Worst-Case Optimal Investment and Consumption -A Study with Stochastic Interest Rates-

Dissertation

zur Erlangung des Doktorgrades der Naturwissenschaften
(Dr. rer. nat.)

der

Naturwissenschaftlichen Fakultät II
Chemie, Physik und Mathematik

der Martin-Luther-Universität
Halle-Wittenberg

vorgelegt von

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geboren am 26.05.1989 in Halle (Saale)

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Tag der Verteidigung: 08.10.2015

Danksagung

Ich möchte Herrn Prof. Dr. Dr. h. c. Wilfried Grecksch für die intensive und zügige Betreuung und die Unterstützung bei der Umsetzung meines Dissertationsvorhabens danken. Ich habe unter anderem sehr von seinen umfassenden Kenntnissen über die Theorie der stochastischen optimalen Steuerung gelernt und profitiert. Die zahlreichen Treffen haben mich sowohl fachlich als auch persönlich vorangebracht.

Ich danke Herrn Prof. Dr. Ralf Korn für die fachlichen Diskussionen, die sehr gute Zusammenarbeit sowie für das Heranführen an das Thema der Worst-case Portfolio Optimierung. Großer Dank gilt auch der Abteilung Finanzmathematik des Fraunhofer-Instituts für Techno- und Wirtschaftsmathematik in Kaiserslautern für die entgegengebrachte Gastfreundschaft bei meinen Forschungsaufenthalten.

Ein herzlicher Dank gilt auch Herrn Prof. Dr. Ralf Wunderlich für die Übernahme und Erstellung des weiteren Gutachtens.

Weiterhin danke ich dem Leibniz Institut für Agrarentwicklung in Transformationsökonomien, insbesondere Herrn Prof. Dr. Alfons Balmann für seine Unterstützung und den Freiraum in der Gestaltung meiner Forschung.

Ich danke der Graduiertenförderungskommission der Martin-Luther-Universität Halle-Wittenberg für die Unterstützung dieser Arbeit durch das Graduiertenstipendium des Landes Sachsen-Anhalt, sowie der Stiftung Theoretische Physik/Mathematik für die Vergabe eines Reisestipendiums, mit dem es mir ermöglicht war auf zahlreichen wissenschaftlichen Tagungen teilzunehmen und mehrere Forschungsaufenthalte durchzuführen.

Herzlicher Dank gilt meiner Familie für die Unterstützung, die Fürsorge und den Zuspruch während des Studiums und der Promotion.

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CHAPTER 1

Introduction

The field of portfolio optimization deals with the problem of the optimal allocation of wealth between different financial assets which are traded on a given financial market. A risk averse investor, endowed with a certain initial wealth, is allowed to continuously invest on such assets and/or to consume in order to maximize his expected utility of consumption over the planning horizon or the expected utility of wealth at the terminal time, or some combination of both. Typically, investment or consumption decisions have to be made without any knowledge about future evolution of asset prices.

This type of continuous-time portfolio optimization problem was first investigated in a pioneering work by Merton [35]. He assumed that the prices of the risky asset follow a stochastic process, namely a geometric Brownian motion, which satisfies a linear stochastic differential equation. This implies that the paths of the price process are continuous and the asset prices are logarithmic normally distributed random variables. Moreover, the investor can invest in a non-risky asset, e.g. a savings account. For a special choice of the investor's utility function, Merton derived the optimal investment and consumption strategy in an explicit form by applying stochastic optimal control theory, by applying Bellman's principle and by solving the Hamilton-Jacobi-Bellman (HJB) equation which is a nonlinear partial differential equation. We refer to Fleming and Soner [16] for an introduction into the subject of stochastic optimal control. In Merton [36] the HJB equation was derived when the asset price dynamics are modeled by a more general stochastic process than the geometric Brownian motion. Thus, the results in Merton [35, 36] constitute the starting point of continuous-time portfolio optimization. Thereafter, there has been a vast stream of literature containing generalization models and methods in the field of portfolio optimization. To mention all of them is beyond the scope of this introduction. A survey can be found for example in [11, 22].

Amongst others, one generalization of Merton's portfolio optimization problem concerns the asset price dynamics. By assuming a geometric Brownian motion for the evolution of asset prices, one precludes the possibility of large price jumps because the price process has continuous paths. This implies that prices cannot change in an extraordinary magnitude within a small time interval. But, historical events and empirical studies have shown that they indeed show jumps, for example, a market crash as the financial crisis starting in 2008 induced a sudden downward jump in prices. The first ideas to overcome this modeling drawback came from Merton [37] in the context of option pricing and from Aase [1] in the context of optimal portfolio selection. Aase [1] extended Merton's portfolio optimization problem by using an additional point process. This point process then allows the prices to jump at random times. Further examples for using a so-called jump-diffusion process in portfolio optimization problems can be found in Jeanblanc-Picqué and Pontier [19] and Jonek [20] where the price process is a solution of a linear stochastic differential equation driven by a Brownian motion and a Poisson process. From today's perspective this is the standard way to introduce the possibility of price jumps. The resulting wealth process is again a process with discontinuous paths which can be controlled by the investor's investment and consumption decisions in order to maximize the expected utility of consumption and terminal wealth. Even Lévy processes have been used in previous work for the modeling of asset prices. Here, we refer to [8] for a survey of financial modeling with jump processes. If practitioners model a market with jump processes they have to know the distribution of jump times and jump heights. For example, using a Poisson process implies that the inter-arrival times of the jumps are exponentially distributed random variables. In practice, it is not easy to verify the distribution and corresponding parameters of price jumps since e.g. market crashes are rather rare events.

Instead, Korn and Wilmott [29] proposed to model market crashes without any distributional assumptions on crash times and heights, that is, they modeled crashes as uncertain, rather than risky events as in the jump-diffusion framework above. The idea is only to assume that the maximum number of crashes, which can occur on a given time interval, and the maximum crash size l^* are known in advance. At so-called 'normal' times between two jumps, the prices of the risky assets follow a geometric Brownian motion, whereas at the crash time the asset prices become highly correlated and lose a fraction $l \in [0, l^*]$ of their value. The investor takes an extremely cautious attitude towards the crash uncertainty and aims to maximize his expected utility of terminal wealth in the worst-case crash scenario. In the jump-diffusion framework, the investor chooses strategies which hedge a crash in mean. Korn and Wilmott [29, p.1] argued that this 'is no real protection against the consequences of a jump at all'. In contrast, the worst-case approach protects the investor from the worst possible crash that can happen. The investor's risk preferences in portfolio optimization problems are often modeled by utility functions $U : (0, \infty) \to \mathbb{R}$, that is, strictly concave, monotonously increasing and continuously differentiable functions. Then, on the time interval $[0, T]$, the worst-case portfolio optimization problem by Korn and Wilmott [29] has the form:

$$\sup_{k \in \Pi(x)} \inf_{0 \le \tau \le T, 0 \le l \le l^*} \mathbb{E}\left(U(X_T^k)\right),$$

where $k = \{k_t\}_{t \in [0,T]}$ denotes the fraction of wealth invested in a stock, $\Pi(x)$ is the set of admissible controls under the condition that the wealth process $X^k = \{X_t^k\}_{t \in [0,T]}$ starts in $X_0^k = x > 0$. The stopping time τ and the random variable l denote the crash time and height, respectively. In the problem above, note that it is assumed that at most one market crash can happen on $[0, T]$. Korn and Wilmott [29] obtained the worst-case optimal investment strategy for an investor with utility function $U(x) = \log(x)$. The worst-case optimal strategy depends on time t and is a solution of an ordinary differential equation. In Korn and Menkens [25] the assumption of a logarithmic utility function was relaxed by considering a more general class of utility functions, so-called HARA (hyperbolic absolute risk aversion) utility functions. Therein, the authors used an analogue method to the Bellman principle and the classical HJB equation to determine the optimal investment strategy. Korn and Steffensen [28] applied a

method based on HJB-type inequalities to determine worst-case optimal portfolios in a market model with at most n crashes. Menkens [34] considered the case where the expected return on the risky asset is smaller than the return on the non-risky asset. Based on interpreting the worst-case optimization problem as a controller vs. stopper game, Seifried [44] introduced a new martingale approach and applied it to a worst-case portfolio problem for rather general asset price dynamics. For a survey about ideas, results and methods behind the worst-case approach in portfolio optimization with a finite time horizon we refer to [26]. Recently, further generalizations of the worst-case optimization problem above were considered. For example, Belak et al. [2] considered a problem with a random number of crashes and Belak et al. [3] extended the model by introducing proportional transaction costs.
The martingale approach developed by Seifried [44] was extended by Desmettre et al. [10] in order to solve an infinite horizon worst-case investment and consumption problem where at most one market crash can happen. In this case the market crash is interpreted as a once-in-a-life time event. The investor aims to maximize his expected discounted utility of consumption in the worst-case crash scenario. The problem reads as follows:

$$\sup_{(k,c)\in\Pi} \inf_{(\tau,l)\in\mathcal{C}} \mathbb{E}\left(\int_0^\infty e^{-\varepsilon t} U(c_t X_t^{k,c})\, dt\right),$$

where $c = \{c_t\}_{t\geq 0}$ denotes the rate at which the investor consumes. $X^{k,c} = \{X_t^{k,c}\}_{t\geq 0}$ describes the wealth process controlled by the investment and consumption strategy (k, c), ε is the discount factor and $\mathcal{C}$ describes the set of possible crash scenarios.
The worst-case approach introduced by Korn and Wilmott [29] was also applied in the context of actuarial sciences, see for example Korn [23] and Korn et al. [30].

The worst-case portfolio optimization problems mentioned above allow the investor to invest either in risky assets, which are threatened by one or more market crashes, or to invest in a savings account. A common feature of the literature about worst-case portfolio optimization is that the interest rate of the savings account is constant. This assumption is quite restrictive, since interest rates indeed change randomly from time to time due to fluctuations on financial markets.
For classical portfolio optimization problems (without asset price jumps) this restriction was already relaxed by assuming that the instantaneous interest rate, or briefly short rate, also follows a stochastic process. The short rate is usually denoted by r_t and the value of the savings account is given by

$$B_t = B_0 \exp\left(\int_0^t r_s\, ds\right)$$

for some initial value $B_0 > 0$. Several short rate models, which describe the evolution of the stochastic process $\{r_t\}_{t\geq 0}$, were published in the past and we refer to [5, 6] for an overview. For example, the Vasicek model [45] and the Cox-Ingersoll-Ross model [9] are classical models used for short-rate processes. Amongst others, Korn and Kraft [24] and Kraft [31] considered finite time horizon portfolio optimization problems with stochastic interest rates and determined the optimal investment strategy by applying the stochastic control approach. Therein, they investigated the problem for short rate models by Ho and Lee [18], Vasicek [45], Dothan [12], Black and Karasinski [4] and Cox et al. [9]. Moreover, infinite horizon investment and consumption problems with stochastic interest rates have been investigated in Fleming and Pang [15] and Pang [39]. Therein, a so-called generalized Vasicek model was considered.

The aim of this thesis is to overcome the restriction of constant interest rates within the worst-case optimization framework. We consider worst-case investment and consumption models where the underlying interest rates of the savings account evolve randomly over time and are correlated with the risky asset price. More precisely, we assume that the investor can invest either in a stock or in a savings account. As proposed by Korn and Wilmott [29] the stock price is threatened by one or more market crashes. Our contribution to previous research is that the interest rate of the savings account is stochastic. Basically, we investigate two different worst-case optimization problems.

First, we study a worst-case optimization problem on a finite time interval $[0, T]$ where at most N market crashes can occur and the short rate follows a Vasicek process. We determine the worst-case optimal investment strategy which maximizes the investor's utility of terminal wealth. We restrict our considerations on the class of HARA utility functions. We apply stochastic optimal control theory to determine the strategy which is optimal after the N-th market crash has happened and prove a suitable verification result. In order to determine the optimal pre-crash strategies we apply two alternative methods. The first one is based on solving a HJB-type inequality system which is an analogue to the system considered in Korn and Steffensen [28]. The second one recursively applies the martingale approach by Seifried [44]. Both methods are adapted to the case of stochastic interest rates. Our main findings are an explicit characterization of the worst-case optimal investment strategy and the analysis of its actual form. If the investor's risk preference is modeled by a non-log HARA utility function, the optimal strategies differ from the ones in, e.g. Korn and Steffensen [28] and Seifried [44], due to the influence of the stochastic interest rates. The reason for this is the correlation between the Brownian motions driving the interest rate and the stock prices. Furthermore, we obtain that a logarithmic utility function eliminates the stochastic interest rate risk such that the optimal strategy is the same as for constant interest rates. For the logarithmic utility function we additionally determine the worst-case optimal strategy for a more general short rate model, namely the general affine short rate model, and under the assumption that market parameters change at the crash time.

Second, we consider a worst-case investment and consumption problem with an infinite time horizon, where at most one market crash can happen. Therein, the investor's aim is to maximize his expected discounted logarithmic utility of consumption in the worst-case crash scenario. Within this framework, we assume a generalized Vasicek model (see [15, 39]) and the general affine short rate model. By applying stochastic optimal control theory and by solving the HJB equation with the sub- and supersolution method, we determine the optimal strategy valid after the market crash and the corresponding value function. Then, by applying the martingale approach we derive the worst-case optimal strategy valid before the crash.

The thesis is organised in two main chapters: Chapter 2 contains the worst-case optimal investment problem with a finite time horizon and Chapter 3 is devoted to the study of the worst-case optimal investment and consumption problem with an infinite time horizon.

In Section 2.1 we introduce the financial market model. We give a detailed introduction to the short rate models, we motivate the asset price dynamics and formulate the worst-case optimization problem. In Section 2.2 and in Section 2.3 we derive the worst-case optimal investment strategy for the non-log HARA utility functions and the logarithmic utility function, respectively. In both sections, we solve HJB-type inequalities and prove that their solutions are equal to the value function. Furthermore, in Section 2.4, we provide an alternative way to calculate the value function by recursive application of the martingale approach by Seifried [44]. While Section 2.2- 2.4 contains the Vasicek model, we consider a general affine short rate model in Section 2.5 for logarithmic risk preferences. This section additionally contains the assumption that market parameters, such as excess return and volatility of the stock, change after the market crash has happened.
In Chapter 3 we proceed as follows. In Section 3.1 we explain the financial market model and precisely formulate the worst-case optimization problem. The worst-case optimal investment and consumption strategies for the generalized Vasicek model and the general affine short rate model are derived in Section 3.2 and Section 3.3, respectively. We adapt the martingale approach by Desmettre et al. [10] to stochastic interest rates. Section 3.4 deals with a generalization of the financial market model. We assume that market parameters may change after the market crash, but nothing is known about them in advance, except from the fact that they take values in given intervals. The investor takes a cautious attitude towards this uncertainty and maximizes his expected discounted utility of consumption in the worst-case scenario with respect to the crash and the post-crash parameters.
Finally in Chapter 4, we draw a conclusion and give remarks on possible future research.

