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Hilbert–Poincaré series of parity binomial edge ideals and permanental ideals of complete graphs

Do Trong Hoang[1] · Thomas Kahle[2]

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Abstract
We give an explicit formula for the Hilbert–Poincaré series of the parity binomial edge ideal of a complete graph K_n or equivalently for the ideal generated by all 2×2-permanents of a $2 \times n$-matrix. It follows that the depth and Castelnuovo–Mumford regularity of these ideals are independent of n.

Keywords Betti numbers · Parity binomial edge ideal · Hilbert–Poincaré series

Mathematics Subject Classification 05E40 · 13P10 · 13D02

1 Introduction

Let $R = \Bbbk[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be a standard graded polynomial ring in $2n$ indeterminates. The *parity binomial edge ideal* of an undirected simple graph G on $[n] = \{1, \ldots, n\}$ is

$$\mathcal{I}_G = \left(x_i x_j - y_i y_j \mid \{i,j\} \in E(G)\right) \subset R,$$

where $E(G)$ is the edge set of G. This ideal was defined and studied in [11] in formal similarity to the binomial edge ideals of [7, 13]. If $\mathrm{char}(\Bbbk) \neq 2$, then the linear coordinate change $x_i \mapsto (x_i - y_i)$ and $y_i \mapsto (x_i + y_i)$ turns this ideal into the *permanental edge ideal*

$$\left(x_i y_i + x_j y_j \mid \{i,j\} \in E(G)\right) \subset R.$$

We aim to understand homological properties of these ideals and we view such understanding as helpful in the context of complexity theory and the dichotomy of permanents and determinants. In linear algebra it is known that determinants can be evaluated quickly with

✉ Thomas Kahle
thomas.kahle@ovgu.de

Do Trong Hoang
hoang.dotrong@hust.edu.vn

1 School of Applied Mathematics and Informatics, Hanoi University of Science and Technology, 1 Dai Co Viet, Hai Ba Trung, Hanoi, Vietnam

2 Fakultät für Mathematik, Otto-von-Guericke Universität, Universitätsplatz 2, 39106 Magdeburg, Germany

Gaussian elimination, but permanents are #P-complete and thus NP-hard to evaluate. This complexity distinction is also visible for ideals generated by determinants and permanents, as the permanental versions are often much harder to analyze and have nice properties much more rarely. For details and history we recommend [12] which treats ideals of 2×2 -permanents of $m \times n$-matrices in detail.

2×2-permanental ideals also arise from the study of orthogonal embeddings of graphs in $\mathbb{R}^2$ as the Lovász–Saks–Schrijver ideals of [8]. That paper also contains information about radicality and Gröbner bases of parity binomial edge ideals. Badiane, Burke and Sköldberg proved in [2] that the universal Gröbner basis and the Graver basis coincide for parity binomial edge ideals of complete graphs. The case of bipartite graphs is also special, as then binomial edge ideals and parity binomial edge ideals agree up to a linear coordinate change. A coherent presentation of our knowledge about these binomial ideals can be found in [6], in particular Chapter 7.

In this paper we are concerned with permanental ideals of $2 \times n$-matrices, but switch to the representation as parity binomial edge ideals of complete graphs, as this seems easier to analyze. For example, the permanental ideal contains monomials by [12, Lemma 2.1] and these make the combinatorics more opaque [10]. Due to the linear coordinate change, our computations of homological invariants are valid for both ideals unless $\operatorname{char}(\Bbbk) = 2$, in which case the permanental ideal and the determinantal ideal agree.

The binomial edge ideal of a complete graph, also known as the standard determinantal ideal of a generic $2 \times n$-matrix, is well understood. It has a linear minimal free resolution independent of n, constructed explicitly by Eagon and Northcott [4]. Parity binomial edge ideals of complete graphs do not have a linear resolution and their Betti numbers have no obvious explanation.

Example 1.1 The package BinomialEdgeIdeals in Macaulay2 [5] easily generates the following Betti table of $\mathcal{I}_{K_7}$. The Betti table agrees with the Betti table of a permanental ideal of a generic 2×7-matrix.

