this into the bound in (35) leads to

$$\mathrm{tr}(\Lambda) + \mathrm{tr}(\hat{P}) - 2\mathrm{tr}(S^\top XV^\top) = \mathrm{tr}(\hat{P}-\Lambda_1) + \mathrm{tr}\left(\Lambda_2\left[I + 2A_{12}^\top Y_2 + 2\sum_{i=1}^q N_{i,12}^\top Y\begin{bmatrix}N_{i,12}\\N_{i,22}\end{bmatrix}\right]\right),$$

which proves (32).

Now let us consider the case where $0 \notin \lambda(I\otimes A_{11} + A_{11}\otimes I + \sum_{i=1}^q N_{i,11}\otimes N_{i,11})$, i.e., the reduced system is mean square asymptotically stable by Proposition 4.1 and Lemma 2.2. Therefore, (34) has a unique positive semidefinite solution $\hat{Q}$. Subtracting the left upper $r\times r$ block of (36) from (31b), we find

$$A_{11}(\hat{P}-\Lambda_1) + (\hat{P}-\Lambda_1)A_{11}^\top + \sum_{i=1}^q N_{i,11}(\hat{P}-\Lambda_1)N_{i,11}^\top = \sum_{i=1}^q N_{i,12}\Lambda_2N_{i,12}^\top.$$

Hence, we have

$$\mathrm{tr}(\hat{P}-\Lambda_1) = -\mathrm{tr}\left(\left[A_{11}^\top\hat{Q} + \hat{Q}A_{11} + \sum_{i=1}^q N_{i,11}^\top\hat{Q}N_{i,11}\right](\hat{P}-\Lambda_1)\right)$$
$$= -\mathrm{tr}\left(\hat{Q}\left[A_{11}(\hat{P}-\Lambda_1) + (\hat{P}-\Lambda_1)A_{11}^\top + \sum_{i=1}^q N_{i,11}(\hat{P}-\Lambda_1)N_{i,11}^\top\right]\right)$$
$$= -\mathrm{tr}\left(\Lambda_2\sum_{i=1}^q N_{i,12}^\top\hat{Q}N_{i,12}\right),$$

which concludes the proof of this theorem. □

Theorem 4.8 is a vital since it shows the relation between the truncated eigenvalues contained in Λ_2 and the error of the model reduction procedure. By Corollary 4.6, we know that $\mathrm{tr}(\hat{P}-\Lambda_1) \le 0$ and therefore (32) shows that the error between x and $V\hat{x}$ is small if Λ_2 has small diagonal entries. Consequently, the reduced system is accurate if only the small eigenvalues of P are neglected. Moreover, this tells us that the reduced order dimension r can be chosen based on the eigenvalues of P since their order is a good indicator for the error. Certainly, the error bound representation in Proposition 4.7 is more suitable for practical computations than the one in Theorem 4.8. This is because one only needs to solve for $\hat{P}$ and P_2 satisfying (31b) and (31c) in addition to the Gramian P which is already computed within the model reduction procedure.

5. Full state approximation for general initial conditions

In different applications, it is required to apply MOR to stochastic systems with non-zero initial conditions, see, e.g., [2] for a MOR approach in the context of efficiently solving stochastic optimal control problems. So far, the case of $x_0 = 0$ has only been considered here. However, the above results can be transferred to a scenario of general initial states, since the reduction of the control part with $x_0 = 0$, represented by (1), can be separated from the reduction of the subsystem involving a the non-zero initial data. To illustrate this, let $x(t, x_0, u)$ denote the solution to (1) with general $x_0 = X_0v_0$. Here, we allow investigating several initial states at the same time which are spanned by the columns of a matrix X_0. The matrix/vector multiplication X_0v_0 then expresses the respective linear combination of these columns. We can now see that $x(t, x_0, u) =$

$x(t)+x_{x_0}(t)$, where $x(t)=x(t,0,u)$ is the solution to (1) with $x(0)=0$ and $x_{x_0}(t)=x(t,x_0,0)$ is the homogeneous part of the equation involving the initial state, i.e.,

$$dx_{x_0}(t)=Ax_{x_0}(t)dt+\sum_{i=1}^{q}N_i x_{x_0}(t)dW_i(t),\quad x_{x_0}(0)=x_0=X_0v_0,\quad t\geq 0. \tag{38}$$

Consequently, a reduced order approximation can be found by $x(t,x_0,u)\approx\hat{x}(t,x_0,u):=V\hat{x}(t)+V_{x_0}\hat{x}_{x_0}(t)$, where $\hat{x}$ is the reduced state associated to x (see Section 3.2) and $\hat{x}_{x_0}$ is a low dimensional approximation of x_{x_0}. How V_{x_0} and $\hat{x}_{x_0}$ are constructed will be discussed in the following and relies on the same ideas used in the previous sections. A modification of the Gramian and the error norm then allows to establish similar results as proved in Section 4.

Let us introduce the Gramian

$$P_{x_0}:=\mathbb{E}\int_0^\infty \Phi(s)X_0X_0^\top\Phi^\top(s)ds$$

for (38) which, by Section 3.1, satisfies

$$AP_{x_0}+P_{x_0}A^\top+\sum_{i=1}^{q}N_iP_{x_0}N_i^\top=-X_0X_0^\top. \tag{39}$$

Given an orthonormal basis $(p_{x_0,k})_{k=1,\dots,n}$ of eigenvectors of P_{x_0} with associated eigenvalues $(\lambda_{x_0,k})_{k=1,\dots,n}$, we have

$$\begin{aligned}\int_0^T\mathbb{E}\left|\langle x_{x_0}(t),p_{x_0,k}\rangle_2\right|^2dt&=\int_0^T\mathbb{E}\left|\langle\Phi(t)X_0v_0,p_{x_0,k}\rangle_2\right|^2dt\\&=\int_0^T\mathbb{E}\left|\langle v_0,X_0^\top\Phi^\top(t)p_{x_0,k}\rangle_2\right|^2dt\\&\leq p_{x_0,k}^\top\int_0^\infty\mathbb{E}\left[\Phi(t)X_0X_0^\top\Phi^\top(t)\right]dt\;p_{x_0,k}\|v_0\|_2^2=\lambda_{x_0,k}\|v_0\|_2^2,\end{aligned} \tag{40}$$

exploiting that $x_{x_0}(t)=\Phi(t)x_0$ and using the inequality of Cauchy-Schwarz. Inequality (40) shows that eigenspaces corresponding to small $\lambda_{x_0,k}$ are less relevant in the system dynamics motivating the same type of ROM like in Section 3.2. Let us introduce the eigenvalue decomposition of the Gramian