CHAPTER 2

Worst-Case Optimal Investment with a Finite Time Horizon

2.1. Introduction of the financial market model

On the financial market, which we consider here, the investor is allowed to invest both in a savings account and in a stock. We assume that the investor is acting on a given finite time interval $[0,T]$. In contrast to the classical investment model by Merton [35], where a diffusion process with continuous paths is used to model the stock price evolution, here, we assume that the stock price evolution may have discontinuities, which represent crashes on the financial market. This means, instead of a stochastic process with continuous paths, we use a stochastic process whose paths may have sudden downward jumps. The market crashes are modeled as uncertain events, as first Korn and Wilmott [29] proposed in their work about optimal portfolios under the threat of a crash. Moreover, the instantaneous interest rate of the savings account is also assumed to follow a stochastic process which is not affected by the market crashes. The investor's aim is to choose his investment strategy such that the expected utility of terminal wealth is maximized in the worst-case crash scenario. In Section 2.1.1, we model the value of the savings account and motivate the use of stochastic interest rates. In Section 2.1.2, we introduce the stock price equation and then, in Section 2.1.3, we derive the investor's wealth equation and formulate the corresponding worst-case optimization problem.
Throughout the thesis we make the following basic assumptions:
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given complete probability space with filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$. All processes below are defined on this probability space. Moreover, $\{\mathcal{F}_t\}_{t\in[0,T]}$ is extended to $[0,T]\cup\{\infty\}$ by setting $\mathcal{F}_\infty := \mathcal{F}_T$ and a process $\{Y_t\}_{t\in[0,T]}$ is extended to $[0,T]\cup\{\infty\}$ by letting $Y_\infty := Y_T$.

2.1.1. The savings account with stochastic interest rates. For $t \in [0,T]$ let B_t denote the value of the savings account at time t. As usual, under time-continuous interest payments we assume that

$$B_t = B_0 \exp\left(\int_0^t r_s\, ds\right),$$

where B_0 is the given price at $t = 0$ and r_s denotes the instantaneous interest rate, which is briefly referred to as short rate. Obviously, $B = \{B_t\}_{t\in[0,T]}$ is the solution of the following differential equation:

$$dB_t = r_t B_t\, dt. \tag{1}$$

In contrast to previous financial market models within the worst-case optimization framework, where $r_t \equiv r$ for some given constant $r > 0$, we assume here that the short rate $r = \{r_t\}_{t\in[0,T]}$ is modeled as a stochastic process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In the context of pricing financial products, for example in bond pricing, different short rate models have been established in the past. In this thesis, we consider short rate models which belong to the class of affine term structure models. Note that a short rate model is called affine if the price of a corresponding zero coupon bond at time t with maturity T takes the form

$$p(t,T) = e^{A(t,T)-B(t,T)r_t},$$

where $A(t,T)$ and $B(t,T)$ are some deterministic functions. In general, a sufficient condition for a model to display this affine term structure is that it follows a stochastic differential equation (SDE) of the form (see e.g. [5, Chp.3.2.4]):

$$dr_t = (\lambda_1(t)r_t + \lambda_2(t))\,dt + \sqrt{\xi_1(t)r_t + \xi_2(t)}\,d\tilde{w}_t,$$

for suitable deterministic functions $\lambda_1, \lambda_2, \xi_1, \xi_2$ and a Wiener process $\tilde{w} = \{\tilde{w}_t\}_{t\in[0,T]}$.
In Sections 2.2, 2.3 and 2.4, we consider a famous special case of the affine term structure model, namely the Vasicek model [45], where the short rate evolves as an Ornstein-Uhlenbeck process with constant coefficients. Thus, we assume that $\lambda_1(t) \equiv -a$, $\lambda_2(t) \equiv ar_M$, $\xi_1(t) \equiv 0$ and $\xi_2(t) \equiv \sigma_2^2$ for some positive constants a, r_M, σ_2. More precisely, in these sections the short rate is assumed to be a solution of the following SDE:

$$\begin{aligned} dr_t &= a(r_M - r_t)\,dt + \sigma_2\,d\tilde{w}_t, \\ r_0 &= r^0 > 0, \end{aligned} \tag{2}$$

where r^0 is some positive constant. This model was first proposed by Vasicek [45] and therefore the solution of SDE (2) is often called *Vasicek process*. Using Ito's formula, we can calculate a closed form solution of (2) such that r_t is given by

$$r_t = r_0e^{-at} + r_M(1-e^{-at}) + \sigma_2\int_0^t e^{-a(t-u)}\,d\tilde{w}_u.$$

Since the stochastic integral is a normally distributed random variable, r_t is normally distributed with mean

$$\mathbb{E}(r_t) = r^0e^{-at} + r_M\left(1-e^{-at}\right),$$

and variance

$$Var(r_t) = \frac{\sigma_2^2}{2a}\left(1-e^{-2at}\right).$$

Obviously, it holds:

$$\lim_{t\to\infty}\mathbb{E}(r_t) = r_M.$$

Hence r_M is called *long term mean level* of the short rate. Moreover, $a > 0$ denotes the speed of reversion to r_M and $\sigma_2 > 0$ describes the volatility. Furthermore, the Vasicek process has the mean reverting property, that means if $r_t < r_M$ ($r_t > r_M$), then the dt term in (2) is positive (negative), such that r_t is pushed closer to r_M. The Vasicek model is not only used due to its means reversion property but also due to its analytic tractability in the context of bond pricing. But, the model has one major drawback. Since r_t is normally distributed, the short rate r_t can become negative with positive probability. Nevertheless, the Vasicek process is often used in the literature to model short rate dynamics and can be used to approximate more realistic short rate models (see e.g. [7]).

In Section 2.5, we consider a financial market model with a more general short rate dynamics than the Vasicek process. We assume an affine term structure model of the form:

$$\begin{aligned} dr_t &= (\lambda_1 r_t + \lambda_2)\, dt + \sqrt{\xi_1 r_t + \xi_2}\, d\tilde{w}_t \\ r_0 &= r^0 > 0, \end{aligned} \tag{3}$$

where $\lambda_1, \lambda_2, \xi_1, \xi_2, r^0$ are given constants. For $\lambda_1 = -a$, $\lambda_2 = ar_M$, $\xi_1 = \sigma_2^2$ and $\xi_2 = 0$, the short rate above refers to the *Cox-Ingersoll-Ross process* [9] which is a solution of the SDE:

$$dr_t = a(r_M - r_t)\, dt + \sigma_2 \sqrt{r_t}\, d\tilde{w}_t.$$

In this case r_t is noncentral chi-squared distributed with mean

$$\mathbb{E}(r_t) = r^0 e^{-at} + r_M \left(1 - e^{-at}\right)$$

and variance

$$Var(r_t) = r^0 \frac{\sigma_2^2}{a}\left(e^{-at} - e^{-2at}\right) + r_M \frac{\sigma_2^2}{2a}\left(1 - e^{-at}\right)^2.$$

The Cox-Ingersoll Ross process also has the mean reversion property with long term mean level r_M and speed of reversion a. In contrast to the Vasicek model, the diffusion coefficient contains a square root term and the short rate is positive with probability one if $2ar_M > \sigma_2^2$ and always nonnegative if we have the opposite inequality. Thus, in Section 2.5, by considering a short rate of the more general form (3), we also cover the Cox-Ingersoll-Ross model.
Since we restrict our considerations in this thesis on the models explained above, we refer to Brigo and Mercurio [5] and Cairns [6] for other short rate models.

2.1.2. The stock price process and the modeling of market crashes. In addition to the investment in the savings account, the investor can invest in a stock which is threatened by significant market fluctuations, that means price jumps of extraordinary magnitude. After his pioneering work about continuous-time portfolio optimization in [35], Merton later proposed to use jump-diffusion processes to allow large price changes with a positive probability. This approach assumes that market crashes are risky events. For example a Poisson process or, more generally, a Lévy process can be used to extend classical diffusion processes. We refer to Cont and Tankov [8] for an overview of financial modeling with jump processes. All these models have a common assumption: the distribution of the crash time and the crash size is known. If these information about the market crashes would be available, one could model the stock price process, denoted by $P = \{P\}_{t\in[0,T]}$, as the solution of the following jump-diffusion SDE:

$$\begin{aligned} dP_t &= P_{t-}\left[(\mu + r_t)\, dt + \sigma_1\, dw_{1,t} + dQ_t\right], \\ P_0 &= p^0 > 0, \end{aligned}$$

where μ and σ_1 are positive constants and $w_1 = \{w_{1,t}\}_{t\in[0,T]}$ is another Wiener process. Within this framework, one could assume that $Q = \{Q_t\}_{t\in[0,T]}$ is a compound Poisson process with

$$Q_t := \sum_{i=1}^{N_t} Y_i,$$

where $\{N_t\}_{t\in[0,T]}$ is a Poisson process with a given intensity and independent of w_1, which counts the number of jumps and $Y_i > -1$ are i.i.d. random variables which describe the jump size of the i-th jump. Using Ito's formula one can derive the solution of the jump-diffusion SDE above and obtain:

$$P_t = p^0 \cdot \exp\left\{ \int_0^t \mu + r_s - \frac{\sigma_1^2}{2}\, ds + \sigma_1 w_{1,t} \right\} \cdot \prod_{i=1}^{N_t} (1 + Y_i). \tag{4}$$

Thus, at 'normal times' the stock price evolves as a geometric Brownian motion with drift $(\mu + r_t)$ and diffusion coefficient σ_1, and at jump times τ_i the stock price changes in the following way

$$P_{\tau_i} = P_{\tau_i -}(1 + Y_i), \qquad i = 1, \dots, N_t.$$

Therefore, there is a market crash if Y_i is negative. Then, the stock price loses a fraction $(1+Y_i)$ at the crash time. Since Q is a Poisson process, the time between two market crashes is assumed to be exponentially distributed. The jump sizes Y_i are random variables with a given distribution. But often, these information about the time and size of a crash are not available. Korn and Wilmott [29] argued that market crashes are rare such that the distributions of crash times and crash sizes are difficult to quantify. Instead, they proposed to model market crashes as uncertain events, that means they assumed that there can happen a maximum number N of market crashes on the time interval $[0, T]$. A second assumption is that the crash sizes are bounded from above by some given constant. Thus, no distributional assumptions about the event 'market crash' were imposed. Without any distributional assumptions on the crash time and size it is not meaningful to solve a classical utility maximization problem as did in [35]. The idea of Korn and Wilmott [29] was to assume that the investor takes a very cautious attitude towards the uncertain event 'market crash'. Thus, a worst-case optimization problem was formulated. After that, the worst-case approach was extended in several directions, but all assumed a constant interest rate of the savings account. Here, we adopt the modeling of market crashes for a financial market model with stochastic instantaneous interest rates.

As in Korn and Steffensen [28], we assume that there can happen at most N market crashes on $[0, T]$, where N is a given positive number. The i-th market crash is denoted by a pair (τ_i, l_i), where the crash time of the first market crash τ_1 is a $[0, T] \cup \{\infty\}$-valued stopping time and the crash time of the i-th market crash τ_i is a $(\tau_{i-1}, T] \cup \{\infty\}$-valued stopping time (for $i = 2, \dots, N$). Here, the event $\tau_i = \infty$ describes the case if no crash occurs at all. In this setting we assume that market crashes cannot happen at the same time. The $l_i \in [0, l^*]$ denotes the crash size, which is a $\mathcal{F}_{\tau_i}$-measurable random variable. The maximum crash size $l^* < 1$ is assumed to be given and equal for each market crash. Now, the price of the stock at time t, denoted by P_t, follows a geometric Brownian motion at normal times $t \in (\tau_i, \tau_{i+1})$ and at the i-th market crash it loses a fraction l_i of its value. Given a crash sequence of $n \le N$ market crashes, denoted by $(\tau_i, l_i)_{i \in \{1,\dots,n\}}$, we assume that the stock price process $P = \{P_t\}_{t \in [0,T]}$ fulfills the following equations:

$$\begin{aligned} P_0 &= p^0 > 0, \\ dP_t &= P_t \left[(\mu + r_t)\, dt + \sigma_1\, dw_{1,t} \right], \qquad && t \in (\tau_i, \tau_{i+1}), \quad i = 0, \dots, n, \\ P_{\tau_i} &= P_{\tau_i -}(1 - l_i), && i = 1, \dots, n, \end{aligned} \tag{5}$$

where $\tau_0 := 0$ and $\tau_{n+1} := T$ and μ, σ_1 and p^0 are some positive constants, and $P_{\tau_i -} := \lim_{s \nearrow \tau_i} P_s$. In comparison to the jump-diffusion model (4), the jump sizes Y_i correspond to $-l_i$

and the jump times τ_i correspond to the crash times, and the main difference is that nothing is assumed about the distribution of these variables.
Note that the excess return in the financial market model is given by the constant $\mu > 0$. Moreover σ_1 denotes the volatility of the stock price. $w_1 = \{w_{1,t}\}_{t\in[0,T]}$ is a Wiener process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which may be correlated with the Wiener process $\tilde{w}$. Thus, we have that

$$\mathbb{E}(w_{1,t} \cdot \tilde{w}_t) = \rho t, \qquad \forall\, t \in [0,T],$$

where $\rho \in [-1,1]$ denotes the correlation coefficient. In order to encode the information which is available at time t to the investor, we define

$$\tilde{N}_t := \#\{0 < s \leq t : P_s \neq P_{s-}\}, \tag{6}$$

which counts the market crashes until time t and assume that the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ is generated by the processes w_1, $\tilde{w}$ and $\tilde{N} = \{\tilde{N}_t\}_{t\in[0,T]}$.

2.1.3. Admissible controls and the worst-case optimization problem. Now the investor's behavior is described by a self-financing portfolio process $k = \{k_t\}_{t\in[0,T]}$ which denotes the fraction of his wealth invested in the stock. Accordingly, $1 - k_t$ describes the fraction of wealth invested in the savings account at time t. Below, we use the notation

$$k = \left(k^{(0)}, k^{(1)}, \ldots, k^{(N)}\right),$$

where $k^{(j)}$ denotes the investment strategy if j market crashes can still occur, that means $k_t^{(j)}$ is valid for $t \in (\tau_{N-j}, \tau_{N-j+1}]$. According to this notation, we define the set of admissible controls.

Definition 2.1.1 (Admissible Control). *A process $k = \left(k^{(0)}, k^{(1)}, \ldots, k^{(N)}\right)$, where $k^{(j)} = \{k_t^{(j)}\}_{t\in[0,T]}$ denotes the strategy which is valid on the interval $(\tau_{N-j}, \tau_{N-j+1}]$, is called admissible control if it fulfills the following conditions:*

(1) k is a $\mathbb{F}$-adapted process,
(2) $k_t \in A$ for $t \in [0,T]$, where $A \subset \mathbb{R}$ is compact,
(3) $k_t^{(j)} < \frac{1}{l^*}$ for $j = 1, \ldots N, t \in [0,T]$,
(4) For $j = 0, \ldots, N$: $k^{(j)}$ has continuous paths.

The set of admissible controls is denoted by Π.