	0	1	2	3	4	5	6	7	8	9	10	11
total:	1	21	455	1925	4256	6111	6160	4466	2289	784	161	15
0:	1	·	·	·	·	·	·	·	·	·	·	·
1:	·	21	·	·	·	·	·	·	·	·	·	·
2:	·	·	455	1890	3976	5166	4410	2520	945	210	21	·
3:	·	·	·	35	280	945	1750	1946	1344	574	140	15

From computations for the first few n one can observe that the Castelnuovo–Mumford regularity (the index of the last row of the Betti table, see Sect. 2 for definitions) of $R/\mathcal{I}_{K_n}$ appears to be independent of $n \geq 4$. That $\operatorname{reg}(R/\mathcal{I}_{K_n}) = 3$ was conjectured by the second author and Krüsemann [9, Remark 2.15] and is now our Theorem 3.6. Our main results are explicit formulas for the Hilbert–Poincaré series, the depth, the Castelnuovo–Mumford regularity, and some extremal Betti numbers in the case of a complete graph. The proof of our theorem relies on good knowledge of the primary decomposition of $\mathcal{I}_{K_n}$ from [11] and the resulting exact sequences. At the moment it is not clear if the techniques can be generalized to other graphs or maybe even yield the conjectured upper bound $\operatorname{reg}(R/\mathcal{I}_G) \leq n$ from [9, Remark 2.15].

2 Basics of (parity) binomial edge ideals

Throughout this paper, let G be a simple (i.e. finite, undirected, loopless and without multiple edges) graph on the vertex set $V(G) = [n] := \{1, \dots, n\}$. Let $E(G)$ denote the set of edges of G. Each graded R-module and in particular $R/\mathcal{I}_G$ has a minimal graded free resolution

$$0 \leftarrow R/\mathcal{I}_G \leftarrow \bigoplus_j R(-j)^{\beta_{0,j}(R/\mathcal{I}_G)} \leftarrow \cdots \leftarrow \bigoplus_j R(-j)^{\beta_{p,j}(R/\mathcal{I}_G)} \leftarrow 0.$$

where $R(-j)$ denotes the free R-module obtained by shifting the degrees of R by j. The number $\beta_{i,j}(R/\mathcal{I}_G)$ is the (i, j)-*th graded Betti number* of $R/\mathcal{I}_G$. Let $H_{R/\mathcal{I}_G}$ be the Hilbert function of $R/\mathcal{I}_G$. The *Hilbert–Poincaré series* of the R-module $R/\mathcal{I}_G$ is

$$HP_{R/\mathcal{I}_G}(t) = \sum_{i \geq 0} H_{R/\mathcal{I}_G}(i) t^i.$$

By [14, Theorem 16.2], this series has a rational expression

$$HP_{R/\mathcal{I}_G}(t) = \frac{P_{R/\mathcal{I}_G}(t)}{(1-t)^{2n}}.$$

The numerator is the *Hilbert–Poincaré polynomial* of $R/\mathcal{I}_G$ and has the form

$$P_{R/\mathcal{I}_G}(t) = \sum_{i=0}^{p} \sum_{j=0}^{p+r} (-1)^i \beta_{i,j}(R/\mathcal{I}_G) t^j.$$

It encodes different homological invariants of $R/\mathcal{I}_G$ of which we are particulary interested in the *Castelnuovo–Mumford regularity*

$$\operatorname{reg}(R/\mathcal{I}_G) = \max\{j - i \mid \beta_{i,j}(R/\mathcal{I}_G) \neq 0\}$$

and the *projective dimension* of $R/\mathcal{I}_G$:

$$\operatorname{pdim}(R/\mathcal{I}_G) = \max\{i \mid \beta_{i,j}(R/\mathcal{I}_G) \neq 0 \text{ for some } j\}.$$

In terms of Betti tables, the regularity is the index of the last non-vanishing row, while the projective dimension is the index of the last non-vanishing column of the Betti table. Both are finite for any R-module as R is a regular ring.

The Auslander–Buchsbaum formula [6, Theorem 2.15] relates depth and projective dimension over R as $\operatorname{depth}(R/\mathcal{I}_G) = 2n - \operatorname{pdim}(R/\mathcal{I}_G)$. The Castelnuovo–Mumford regularity and depth could also be computed from vanishing of local cohomology. Using that definition allows to easily deduce some basic properties of the regularity and depth. For instance, the regularity and depth behave well in a short exact sequence. The following lemma appears as [14, Corollary 18.7].