$$P_{x_0}=S_{x_0}^\top\Lambda_{x_0}S_{x_0},\quad S_{x_0}^\top=\begin{bmatrix}V_{x_0}&\star\end{bmatrix},\quad\Lambda_{x_0}=\begin{bmatrix}\Lambda_{x_0,1}&\\&\Lambda_{x_0,2}\end{bmatrix}$$

with $V_{x_0}\in\mathbb{R}^{n\times r_{x_0}}$, $\Lambda_{x_0,1}\in\mathbb{R}^{r_{x_0}\times r_{x_0}}$ and with $\Lambda_{x_0,2}\in\mathbb{R}^{(n-r_{x_0})\times(n-r_{x_0})}$ being the matrix of small eigenvalues of P_{x_0}. Now, we find a good approximation of x_{x_0} by $V_{x_0}\hat{x}_{x_0}$, where $\hat{x}_{x_0}$ is the r_{x_0}-dimensional solution to

$$d\hat{x}_{x_0}(t)=\hat{A}_{x_0}\hat{x}_{x_0}(t)dt+\sum_{i=1}^{q}\hat{N}_{x_0,i}\hat{x}_{x_0}(t)dW_i(t),\quad\hat{x}_{x_0}(0)=\hat{X}_0v_0,\quad t\geq 0, \tag{41}$$

where the ingredients of (41) are given by

$$\hat{A}_{x_0}=A_{11}^{(x_0)}=V_{x_0}^\top AV_{x_0},\quad\hat{X}_0=X_{0,1}=V_{x_0}^\top X_0,\quad\hat{N}_{x_0,i}=N_{i,11}^{(x_0)}=V_{x_0}^\top N_iV_{x_0}. \tag{42}$$

The reduced order matrices in (42) result from the transformed coefficients

$$S_{x_0}AS_{x_0}^\top=\begin{bmatrix}A_{11}^{(x_0)}&\star\\\star&\star\end{bmatrix},\quad S_{x_0}X_0=\begin{bmatrix}X_{0,1}\\\star\end{bmatrix},\quad S_{x_0}N_iS_{x_0}^\top=\begin{bmatrix}N_{i,11}^{(x_0)}&N_{i,12}^{(x_0)}\\\star&\star\end{bmatrix}. \tag{43}$$

Now, we are able to conduct a stability and error analysis for (38). We start the preservation of stability as a consequence of Theorem 4.4.

Theorem 5.1. *Given the solution $\hat{x}_{x_0}$ to the reduced system (41) with matrices $\hat{A}_{x_0}=A_{11}^{(x_0)}$, $\hat{N}_{x_0,i}=N_{i,11}^{(x_0)}$ and initial conditions spanned by the columns of $\hat{X}_0=X_{0,1}$. If $\Lambda_{x_0,1}>0$, then there is a constant $c>0$ such that $\mathbb{E}\left\|\hat{x}_{x_0}(t)\right\|_2^2\lesssim e^{-ct}$.*

Proof. With the arguments of Section 3.2, it is known that the realization in (43) has the Gramian Λ_{x_0}, i.e.,

$$\begin{aligned}-\begin{bmatrix}X_{0,1}X_{0,1}^\top&\star\\\star&\star\end{bmatrix}=&\begin{bmatrix}A_{11}^{(x_0)}&\star\\\star&\star\end{bmatrix}\begin{bmatrix}\Lambda_{x_0,1}&\\&\Lambda_{x_0,2}\end{bmatrix}+\begin{bmatrix}\Lambda_{x_0,1}&\\&\Lambda_{x_0,2}\end{bmatrix}\begin{bmatrix}A_{11}^{(x_0)\top}&\star\\\star&\star\end{bmatrix}\\&+\sum_{i=1}^{q}\begin{bmatrix}N_{i,11}^{(x_0)}&N_{i,12}^{(x_0)}\\+\star&\star\end{bmatrix}\begin{bmatrix}\Lambda_{x_0,1}&\\&\Lambda_{x_0,2}\end{bmatrix}\begin{bmatrix}N_{i,11}^{(x_0)\top}&\star\\N_{i,12}^{(x_0)\top}&\star\end{bmatrix}.\end{aligned}$$

The left upper block of this equation yields

$$A_{11}^{(x_0)}\Lambda_{x_0,1}+\Lambda_{x_0,1}A_{11}^{(x_0)\top}+\sum_{i=1}^{q}N_{i,11}^{(x_0)}\Lambda_{x_0,1}N_{i,11}^{(x_0)\top}=-X_{0,1}X_{0,1}^{\top}-\sum_{i=1}^{q}N_{i,12}^{(x_0)}\Lambda_{x_0,2}N_{i,12}^{(x_0)\top}.$$

This is the same type of equation like in (14) allowing to conduct the proof with exactly the same arguments as in Theorem 4.4. Therefore, we know that $\mathbb{E}\left\|\hat{\Phi}(t)X_{0,1}\right\|_F^2\lesssim \mathrm{e}^{-ct}$, where $\hat{\Phi}$ is the fundamental solution to the ROM. This implies the result of this theorem. □

We can see that Theorem 5.1 ensures mean square asymptotic stability of the ROM at least for the initial conditions of interest. This result is also essential for the existence of the reduced order Gramian needed for the error bound that follows next. We now analyze the error between (38) and (41). Following the steps of the proof in Proposition 4.7, we have

$$\begin{aligned}\int_0^T\mathbb{E}\left\|x_{x_0}(t)-V_{x_0}\hat{x}_{x_0}(t)\right\|_2^2dt&=\int_0^T\mathbb{E}\left\|\left(\Phi(t)X_0-V_{x_0}\hat{\Phi}(t)\hat{X}_0\right)v_0\right\|_2^2dt\\&\le\int_0^T\mathbb{E}\left\|\Phi(t)X_0-V_{x_0}\hat{\Phi}(t)\hat{X}_0\right\|_F^2dt\,\|v_0\|_2^2\\&\le\left(\mathrm{tr}(P_{x_0})+\mathrm{tr}(\hat{P}_{x_0})-2\mathrm{tr}(P_{x_0,2}V_{x_0}^{\top})\right)\|v_0\|_2^2,\end{aligned}\tag{44}$$

where $\hat{P}_{x_0}:=\mathbb{E}\int_0^\infty\hat{\Phi}(s)X_{0,1}X_{0,1}^{\top}\hat{\Phi}^{\top}(s)ds$ and $P_{x_0,2}:=\mathbb{E}\int_0^\infty\Phi(s)X_0X_{0,1}^{\top}\hat{\Phi}^{\top}(s)ds$. Based on Theorem 4.8, we can express (44) using the matrix of truncated eigenvalues $\Lambda_{x_0,2}$ of P_{x_0}. Consequently, there is a matrix $\mathcal{W}_{x_0}$ such that

$$\int_0^T\mathbb{E}\left\|x_{x_0}(t)-V_{x_0}\hat{x}_{x_0}(t)\right\|_2^2dt\le\mathrm{tr}\left(\Lambda_{x_0,2}\mathcal{W}_{x_0}\right)\|v_0\|_2^2.\tag{45}$$

This indicates that the truncated eigenvalues of P_{x_0} determine the error in approximating x_{x_0} by $V_{x_0}\hat{x}_{x_0}$. Therefore, the reduced dimension r_{x_0} is supposed to be chosen such that $\Lambda_{x_0,2}$ has small entries only. This then leads to a small error according to (45).