Remark 2.1.2. *(i) We assume that the investment strategy has to be $\mathbb{F}$-adapted, which means that the investors decides on his strategies at time t based on information until time t. Thus, for every $t \in [0,T]$ he can conclude how many crashes still can occur. Moreover, we do not restrict the strategies to be nonnegative. This is mainly due to the fact that the optimal strategy after the N-th market crash can indeed be negative (see Theorem 2.2.2 below). It would therefore be conceptually bad to exclude negative strategies. Thus, in comparison to the literature about worst-case optimization with constant interest rates, we allow short selling of the stock.*
(ii) Note that condition 1 together with condition 4 implies that $k_t^{(j)}$ is progressively measurable with respect to $\mathbb{F}$.

Let $\mathcal{N}(t,n)$ be the set of possible crash sequences $M = (\tau_j, l_j)_{j\leq n}$ on $[t,T]$, if there are at most n crashes left at time t. Moreover, given a strategy $k \in \Pi$ and a crash sequence $M = (\tau_j, l_j)_{j\leq n}$

with $n \leq N$ market crashes, we denote by $X_t^{k,M}$ the investor's wealth at time t. Using the stock price equation (5), we can derive the SDE for the wealth process $X^{k,M} = \{X_t^{k,M}\}_{t\in[0,T]}$:

$$\begin{aligned} X_0^{k,M} &= x^0 > 0, \\ dX_t^{k,M} &= X_t^{k,M}\left[r_t + \mu k_t^{(j)}\right] dt + X_t^{k,M}\sigma_1 k_t^{(j)}\, dw_{1,t}, \quad && t \in (\tau_{n-j}, \tau_{n-j+1}), \\ & && j = 0, \dots, n, \\ X_{\tau_j}^{k,M} &= (1 - l_j k_{\tau_j}^{(n-j+1)}) X_{\tau_{j-}}^{k,M}, && j = 1, \dots, n, \end{aligned} \tag{7}$$

where $\tau_0 := 0$ and $\tau_{n+1} := T$.

Remark 2.1.3. *Condition 2 in Definition 2.1.1 implies that*

$$\mathbb{E}\left(\int_0^T |k_t^{(j)}|^m \, dt\right) < \infty \qquad \textit{for} \quad m = 1, 2, \dots, \; j = 1, \dots, N.$$

This, together with condition 3, ensures that the wealth stays nonnegative for all $t \in [0, T]$ $\mathbb{P}$-a.s.

As in [26, 28], the investor's aim is to maximize his expected utility of wealth at the terminal time T in the worst-case scenario. Thus, the investor is extremely cautious towards the uncertainty about the market crashes. Using the notations and definitions above, we formulate the worst-case optimization problem:

$$\sup_{k\in\Pi(0,x^0,r^0)} \inf_{M\in\mathcal{N}(0,N)} \mathbb{E}\left(U(X_T^{k,M})\right), \tag{8}$$

where $U : (0, \infty) \to \mathbb{R}$ denotes the investor's utility function, which is assumed to be strictly concave, continuously differentiable and

$$\lim_{x\to 0+} U'(x) = \infty \qquad \text{and} \qquad \lim_{x\to\infty} U'(x) = 0.$$

Moreover, $\Pi(t, x, r)$ denotes the set of admissible controls corresponding to the condition that $X_t^{k,M} = x$ and $r_t = r$.

In this thesis we model the investor's risk preferences by so-called HARA (hyperbolic absolute risk aversion) utility functions U. In Sections 2.2 and 2.4, we assume that

$$U(x) = \frac{1}{\gamma} x^\gamma, \qquad \gamma < 1, \gamma \neq 0. \tag{9}$$

If the investor chooses a utility function of this class for a certain $\gamma < 1$, then he has a hyperbolic absolute risk aversion of $(1-\gamma)x^{-1}$ and a constant relative risk aversion (CRRA for short) of $1-\gamma$. Thus, these utility functions are also of CRRA-type. The higher the CRRA, the higher the investor's risk aversion. The measures of absolute and relative risk aversion are invariant under positive linear transformation of the utility function U. Considering a transformed version of (9) given by

$$U^{trans}(x) := \frac{1}{\gamma} x^\gamma - \frac{1}{\gamma}, \qquad \gamma < 1, \gamma \neq 0,$$

one easily obtains that

$$\lim_{\gamma\to 0} U^{trans}(x) = \log(x).$$

Thus, $U(x) = \log(x)$ corresponds to the limit case of $\gamma \to 0$ and will be considered in Sections 2.3 and 2.5.

As in [28], we can interpret the worst-case optimization problem (8) as a game between the investor and the market. Here, the investor chooses the strategy k and the market chooses the crash sequence M with at most N market crashes (τ_i, l_i). Due to the stochastic interest rate of the savings account, our financial market model provides a generalization of models considered in [25, 26, 28, 29].

Now, let us define the value function for $n = 0, 1, \dots, N$:

$$V^n(t,x,r) := \sup_{k \in \Pi(t,x,r)} \inf_{M \in \mathcal{N}(t,n)} \mathbb{E}^{t,x,r,n}\left(U(X_T^{k,M})\right), \tag{10}$$

where $\mathbb{E}^{t,x,r,n}$ denotes the conditional expectation given that $X_t^{k,M} = x$, $r_t = r$ and there are at most n crashes left at time $t \in [0,T]$. Hence, $V^n : [0,T] \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ gives the value of the worst-case optimal expected utility of terminal wealth if the wealth process and the short rate process start at time t with x and r, respectively, and there can happen at most n market crashes on $[t,T]$.

Analogously to the approach of [28], in Sections 2.2 and 2.3, we try to solve a so-called *HJB-inequality system* in order to obtain the value function V^n and the worst-case optimal strategy for problem (8). First, we will use this approach to solve the worst-case optimization problem for the class of non-log HARA utility functions in Section 2.2. Afterwards, in Section 2.3, we investigate the problem for a logarithmic utility function. In Sections 2.4 and 2.5, we show how to proceed using the martingale approach, which was recently developed in [44].

In what follows, we need the following operator. For each $v \in C^{1,2,2}([0,T] \times \mathbb{R}_+ \times \mathbb{R})$ we define the operator $\mathcal{L}^k$ by

$$\begin{aligned}\mathcal{L}^k v(t,x,r) =& v_t(t,x,r) + (\mu k + r)x\, v_x(t,x,r) + \frac{\sigma_1^2}{2} k^2 x^2\, v_{xx}(t,x,r) \\ &+ \rho\sigma_1\sigma_2 kx\, v_{xr}(t,x,r) + a(r_M - r)\, v_r(t,x,r) + \frac{\sigma_2^2}{2} v_{rr}(t,x,r).\end{aligned}$$

2.2. The N-crash market with HARA utility (non-log utility)

In this section, the aim is to determine the worst-case optimal investment strategy for an investor with a non-log utility function of HARA-type under the assumption that the short rate dynamics is given by a Vasicek process of the form (2).

First of all, we give a Corollary which ensures that the value function, defined in (10), is well-defined.

Corollary 2.2.1. *For $(t,x,r) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}$ and $n \leq N$, let $k \in \Pi(t,x,r)$ be an arbitrary admissible strategy and let M be an arbitrary crash sequence of length n on $[t,T]$, which fulfills the assumptions above. Moreover, let $\{r_t\}_{t \in [0,T]}$ and $X^{k,M} = \{X_t^{k,M}\}_{t \in [0,T]}$ be given by (2) and (7), respectively. Then*

$$\mathbb{E}^{t,x,r,n}\left(\left|\frac{1}{\gamma}(X_T^{k,M})^\gamma\right|\right) < \infty.$$

PROOF. Let $(t,x,r) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}$ be arbitrary but fixed. For $s \in (\tau_j, \tau_{j+1})$ for some $j \in \{0,\dots,n\}$, where $\tau_0 = t$ and $\tau_{n+1} = T$ we have

$$X_s^{k,M} = x \exp\left(\int_t^s \mu k_u - \frac{\sigma_1^2}{2}(k_u)^2 + r_u \, du + \int_t^s \sigma_1 k_u \, dw_{1,u} \right) \prod_{i=1}^{j} (1 - l_i k_{\tau_i}^{(n-i+1)}).$$

Thus, with $x > 0$ fixed, we obtain

$$\left| \frac{1}{\gamma} (X_T^{k,M})^\gamma \right| = |\gamma^{-1}| x^\gamma \exp\left(\gamma \int_t^T \mu k_u - \frac{\sigma_1^2}{2}(k_u)^2 + r_u \, du + \gamma \int_t^T \sigma_1 k_u \, dw_{1,u} \right) \cdot \prod_{i=1}^{n} |1 - l_i k_{\tau_i}^{(n-i+1)}|^\gamma.$$

By assumption, it holds $l_i \in [0, l^*]$ with $l^* < 1$, $k_u^{(j)} < \frac{1}{l^*}$ for all $j = 1,\dots,n$, $u \in [0,T]$ and therefore $1 - l_i k_{\tau_i}^{(n-i+1)} > 0$ for $i = 1,\dots,n$. Moreover, we assumed that $k_u \in A$ where $A \subset \mathbb{R}$ is compact and therefore the product is bounded

$$0 < \prod_{i=1}^{n} (1 - l_i k_{\tau_i}^{(n-i+1)})^\gamma \le K_1$$

for some constant $K_1 > 0$. Moreover, by Proposition A.1.1 in Appendix A, we have that

$$\int_t^T r_u \, du = \frac{r}{a}(1 - e^{-a(T-t)}) + r_M \left((T-t) - \frac{1 - e^{-a(T-t)}}{a} \right) + \sigma_2 \int_t^T \frac{1 - e^{-a(T-u)}}{a} \, d\tilde{w}_u.$$

Since $\tilde{w}$ and w_1 are correlated with coefficient ρ, we can replace $d\tilde{w}_t$ by

$$\rho \, dw_{1,t} + \sqrt{1-\rho^2} \, dw_{2,t},$$

where w_2 is a Wiener process independent of w_1. Since $k_u \in A$ for $A \subset \mathbb{R}$ compact, we obtain

$$\begin{aligned} &\left| \frac{1}{\gamma} (X_T^{k,M})^\gamma \right| \\ &\le K \exp\left(\int_t^T \underbrace{\frac{\gamma \sigma_2}{a}(1 - e^{-a(T-u)})}_{=:\tilde{I}(u)} \, d\tilde{w}_u + \int_t^T \underbrace{\gamma \sigma_1 k_u}_{=:I(u)} \, dw_{1,u} \right) \\ &= K \exp\left(\int_t^T [I(u) + \rho \tilde{I}(u)] \, dw_{1,u} + \int_t^T \sqrt{1-\rho^2} \tilde{I}(u) \, dw_{2,u} \right) \le K Z_T, \end{aligned}$$

where $K > 0$ is a universal constant and

$$Z_s := \exp\left(\int_t^s [I(u) + \rho \tilde{I}(u)] \, dw_{1,u} + \int_t^s \sqrt{1-\rho^2} \tilde{I}(u) \, dw_{2,u} - \frac{1}{2} \int_t^s [I(u) + \rho \tilde{I}(u)]^2 \, du - \frac{1}{2} \int_t^s (1-\rho^2) \tilde{I}^2(u) \, du \right).$$

Then, $\{Z_s\}_{s \in [t,T]}$ is the uniquely determined solution of

$$dZ_s = Z_s \left[(I(s) + \rho \tilde{I}(s)) \, dw_{1,s} + \sqrt{1-\rho^2} \tilde{I}(s) \, dw_{2,s} \right], \quad Z_t = 1.$$

Then, Krylov [32, Chp. 5.2, Cor.12] implies

$$\mathbb{E}\left(\sup_{s\in[t,T]}|Z_s|\right) < \infty.$$

Finally, taking the expectation on both sides of the inequality above implies that

$$\mathbb{E}^{t,x,r,n}\left(\left|\frac{1}{\gamma}(X_T^{k,M})^\gamma\right|\right) \le K\mathbb{E}^{t,x,r,n}\left(|Z_T|\right) < \infty.$$

□

The following theorem is our main result of this chapter which determines the worst-case optimal investment strategy in the N-crash market in an explicit form:

Theorem 2.2.2 (Worst-Case Optimal Investment Strategy).
Assume that the short rate process $\{r_t\}_{t\in[0,T]}$ is a Vasicek process of the form (2) and assume that the wealth process $X^{k,M}$ is given by (7).
a) Let $k_t^{(0)*}$ be given by

$$k_t^{(0)*} = \frac{\mu}{(1-\gamma)\sigma_1^2} + \frac{\rho\sigma_2\beta(t)}{(1-\gamma)\sigma_1}, \quad \text{where} \quad \beta(t) = \frac{\gamma}{a}\left[1-\exp(-a(T-t))\right] \tag{11}$$

and let

$$v^0(t,x,r) = \frac{1}{\gamma}x^\gamma g^{(0)}(t)\exp(\beta(t)r) \tag{12}$$

where $g^{(0)}(t)$ solves the ordinary differential equation (ODE for short):

$$\begin{aligned}\dot{g}^{(0)}(t) + g^{(0)}(t)\Big(&\gamma(\mu+\rho\sigma_1\sigma_2\beta(t))k_t^{(0)*} - \frac{\sigma_1^2}{2}\gamma(1-\gamma)(k_t^{(0)*})^2 \\ &+ ar_M\beta(t) + \frac{\sigma_2^2}{2}\beta^2(t)\Big) = 0,\\ g^{(0)}(T) = 1.&\end{aligned}$$

Then, $V^0(t,x,r) = v^0(t,x,r)$ and $k_t^{(0)*}$ is the optimal strategy if no crash can occur anymore.
b) Moreover, for $n \in \{1,\dots,N\}$, define $k_t^{(n)*} := \hat{k}_t^{(n)} \wedge k_t^{(0)*}$, where $\hat{k}_t^{(n)}$ is the uniquely determined solution of

$$\begin{aligned}\dot{k}_t^{(n)} &= \frac{1-l^*k_t^{(n)}}{l^*}\left(\phi(t,k_t^{(n)}) - \phi(t,k_t^{(n-1)*})\right), \qquad k_T^{(n)} = 0, \\ \phi(t,k) &:= (\mu+\rho\sigma_1\sigma_2\beta(t))k - \frac{\sigma_1^2}{2}(1-\gamma)k^2,\end{aligned} \tag{13}$$

and let

$$v^n(t,x,r) = \frac{1}{\gamma}x^\gamma g^{(n)}(t)\exp(\beta(t)r), \tag{14}$$

where $g^{(n)}(t)$ solves

$$\begin{aligned}\dot{g}^{(n)}(t) + g^{(n)}(t)\Big(&\gamma(\mu+\rho\sigma_1\sigma_2\beta(t))k_t^{(n)*} - \frac{\sigma_1^2}{2}\gamma(1-\gamma)(k_t^{(n)*})^2 \\ &+ ar_M\beta(t) + \frac{\sigma_2^2}{2}\beta^2(t)\Big) = 0, \qquad g^{(n)}(T) = 1.\end{aligned} \tag{15}$$

Then, $V^n(t,x,r) = v^n(t,x,r)$ *and* $k_t^{(n)*}$ *for* $n \in \{1,\dots,N\}$ *is the worst-case optimal investment strategy for the problem* (8).