Lemma 2.1 *If* $0 \to A \to B \to C \to 0$ *is a short exact sequence of finitely generated graded R-modules with homomorphisms of degree* 0, *then*

$$P_B(t) = P_A(t) + P_C(t), \text{ and}$$

(1) $\operatorname{reg}(B) \leq \max\{\operatorname{reg}(A), \operatorname{reg}(C)\}$,
(2) $\operatorname{reg}(A) \leq \max\{\operatorname{reg}(B), \operatorname{reg}(C)+1\}$,
(3) $\operatorname{reg}(C) \leq \max\{\operatorname{reg}(A)-1, \operatorname{reg}(B)\}$,
(4) $\operatorname{depth}(B) \geq \min\{\operatorname{depth}(A), \operatorname{depth}(C)\}$,
(5) $\operatorname{depth}(A) \geq \min\{\operatorname{depth}(B), \operatorname{depth}(C)+1\}$,
(6) $\operatorname{depth}(C) \geq \min\{\operatorname{depth}(A)-1, \operatorname{depth}(B)\}$.

As with any binomial ideal, the saturation at the coordinate hyperplanes plays a central role. To this end, let $g = \prod_{i \in [n]} x_i y_i$ and let

$$\mathcal{J}_G := \left(\mathcal{I}_G : g^\infty\right) := \bigcup_{t \geq 1} \left(\mathcal{I}_G : g^t\right).$$

By [11, Proposition 2.7], the generators of the saturation $\mathcal{J}_G$ can be explained using walks in G. For our purposes it suffices to know the following generating set which can be derived from [11, Section 2].

Proposition 2.2 *If G is a non-bipartite connected graph, then*

$$\mathcal{J}_G = (x_i^2 - y_i^2 \mid 1 \leq i \leq n) + (x_i y_j - x_j y_i,\ x_i x_j - y_i y_j \mid 1 \leq i < j \leq n).$$

3 Parity binomial edge ideals of complete graphs

We now consider the parity binomial edge ideal $\mathcal{I}_{K_n}$ of a complete graph K_n on $n \geq 3$ vertices. For $1 \leq i < j \leq n$, let

$$f_{ij} := x_i y_j - x_j y_i \quad \text{and} \quad g_{ij} := x_i x_j - y_j y_i.$$

The parity binomial edge ideal of the complete graph is $\mathcal{I}_{K_n} = \left(g_{ij} \mid 1 \leq i < j \leq n\right)$.

We need some further notation. For any $I \subseteq [n]$ we denote $\mathfrak{m}_I := (x_i, y_i \mid i \in I)$. Let $\mathfrak{p}^+ := (x_i + y_i \mid i \in [n])$ and $\mathfrak{p}^- := (x_i - y_i \mid i \in [n])$. Denote $P_{ij} := (g_{ij}) + \mathfrak{m}_{[n]\setminus\{i,j\}}$. By [11, Theorem 5.9], there is a decomposition of $\mathcal{I}_{K_n}$ as follows.

Proposition 3.1 *For $n \geq 3$, we have*

$$\mathcal{I}_{K_n} = \mathcal{J}_{K_n} \cap \bigcap_{1 \leq i < j \leq n} P_{ij}.$$

In particular, $\dim(R/\mathcal{I}_{K_n}) = n$.

We analyze $\mathcal{I}_{K_n}$ by regular sequences arising from successively adding the polynomials f_{kn} or saturating with respect to them. Let $I_0 := \mathcal{I}_{K_n}$ and, inductively for $1 \leq k \leq n-1$, $I_k := I_{k-1} + (f_{kn})$.

Lemma 3.2 *For $1 \leq k \leq n-1$, we have*

$$I_{k-1} \subseteq \bigcap_{1\le i<j\le n-1} P_{ij} \cap \mathcal{J}_{K_n} \cap \bigcap_{t=k}^{n-1} P_{tn}.$$

Proof By Proposition 2.2, $f_{1n}, \dots, f_{(k-1)n} \in \mathcal{J}_{K_n}$. Moreover, for all $(\ell, n) \neq (i,j)$ we have $f_{\ell n} \in P_{ij}$. Thus

$$(f_{1n}, \dots, f_{(k-1)n}) \subseteq \bigcap_{1\le i<j\le n-1} P_{ij} \cap \mathcal{J}_{K_n} \cap \bigcap_{t=k}^{n-1} P_{tn}.$$

Together with Proposition 3.1 the lemma is proven. □

Lemma 3.3 *For* $1 \le k \le n-1$, *we have*

$$I_{k-1} : f_{kn} = P_{kn}.$$

In particular, $\operatorname{depth}(R/(I_{k-1} : f_{kn})) = 3$, $\operatorname{reg}(R/(I_{k-1} : f_{kn})) = 1$ *and* $P_{R/(I_{k-1}:f_{kn})}(t) = (1-t)^{2n-3}(1+t)$.