6. Full state approximation for bilinear systems

Besides the above extension to arbitrary initial states in stochastic systems, a discussion of the proposed results for the class of bilinear systems follows. We consider the Galerkin projection based model reduction scheme that was studied in Section 3.2 for deterministic bilinear dynamical systems governed by

$$\dot{z}(t)=Az(t)+Bu(t)+\sum_{i=1}^{m}N_iz(t)u_i(t),\quad t\ge 0.\tag{46}$$

Roughly speaking, (46) is obtained by replacing the white noise processes $\frac{dW_i}{dt}$ in (1) ($q=m$) by the ith component u_i of the control vector $u\in L_T^2$, which we assume henceforth to be deterministic. Transferring the results from the linear stochastic to the deterministic bilinear case is not trivial, since from the theoretical point of view (46) and (1) are very different, since white noise is not a function. However, due to the recently shown relation between stochastic and bilinear systems in [26], we are able to establish the results of the previous sections for (46) in a similar manner. Let us assume that the matrix A is Hurwitz, i.e., $\lambda(A)\subset\mathbb{C}_-$. Writing the solution $z=z(\cdot,z_0,B)$ to (46) dependent on the initial state z_0 and the input matrix B, $\lambda(A)\subset\mathbb{C}_-$ implies

$$\|z(t,z_0,0)\|_2\lesssim \mathrm{e}^{-ct},\quad c>0,$$

for all $z_0\in\mathbb{R}^n$ if $\int_0^\infty\|u(s)\|_2^2ds<\infty$, i.e., the homogeneous equation is asymptotically stable with exponential decay, see [26]. If N_i for all $i=1,\dots,m$ is sufficiently small, A being Hurwitz implies mean square asymptotic stability in the sense of (4). This can be, e.g., seen by the sufficient condition for (4) in [13], Corollary 3.6.3, see also [32]. We can now control the matrices N_i by recalling (46) with $\gamma>0$ resulting in

$$\dot{z}(t)=Az(t)+[\frac{1}{\gamma}B][\gamma u(t)]+\sum_{i=1}^{m}[\frac{1}{\gamma}N_i]z(t)[\gamma u_i(t)],\tag{47}$$

compare also with [3,12], where this technique has also been used. If γ is sufficiently large, (4) can be guaranteed for the pair $(A,\frac{1}{\gamma}N_i)$ which provides the existence of a unique solution to

$$AP_\gamma+P_\gamma A^{\top}+\frac{1}{\gamma^2}\sum_{i=1}^{m}N_iP_\gamma N_i^{\top}=-\frac{1}{\gamma^2}BB^{\top}.\tag{48}$$

According to Section 3.1, P_γ is the reachability Gramian of the stochastic system (1) with coefficients $(A,\frac{1}{\gamma}B,\frac{1}{\gamma}N_i)$. Choosing P_γ for $\gamma=1$ as a reachability Gramian in the context of model reduction for bilinear systems was first proposed in [1] and, e.g., further investigated in [3]. By [26] we know that P_γ takes a similar role as in the stochastic case (compare with (7)), i.e., it characterizes redundant information in (46) by

$$\sup_{t\in[0,T]} \left|\langle z(t,0,B), p_{\gamma,k}\rangle_2\right| \leq \lambda_{\gamma,k}^{\frac{1}{2}} \exp\left\{0.5\gamma^2 \left\|u^0\right\|_{L_T^2}^2\right\}\gamma \left\|u\right\|_{L_T^2}, \tag{49}$$

where $(p_{\gamma,k})$ is an orthonormal basis of eigenvector of P_γ with associated eigenvalues $(\lambda_{\gamma,k})$ and u^0 the vector of controls entering the bilinear part of the equation, i.e.,

$$u^0 = (u_1^0\ u_2^0\ \dots\ u_m^0)^\top \quad \text{with} \quad u_i^0 \equiv \begin{cases} 0, & \text{if } N_i = 0 \\ u_i, & \text{else.} \end{cases} \tag{50}$$

Therefore, the eigenspaces of P_γ corresponding to the zero eigenvalues are irrelevant for the system dynamics. Moreover, assuming that the control energy is sufficiently small, (49) tells us that $z(\cdot,0,B)$ is small in the direction of $p_{\gamma,k}$ if $\lambda_{\gamma,k}$ is small. Therefore, these eigenspaces can also be seen as less relevant in (46) and can hence be removed leading to ROMs. A somehow different way of characterizing unimportant states in a bilinear equation was discussed in [3,16], where local estimates for the reachability energy based on P_γ, $\gamma = 1$ have been shown.

Remark 3. So far, we observed some essential differences between stochastic and bilinear systems. System (46) only requires A to be Hurwitz instead of (4). On the other hand, we consider a family of Gramians for the bilinear case depending on γ rather than a fixed Gramian. Although the characterization of irrelevant states are similar in both cases, the exponential in (49) indicates that we need a certain smallness assumption on u^0 and γ in order to make our arguments valid.