Before proving Theorem 2.2.2, we state some auxiliary results which are used in the proof. First, the following Lemma is an analogue to [28, Lemma 3] for the case of stochastic interest rates.

LEMMA 2.2.3. *Let* $V^n(t,x,r)$ *be given by* (10) *and let* (τ, l) *be the first intervention of the market, that means the first market crash, after time* t*. Then, it holds*

$$\begin{aligned} V^n(t,x,r) &= \sup_{k\in\Pi(t,x,r)} \inf_{M\in\mathcal{N}(t,n)} \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right] \\ &= \inf_{M\in\mathcal{N}(t,n)} \sup_{k\in\Pi(t,x,r)} \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right] \\ &= \sup_{k\in\Pi(t,x,r)} \inf_{(\tau,l)} \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right] \\ &= \inf_{(\tau,l)} \sup_{k\in\Pi(t,x,r)} \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right]. \end{aligned}$$

PROOF. Analogously to the literature [28, Lemma 3], we prove the result for our model with stochastic interest rates. Since it is rather technical, we refer to Appendix 2.6.1 for details. □

By definition, we have that $k_t^{(n)*} = \hat{k}_t^{(n)} \wedge k_t^{(0)*}$, where $\hat{k}_t^{(n)}$ is the solution of the backward ODE (13). By time reversion $t \to T-t$, this ODE takes the form

$$\begin{aligned} \dot{h}_t^{(n)} = -\frac{1-l^*h_t^{(n)}}{l^*}\Bigg[&(\mu+\rho\sigma_1\sigma_2\beta(T-t))\left(h_t^{(n)} - k_{T-t}^{(n-1)*}\right) \\ &- \frac{\sigma_1^2}{2}(1-\gamma)\left(h_t^{(n)2} - (k_{T-t}^{(n-1)*})^2\right)\Bigg], \qquad h_0^{(n)} = 0. \end{aligned} \tag{16}$$

where $h_t^{(n)} := \hat{k}_{T-t}^{(n)}$. By definition, we obtain

$$k_{T-t}^{(n-1)*} = \hat{k}_{T-t}^{(n-1)} \wedge k_{T-t}^{(0)*} = h_t^{(n-1)} \wedge k_{T-t}^{(0)*}.$$

Let us define

$$\begin{aligned} f^{(n)}(t,h^{(n)}) := -\frac{1-l^*h^{(n)}}{l^*}\Bigg[&(\mu+\rho\sigma_1\sigma_2\beta(T-t))\left(h^{(n)} - k_{T-t}^{(n-1)*}\right) \\ &- \frac{\sigma_1^2}{2}(1-\gamma)\left(h^{(n)2} - k_{T-t}^{(n-1)*2}\right)\Bigg]. \end{aligned}$$

Using the forward ODE (16), we can prove the following auxiliary result to ensure that there exists a uniquely determined solution $\hat{k}_t^{(n)}$ of the backward ODE (13).

PROPOSITION 2.2.4. *Let* $n \in \{1,\dots,N\}$ *be arbitrary but fixed. Then, there exists a uniquely determined solution* $\hat{k}_t^{(n)}$ *of* (13) *and it holds* $\hat{k}_t^{(n)} \in [0, \frac{1}{l^*})$ *for all* $t \in [0,T]$, $n = 1\dots,N$.

PROOF. We refer to Appendix 2.6.2 for the proof. □

REMARK 2.2.5. *By definition, we immediately obtain that* $k_t^{(n)*} = \hat{k}_t^{(n)} \wedge k_t^{(0)*} < \frac{1}{l^*}$ *for all* $t \in [0,T]$, $n \in \{1,\dots,N\}$*. Since* $\hat{k}_t^{(n)}$ *is a solution of the ODE* (13)*, we have that* $k_t^{(n)*}$ *is a deterministic function in* t*, and therefore* $k^* = (k^{(0)*},\dots,k^{(N)*})$ *is an admissible strategy in the sense of Definition 2.1.1.*

PROPOSITION 2.2.6. *For $n = 2, \dots, N$ we define $u_t^{(n)} := h_t^{(n)} - h_t^{(n-1)}$, where $h^{(n)}$ is the solution of the corresponding equation* (16). *Then, $u_t^{(n)} \leq 0$ for all $t \in [0, T]$ and for all $n \in \{2, 3, \dots, N\}$.*

PROOF. We refer to Appendix 2.6.3 for the proof. □

PROPOSITION 2.2.7. *Let $\gamma\rho \geq 0$. Then $u_t^{(1)} := h_t^{(1)} - h_t^{(0)} \leq 0$ for all $t \in [0, T]$, where $h_t^{(0)} := k_{T-t}^{(0)*}$ and $h_t^{(1)}$ is the uniquely determined solution of* (16) *for $n = 1$.*

PROOF. We refer to Appendix 2.6.4 for the proof. □

REMARK 2.2.8. *By Proposition 2.2.6, we obtained that $h_t^{(n)} \leq h_t^{(n-1)} \leq \cdots \leq h_t^{(1)}$ for all $t \in [0, T]$. By time reversion, this is equivalent to*

$$\hat{k}_t^{(n)} \leq \hat{k}_t^{(n-1)} \leq \cdots \leq \hat{k}_t^{(1)}, \quad \forall t \in [0, T].$$

Note that, Proposition 2.2.7 implies that $h_t^{(1)} \leq h_t^{(0)}$ if $\gamma\rho \geq 0$ and therefore

$$\hat{k}_t^{(n)} \leq \hat{k}_t^{(n-1)} \leq \cdots \leq \hat{k}_t^{(1)} \leq k_t^{(0)*}, \quad \forall t \in [0, T].$$

Now, we prove Theorem 2.2.2 and obtain that $k_t^{(n)*} = \hat{k}_t \wedge k_t^{(0)*}$ is indeed the worst-case optimal investment strategy.

PROOF OF THEOREM 2.2.2.
a) Here, we investigate how the investor has to choose his strategy immediately after the N-th market crash. Thereafter, the investor is faced with a classical stochastic optimal control problem with a finite time horizon because no market crash can occur anymore. We solve this problem by Dynamic Programming Principle (DPP). First, we solve the corresponding HJB equation and prove that the solution is equal to the value function

$$V^0(t, x, r) = \sup_{k \in \Pi(t,x,r)} \inf_{M \in \mathcal{N}(t,0)} \mathbb{E}^{t,x,r,0}\left(U(X_T^{k,M})\right) = \sup_{k^{(0)} \in \Pi(t,x,r)} \mathbb{E}^{t,x,r}\left(\frac{1}{\gamma}\overline{X}_T^{\gamma}\right), \tag{17}$$

where $\overline{X}_s$ denotes the wealth at time $s \geq t$ if no crash can occur anymore, that means $\overline{X}_s$ solves the classical wealth equation controlled by $k^{(0)}$ starting at time t in $(x, r) \in \mathbb{R}_+ \times \mathbb{R}$:

$$d\overline{X}_s = \overline{X}_s\left[\overline{r}_s + \mu k_s^{(0)}\right] ds + \overline{X}_s \sigma_1 k_s^{(0)}\, dw_{1,s}, \qquad \overline{X}_t = x,$$
$$d\overline{r}_s = a(r_M - \overline{r}_s)\, ds + \sigma_2\, d\tilde{w}_s, \qquad \overline{r}_t = r,$$

for $s \geq t$. The corresponding HJB equation is given by

$$\sup_{k^{(0)} \in A} \mathcal{L}^{k^{(0)}} v^0(t, x, r) = 0, \qquad (t, x, r) \in [0, T) \times \mathbb{R}_+ \times \mathbb{R}, \tag{18}$$
$$v^0(T, x, r) = \frac{1}{\gamma} x^{\gamma}, \qquad (x, r) \in \mathbb{R}_+ \times \mathbb{R}.$$

By applying a standard separation method for the case of non-log HARA utility functions, where we assume that the solution of (18) takes the form $v^0(t, x, r) = \frac{1}{\gamma} x^{\gamma} W(t, r)$ and $W(T, r) = 1$ for all $r \in \mathbb{R}$, we obtain

$$\begin{aligned} &W_t(t,r) + \gamma \sup_{k^{(0)} \in A} \left[\mu k^{(0)} W(t,r) - \frac{\sigma_1^2}{2}(1-\gamma)(k^{(0)})^2 W(t,r) + \rho\sigma_1\sigma_2 k^{(0)} W_r(t,r) \right] \\ &+ \gamma r W(t,r) + a(r_M - r) W_r(t,r) + \frac{\sigma_2^2}{2} W_{rr}(t,r) = 0, \qquad (t,r) \in [0,T) \times \mathbb{R}, \\ &W(T,r) = 1, \qquad r \in \mathbb{R}. \end{aligned}$$

Assuming that $W(t,r) > 0$ for all $(t,r) \in [0,T] \times \mathbb{R}$, we obtain the candidate for the optimal control by the first order optimality condition:

$$k^{(0)*}(t,r) = \frac{\mu}{(1-\gamma)\sigma_1^2} + \frac{\rho\sigma_2}{(1-\gamma)\sigma_1} \cdot \frac{W_r(t,r)}{W(t,r)},$$

and, by inserting, we get the following second order partial differential equation for W:

$$\begin{aligned} &W_t(t,r) + \gamma\left(\mu W(t,r) + \rho\sigma_1\sigma_2 W_r(t,r)\right) k^{(0)*}(t,r) - \frac{\sigma_1^2}{2}\gamma(1-\gamma)(k^{(0)*}(t,r))^2 W(t,r) \\ &+ \gamma r W(t,r) + a(r_M - r) W_r(t,r) + \frac{\sigma_2^2}{2} W_{rr}(t,r) = 0, \qquad (t,r) \in [0,T) \times \mathbb{R} \qquad (19) \\ &W(T,r) = 1, \qquad r \in \mathbb{R}. \end{aligned}$$

By a further separation approach of the form $W(t,r) = g^{(0)}(t)\exp(\beta(t)r)$ with $g^{(0)}(T) = 1$ and $\beta(T) = 0$ we arrive at:

$$\begin{aligned} &\dot{g}^{(0)}(t) + g^{(0)}(t) r \left(\dot{\beta}(t) - a\beta(t) + \gamma\right) \\ &+ g^{(0)}(t)\left(\gamma(\mu + \rho\sigma_1\sigma_2\beta(t))k_t^{(0)*} - \frac{\sigma_1^2}{2}\gamma(1-\gamma)(k_t^{(0)*})^2 + a r_M \beta(t) + \frac{\sigma_2^2}{2}\beta^2(t)\right) = 0, \\ &g^{(0)}(T) = 1, \beta(T) = 0. \end{aligned}$$

In order to eliminate the state variable r from the equation above, $\beta(t)$ has to fulfill

$$\begin{aligned} &\dot{\beta}(t) - a\beta(t) + \gamma = 0, \qquad \beta(T) = 0, \\ &\Rightarrow \beta(t) = \frac{\gamma}{a}\left[1 - \exp(-a(T-t))\right]. \end{aligned}$$

Thus, we obtain a linear ODE for $g^{(0)}(t)$:

$$\dot{g}^{(0)}(t) + g^{(0)}(t)\alpha^{(0)}(t) = 0, \qquad g^{(0)}(T) = 1,$$

where

$$\alpha^{(0)}(t) := \gamma(\mu + \rho\sigma_1\sigma_2\beta(t))k_t^{(0)*} - \frac{\sigma_1^2}{2}\gamma(1-\gamma)(k_t^{(0)*})^2 + a r_M \beta(t) + \frac{\sigma_2^2}{2}\beta^2(t).$$

Finally, this leads to an explicit formula

$$v^0(t,x,r) = \frac{1}{\gamma}x^\gamma W(t,r) = \frac{1}{\gamma}x^\gamma g^{(0)}(t)\exp(\beta(t)r) \qquad (20)$$

which solves the HJB equation, and since $\frac{W_r(t,r)}{W(t,r)} = \beta(t)$ for all $(t,r) \in [0,T] \times \mathbb{R}$, the candidate $k^{(0)*}$ is given by

$$k_t^{(0)*} = \frac{\mu}{(1-\gamma)\sigma_1^2} + \frac{\rho\sigma_2\beta(t)}{(1-\gamma)\sigma_1}. \qquad (21)$$

Since $g^{(0)}(t) = \exp(\int_t^T \alpha^{(0)}(s)\,ds) > 0$, we obtain that $W(t,r) = g^{(0)}(t)\exp(\beta(t)r) > 0$ for all $(t,r) \in [0,T] \times \mathbb{R}$. Now, it remains to show that the solution (20) of the HJB equation is indeed equal to the value function $V^0(t,x,r)$, defined in (17), and that the candidate $k^{(0)*}$ in (21) is the optimal strategy. This can be done by proving the assumptions of [24, Corollary 3.2]), which provides such verification result. These proofs are rather technical, but standard, and therefore, we refer to Appendix 2.6.5. Now, we conclude that the optimal strategy after the N-th market crash is given by $k^{(0)*}$ and that $v^0(t,x,r) = V^0(t,x,r)$ and therefore, a) holds.

b) In order to show the assertion for $n \in \{1,\dots,N\}$, we adapt the system of variational inequalities from [28, Thm.2] for the case of stochastic interest rates. First, following the notation of the literature, we define for $n \in \{1,\dots,N\}$:

$$\mathcal{A}'_n(t,x,r) := \left\{k \in A : 0 \le \mathcal{L}^k v^n(t,x,r)\right\},$$
$$\mathcal{A}''_n(t,x,r) := \left\{k \in A : 0 \le v^{n-1}\left(t, x(1-l^*k^+), r\right) - v^n(t,x,r)\right\}.$$

We consider the following system of variational inequalities:

$$0 \le \sup_{k \in \mathcal{A}''_n(t,x,r)} \left[\mathcal{L}^k v^n(t,x,r)\right], \tag{22}$$
$$0 \le \sup_{k \in \mathcal{A}'_n(t,x,r)} \left[v^{n-1}\left(t, x(1-l^*k^+), r\right) - v^n(t,x,r)\right], \tag{23}$$
$$0 = \sup_{k \in \mathcal{A}''_n(t,x,r)} \left[\mathcal{L}^k v^n(t,x,r)\right] \sup_{k \in \mathcal{A}'_n(t,x,r)} \left[v^{n-1}\left(t, x(1-l^*k^+), r\right) - v^n(t,x,r)\right], \tag{24}$$
$$v^n(T,x,r) = \frac{1}{\gamma}x^\gamma, \quad (x,r) \in \mathbb{R}_+ \times \mathbb{R}. \tag{25}$$

and define

$$p^n(t,x,r) := \arg \sup_{k \in \mathcal{A}''_n(t,x,r)} \left[\mathcal{L}^k v^n(t,x,r)\right],$$
$$\theta^n(t,x,r) := \inf_{s:s\ge t} \left[v^{n-1}(s, X_s^{k,M}(1-l^*k_s^+), r_s) - v^n(s, X_s^{k,M}, r_s) \le 0\right], \tag{26}$$
$$l^{(n)}(k) := l^* \mathbb{1}_{k \ge 0}, \tag{27}$$

where $X_t^{k,M} = x$ and $r_t = r$.
By the heuristic construction of $v^n(t,x,r)$ (see Appendix 2.6.6), we have that $v^n(t,x,r)$, given by (14), indeed solves the system of inequalities (22)-(25) for $n \in \{1,\dots,N\}$.
Using that $v^n(t,x,r)$ is a solution of the system above, we prove that $v^n(t,x,r) = V^n(t,x,r)$ and that $k_t^{(n)*} = \hat{k}_t^{(n)} \wedge k_t^{(0)*}$ is the worst-case optimal strategy. In contrast to [28, Thm. 2], where the verification theorem is proved for constant interest rates and general utility functions U, here, we prove the assertion of Theorem 2.2.2 for the Vasicek short rate model and the special class of HARA utility functions using the explicit form of the solution v^n of the system of inequalities. We prove that $v^n(t,x,r) = V^n(t,x,r)$ via induction.