Proof One can check that $I_{k-1} : f_{kn} \supseteq P_{kn}$ (in fact $\mathcal{I}_{K_n} : f_{kn} \supseteq P_{kn}$) by simple calculations like $x_1 f_{kn} \equiv -y_k g_{1n} \mod \mathcal{I}_{K_n}$. Now, for all $(k,n) \neq (i,j)$, one can see that f_{kn} is contained in both P_{ij} and $\mathcal{J}_{K_n}$. By [1, Lemma 4.4], $P_{ij} : f_{kn} = \mathcal{J}_{K_n} : f_{kn} = R$ and $P_{kn} : f_{kn} = P_{kn}$ because P_{kn} is a prime that does not contain f_{kn}. Hence by Lemma 3.2, we have $I_{k-1} : f_{kn} \subseteq P_{kn}$ and thus $I_{k-1} : f_{kn} = P_{kn}$.

Using this result, the invariants can be computed for the prime P_{kn} as follows: $\operatorname{depth}(R/(I_{k-1} : f_{kn})) = \operatorname{depth}(R/P_{kn}) = 3$, $\operatorname{reg}(R/(I_{k-1} : f_{kn})) = \operatorname{reg}(R/P_{kn}) = 1$, and $P_{R/(I_{k-1}:f_{kn})}(t) = P_{R/P_{kn}}(t) = (1-t)^{2n-3}(1+t)$. □

Lemma 3.4

$$I_{n-2} : (x_n + y_n) = \mathfrak{p}^- \cap P_{n-1,n}.$$

In particular, $\operatorname{depth}(R/(I_{n-2} : (x_n + y_n))) \ge 3$, $\operatorname{reg}(R/(I_{n-2} : (x_n + y_n))) \le 1$ and $P_{R/(I_{n-2}:(x_n+y_n))}(t) = (1-t)^n + 2t(1-t)^{2n-3}$.

Proof For the lexicographic ordering on $\Bbbk[x_1, \dots, x_n, y_1, \dots, y_n, t]$ induced by $x_1 > \dots > x_n > y_1 > \dots > y_n > t$, the Gröbner basis for $J = t\mathfrak{p}^- + (1-t)P_{n-1,n}$ is

$$\begin{aligned}\mathcal{G} = \{&(x_{n-1} - y_{n-1})t,\ (x_n - y_n)t),\ x_{n-1}x_n - y_{n-1}y_n,\\ &x_i - y_i,\ (x_{n-1} - y_{n-1})y_i,\ (x_n - y_n)y_i,\ (t-1)y_i \mid 1 \le i \le n-2\}.\end{aligned}$$

Thus,

$$\begin{aligned}&\mathfrak{p}^- \cap P_{n-1,n}\\ &\quad = \big(x_{n-1}x_n - y_{n-1}y_n,\ x_i - y_i,\ (x_{n-1} - y_{n-1})y_i,\ (x_n - y_n)y_i \mid 1 \le i \le n-2\}\big).\end{aligned}$$

This implies the containment $\mathfrak{p}^- \cap P_{n-1,n} \subseteq I_{n-2} : (x_n + y_n)$. Conversely, by Lemma 3.2

$$I_{n-2} \subseteq \bigcap_{1\le i<j\le n-1} P_{ij} \cap \mathcal{J}_{K_n} \cap P_{n-1,n}.$$

For all $1 \le i < j \le n-1$, it is clear that $x_n + y_n \in P_{ij}$ and so $P_{ij} : (x_n + y_n) = R$. By [1, Lemma 4.4], $P_{n-1,n} : (x_n + y_n) = P_{n-1,n}$. Moreover, by Proposition 2.2, we obtain that $\mathcal{J}_{K_n} : (x_n + y_n) = \mathfrak{p}^-$. This implies that $I_{n-2} : (x_n + y_n) \subseteq \mathfrak{p}^- \cap P_{n-1,n}$ and thus the conclusion $I_{n-2} : (x_n + y_n) = \mathfrak{p}^- \cap P_{n-1,n}$.