The above considerations motivate to conduct the same reduced order modeling procedure as explained in Section 3.2. We introduce the eigenvalue decomposition of

$$P_\gamma = S_\gamma^\top \begin{bmatrix} \Lambda_{\gamma,1} & 0 \\ 0 & \Lambda_{\gamma,2} \end{bmatrix} S_\gamma,$$

where $\Lambda_{\gamma,1} > 0$ contains the large and $\Lambda_{\gamma,2}$ the small ordered eigenvalues of P_γ. Using the partition

$$S_\gamma A S_\gamma^\top = \begin{bmatrix} A_{11}^{(\gamma)} & A_{12}^{(\gamma)} \\ A_{21}^{(\gamma)} & A_{22}^{(\gamma)} \end{bmatrix}, \quad S_\gamma B = \begin{bmatrix} B_1^{(\gamma)} \\ B_2^{(\gamma)} \end{bmatrix}, \quad S_\gamma N_i S_\gamma^\top = \begin{bmatrix} N_{i,11}^{(\gamma)} & N_{i,12}^{(\gamma)} \\ N_{i,21}^{(\gamma)} & N_{i,22}^{(\gamma)} \end{bmatrix} \tag{51}$$

the eigenvectors associated to small eigenvalues of P_γ are then truncated, resulting in the reduced model

$$\dot{\hat{z}}_\gamma(t) = A_{11}^{(\gamma)}\hat{z}_\gamma(t) + B_1^{(\gamma)}u(t) + \sum_{i=1}^m N_{i,11}^{(\gamma)}\hat{z}_\gamma(t)u_i(t), \quad t \geq 0. \tag{52}$$

The properties of (52) can now be immediately transferred from the considerations in the stochastic case. By Proposition 4.1, we have

$$\lambda(I\otimes A_{11}^{(\gamma)} + A_{11}^{(\gamma)}\otimes I + \frac{1}{\gamma^2}\sum_{i=1}^m N_{i,11}^{(\gamma)}\otimes N_{i,11}^{(\gamma)}) \subset \overline{\mathbb{C}_-}. \tag{53}$$

Example 4.2 shows that eigenvalues on the imaginary axis can occur in (53), but they can be excluded by Lemma 2.2 if $0 \notin \lambda(I\otimes A_{11}^{(\gamma)} + A_{11}^{(\gamma)}\otimes I + \frac{1}{\gamma^2}\sum_{i=1}^m N_{i,11}^{(\gamma)}\otimes N_{i,11}^{(\gamma)})$. However, even though there is a zero eigenvalue, the existence of the reduced order Gramian can be guaranteed using the arguments of Theorem 4.4.

In order to keep the discussion around the error bound for bilinear systems short, we do not discuss the scenario of a zero eigenvalue (53) in detail. Therefore, let us exclude this case below. We can now transfer the result of Proposition 4.7 to the bilinear case by the results of [26].

Proposition 6.1. *Let z be the solution to (46) with $\lambda(A)\subset\mathbb{C}_-$ and let $\hat{z}_\gamma$ represent the solution to (52). Moreover, let $\gamma > 0$ such that*

$$\lambda(I\otimes A + A\otimes I + \frac{1}{\gamma^2}\sum_{i=1}^m N_i\otimes N_i) \subset \mathbb{C}_-$$

and that the ROM coefficients satisfy

$$0 \notin \lambda\left(I\otimes A_{11}^{(\gamma)} + A_{11}^{(\gamma)}\otimes I + \frac{1}{\gamma^2}\sum_{i=1}^m N_{i,11}^{(\gamma)}\otimes N_{i,11}^{(\gamma)}\right).$$

Given zero initial states to both equations and $V_\gamma \in \mathbb{R}^{n\times r}$ being the first r columns of the factor $S_\gamma^\top$ of the eigenvalue decomposition of P_γ (unique solution to (48)), we have

$$\sup_{t\in[0,T]} \left\|z(t) - V_\gamma\hat{z}_\gamma(t)\right\|_2 \leq \left(\text{tr}(P_\gamma) + \text{tr}(\hat{P}_\gamma) - 2\text{tr}(P_{\gamma,2}V_\gamma^\top)\right)^{\frac{1}{2}} \exp\left\{0.5\gamma^2\left\|u^0\right\|_{L_T^2}^2\right\}\gamma\left\|u\right\|_{L_T^2},$$

where $P_{\gamma,2}$ and $\hat{P}_\gamma$ are the unique solutions to

$$AP_{\gamma,2} + P_{\gamma,2}A_{11}^{(\gamma)^\top} + \frac{1}{\gamma^2}\sum_{i=1}^m N_i P_{\gamma,2} N_{i,11}^{(\gamma)^\top} = -\frac{1}{\gamma^2} BB_1^{(\gamma)^\top},$$

$$A_{11}^{(\gamma)}\hat{P}_\gamma + \hat{P}_\gamma A_{11}^{(\gamma)^\top} + \frac{1}{\gamma^2}\sum_{i=1}^m N_{i,11}^{(\gamma)}\hat{P}_\gamma N_{i,11}^{(\gamma)^\top} = -\frac{1}{\gamma^2} B_1^{(\gamma)} B_1^{(\gamma)^\top}.$$

Proof. Given the assumptions P_γ, $P_{\gamma,2}$ and $\hat{P}_\gamma$ exist. The result is then a direct consequence of Corollary 4.3 in [26]. □

Theorem 6.2. *Under the assumptions of Proposition 6.1, we have*

$$\sup_{t\in[0,T]} \left\| z(t) - V_\gamma \hat{z}_\gamma(t)\right\|_2 \leq \left(\mathrm{tr}(\Lambda_{\gamma,2}\mathcal{W}_\gamma)\right)^{\frac{1}{2}} \exp\left\{0.5\gamma^2 \left\|u^0\right\|_{L_T^2}^2\right\}\gamma \|u\|_{L_T^2}, \tag{54}$$

where the weight is

$$\mathcal{W}_\gamma = I + 2A_{12}^{(\gamma)^\top} Y_{\gamma,2} + \frac{1}{\gamma^2}\sum_{i=1}^m N_{i,12}^{(\gamma)^\top}\left(2Y_\gamma \begin{bmatrix} N_{i,12}^{(\gamma)} \\ N_{i,22}^{(\gamma)}\end{bmatrix} - \hat{Q}_\gamma N_{i,12}^{(\gamma)}\right).$$

Above, $Y_\gamma = \begin{bmatrix} Y_{\gamma,1} & Y_{\gamma,2}\end{bmatrix}$ and $\hat{Q}_\gamma$ are defined as the unique solutions to

$$A_{11}^{(\gamma)^\top} Y_\gamma + Y_\gamma A_b^{(\gamma)} + \frac{1}{\gamma^2}\sum_{i=1}^m N_{i,11}^{(\gamma)^\top} Y_\gamma N_{i,b}^{(\gamma)} = -\begin{bmatrix} I & 0\end{bmatrix},$$

$$A_{11}^{(\gamma)^\top}\hat{Q}_\gamma + \hat{Q}_\gamma A_{11}^{(\gamma)} + \frac{1}{\gamma^2}\sum_{i=1}^m N_{i,11}^{(\gamma)^\top}\hat{Q}_\gamma N_{i,11}^{(\gamma)} = -I,$$

where we set $A_b^{(\gamma)} := S_\gamma A S_\gamma^\top$ and $N_{i,b}^{(\gamma)} := S_\gamma N_i S_\gamma^\top$.