First, we show that $v^1(t,x,r) = V^1(t,x,r)$:
Let $(t,x,r) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}$ be arbitrary but fixed. We denote by (τ, l) the first intervention of the market after time t (that means the first market crash after time t at time τ with jump

size l). Let $(k,(\tau,l))$ be an arbitrary but fixed strategy, where k is chosen by the investor and (τ,l) is chosen by the market. Here, we only have to consider the wealth $X^{k,M}$ until the first intervention time τ and therefore we can just denote the argument M in (7) by (τ,l), such that the wealth and the short rate dynamics are described by:

$$\begin{aligned}
X_t^{k,(\tau,l)} &= x,\\
dX_s^{k,(\tau,l)} &= X_s^{k,(\tau,l)}\left[r_s+\mu k_s\right]ds + X_s^{k,(\tau,l)}\sigma_1 k_s\,dw_{1,s}, \qquad t<s<\tau\\
X_\tau^{k,(\tau,l)} &= X_{\tau-}^{k,(\tau,l)}(1-lk_\tau),\\
r_t &= r,\\
dr_s &= a(r_M-r_s)\,ds+\sigma_2 d\tilde{w}_s, \qquad t<s\le\tau.
\end{aligned}$$

Now, by applying multidimensional Ito's formula, we obtain

$$\begin{aligned}
dv^1(s,X_s^{k,(\tau,l)},r_s) =&\mathcal{L}^{k_s}v^1(s,X_s^{k,(\tau,l)},r_s)\,ds + v_x^1(s,X_s^{k,(\tau,l)},r_s)\sigma_1 k_s X_s^{k,(\tau,l)}\,dw_{1,s}\\
&+v_r^1(s,X_s^{k,(\tau,l)},r_s)\rho\sigma_2\,dw_{1,s}\\
&+v_r^1(s,X_s^{k,(\tau,l)},r_s)\sqrt{1-\rho^2}\sigma_2\,dw_{2,s}, \quad t<s<\tau,
\end{aligned}\tag{28}$$

with

$$\begin{aligned}
v^1(t,X_t^{k,(\tau,l)},r_t) =&v^1(t,x,r),\\
dv^1(\tau,X_\tau^{k,(\tau,l)},r_\tau) =&v^1(\tau,X_\tau^{k,(\tau,l)},r_\tau)-v^1(\tau-,X_{\tau-}^{k,(\tau,l)},r_{\tau-})\\
=&v^1(\tau,X_{\tau-}^{k,(\tau,l)}(1-lk_\tau),r_\tau)-v^1(\tau-,X_{\tau-}^{k,(\tau,l)},r_{\tau-}).
\end{aligned}$$

Integrating on both sides of (28), leads to

$$\begin{aligned}
&v^1(\tau-,X_{\tau-}^{k,(\tau,l)},r_{\tau-})-v^1(t,x,r)\\
&=\int_t^\tau \mathcal{L}^{k_s}v^1(s,X_s^{k,(\tau,l)},r_s)\,ds\\
&\quad+\int_t^\tau\left(v_x^1(s,X_s^{k,(\tau,l)},r_s)\sigma_1 k_s X_s^{k,(\tau,l)}+v_r^1(s,X_s^{k,(\tau,l)},r_s)\rho\sigma_2\right)dw_{1,s}\\
&\quad+\int_t^\tau v_r^1(s,X_s^{k,(\tau,l)},r_s)\sqrt{1-\rho^2}\sigma_2\,dw_{2,s}.
\end{aligned}\tag{29}$$

Equation (29) holds for an arbitrary but fixed strategy $(k,(\tau,l))$.
Now, we fix $k_s=p^1(s,X_{s-}^{k,(\tau,l)},r_s)=k_s^{(1)*}$ for $t\le s\le\tau$ and let (τ,l) be an arbitrary but fixed intervention by the market, then we have, by construction, that

$$\mathcal{L}^{k_s^{(1)*}}v^1(s,X_s^{k^{(1)*},(\tau,l)},r_s)=0, \qquad t\le s\le\tau,$$

and $k_s^{(1)*}\in\mathcal{A}_1''(s)$, that is

$$\begin{aligned}
0&\le v^0\left(s,X_s^{k^{(1)*},(\tau,l)}(1-l^*(k_s^{(1)*})^+),r_s\right)-v^1\left(s,X_s^{k^{(1)*},(\tau,l)},r_s\right),\\
&\le v^0\left(s,X_s^{k^{(1)*},(\tau,l)}(1-lk_s^{(1)*}),r_s\right)-v^1\left(s,X_s^{k^{(1)*},(\tau,l)},r_s\right), \quad t\le s\le\tau.
\end{aligned}$$

Equation (29) together with $k_s = k_s^{(1)*}$ implies

$$\begin{aligned}
&v^1(t,x,r)\\
&= v^1(\tau-, X_{\tau-}^{k^{(1)*},(\tau,l)}, r_{\tau-})\\
&\quad - \int_t^\tau \mathcal{L}^{k_s^{(1)*}} v^1(s, X_s^{k^{(1)*},(\tau,l)}, r_s)\, ds\\
&\quad - \int_t^\tau \Big(v_x^1(s, X_s^{k^{(1)*},(\tau,l)}, r_s)\sigma_1 k_s^{(1)*} X_s^{k^{(1)*},(\tau,l)} + v_r^1(s, X_s^{k^{(1)*},(\tau,l)}, r_s)\rho\sigma_2\Big)\, dw_{1,s}\\
&\quad - \int_t^\tau v_r^1(s, X_s^{k^{(1)*},(\tau,l)}, r_s)\sqrt{1-\rho^2}\sigma_2\, dw_{2,s}\\
&\le v^0\left(\tau-, X_{\tau-}^{k^{(1)*},(\tau,l)}(1 - lk_\tau^{(1)*}), r_{\tau-}\right)\\
&\quad - \int_t^\tau \Big(v_x^1(s, X_s^{k^{(1)*},(\tau,l)}, r_s)\sigma_1 k_s^{(1)*} X_s^{k^{(1)*},(\tau,l)} + v_r^1(s, X_s^{k^{(1)*},(\tau,l)}, r_s)\rho\sigma_2\Big)\, dw_{1,s}\\
&\quad - \int_t^\tau v_r^1(s, X_s^{k^{(1)*},(\tau,l)}, r_s)\sqrt{1-\rho^2}\sigma_2\, dw_{2,s}.
\end{aligned} \tag{30}$$

Using the fact that

$$v_x^1(s,x,r)x = v^1(s,x,r)\gamma, \qquad v_r^1(s,x,r) = v^1(s,x,r)\beta(s),$$

the stochastic integrals are equal to the term

$$\begin{aligned}
&- \int_t^\tau v^1(s, X_s^{k^{(1)*},(\tau,l)}, r_s)\left[\Big(\gamma\sigma_1 k_s^{(1)*} + \beta(s)\rho\sigma_2\Big)\, dw_{1,s} + \beta(s)\sqrt{1-\rho^2}\sigma_2\, dw_{2,s}\right]\\
&= -\int_t^\tau f^{(\tau,l)}(s) dw_s,
\end{aligned}$$

where $f^{(\tau,l)}(s) := (f_1^{(\tau,l)}(s), f_2^{(\tau,l)}(s)), dw_s := (dw_{1,s}, dw_{2,s})^T$ with

$$\begin{aligned}
f_1^{(\tau,l)}(s) &:= v^1(s, X_s^{k^{(1)*},(\tau,l)}, r_s)\left(\gamma\sigma_1 k_s^{(1)*} + \beta(s)\rho\sigma_2\right),\\
f_2^{(\tau,l)}(s) &:= v^1(s, X_s^{k^{(1)*},(\tau,l)}, r_s)\sqrt{1-\rho^2}\sigma_2\beta(s).
\end{aligned}$$

In contrast to [28, Thm. 2], we can use the knowledge about the explicit form of the function $v^1(t,x,r)$ in order to show that

$$\mathbb{E}^{t,x,r,1}\left[-\int_t^\tau f^{(\tau,l)}(s) dw_s\right] = 0. \tag{31}$$

For the proof of (31), we refer to Appendix 2.6.8.
Taking the expectation on both sides of (30), leads to

$$v^1(t,x,r) \le \mathbb{E}^{t,x,r,1}\left[v^0\left(\tau, X_{\tau-}^{k^{(1)*},(\tau,l)}(1 - lk_\tau^{(1)*}), r_\tau\right)\right]. \tag{32}$$

Now, the following steps are similar to the case of constant interest rates in the literature. By (32) we have

$$v^1(t,x,r) \le \sup_{k\in\Pi(t,x,r)} \mathbb{E}^{t,x,r,1}\left[v^0\left(\tau, X_{\tau-}^{k,(\tau,l)}(1 - lk_\tau), r_\tau\right)\right]. \tag{33}$$

Now, taking the infimum over (τ, l) on both sides of (33) leads to

$$v^1(t,x,r) \leq \inf_{(\tau,l)} \sup_{k\in\Pi(t,x,r)} \mathbb{E}^{t,x,r,1}\left[v^0\left(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau\right)\right]. \tag{34}$$

Moreover, taking the infimum over (τ, l) on both sides of (32), we have

$$v^1(t,x,r) \leq \inf_{(\tau,l)} \mathbb{E}^{t,x,r,1}\left[v^0\left(\tau, X_{\tau-}^{k^{(1)*},(\tau,l)}(1-lk_\tau^{(1)*}), r_\tau\right)\right] \tag{35}$$

and therefore

$$v^1(t,x,r) \leq \sup_{k\in\Pi(t,x,r)} \inf_{(\tau,l)} \mathbb{E}^{t,x,r,1}\left[v^0\left(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau\right)\right]. \tag{36}$$

Now let $k \in \Pi(t,x,r)$ be arbitrary but fixed. We fix the strategy $(\tau,l) = (\theta,\tilde{l})$, where $\theta := \theta^1(t,x,r)$ and $\tilde{l} = l^{(1)}(k_\theta)$. Then, by definition of θ, it holds

$$v^0(s, X_{s-}^{k,(\theta,\tilde{l})}(1-l^{(1)}(k_s)k_s), r_s) - v^1(s, X_{s-}^{k,(\theta,\tilde{l})}, r_s) > 0, \qquad \text{for} \quad t \leq s < \theta, \tag{37}$$

$$v^0(\theta, X_{\theta-}^{k,(\theta,\tilde{l})}(1-\tilde{l}k_\theta), r_\theta) - v^1(\theta, X_{\theta-}^{k,(\theta,\tilde{l})}, r_\theta) \leq 0. \tag{38}$$

For $t \leq s < \theta$, k_s either fulfills

$$0 > \mathcal{L}^{k_s} v^1(s, X_s^{k,(\theta,\tilde{l})}, r_s), \quad \text{or} \quad 0 \leq \mathcal{L}^{k_s} v^1(s, X_s^{k,(\theta,\tilde{l})}, r_s). \tag{39}$$

Assume the last inequality holds for k, then $k_s \in \mathcal{A}_1'(s, X_s^{k,(\theta,\tilde{l})}, r_s)$. For the sake of brevity, we write $\mathcal{A}_1'(s)$ and $\mathcal{A}_1''(s)$ instead of $\mathcal{A}_1'(s, X_s^{k,(\theta,\tilde{l})}, r_s)$ and $\mathcal{A}_1''(s, X_s^{k,(\theta,\tilde{l})}, r_s)$, respectively. Together with (37), it follows

$$\begin{aligned}
0 &< v^0(s, X_{s-}^{k,(\theta,\tilde{l})}(1-l^{(1)}(k_s)k_s), r_s) - v^1(s, X_{s-}^{k,(\theta,\tilde{l})}, r_s) \\
&\leq \sup_{k_s\in\mathcal{A}_1'(s)} v^0(s, X_{s-}^{k,(\theta,\tilde{l})}(1-l^{(1)}(k_s)k_s), r_s) - v^1(s, X_{s-}^{k,(\theta,\tilde{l})}, r_s) \\
\Rightarrow 0 &< \sup_{k_s\in\mathcal{A}_1'(s)} v^0(s, X_{s-}^{k,(\theta,\tilde{l})}(1-l^{(1)}(k_s)k_s), r_s) - v^1(s, X_{s-}^{k,(\theta,\tilde{l})}, r_s) \\
&= \sup_{k_s\in\mathcal{A}_1'(s)} v^0(s, X_{s-}^{k,(\theta,l^*)}(1-l^*k_s^+), r_s) - v^1(s, X_{s-}^{k,(\theta,l^*)}, r_s).
\end{aligned}$$

Here, we used that $l^{(1)}(k_s)k_s = l^*\mathbb{1}_{k_s\geq 0}k_s = l^*k_s^+$. Since v^1 is a solution of the system of inequalities, it follows by (24), that

$$\sup_{k_s\in\mathcal{A}_1''(s)} \left[\mathcal{L}^{k_s} v^1(s, X_s^{k,(\theta,\tilde{l})}, r_s)\right] = 0.$$