In order to prove the second part, note that

$$\mathfrak{p}^- + P_{n-1,n} = (x_{n-1} + y_{n-1}, x_n + y_n) + \mathfrak{m}_{[n-2]}.$$

Therefore one reads off $\operatorname{depth}(R/(\mathfrak{p}^- + P_{n-1,n})) = 2$ and $\operatorname{reg}(R/(\mathfrak{p}^- + P_{n-1,n})) = 0$. It is clear that $\operatorname{depth}(R/\mathfrak{p}^-) = n$ and $\operatorname{reg}(R/\mathfrak{p}^-) = 0$. From the exact sequence

$$0 \longrightarrow R/(\mathfrak{p}^- \cap P_{n-1,n}) \longrightarrow R/\mathfrak{p}^- \oplus R/P_{n-1,n} \longrightarrow R/(\mathfrak{p}^- + P_{n-1,n}) \longrightarrow 0,$$

we obtain, using Lemma 2.1, that

$$\begin{aligned}\operatorname{depth}(R/I_{n-2} : (x_n + y_n)) &= \operatorname{depth}(R/(\mathfrak{p}^- \cap P_{n-1,n})) \ge \min\{n, 3, 2+1\} = 3,\\ \operatorname{reg}(R/I_{n-2} : (x_n + y_n)) &= \operatorname{reg}(R/(\mathfrak{p}^- \cap P_{n-1,n})) \le \max\{0, 1, 0+1\} = 1,\end{aligned}$$

and furthermore,

$$\begin{aligned}P_{R/I_{n-2}:(x_n+y_n)}(t) &= P_{R/\mathfrak{p}^-}(t) + P_{R/P_{n-1,n}}(t) - P_{R/(\mathfrak{p}^- + P_{n-1,n})}(t)\\ &= (1-t)^n + (1-t)^{2n-3}(1+t) - (1-t)^{2n-2}\\ &= (1-t)^n + 2t(1-t)^{2n-3}.\end{aligned}$$

□

Lemma 3.5 *Let $J := (x_n + y_n, I_{n-2})$. Then*

$$\begin{aligned}\operatorname{depth}(R/J) &\ge \min\{n, \operatorname{depth}(S/\mathcal{I}_{K_{n-1}})\},\\ \operatorname{reg}(R/J) &\le \max\{1, \operatorname{reg}(S/\mathcal{I}_{K_{n-1}})\},\end{aligned}$$

and $P_{R/J}(t) = t(1-t)^n + (1-t)^2 P_{S/\mathcal{I}_{K_{n-1}}}(t)$, where $S = \Bbbk[x_i, y_i \mid 1 \le i \le n-1]$.

Proof In order to prove the lemma, we first check two following claims:

Claim 1 $(J, x_n) = (x_n, y_n, \mathcal{I}_{K_{n-1}})$.

Since $y_n = (x_n + y_n) - x_n \in (x_n, J)$ and $\mathcal{I}_{K_{n-1}} \subseteq I_{n-2}$, we have $(x_n, y_n, \mathcal{I}_{K_{n-1}}) \subseteq (J, x_n)$. Conversely, $x_n + y_n, g_{in}, f_{in} \in (x_n, y_n)$ for $1 \le i \le n-1$ and thus $(J, x_n) \subseteq (x_n, y_n, \mathcal{I}_{K_{n-1}})$.

Claim 2 $J : x_n = \mathfrak{p}^+$.

One can compute $x_n(x_i + y_i) = (x_i x_n - y_i y_n) + y_i(x_n + y_n) \in J$ for $1 \le i \le n$, so that $x_n \mathfrak{p}^+ \subseteq J$ which implies that $\mathfrak{p}^+ \subseteq J : x_n$. Conversely, for $1 \le i < j \le n$, we have

$$\begin{aligned}g_{ij} &= x_i x_j - y_i y_j = (x_i - y_i)x_j + y_i(x_j - y_j) = (x_i + y_i)x_j - y_i(x_j + y_j),\\ f_{ij} &= x_i y_j - x_j y_i = (x_i + y_i)y_j - y_i(x_j + y_j) = (x_i - y_i)y_j - y_i(x_j - y_j).\end{aligned}$$

Thus, by Proposition 2.2, $\mathcal{J}_{K_n} \subseteq \mathfrak{p}^+ \cap (x_1 - y_1, \dots, x_{n-1} - y_{n-1}, x_n, y_n)$ and $f_{kn} \in \mathfrak{p}^+ \cap (x_1 - y_1, \dots, x_{n-1} - y_{n-1}, x_n, y_n)$ for all $1 \le k \le n-2$. Together with Proposition 3.1,

$$J \subseteq \bigcap_{1 \le i < j \le n-1} P_{ij} \cap \mathfrak{p}^+ \cap (x_1 - y_1, \dots, x_{n-1} - y_{n-1}, x_n, y_n).$$

By [1, Lemma 4.4], $J : x_n \subseteq \mathfrak{p}^+$ and thus the claim holds.