Proof. The result directly follows from the proof of Theorem 4.8 in which B and N_i need to be replaced by $\frac{1}{\gamma}B$ and $\frac{1}{\gamma}N_i$. □

As in the stochastic framework, we can conclude that truncating the small eigenvalues of P_γ leads to small diagonal entries of $\Lambda_{\gamma,2}$ and hence to a small error in the dimension reduction according to Theorem 6.2 given that the exponential in (54) is not too dominant. Therefore, the eigenvalues of P_γ can be used as a criterion to determine a suitable reduced order dimension r.

Notice that the above results can be generalized to non-zero initial states, since a general system can be decomposed into (46) with zero initial data and

$$\dot{z}_{z_0}(t) = Az_{z_0}(t) + \sum_{i=1}^m N_i z_{z_0}(t)u_i(t), \quad z(0) = z_0 = X_0 v_0. \tag{55}$$

Now, establishing the MOR scheme for (55) the same way as for the stochastic setting in Section 5 provides the desired extension.

7. Numerical experiments

In this section, we test the efficiency of the proposed method (see Sections 3.2 and 6), denoted here by OS, in some numerical examples. We compare the results with the ones obtained by applying the standard balanced truncation method for a full state approximation, denoted here by BT (see, e.g., [3] for the bilinear and [7] for the stochastic case). All the simulations are done on a CPU 2.6 GHz Intel® Core™i5, 8 GB 1600 MHz DDR3, MATLAB® 9.1.0.441655 (R2016b).

For this study, we consider a standard test example representing a 2D boundary controlled heat transfer system; see, e.g., [3]. Its dynamics is governed by the heat equation subject to Dirichlet and Robin boundary conditions, i.e., the following boundary value problem

$$\begin{array}{ll} \partial_t x = \Delta x, & \text{in } (0,1)\times(0,1), \\ n\cdot\nabla x = 0.8u_1 x & \text{on } \Gamma_1, \\ x = u_2, & \text{on } \Gamma_2, \\ x = 0, & \text{on } \Gamma_3, \Gamma_4, \end{array}$$

where $\Gamma_1 = \{0\}\times(0,1)$, $\Gamma_2 = (0,1)\times\{0\}$, $\Gamma_3 = \{1\}\times(0,1)$ and $\Gamma_4 = (0,1)\times\{1\}$. In this system, there are two source terms, namely u_1 and u_2, which are applied at the boundaries Γ_1 and Γ_2, respectively. A semi-discretization in space using finite differences with $k = 20$ grid points results in a control system of dimension $n = 400$ of the form

$$\dot{x} = Ax(t) + Nx(t)u_1(t) + Bu_2(t). \tag{56}$$

We refer to [3] for more details on the matrices in (56).

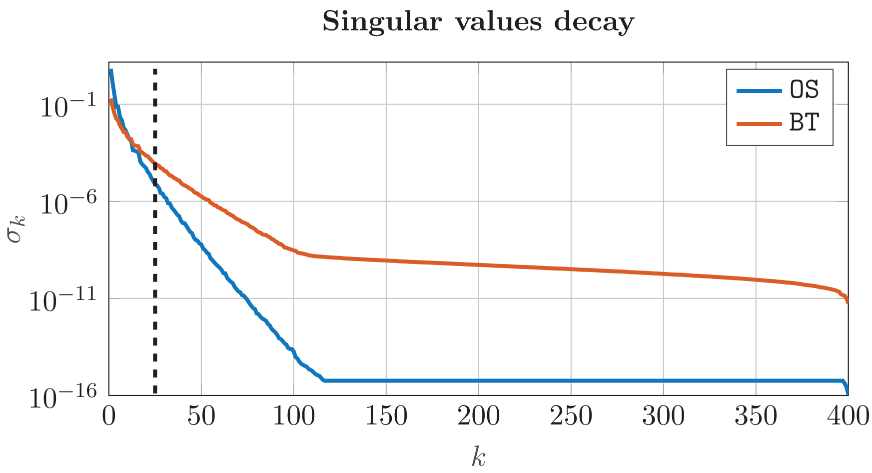

Fig. 1. Decay of singular values σ_k: the blue curve corresponds to eig$P$. The red curve corresponds to $\sqrt{\text{eig}PQ}$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Table 1
Stochastic example: Error bounds and the max value of the mean error for OS and BT for the simulation presented in Fig. 2 with zero initial condition.

Method	Error bound	max mean error
OS	$5.46 \cdot 10^{-3}$	$3.56 \cdot 10^{-4}$
BT	$5.63 \cdot 10^{-3}$	$2.05 \cdot 10^{-4}$

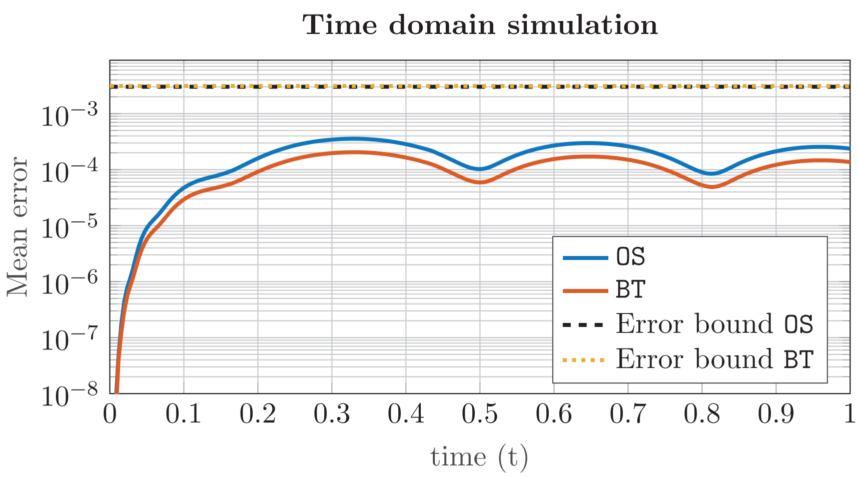

Fig. 2. Stochastic simulation: pointwise mean error between the original model and the ROMs for the input $u_2(t) = u(t) = e^{-\frac{1}{2}t}\sin(10t)$ and zero initial condition.