But $k_s \in \mathcal{A}_1''(s)$ by (37), and therefore

$$0 = \sup_{k_s\in\mathcal{A}_1''(s)} \left[\mathcal{L}^{k_s} v^1(s, X_s^{k,(\theta,\tilde{l})}, r_s)\right] \geq \mathcal{L}^{k_s} v^1(s, X_s^{k,(\theta,\tilde{l})}, r_s),$$

which is a contradiction to the assumption that the second inequality in (39) holds. Thus, we have $\mathcal{L}^{k_s} v^1(s, X_s^{k,(\theta,\tilde{l})}, r_s) \leq 0$ for $t \leq s < \theta$. Now, by inserting $(\theta, \tilde{l})$ in (29), we obtain by (38)

$$\begin{aligned}
&v^1(t,x,r)\\
&= v^1(\theta, X_{\theta-}^{k,(\theta,\tilde{l})}, r_\theta)\\
&- \int_t^\theta \mathcal{L}^{k_s} v^1(s, X_s^{k,(\theta,\tilde{l})}, r_s)\, ds\\
&- \int_t^\theta \left(v_x^1(s, X_s^{k,(\theta,\tilde{l})}, r_s)\sigma_1 k_s X_s^{k,(\theta,\tilde{l})} + v_r^1(s, X_s^{k,(\theta,\tilde{l})}, r_s)\rho\sigma_2 \right)\, dw_{1,s}\\
&- \int_t^\theta v_r^1(s, X_s^{k,(\theta,\tilde{l})}, r_s)\sqrt{1-\rho^2}\sigma_2\, dw_{2,s}\\
&\geq v^0(\theta, X_{\theta-}^{k,(\theta,\tilde{l})}(1-\tilde{l}k_\theta), r_\theta)\\
&- \int_t^\theta \left(v_x^1(s, X_s^{k,(\theta,\tilde{l})}, r_s)\sigma_1 k_s X_s^{k,(\theta,\tilde{l})} + v_r^1(s, X_s^{k,(\theta,\tilde{l})}, r_s)\rho\sigma_2 \right)\, dw_{1,s}\\
&- \int_t^\theta v_r^1(s, X_s^{k,(\theta,\tilde{l})}, r_s)\sqrt{1-\rho^2}\sigma_2\, dw_{2,s}.
\end{aligned} \tag{40}$$

In Appendix 2.6.8 we have shown that

$$\mathbb{E}^{t,x,r,1}\left[-\int_t^\theta f^{(k)}(s) dw_s \right] = 0, \tag{41}$$

where $f^{(k)}(s) := (f_1^{(k)}(s), f_2^{(k)}(s)), dw_s := (dw_{1,s}, dw_{2,s})^T$ with

$$\begin{aligned}
f_1^{(k)}(s) &:= v^1(s, X_s^{k,(\theta,\tilde{l})}, r_s)\left(\gamma\sigma_1 k_s + \beta(s)\rho\sigma_2\right),\\
f_2^{(k)}(s) &:= v^1(s, X_s^{k,(\theta,\tilde{l})}, r_s)\sqrt{1-\rho^2}\beta(s).
\end{aligned}$$

Now, taking the expectation on both sides of (40) leads to

$$v^1(t,x,r) \geq \mathbb{E}^{t,x,r,1}\left[v^0(\theta, X_{\theta-}^{k,(\theta,\tilde{l})}(1-\tilde{l}k_\theta), r_\theta) \right] \tag{42}$$

$$\Rightarrow v^1(t,x,r) \geq \inf_{(\tau,l)} \mathbb{E}^{t,x,r,1}\left[v^0(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau) \right]. \tag{43}$$

Taking the supremum on both sides of (43) implies

$$v^1(t,x,r) \geq \sup_{k\in\Pi(t,x,r)} \inf_{(\tau,l)} \mathbb{E}^{t,x,r,1}\left[v^0(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau) \right]. \tag{44}$$

If we take the supremum on both sides of (42), then

$$v^1(t,x,r) \geq \sup_{k\in\Pi(t,x,r)} \mathbb{E}^{t,x,r,1}\left[v^0(\theta, X_{\theta-}^{k,(\theta,\tilde{l})}(1-\tilde{l}k_\theta), r_\theta) \right] \tag{45}$$

$$\Rightarrow v^1(t,x,r) \geq \inf_{(\tau,l)} \sup_{k\in\Pi(t,x,r)} \mathbb{E}^{t,x,r,1}\left[v^0(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau) \right]. \tag{46}$$

Summing up, we finally obtain

$$\begin{aligned}
v^1(t,x,r) &= \inf_{(\tau,l)} \sup_{k\in\Pi(t,x,r)} \mathbb{E}^{t,x,r,1}\left[v^0(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right] \\
&= \sup_{k\in\Pi(t,x,r)} \inf_{(\tau,l)} \mathbb{E}^{t,x,r,1}\left[v^0(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right] \\
&= \sup_{k\in\Pi(t,x,r)} \inf_{(\tau,l)} \mathbb{E}^{t,x,r,1}\left[V^0(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right] \\
&= V^1(t,x,r),
\end{aligned}$$

where the first equality follows by (34) and (46) and the second equality follows by (36) and (44). The third equality follows by the fact that $V^0(t,x,r) = v^0(t,x,r)$ and the fourth equality follows by Lemma 2.2.3. Thus, we have shown that $v^1(t,x,r) = V^1(t,x,r)$, where v^1 is given by (14)

Finally, with (45), we have

$$\begin{aligned}
V^1(t,x,r) &\geq \sup_{k\in\Pi(t,x,r)} \mathbb{E}^{t,x,r,1}\left[V^0(\theta, X_{\theta-}^{k,(\theta,\tilde{l})}(1-\tilde{l}k_\theta), r_\theta)\right] \\
&\geq \mathbb{E}^{t,x,r,1}\left[V^0(\theta, X_{\theta-}^{k^{(1)*},(\theta,\tilde{l})}(1-\tilde{l}k_\theta^{(1)*}), r_\theta)\right] \\
&\geq \inf_{(\tau,l)} \mathbb{E}^{t,x,r,1}\left[V^0(\tau, X_{\tau-}^{k^{(1)*},(\tau,l)}(1-lk_\tau^{(1)*}), r_\tau)\right],
\end{aligned}$$

and together with (35), we have

$$V^1(t,x,r) = \inf_{(\tau,l)} \mathbb{E}^{t,x,r,1}\left[V^0(\tau, X_{\tau-}^{k^{(1)*},(\tau,l)}(1-lk_\tau^{(1)*}), r_\tau)\right]$$

and therefore, $k_t^{(1)*} = \hat{k}_t^{(1)} \wedge k_t^{(0)*}$ is the optimal strategy if at most one crash still can happen. Now, assume that $v^{n-1}(t,x,r) = V^{n-1}(t,x,r)$. Then, we can show that $v^n(t,x,r) = V^n(t,x,r)$. Let $(t,x,r) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}$ and assume that n market crashes are left. Again we denote by (τ,l) the first intervention of the market after time t and let $(k,(\tau,l))$ be an arbitrary but fixed strategy. Again by using Ito's formula, we obtain that

$$\begin{aligned}
v^n(t,x,r) =& v^n(\tau-, X_{\tau-}^{k,(\tau,l)}, r_{\tau-}) \\
&- \int_t^\tau \mathcal{L}^{k_s} v^1(s, X_s^{k,(\tau,l)}, r_s)\, ds \\
&- \int_t^\tau \left(v_x^n(s, X_s^{k,(\tau,l)}, r_s)\sigma_1 k_s X_s^{k,(\tau,l)} + v_r^n(s, X_s^{k,(\tau,l)}, r_s)\rho\sigma_2\right) dw_{1,s} \\
&- \int_t^\tau v_r^n(s, X_s^{k,(\tau,l)}, r_s)\sqrt{1-\rho^2}\sigma_2\, dw_{2,s}.
\end{aligned} \tag{47}$$

First, we fix the strategy $k_s = p^n(s, X_{s-}^{k,(\tau,l)}, r_s) = k_s^{(n)*}$ for $t \leq s \leq \tau$ and obtain by the same arguments as above that

$$v^n(t,x,r) \leq \inf_{(\tau,l)} \sup_{k\in\Pi(t,x,r)} \mathbb{E}^{t,x,r,n}\left[v^{n-1}\left(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau\right)\right]$$

and

$$v^n(t,x,r) \leq \sup_{k\in\Pi(t,x,r)} \inf_{(\tau,l)} \mathbb{E}^{t,x,r,n}\left[v^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right].$$

By fixing the strategy $(\tau, l) = (\theta, \tilde{l})$, with $\theta = \theta^n(t,x,r)$ and $\tilde{l} := l^{(n)}(k_\theta)$, where $k \in \Pi(t,x,r)$ is arbitrary but fixed, we use again the same arguments as above and obtain

$$v^n(t,x,r) \geq \sup_{k\in\Pi(t,x,r)} \inf_{(\tau,l)} \mathbb{E}^{t,x,r,n}\left[v^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right],$$

and

$$v^n(t,x,r) \geq \inf_{(\tau,l)} \sup_{k\in\Pi(t,x,r)} \mathbb{E}^{t,x,r,n}\left[v^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right].$$

Both together lead to

$$\begin{aligned} v^n(t,x,r) &= \inf_{(\tau,l)} \sup_{k\in\Pi(t,x,r)} \mathbb{E}^{t,x,r,n}\left[v^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right] \\ &= \sup_{k\in\Pi(t,x,r)} \inf_{(\tau,l)} \mathbb{E}^{t,x,r,n}\left[v^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right] \\ &= \sup_{k\in\Pi(t,x,r)} \inf_{(\tau,l)} \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right] \\ &= V^n(t,x,r), \end{aligned}$$

where the third equality follows by the assumption that $v^{n-1}(t,x,r) = V^{n-1}(t,x,r)$ and the fourth equality follows by Lemma 2.2.3. Thus,

$$v^n(t,x,r) = V^n(t,x,r).$$

Moreover, it follows that $k_t^{(n)*} = \hat{k}_t^{(n)} \wedge k_t^{(0)*}$ is the worst-case optimal investment strategy if at most n market crashes still can happen. □

2.2.1. Conclusion from Theorem 2.2.2 and comparison with the case $r_t \equiv r$. Theorem 2.2.2 implies that the worst-case optimal strategies can be calculated numerically. First, the strategy after the N-th market crash is given by the solution of the classical stochastic optimal control problem:

$$k_t^{(0)*} = \frac{\mu}{(1-\gamma)\sigma_1^2} + \frac{\rho\sigma_2\beta(t)}{(1-\gamma)\sigma_1}.$$

Note that $k^{(0)*}$ is independent of $\omega \in \Omega$ as it does not depend on the short rate $r_t(\omega)$. This is due to the fact that the access return μ in our financial market model is assumed to be constant, and therefore, the control variable k is not coupled with the short rate r_t in the wealth equation. Nevertheless, $k^{(0)*}$ depends on the parameters a and σ_2, which determine the Vasicek process in (2). The same is true for the worst-case optimal strategies $k_t^{(n)*} = \hat{k}_t^{(n)*} \wedge k_t^{(0)*}$, $n = 1, \ldots, N$. Here, $\hat{k}_t^{(n)}$ can be calculated by solving the corresponding nonlinear non-autonomous ODE of the form

$$\dot{k}_t^{(n)} = \frac{1-l^*k_t^{(n)}}{l^*}\left(\phi(t,k_t^{(n)}) - \phi(t,k_t^{(n-1)*})\right), \qquad k_T^{(n)} = 0,$$

where $\phi(t,k) = (\mu + \rho\sigma_1\sigma_2\beta(t))k - \frac{\sigma_1^2}{2}(1-\gamma)k^2$. If we would assume that $r_t \equiv r$ for some constant $r > 0$, then the ODE above reduces to a nonlinear autonomous ODE, which is already published in the previous literature, see e.g. [28, 33, 44]. The same applies if we especially assume that $\rho = 0$, that means that w_1 and $\tilde{w}$ are uncorrelated and therefore independent processes. Then, the strategy $k_t^{(0)*}$ reduces to

$$k_t^{(0)*} \equiv \frac{\mu}{(1-\gamma)\sigma_1^2},$$

and ϕ reduces to $\phi(t,k) = \mu k - \frac{\sigma_1^2}{2}(1-\gamma)k^2$. Therefore, in these two special cases ODE (13) reduces to an autonomous ODE and the worst-case optimal strategies do not depend on the short rate parameters anymore. Thus, the worst-case optimal strategies do not differ from the worst-case optimal strategies for the case of constant interest rates (see e.g. [28, 33, 44]).

2.3. The N-crash market with Log utility

Here, we consider the logarithmic utility function which corresponds to the case of $\gamma = 0$ of the section before. Thus, we consider the worst-case optimization problem (8) with

$$U(x) = \log(x)$$

and with Vasicek short rate dynamics of the form (2). The logarithmic utility function also belongs to the class of HARA utility functions. First of all, we obtain that the value function, defined in (10), is well-defined for $U(x) = \log(x)$.

Corollary 2.3.1. *For $(t,x,r) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}$ and $n \le N$, let $k \in \Pi(t,x,r)$ be an arbitrary admissible strategy and let M be an arbitrary crash sequence of length n on $[t,T]$, which fulfills the assumptions given in Section 2.1. Moreover, let $\{r_t\}_{t\in[0,T]}$ and $X^{k,M} = \{X_t^{k,M}\}_{t\in[0,T]}$ be given by* (2) *and* (7)*, respectively. Then,*

$$\mathbb{E}^{t,x,r,n}\left(\left|\log(X_T^{k,M})\right|\right) < \infty.$$

Proof. Let $(t,x,r) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}$ be arbitrary but fixed. Then, we have

$$X_T^{k,M} = x\exp\left(\int_t^T \mu k_u - \frac{\sigma_1^2}{2}(k_u)^2 + r_u\,du + \int_t^T \sigma_1 k_u\,dw_{1,u}\right)\prod_{i=1}^{n}(1 - l_i k_{\tau_i}^{(n-i+1)}).$$

By $k \in \Pi(t,x,r)$ we have that k is bounded. Moreover by Proposition A.1.1 and by triangle inequality, we obtain

$$\begin{aligned}\left|\log(X_T^{k,M})\right| \le & K + \frac{1}{2}\left|\int_t^T \left(\frac{\sigma_2}{a}\rho(1-e^{-a(T-u)}) + \sigma_1 k_u\right)\,dw_{1,u}\right|^2 \\ & + \frac{1}{2}\left|\int_t^T \frac{\sigma_2}{a}\sqrt{1-\rho^2}(1-e^{-a(T-u)})\,dw_{2,u}\right|^2,\end{aligned}$$

for a sufficiently large constant $K > 0$. Taking the expectation on both sides and using Ito isometry, leads to

$$\mathbb{E}^{t,x,r,n}\left(\left|\log(X_T^{k,M})\right|\right) < \infty.$$

□

Analogously to Theorem 2.2.2, we determine the worst-case optimal strategy for problem (8) with logarithmic utility function by the following theorem.