Now, we turn to the proof of the lemma. By Claim 1,

$$\operatorname{depth}(R/(J,x_n)) = \operatorname{depth}(S/\mathcal{I}_{K_{n-1}}) \text{ and } \operatorname{reg}(R/(J,x_n)) = \operatorname{reg}(S/\mathcal{I}_{K_{n-1}}).$$

Moreover, by Claim 2, we have

$$\operatorname{depth}(R/J : x_n) = \operatorname{depth}(R/\mathfrak{p}^+) = n \text{ and } \operatorname{reg}(R/J : x_n) = \operatorname{reg}(R/\mathfrak{p}^+) = 0.$$

From the exact sequence

$$0 \longrightarrow R/(J : x_n)(-1) \longrightarrow R/J \longrightarrow R/(J,x_n) \longrightarrow 0$$

we obtain

$$\operatorname{depth}(R/J) \geq \min\{n, \operatorname{depth}(S/\mathcal{I}_{K_{n-1}})\} \text{ and } \operatorname{reg}(R/J) \leq \max\{1, \operatorname{reg}(S/\mathcal{I}_{K_{n-1}})\}.$$

Moreover,

$$\begin{aligned} P_{R/J}(t) &= tP_{R/J:x_n}(t) + P_{R/(J,x_n)}(t) = tP_{R/\mathfrak{p}^+}(t) + P_{R/(x_n,y_n,\mathcal{I}_{K_{n-1}})}(t) \\ &= t(1-t)^n + (1-t)^2 P_{S/\mathcal{I}_{K_{n-1}}}(t), \end{aligned}$$

as required. □

Theorem 3.6 *The Hilbert–Poincaré polynomial of* $R/\mathcal{I}_{K_n}$ *is*

$$P_{R/\mathcal{I}_{K_n}}(t) = 2(1-t)^n + \left[-1 + 3t + \left(\frac{n^2+n-6}{2}\right)t^2 + \left(\frac{n^2-3n+2}{2}\right)t^3\right](1-t)^{2n-3}.$$

In particular, $\operatorname{depth}(R/\mathcal{I}_{K_n}) \geq 3$ *and* $\operatorname{reg}(R/\mathcal{I}_{K_n}) \leq 3$.

Proof The proof is by induction on n. If $n = 3$, then a simple calculation (e.g. in Macaulay2) gives the result. Now assume $n \geq 4$. For any $1 \leq k \leq n-1$ there is an exact sequence

$$0 \longrightarrow R/(I_{k-1} : f_{kn})(-2) \xrightarrow{\cdot f_{kn}} R/I_{k-1} \longrightarrow R/I_k \longrightarrow 0.$$

By Lemmas 2.1 and 3.3, $\operatorname{depth}(R/I_{k-1}) \geq \min\{3, \operatorname{depth}(R/I_k)\}$, $\operatorname{reg}(R/I_{k-1}) \leq \max\{3, \operatorname{reg}(R/I_k)\}$ and $P_{R/I_{k-1}}(t) = t^2(1-t)^{2n-3}(1+t) + P_{R/I_k}(t)$. This implies that $\operatorname{depth}(R/I_0) \geq \min\{3, \operatorname{depth}(R/I_{n-2})\}$, $\operatorname{reg}(R/I_0) \leq \max\{3, \operatorname{reg}(R/I_{n-2})\}$ and

$$P_{R/I_0}(t) = (n-2)t^2(1-t)^{2n-3}(1+t) + P_{R/I_{n-2}}(t).$$

Now consider the following exact sequence

$$0 \longrightarrow R/(I_{n-2} : (x_n + y_n))(-1) \longrightarrow R/I_{n-2} \longrightarrow R/(x_n + y_n, I_{n-2}) \longrightarrow 0.$$