7.1. Stochastic example

7.1.1. Zero initial condition

First, we consider that the boundary Γ_1 is a perturbed by noise, i.e., $u_1 = \frac{dW}{dt}$ with W being a standard Wiener process. Hence the resulting dynamical stochastic system is of the form

$$dx(t) = [Ax(t) + Bu_2(t)]dt + Nx(t)dW(t), \quad t \geq 0.$$

Additionally, we first assume that $x(0) = 0$ before we study a non-zero initial state. In order to apply BT, we additionally need to compute the observability Gramian Q, which satisfies the following Lyapunov equation

$$A^\top Q + QA + \sum_{i=1}^{q} N_i^\top Q N_i = -I \tag{57}$$

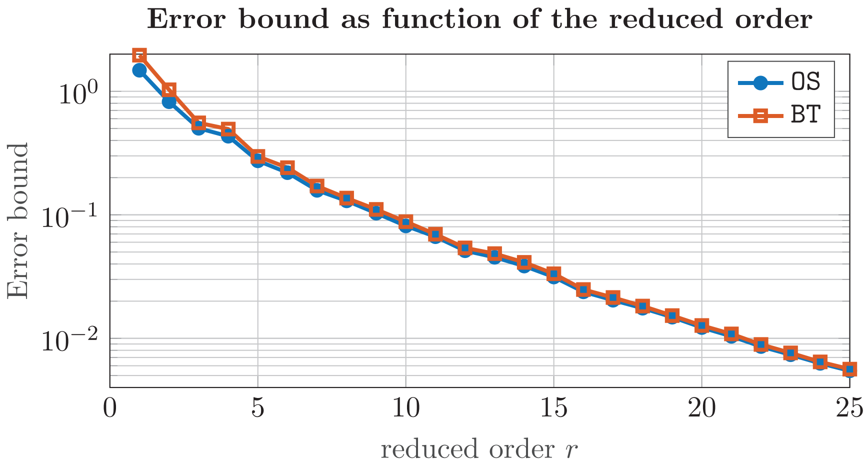

Fig. 3. Decay input-independent part of error bound for OS and BT computed for different orders $r = 1, \dots, 25$.

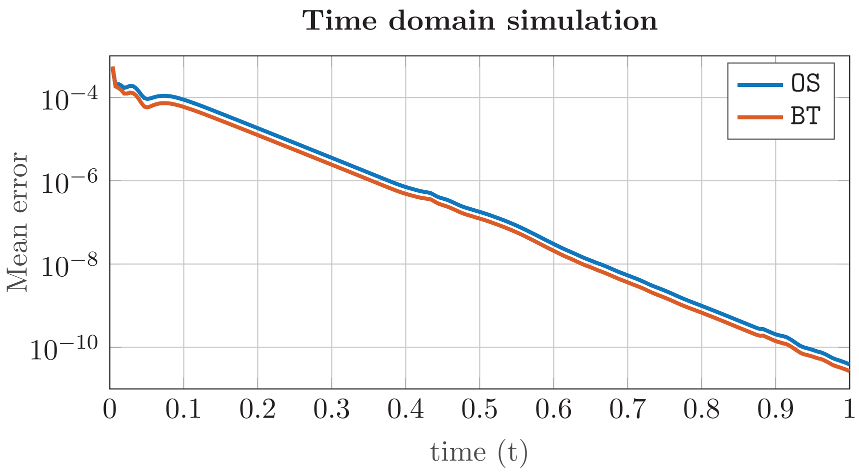

Fig. 4. Stochastic simulation: pointwise mean error between the original model and the ROMs for the initial condition $x(0) = 0.1/[1 \quad \dots \quad 1/]^\top$ and input $u_2 \equiv 0$.

with $q = 1$ and $N_1 = N$. This method was studied in detail in [7]. However, solving (57) leads to much higher computational cost especially due to the full-rank right hand side which does not allow the usage of low-rank solvers. Fig. 1 depicts the decay of the eigenvalues/singular values of P as well as the decay of square root of the eigenvalues of PQ (Hankel singular values). As shown in Theorem 4.8, the eigenvalues of P play an important role in the error bound for OS and provide an intuition for the expected error. Similarly, the Hankel singular values are also associated the error bound for BT, see [7]. The decay of both curves in Fig. 1 indicates that a small reduction error can already be achieved for small r.

For this example, we compute reduced systems of order $r = 25$ for both OS and BT. As a next step, we compare the quality of the reduced-order systems by simulating their responses for the input $u_2(t) = u(t) = e^{-\frac{1}{2}t}\sin(10t)$. To determine the transient response, we apply a semi-implicit Euler-Maruyama scheme with step size $h = 1/256$ and simulate the original system and the reduced-order models in the time interval [0,1]. Additionally, those simulations are done using 10^5 samples. The mean error between the original and the reduced models are depicted in Fig. 2 as well as the error bounds from Proposition 4.7. Table 1 presents the numerical values for the error bounds and max mean error for both methods. We notice that both reduced models are able to follow the behavior of the original system. Furthermore, this figure shows that the two methods, BT and OS, provide very similar quality reduced models in terms of the magnitude of the error, an observation we also made with other test examples. However, we note that BT is a numerically more expensive method, since one needs to additionally solve for Q.

Additionally, for different reduced orders varying in the range $r = 1, \dots, 25$, the input-independent part of the error bound given in Proposition 4.7 is computed in Fig. 3, i.e., for each reduced order r we plot the value

$$\mathcal{E}(r) = \left(\operatorname{tr}(P) + \operatorname{tr}(\hat{P}(r)V(r)^\top V(r)) - 2\operatorname{tr}(P_2(r)V(r)^\top)\right)^{\frac{1}{2}},$$

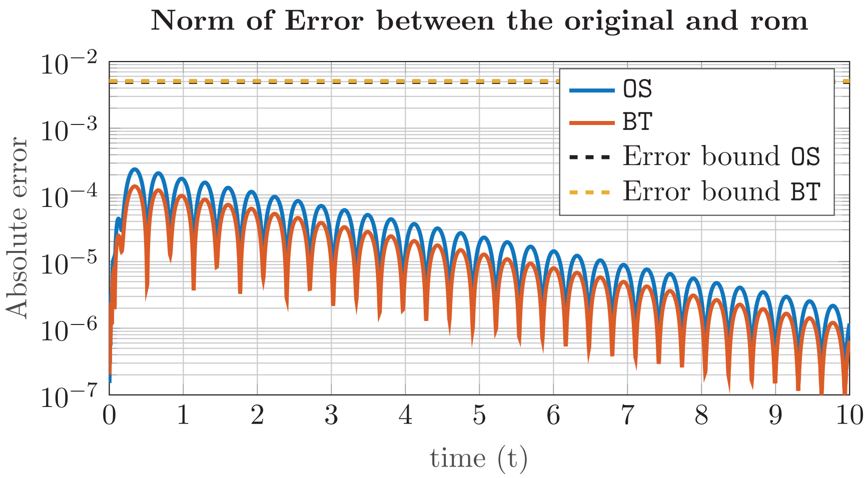

Fig. 5. Bilinear simulation: pointwise error between the original model and the ROMs for the input $u(t) = e^{-\frac{1}{2}t}\sin(10t)$.