THEOREM 2.3.2.
Assume that the short rate process $\{r_t\}_{t\in[0,T]}$ *is given by* (2) *and assume that* $X^{k,M}$ *is given by* (7)*. Moreover, let* $U(x) = \log(x)$*.*
a) Let $k_t^{(0)*} \equiv \frac{\mu}{\sigma_1^2}$ *and let*

$$v^0(t,x,r) = \log(x) + W^{(0)}(t,r),$$

where

$$W^{(0)}(t,r) = \left(\frac{\mu^2}{2\sigma_1^2} + r_M\right)(T-t) + \frac{1}{a}\left(r - r_M + e^{-a(T-t)}(r_M - r)\right).$$

Then, $V^0(t,x,r) = v^0(t,x,r)$*, where* V^0 *is defined in* (10)*, and* $k_t^{(0)*}$ *is the optimal strategy if no crash can occur anymore.*
b) Moreover, for $n \in \{1,\dots,N\}$*, let* $\hat{k}_t^{(n)}$ *be the uniquely determined solution of*

$$\dot{k}_t^{(n)} = \frac{1-l^* k_t^{(n)}}{l^*}\left(\phi(k_t^{(n)}) - \phi(\hat{k}_t^{(n-1)})\right), \qquad k_T^{(n)} = 0, \tag{48}$$
$$\phi(k) := \mu k - \frac{\sigma_1^2}{2}k^2,$$

with $\hat{k}_t^{(0)} := \frac{\mu}{\sigma_1^2}$*, and let*

$$v^n(t,x,r) = \log(x) + W^{(n)}(t,r), \tag{49}$$

where

$$\begin{aligned} W^{(n)}(t,r) &= g^{(n)}(t) + h(t,r),\\ g^{(n)}(t) &= \int_t^T \mu\hat{k}_s^{(n)} - \frac{\sigma_1^2}{2}(\hat{k}_s^{(n)})^2\,ds,\\ h(t,r) &= r_M(T-t) + \frac{1}{a}\left(r - r_M + e^{-a(T-t)}(r_M - r)\right). \end{aligned}$$

Then, $V^n(t,x,r) = v^n(t,x,r)$ *and* $\hat{k}_t^{(n)}$ *is the worst-case optimal strategy if* n *crashes still can happen.*

In the Theorem above, we assume that there exists a uniquely determined solution of (48). This assumption can be verified by the following Proposition.

PROPOSITION 2.3.3. *Let* $n \in \{1,\dots,N\}$*, then ODE* (48) *has a uniquely determined solution* $\hat{k}_t^{(n)}$ *and for all* $t \in [0,T]$ *it holds*

(1) $\hat{k}_t^{(n)} \in [0, \frac{1}{l^*})$*,*
(2) $\hat{k}_t^{(n)} \le \hat{k}_t^{(n-1)} \le \dots \le \hat{k}_t^{(1)} \le k^{(0)*}$*.*

PROOF. First, note that (13) reduces to (48) for $\gamma = 0$. By applying Proposition 2.2.4 for the special choice of $\gamma = 0$, we immediately obtain that (48) has a uniquely determined solution $\hat{k}_t^{(n)} \in [0, \frac{1}{l^*})$, $t \in [0,T]$. Moreover, Proposition 2.2.6, Proposition 2.2.7 and Remark 2.2.8 are especially true for $\gamma = 0$, and therefore the assertion holds. □

PROOF OF THEOREM 2.3.2. a) Analogously to the proof of Theorem 2.2.2, we first investigate how the investor has to choose his strategy immediately after the N-th market crash. After the N-th crash the investor is faced with a classical stochastic optimal control problem with logarithmic utility function. By definition, the value function is given by

$$V^0(t,x,r) = \sup_{k^{(0)}\in\Pi(t,x,r)} \mathbb{E}^{t,x,r}\left(\log(\overline{X}_T)\right),$$

where $\overline{X}_s$ denotes the wealth at time $s \geq t$, that means:

$$\begin{aligned} d\overline{X}_s &= \overline{X}_s\left[\overline{r}_s + \mu\, k_s^{(0)}\right] ds + \overline{X}_s\sigma_1 k_s^{(0)}\, dw_{1,s}, \qquad && \overline{X}_t = x,\\ d\overline{r}_s &= a(r_M - \overline{r}_s)\, ds + \sigma_2\, d\tilde{w}_s, && \overline{r}_t = r. \end{aligned}$$

The corresponding HJB equation is given by

$$\begin{aligned} \sup_{k^{(0)}\in A} \mathcal{L}^{k^{(0)}} v^0(t,x,r) &= 0, \qquad (t,x,r)\in[0,T)\times\mathbb{R}_+\times\mathbb{R}, \\ v^0(T,x,r) &= \log(x), \qquad (x,r)\in\mathbb{R}_+\times\mathbb{R}. \end{aligned} \tag{50}$$

The usual separation ansatz to find a solution of this HJB equation is to assume that $v^0(t,x,r) = \log(x) + W^{(0)}(t,r)$, where $W^{(0)}(T,r) = 0$ for all $r\in\mathbb{R}$. Thus, the equation above reduces to

$$\begin{aligned} &W_t^{(0)}(t,r) + \sup_{k^{(0)}\in A}\left[\mu k^{(0)} - \frac{\sigma_1^2}{2}(k^{(0)})^2\right] + r \\ &+ a(r_M - r)W_r^{(0)}(t,r) + \frac{\sigma_2^2}{2}W_{rr}^{(0)}(t,r) = 0, \qquad (t,r)\in[0,T)\times\mathbb{R}, \\ &W^{(0)}(T,r) = 0, \qquad r\in\mathbb{R}. \end{aligned}$$

Now, we obtain the optimal candidate by the first order optimality condition:

$$k_t^{(0)*} \equiv \frac{\mu}{\sigma_1^2},$$

and it remains to find a solution of the following partial differential equation (PDE)

$$\begin{aligned} &W_t^{(0)}(t,r) + \frac{\sigma_2^2}{2}W_{rr}^{(0)}(t,r) + a(r_M - r)W_r^{(0)}(t,r) + r + \frac{\mu^2}{2\sigma_1^2} = 0, \quad (t,r)\in[0,T)\times\mathbb{R}, \\ &W^{(0)}(T,r) = 0, \qquad r\in\mathbb{R}. \end{aligned} \tag{51}$$

Now, the Feynman-Kac Theorem (see Appendix A,Theorem A.4.1) tells us that the unique solution of the PDE (51) can be written as a conditional expectation and we obtain that

$$\begin{aligned} W^{(0)}(t,r) &= \frac{\mu^2}{2\sigma_1^2}(T-t) + \mathbb{E}^{t,r}\left(\int_t^T \overline{r}_s\, ds\right) \\ &= \left(\frac{\mu^2}{2\sigma_1^2} + r_M\right)(T-t) + \frac{1}{a}\left(r - r_M + e^{-a(T-t)}(r_M - r)\right). \end{aligned}$$

For the calculation of $\mathbb{E}^{t,r}\left(\int_t^T \overline{r}_s\, ds\right)$ we refer to Proposition A.1.1 in Appendix A. Thus, we determined a solution $v^0(t,x,r) = \log(x) + W^{(0)}(t,r)$ of the HJB equation (50). Again, by proving the assumptions of the Verification Theorem A.5.2 in Appendix A, we obtain that $V^0(t,x,r) = v^0(t,x,r)$ and that $k_t^{(0)*} \equiv \frac{\mu}{\sigma_1^2}$ is the optimal strategy after the N-th market crash.

b) Analogously to the non-log HARA utility case, we can show that $v^n(t,x,r)$ given by (49) is a solution of the system of variational inequalities:

$$
\begin{aligned}
&0 \leq \sup_{k \in \mathcal{A}_n''(t,x,r)} \left[\mathcal{L}^k v^n(t,x,r)\right], \\
&0 \leq \sup_{k \in \mathcal{A}_n'(t,x,r)} \left[v^{n-1}\left(t, x(1-l^*k^+), r\right) - v^n(t,x,r)\right], \\
&0 = \sup_{k \in \mathcal{A}_n''(t,x,r)} \left[\mathcal{L}^k v^n(t,x,r)\right] \sup_{k \in \mathcal{A}_n'(t,x,r)} \left[v^{n-1}\left(t, x(1-l^*k^+), r\right) - v^n(t,x,r)\right], \\
&v^n(T,x,r) = \log(x), \quad \forall (x,r) \in \mathbb{R}_+ \times \mathbb{R}.
\end{aligned}
$$

Moreover, we have

$$
\hat{k}_t^{(n)} = p^n(t,x,r) := \arg \sup_{k \in \mathcal{A}_n''(t,x,r)} \left[\mathcal{L}^k v^n(t,x,r)\right],
$$

because the supremum is attained for k such that the condition $v^n(t,x,r) = v^{n-1}(t, x(1-l^*k^+), r)$ is fulfilled. We refer to the Appendix 2.6.7 for details about finding the solution $v^n(t,x,r)$ and $\hat{k}_t^{(n)}$. We prove that $v^1(t,x,r) = V^1(t,x,r)$ along the lines of the proof of Theorem 2.2.2.
First, for arbitrary but fixed $(t,x,r) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}$ and for an arbitrary but fixed strategy $(k,(\tau,l))$ we have

$$
\begin{aligned}
&v^1(\tau-, X_{\tau-}^{k,(\tau,l)}, r_{\tau-}) - v^1(t,x,r) \\
&= \int_t^\tau \mathcal{L}^{k_s} v^1(s, X_s^{k,(\tau,l)}, r_s)\, ds \\
&\quad + \int_t^\tau \left(v_x^1(s, X_s^{k,(\tau,l)}, r_s)\sigma_1 k_s X_s^{k,(\tau,l)} + v_r^1(s, X_s^{k,(\tau,l)}, r_s)\rho\sigma_2\right) dw_{1,s} \\
&\quad + \int_t^\tau v_r^1(s, X_s^{k,(\tau,l)}, r_s)\sqrt{1-\rho^2}\sigma_2\, dw_{2,s}.
\end{aligned}
$$

By fixing $k_s = p^1(s, X_{s-}^{k,(\tau,l)}, r_s) = \hat{k}_s^{(1)}$ for $t \leq s \leq \tau$, where $\hat{k}_s^{(1)}$ solves (48), and by the same arguments as in the proof of Theorem 2.2.2, we obtain by that

$$
\begin{aligned}
v^1(t,x,r) \leq\, & v^0\left(\tau-, X_{\tau-}^{\hat{k}^{(1)},(\tau,l)}(1 - l\hat{k}_\tau^{(1)}), r_{\tau-}\right) \\
&- \int_t^\tau \left(v_x^1(s, X_s^{\hat{k}^{(1)},(\tau,l)}, r_s)\sigma_1 \hat{k}_s^{(1)} X_s^{\hat{k}^{(1)},(\tau,l)} + v_r^1(s, X_s^{\hat{k}^{(1)},(\tau,l)}, r_s)\rho\sigma_2\right) dw_{1,s} \\
&- \int_t^\tau v_r^1(s, X_s^{\hat{k}^{(1)},(\tau,l)}, r_s)\sqrt{1-\rho^2}\sigma_2\, dw_{2,s}.
\end{aligned}
$$

Using the explicit form of the function $v^1(t,x,r) = \log(x) + W^{(1)}(t,r)$, we easily obtain that

$$
v_x^1(s,x,r)x = 1, \qquad v_r^1(s,x,r) = W_r^{(1)}(s,r) = \frac{1}{a}\left(1 - e^{-a(T-s)}\right),
$$

and it follows that

$$
\mathbb{E}^{t,x,r,1}\left[\int_t^\tau \left(\sigma_1 \hat{k}_s^{(1)} + W_r(s,r_s)\rho\sigma_2\right) dw_{1,s} + W_r(s,r_s)\sqrt{1-\rho^2}\sigma_2\, dw_{2,s}\right] = 0.
$$

Therefore, we have

$$v^1(t,x,r) \le \inf_{(\tau,l)} \sup_{k\in\Pi(t,x,r)} \mathbb{E}^{t,x,r,1}\left[v^0\left(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau\right)\right],$$
$$v^1(t,x,r) \le \sup_{k\in\Pi(t,x,r)} \inf_{(\tau,l)} \mathbb{E}^{t,x,r,1}\left[v^0\left(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau\right)\right].$$

Now, let $k \in \Pi(t,x,r)$ be arbitrary but fixed. We fix the strategy $(\tau,l) = (\theta,\tilde{l})$, where $\theta = \theta^1(t,x,r)$ and $\tilde{l} = l^{(1)}(k_\theta)$. Note that $\theta^n(t,x,r)$ and $l^{(1)}(k)$ are defined in (26) and (27), respectively. Then, by the same arguments as in the proof of Theorem 2.2.2, we obtain that

$$\begin{aligned} v^1(t,x,r) \ge{}& v^0(\theta, X_{\theta-}^{k,(\theta,\tilde{l})}(1-\tilde{l}k_\theta), r_\theta) \\ &- \int_t^\theta \left(v_x^1(s, X_s^{k,(\theta,\tilde{l})}, r_s)\sigma_1 k_s X_s^{k,(\theta,\tilde{l})} + v_r^1(s, X_s^{k,(\theta,\tilde{l})}, r_s)\rho\sigma_2\right) dw_{1,s} \\ &- \int_t^\theta v_r^1(s, X_s^{k,(\theta,\tilde{l})}, r_s)\sqrt{1-\rho^2}\sigma_2\, dw_{2,s}. \end{aligned}$$

Now, by the fact that the expectation of the stochastic integrals vanish, we obtain

$$v^1(t,x,r) \ge \sup_{k\in\Pi(t,x,r)} \inf_{(\tau,l)} \mathbb{E}^{t,x,r,1}\left[v^0(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right],$$
$$v^1(t,x,r) \ge \inf_{(\tau,l)} \sup_{k\in\Pi(t,x,r)} \mathbb{E}^{t,x,r,1}\left[v^0(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right].$$

Summing up, we obtain

$$\begin{aligned} v^1(t,x,r) &= \inf_{(\tau,l)} \sup_{k\in\Pi(t,x,r)} \mathbb{E}^{t,x,r,1}\left[v^0(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right] \\ &= \sup_{k\in\Pi(t,x,r)} \inf_{(\tau,l)} \mathbb{E}^{t,x,r,1}\left[v^0(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right] \\ &= \sup_{k\in\Pi(t,x,r)} \inf_{(\tau,l)} \mathbb{E}^{t,x,r,1}\left[V^0(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right] \\ &= V^1(t,x,r). \end{aligned}$$

Note that the third equality follows by part a) and fourth equality holds because Lemma 2.2.3 also holds for the logarithmic utility function. Therefore, $v^1(t,x,r) = V^1(t,x,r)$. Using the same arguments as above, and assuming that $v^{n-1}(t,x,r) = V^{n-1}(t,x,r)$, we arrive at $v^n(t,x,r) = V^n(t,x,r)$ and it follows that $\hat{k}_t^{(n)}$, which is the solution of (48), is the worst-case optimal investment strategy. □

REMARK 2.3.4. *In order to find a solution $W^{(0)}$ of the PDE (51) in part a) of the proof, we could also use the separation ansatz $W^{(0)}(t,r) = g^{(0)}(t) + \beta(t)r$, where we conclude that $\beta(t)$ and $g^{(0)}(t)$ are given by*

$$\beta(t) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right], \quad g^{(0)}(t) = \left(\frac{\mu}{2\sigma_1^2} + r_M\right)(T-t) - \frac{r_M}{a}\left(1 - e^{-a(T-t)}\right).$$

Obviously, this separation method leads to the same result

$$W^{(0)}(t,r) = \left(\frac{\mu^2}{2\sigma_1^2} + r_M\right)(T-t) + \frac{1}{a}\left(r - r_M + e^{-a(T-t)}(r_M - r)\right).$$

2.3.1. Practical implication of Theorem 2.3.2. By Theorem 2.3.2, the optimal investment strategies are determined by $k_t^{(0)*} \equiv \frac{\mu}{\sigma_1^2}$ after the N-th crash and by $\hat{k}_t^{(n)}$, which solves (48), if n market crashes still can happen. Here, we can see that the worst-case optimal strategies do not depend on the short rate as they do not depend on parameters a, r_M and σ_2 determining the short rate dynamics in (2). Thus, in contrast to the non-log HARA utility case in Section 2.2, the logarithmic utility function eliminates the stochastic interest rate risk. We can give a heuristic explanation, for example in the 1-crash market. There, we have for arbitrary $k \in \Pi(t,x,r)$ and an arbitrary crash (τ, l):

$$X_T^{k,(\tau,l)} = (1 - lk_\tau)\tilde{X}_T^k$$

where $\tilde{X}_t^k$ is the wealth process in a crash-free market starting in (t,x) and controlled by $k \in \Pi(t,x,r)$, and therefore

$$\begin{aligned}
&\mathbb{E}^{t,x,r}\left(\log(X_T^{k,(\tau,l)})\right)\\
&= \mathbb{E}\left(\log(1 - lk_\tau)\right) + \mathbb{E}\left(\log(\tilde{X}_T^k)\right)\\
&= \mathbb{E}\left(\log(1 - lk_\tau)\right) + \log(x) + \mathbb{E}\left(\int_t^T \mu k_s - \frac{\sigma_1^2}{2}k_s^2\, ds\right) + \mathbb{E}\left(\int_t^T r_s\, ds\right).
\end{aligned}$$

Obviously, the optimal strategy $k^* \in \Pi(t,x,r)$ which maximizes the worst-case expected utility of terminal wealth

$$\inf_{(\tau,l)} \mathbb{E}^{t,x,r,1}\left(\log(X_T^{k,(\tau,l)})\right)$$

will neither depend on the short rate $r_t(\omega)$ itself, nor on parameters which determine the short rate equation (2), because it is not affected by the market crash or by control k. Moreover, it is important to note that the worst-case optimal strategies for an investor with a logarithmic utility function do not differ from the strategies which are optimal on a financial market with constant interest rates, see e.g. [28, 33]. The main reason is that the logarithmic utility function eliminates the stochastic interest rate risk. As we have mentioned in Section 2.2.1, this elimination is not possible if the investor has a non-log HARA utility function and if $\rho \neq 0$.