Let $S := \Bbbk[x_i, y_i \mid 1 \leq i \leq n-1]$. By Lemmas 3.4 and 3.5, $\operatorname{depth}(R/I_{n-2}) \geq \min\{3, \operatorname{depth}(S/\mathcal{I}_{K_{n-1}})\}$, $\operatorname{reg}(R/I_{n-2}) \leq \max\{1, \operatorname{reg}(S/\mathcal{I}_{K_{n-1}})\}$ and

$$\begin{aligned} P_{R/I_{n-2}}(t) &= tP_{R/(I_{n-2}:(x_n+y_n))}(t) + P_{R/(x_n+y_n,I_{n-2})}(t) \\ &= 2t(1-t)^n + 2t^2(1-t)^{2n-3} + (1-t)^2 P_{S/\mathcal{I}_{K_{n-1}}}(t). \end{aligned}$$

The induction hypothesis yields $\operatorname{depth}(S/\mathcal{I}_{K_{n-1}}) \geq 3$ and $\operatorname{reg}(S/\mathcal{I}_{K_{n-1}}) \leq 3$. Therefore $\operatorname{depth}(R/I_{n-2}) \geq 3$ and $\operatorname{reg}(R/I_{n-2}) \leq 3$. This is enough to conclude that $\operatorname{depth}(R/\mathcal{I}_{K_n}) \geq 3$ and $\operatorname{reg}(R/\mathcal{I}_{K_n}) \leq 3$. Moreover,

$$\begin{aligned}
P_{R/\mathcal{I}_{K_n}}(t) &= 2t(1-t)^n + \Big[(n-2)t^3 + nt^2\Big](1-t)^{2n-3} + (1-t)^2 P_{S/\mathcal{I}_{K_{n-1}}}(t). \\
&= 2t(1-t)^n + \Big[(n-2)t^3 + nt^2\Big](1-t)^{2n-3} \\
&\quad + 2(1-t)^{n+1} + \Big[-1 + 3t + (\frac{n^2-n-6}{2})t^2 + (\frac{n^2-5n+6}{2})t^3\Big](1-t)^{2n-3} \\
&= 2(1-t)^n + \Big[-1 + 3t + (\frac{n^2+n-6}{2})t^2 + (\frac{n^2-3n+2}{2})t^3\Big](1-t)^{2n-3},
\end{aligned}$$

as required. □

If an ideal has a square-free initial ideal, its extremal Betti numbers agree with that of the initial ideal by [3]. Although the parity binomial edge ideal of a complete graph cannot have a square-free initial ideal (see [11, Remark 3.12]), the bottom right Betti number agrees with that of the initial ideal for any term order.

Corollary 3.7 $\beta_{2n-3,2n}(R/\mathcal{I}_{K_n}) = \beta_{2n-3,2n}(R/\operatorname{in}_<(\mathcal{I}_{K_n})) = \dfrac{n^2-3n+2}{2}$.
In particular,

$$\operatorname{reg}(R/\mathcal{I}_{K_n}) = \operatorname{reg}(R/\operatorname{in}_<(\mathcal{I}_{K_n})) = \operatorname{depth}(R/\mathcal{I}_{K_n}) = \operatorname{depth}(R/\operatorname{in}_<(\mathcal{I}_{K_n})) = 3.$$

Proof From Theorem 3.6 we obtain $\beta_{p,p+r}(R/\mathcal{I}_{K_n}) = \frac{n^2-3n+2}{2} \neq 0$, where $p = \operatorname{pdim}(R/\mathcal{I}_{K_n})$ and $r = \operatorname{reg}(R/\mathcal{I}_{K_n})$. Thus, $p + r = 2n$. Since $P_{R/\mathcal{I}_{K_n}}(t) = P_{R/\operatorname{in}_<(\mathcal{I}_{K_n})}(t)$, we get

$$\operatorname{reg}(R/\mathcal{I}_{K_n}) = \operatorname{reg}(R/\operatorname{in}_<(\mathcal{I}_{K_n})), \quad \operatorname{pdim}(R/\mathcal{I}_{K_n}) = \operatorname{pdim}(R/\operatorname{in}_<(\mathcal{I}_{K_n})) \text{ and}$$

$\beta_{p,p+r}(R/\mathcal{I}_{K_n}) = \beta_{p,p+r}(R/\operatorname{in}_<(\mathcal{I}_{K_n}))$. On the other hand, $r \leq 3$ and $p \leq 2n-3$ by the Auslander–Buchsbaum formula. Thus, $r = 3$ and $p = 2n-3$. □

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