Table 2
Stochastic example: Error bounds and the L^2 error for OS and BT for the simulation presented in Fig. 4 with non-zero initial condition.

Method	Error bound	L^2 error
OS	$1.24 \cdot 10^{-4}$	$5.50 \cdot 10^{-5}$
BT	$1.25 \cdot 10^{-4}$	$4.76 \cdot 10^{-5}$

Table 3
Bilinear example: Error bounds and the L^∞ error for OS and BT for the simulation presented in Fig. 5.

Method	Error bound	L^∞ error
OS	$5.46 \cdot 10^{-3}$	$3.56 \cdot 10^{-4}$
BT	$5.63 \cdot 10^{-3}$	$2.05 \cdot 10^{-4}$

where $V(r)$ is the reduced basis of order r, and $\hat{P}(r)$, $P_2(r)$ are the solutions of (31b) and (31c). Notice that we added $V(r)^\top V(r)$ in the second summand of the error bound since $V(r)^\top V(r) \neq I$ for BT. As expected, the bound decays if the reduced order is increased for both OS and BT.

7.1.2. Non-zero initial condition

Now, we consider the same dynamical stochastic system presented in the previous section. Additionally, we assume that the initial temperature constant in space, leading to the initial condition

$$x(0) = X_0 v_0,$$

with $X_0 = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}^\top$ and $v_0 = 0.1$. As shown in Section 5, the control subsystem with zero initial condition can be decoupled from the initial condition subsystem (38). Hence, for the initial condition subsystem, we employ the reduction scheme presented in that section. Firstly, we compute the Gramian P_{x_0} solving the Lyapunov Eq. (39). Based on this Gramian, we are able to construct reduced order models for (38) via OS. Once again we compare the proposed methodology with BT using the Gramians P_{x_0} and Q. For the initial condition subsystem, the singular value curves for OS and BT have a similar behavior as in the case of a zero initial condition (Fig. 1). Therefore, they are omitted here. For this example, we computed reduced models of order $r_{x_0} = 20$. We compare the quality of reduced models (41) by simulating 10^5 samples using a semi-implicit Euler-Maruyama scheme with step size $h = 1/256$. In Fig. 4, we depict the pointwise mean error between the original system (38) and reduced models of the form (41). Also, Table 2 presents the numerical values for the L^2 error bounds (from inequality (44)) and the L^2 error for both methods. Once again, both reduced models are able to follow the behavior of the original system. Furthermore, for this simulation, BT and OS, provide very similar quality reduced models regarding the magnitude of the error. However, BT is more numerically expensive due to the computation of Q.

7.2. Bilinear example

As our second numerical example, we consider the heat transfer system in (56) with $u_2 = u_1 = u$. For the bilinear example, we assume that the initial condition is zero. As a consequence, this leads to a bilinear system having only one input. For

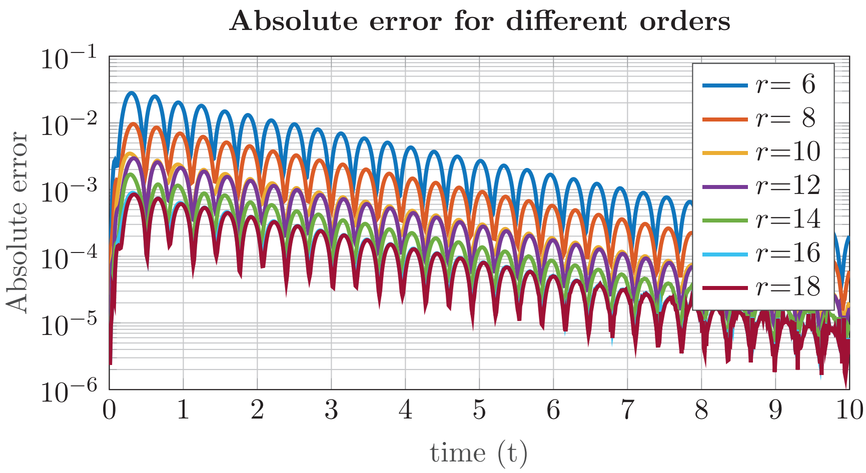

Fig. 6. Bilinear simulation: time domain pointwise error for different reduced orders using OS.

this example, we need to solve the Lyapunov equation in (48). To this aim, we set $\gamma = 1$ leading to the same reachability and observability Gramians as for the stochastic example. Hence, Fig. 1 also gives the decay of singular values for OS and BT. Similarly, Fig. 3 shows the decay of the input-independent part of the error bound from Proposition 6.1.

As in the previous example, we obtain reduced systems of order $r = 25$ by using OS and BT and compare their quality by simulating their responses for the input $u(t) = e^{-\frac{1}{2}t}\sin(10t)$. To determine the transient response, we use the MATLAB®-solver ode45 to simulate the original system and the reduced-order models in the time interval [0,10]. The results are depicted in Fig. 5. Table 5 presents the numerical values for the error bounds and max error for both methods. Similar to the stochastic example, we notice that the two methods, BT and OS, provide very similar quality reduced models in terms of the magnitude of the error. Once again, we note that BT is a computationally more expensive method, since one needs the solution to the additional Lyapunov equation in (57). Finally, Fig. 6 shows the simulation of the error for reduced models obtained by OS with different orders. As expected, the error decays once the order is increased.

Appendix A. Matrix differential equations and their solutions

Lemma A.1. *Let Φ be the fundamental solution of (1) defined in (3) and let $\hat{\Phi}$ be the one of system (2). Suppose that B and $\hat{B}$ are matrices of suitable dimension. Then, the $\mathbb{R}^{n\times r}$-valued function $\mathbb{E}\left[\Phi(t,s)B\hat{B}^\top\hat{\Phi}^\top(t,s)\right]$, $t \geq s$, satisfies*

$$X(t) = B\hat{B}^\top + \int_s^t AX(\tau)d\tau + \int_s^t X(\tau)\hat{A}^\top d\tau + \sum_{i=1}^q \int_s^t N_i X(\tau)\hat{N}_i^\top d\tau. \quad (58)$$

Proof. The result is a direct consequence of [7], Proposition 4.4 or [28], Lemma 2.1. □

Corollary A.2. *Given the assumptions in Lemma A.1, we find that*

$$\mathbb{E}\left[\Phi(t,s)B\hat{B}^\top\hat{\Phi}^\top(t,s)\right] = \mathbb{E}\left[\Phi(t-s)B\hat{B}^\top\hat{\Phi}^\top(t-s)\right], \quad t \geq s. \quad (59)$$

Proof. Setting $X(t) := \mathbb{E}\left[\Phi(t)B\hat{B}^\top\hat{\Phi}^\top(t)\right]$, by Lemma A.1 we find that

$$X(t-s) = B\hat{B}^\top + \int_0^{t-s} AX(\tau)d\tau + \int_0^{t-s} X(\tau)\hat{A}^\top d\tau + \sum_{i=1}^q \int_0^{t-s} N_i X(\tau)\hat{N}_i^\top d\tau.$$

Setting $v = \tau + s$, by substitution, we see that

$$X(t-s) = B\hat{B}^\top + \int_s^t AX(v-s)dv + \int_s^t X(v-s)\hat{A}^\top dv + \sum_{i=1}^q \int_s^t N_i X(v-s)\hat{N}_i^\top dv.$$

Consequently, both sides of (59) satisfy (58). Therefore, they are equal. □

References

[1] S.A. Al-Baiyat, M. Bettayeb, A new model reduction scheme for k–power bilinear systems, Proceedings of the 32nd IEEE Conference on Decision and Control (1993) 22–27.

[2] S. Becker, C. Hartmann, M. Redmann, L. Richter, Feedback control theory & model order reduction for stochastic equations, 2019, ArXiv preprint: 1912.06113.

[3] P. Benner, T. Damm, Lyapunov equations, energy functionals, and model order reduction of bilinear and stochastic systems, SIAM J. Control Optim. 49 (2) (2011) 686–711.
[4] P. Benner, T. Damm, M. Redmann, Y.R.R. Cruz, Positive operators and stable truncation, Linear Algebra Appl 498 (2016) 74–87.
[5] P. Benner, T. Damm, Y.R.R. Cruz, Dual pairs of generalized Lyapunov inequalities and balanced truncation of stochastic linear systems, IEEE Trans. Autom. Contr. 62 (2) (2017) 782–791.
[6] P. Benner, S. Gugercin, K. Willcox, A survey of projection-based model reduction methods for parametric dynamical systems, SIAM Rev. 57 (4) (2015) 483–531.
[7] P. Benner, M. Redmann, Model reduction for stochastic systems, Stoch PDE: Anal Comp 3 (3) (2015) 291–338.
[8] P. Benner, J. Saak, Numerical solution of large and sparse continuous time algebraic matrix Riccati and Lyapunov equations: a state of the art survey, GAMM Mitteilungen 36 (1) (2013) 32–52.
[9] P. Benner, A. Schneider, Balanced Truncation Model Order Reduction for LTI Systems with Many Inputs or Outputs, in: A. Edelmayer (Ed.), Proc. of the 19th International Symposium on Mathematical Theory of Networks and Systems, 1971–1974, 2010. Budapest, Hungary
[10] G. Berkooz, P. Holmes, J.L. Lumley, The proper orthogonal decomposition in the analysis of turbulent flows, Annu Rev Fluid Mech 25 (1) (1993) 539–575.
[11] T. Breiten, T. Damm, Krylov subspace methods for model order reduction of bilinear control systems, Syst. Control. Lett. 59 (8) (2010) 443–450.
[12] M. Condon, R. Ivanov, Nonlinear systems-algebraic Gramians and model reduction, COMPEL 24 (1) (2005) 202–219.
[13] T. Damm, Lecture notes in control and information sciences 297, Rational Matrix Equations in Stochastic Control, Springer, Berlin, 2004.
[14] D.F. Enns, Model Reduction with Balanced Realizations: An Error Bound and a Frequency Weighted Generalization, in: The 23rd IEEE conference on decision and control, IEEE, 1984, pp. 127–132.
[15] R.W. Freund, Model reduction methods based on krylov subspaces, Acta Numerica 12 (2003) 267–319.
[16] W.S. Gray, J. Mesko, Energy functions and algebraic Gramians for bilinear systems, Proceedings of the 4th IFAC Nonlinear Control Systems Design Symposium 31 (17) (1998) 101–106.
[17] R.Z. Khasminskii, Stochastic stability of differential equations, volume 66 of stochastic modelling and applied probability, 2nd, Springer, Heidelberg, 2012.
[18] K. Kunisch, S. Volkwein, Galerkin proper orthogonal decomposition methods for parabolic problems, Numerische Mathematik 90 (1) (2001) 117–148.
[19] B. Moore, Principal component analysis in linear systems: controllability, observability, and model reduction, IEEE Trans Automat Contr 26 (1) (1981) 17–32.
[20] L. Pernebo, L. Silverman, Model reduction via balanced state space representations, IEEE Trans Automat Contr 27 (2) (1982) 382–387.
[21] R.V. Polyuga, A. van der Schaft, Model Reduction of Port-hamiltonian Systems as Structured Systems, in: Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems–MTNS, volume 5, 2010.
[22] S. Prajna, POD Model Reduction with Stability Guarantee, in: 42nd IEEE International Conference on Decision and Control (IEEE Cat. No. 03CH37475), volume 5, IEEE, 2003, pp. 5254–5258.
[23] R. Pulch, Stability preservation in Galerkin-type projection-based model order reduction, Numer. Algebra, Control. Optim. 9 (1) (2019) 23–44.
[24] M. Redmann, Model Order Reduction Techniques Applied to Evolution Equations with Lévy Noise, Otto-von-Guericke-Universität Magdeburg, 2016 Phd thesis.
[25] M. Redmann, Type II singular perturbation approximation for linear systems with Lévy noise, SIAM J. Control Optim. 56 (3) (2018) 2120–2158.
[26] M. Redmann, Bilinear systems – a new link to $\mathcal{H}_2$-norms, relations to stochastic systems and further properties, 2019, ArXiv preprint: 1910.14427v4.
[27] M. Redmann, P. Benner, An h_2-type error bound for balancing-related model order reduction of linear systems with Lévy noise, Syst. Control. Lett. 105 (2017) 1–5.
[28] M. Redmann, M.A. Freitag, Optimization based model order reduction for stochastic systems, Appl. Math. Comput. 398 (2021).
[29] R.C. Selga, B. Lohmann, R. Eid, Stability preservation in projection-based model order reduction of large scale systems, Eur J Control 18 (2) (2012) 122–132.
[30] V. Simoncini, Computational methods for linear matrix equations, SIAM Rev. 38 (3) (2016) 377–441.
[31] D.C. Sorensen, A.C. Antoulas, On Model Reduction of Structured Systems, in: Dimension reduction of large-scale systems, Springer, 2005, pp. 117–130.
[32] W.M. Wonham, On a matrix Riccati equation of stochastic control, SIAM J. Control 6 (4) (1968) 681–697.