2.4. HARA utility via martingale approach

In this section we provide an alternative proof for part b) of Theorem 2.2.2 using the so-called martingale approach. Recently, this method has been introduced by Seifried [44] for worst-case optimization problems in financial markets with constant interest rates. It is based on interpreting the worst-case optimization problem as a controller vs. stopper game. Moreover, the martingale approach was also used by Desmettre et al. [10], where a worst-case lifetime consumption problem is solved. If there can happen at most one market crash on the time interval, the method contains three main steps: First, the post-crash optimization problem is solved by using standard stochastic optimal control theory. In our case, this step corresponds to part a) of Theorem 2.2.2. Afterwards, they can reformulate the problem as a pre-crash problem which can be interpreted as a controller vs. stopper game. Finally, using the notion of *indifference strategies* and *indifference frontier* (for the detailed definition we refer to [44] or to Definition 2.4.1 below), they can determine the optimal pre-crash strategy.

Here, we build on the martingale approach and the ideas in [44] and consider the worst-case optimization problem (8). Note that -in contrast to the literature- we have to handle the influence of the stochastic instantaneous interest rates. First, we demonstrate the three steps of the method for $N = 1$ and, using this result, we show that we can recursively apply the martingale approach in order to provide an alternative proof for Theorem 2.2.2.

2.4.1. Martingale approach for $N = 1$. We consider the financial market model described in Section 2.1 for the special case $N = 1$ and for $U(x) = \frac{1}{\gamma}x^\gamma$. The solution of the corresponding worst-case optimization problem was published by Engler and Korn [14]. We assume that at most one market crash, denoted by (τ, l), can happen on the finite time interval $[0, T]$. Again, τ is a $[0, T] \cup \{\infty\}$-valued stopping time and the crash size $l \in [0, l^*]$ is a $\mathcal{F}_\tau$-measurable random variable. Note that the event $\tau = \infty$ means that no crash happens at all. Analogously to the wealth equation (7), given an admissible control $k = (k^{(0)}, k^{(1)})$ and a crash strategy (τ, l), we formulate the SDE for the investor's wealth $X^{k,(\tau,l)}$ in the following way:

$$
\begin{aligned}
X_0^{k,(\tau,l)} &= x^0 > 0, \\
dX_t^{k,(\tau,l)} &= X_t^{k,(\tau,l)} \left[r_t + \mu k_t^{(1)} \right] dt + X_t^{k,(\tau,l)} \sigma_1 k_t^{(1)} \, dw_{1,t}, \qquad t \in (0, \tau), \\
X_\tau^{k,(\tau,l)} &= (1 - l k_\tau^{(1)}) X_{\tau_-}^{k,(\tau,l)}, \\
dX_t^{k,(\tau,l)} &= X_t^{k,(\tau,l)} \left[r_t + \mu k_t^{(0)} \right] dt + X_t^{k,(\tau,l)} \sigma_1 k_t^{(0)} \, dw_{1,t}, \qquad t \in (\tau, T].
\end{aligned}
$$

Due to the fact that we consider a one-crash market, $k^{(1)}$ is called *pre-crash strategy*, which is valid for $t \in [0, \tau]$ and $k^{(0)}$ is called *post-crash strategy*, valid for $t \in (\tau, T]$. Now, problem (8) simplifies for $N = 1$ to

$$
\sup_{k \in \Pi(0, x^0, r^0)} \inf_{(\tau,l) \in \mathcal{C}} \mathbb{E} \left(\frac{1}{\gamma} \left(X_T^{k,(\tau,l)} \right)^\gamma \right), \quad \gamma < 1, \gamma \neq 0, \tag{52}
$$

where $\mathcal{C}$ denotes the set of crash scenarios (τ, l):

$$
\begin{aligned}
\mathcal{C} := &\{(\tau, l) : \tau \in [0, T] \cup \{\infty\} \text{ stopping time}, \\
&\quad l \in [0, l^*] \, \mathcal{F}_\tau - \text{measurable random variable}\}.
\end{aligned}
$$

In Section 2.2, we applied the classical DPP to prove part a) of Theorem 2.2.2 and to determine the optimal post-crash strategy $k^{(0)*}$ and the corresponding post-crash value function $V^0(t, x, r)$, which are given by:

$$
k_t^{(0)*} = \frac{\mu}{(1-\gamma)\sigma_1^2} + \frac{\rho \sigma_2 \beta(t)}{(1-\gamma)\sigma_1}, \quad V^0(t, x, r) = \frac{1}{\gamma} x^\gamma g^{(0)}(t) \exp(\beta(t) r). \tag{53}
$$

Instead of solving the HJB-inequality system, here, we determine the optimal pre-crash strategy $k^{(1)*}$ by using the post-crash value function $V^0(t, x, r)$ and by applying the following ideas, which have already been applied in the case of constant interest rates (see, e.g. [10, 26, 44]). Using the explicit structure of the post-crash value function V^0, we first reformulate problem (52) as a pre-crash problem, which can be rewritten as a controller vs. stopper game. Afterwards, we identify the optimal pre-crash strategy $k^{(1)*}$ via a combination of the principle of the indifference frontier and the solution of a constrained control problem. These steps lead to an alternative proof of part b) of Theorem 2.2.2 for $N = 1$.

Let $\tilde{X}^k = \{\tilde{X}^k_t\}_{t\in[0,T]}$ be the wealth process in a crash-free market controlled by an arbitrary admissible pre-crash strategy k. That means, $\tilde{X}^k$ is the uniquely determined solution of the following SDE for $t \in [0,T]$:

$$d\tilde{X}^k_t = \tilde{X}^k_t \left[r_t + \mu k_t\right] dt + \tilde{X}^k_t \sigma_1 k_t \, dw_{1,t}, \qquad \tilde{X}^k_0 = x^0,$$

where $r = \{r_t\}_{t\in[0,T]}$ solves (2).
At the crash time τ the investor's wealth equals $x = (1 - lk^{(1)}_\tau)X^{k,(\tau,l)}_{\tau_-} = (1 - lk^{(1)}_\tau)\tilde{X}^{k^{(1)}}_\tau$ and the short rate is denoted by $r = r_\tau$. Then, by Lemma 2.2.3, we can reformulate the worst-case optimization problem (52) as a pre-crash problem:

$$\sup_{k^{(1)}\in\Pi(0,x^0,r^0)} \inf_{(\tau,l)\in\mathcal{C}} \mathbb{E}\left(V^0(\tau, \tilde{X}^{k^{(1)}}_\tau(1 - lk^{(1)}_\tau), r_\tau)\right).$$

Since $V^0(t,x,r)$ is strictly monotone increasing with respect to x, the worst-case crash size is $l = 0$ if $k^{(1)}_\tau < 0$ and it is $l = l^*$ if $k^{(1)}_\tau \geq 0$. This, can also be seen from a practical point of view. If the investor has a negative position in the stock, he would benefit from a positive crash height at the crash time. Thus, the worst-case for the investor is a jump of size zero. On the other hand, if the investor holds the stock at the crash time, then the worst-case crash size is given by the maximum crash size l^*. Thus, we have for a fixed crash time τ:

$$V^0(\tau, \tilde{X}^{k^{(1)}}_\tau(1 - lk^{(1)}_\tau), r_\tau) \geq V^0(\tau, \tilde{X}^{k^{(1)}}_\tau(1 - l^*(k^{(1)}_\tau)^+), r_\tau), \quad \forall l \in [0, l^*].$$

Therefore, the worst-case optimization problem (52) can be rewritten as a controller vs. stopper game of the form:

$$\sup_{k^{(1)}\in\Pi(0,x^0,r^0)} \inf_{\tau\in\mathcal{C}} \mathbb{E}\left(M^{k^{(1)}}_\tau\right), \quad \text{where} \quad M^k_t := V^0(t, \tilde{X}^k_t(1 - l^*(k_t)^+), r_t). \tag{54}$$

Here, the investor takes the role of the controller, who chooses his strategy $k^{(1)}$, and the market takes the role of the stopper, who chooses the crash time τ. Now, the aim is to solve this controller vs. stopper game. As already mentioned above, Korn and Seifried [26] and Seifried [44] used the notion indifference to determine the optimal pre-crash strategy for a model with a constant interest rate. For the reader's convenience we give the definition of an indifference strategy here, which can also be found in [44, Chp. 4.1].

Definition 2.4.1 (Indifference Strategy, cf. Seifried [44]). *A pre-crash strategy k is called indifference strategy if for two stopping times τ, τ' it holds*

$$\mathbb{E}\left(M^k_\tau\right) = \mathbb{E}\left(M^k_{\tau'}\right).$$

If the investor applies an indifference strategy before the market crash, then he is indifferent with respect to the crash time because he always reaches the same performance. In the next step, we show that $\hat{k}^{(1)}$, which is the uniquely determined solution of ODE (13), is an indifference strategy.

LEMMA 2.4.2 (cf. [14]). *Let $\hat{k}^{(1)}$ be the uniquely determined solution of the following ODE:*

$$\dot{k}_t^{(1)} = \frac{1-l^*k_t^{(1)}}{l^*}\left(\phi(t,k_t^{(1)})-\phi(t,k_t^{(0)*})\right), \qquad k_T^{(1)}=0, \tag{55}$$
$$\phi(t,k) = (\mu+\rho\sigma_1\sigma_2\beta(t))k - \frac{\sigma_1^2}{2}(1-\gamma)k^2,$$

and let $M^k=\{M_t^k\}_{t\in[0,T]}$ be given by (54) for $t\in[0,T]$ and $M_\infty^k := V^0(T,\tilde{X}_T^k,r_T)$. Then $M^{\hat{k}^{(1)}}$ is a martingale on $[0,T]\cup\{\infty\}$ and $\hat{k}^{(1)}$ is an indifference strategy for the controller vs. stopper game (54).

REMARK 2.4.3. *Note that (55) is equal to (13) for $n=1$, and obviously, Proposition 2.2.4 remains valid and it holds $\hat{k}_t^{(1)}\in[0,\frac{1}{l^*})$ for all $t\in[0,T]$. Thus, $\hat{k}^{(1)}$ is an admissible strategy in the sense of Definition 2.1.1.*

PROOF OF LEMMA 2.4.2. Throughout the proof we abbreviate $\hat{k}^{(1)}$ by $\hat{k}$. As in [44], we use a martingale argument to prove the assertion. The proof will be divided into two steps. First, we show that $M^{\hat{k}}$ is a martingale on $[0,T]\cup\{\infty\}$, and then, we obtain the assertion by applying Doob's Optional Sampling Theorem.
By applying Ito's formula, by $V^0(t,x,r)=\frac{1}{\gamma}x^\gamma W(t,r)$, with $W(t,r)=g^{(0)}(t)\exp(\beta(t)r)$, and by the fact that $\hat{k}_t^+=\hat{k}_t$ we get:

$$\begin{aligned}
dM_t^{\hat{k}} =& d\left(V^0(t,\tilde{X}_t^{\hat{k}}(1-l^*\hat{k}_t),r_t)\right)\\
=&\frac{1}{\gamma}(\tilde{X}_t^{\hat{k}})^\gamma(1-l^*\hat{k}_t)^\gamma\\
&\quad\left\{\gamma\frac{-l^*}{(1-l^*\hat{k}_t)}\dot{\hat{k}}_t W(t,r_t)\right.\\
&\quad+\gamma\left(\mu\hat{k}_t-\frac{\sigma_1^2}{2}(1-\gamma)\hat{k}_t^2+\rho\sigma_1\sigma_2\hat{k}_t\frac{W_r(t,r_t)}{W(t,r_t)}\right)W(t,r_t)\\
&\quad\left.+W_t(t,r_t)+\frac{\sigma_2^2}{2}W_{rr}(t,r_t)+a(r_M-r_t)W_r(t,r_t)+\gamma r_t W(t,r_t)\right\}dt\\
&+\frac{1}{\gamma}(\tilde{X}_t^{\hat{k}})^\gamma(1-l^*\hat{k}_t)^\gamma\\
&\quad\left\{\left(\gamma\sigma_1\hat{k}_tW(t,r_t)+\rho\sigma_2W_r(t,r_t)\right)dw_{1,t}+\sqrt{1-\rho^2}\sigma_2W_r(t,r_t)\,dw_{2,t}\right\}.
\end{aligned}$$

From the post-crash problem (see proof of part a) of Theorem 2.2.2) we have that $W(t,r)$ solves equation (19), and therefore, we have

$$\begin{aligned}
W_t(t,r_t)+&\frac{\sigma_2^2}{2}W_{rr}(t,r_t)+a(r_M-r_t)W_r(t,r_t)+\gamma r_t W(t,r_t)\\
&=-\gamma\phi(t,k_t^{(0)*})W(t,r_t),
\end{aligned}$$

and

$$\begin{aligned}
dM_t^{\hat{k}} =&\frac{1}{\gamma}(\tilde{X}_t^{\hat{k}})^\gamma(1-l^*\hat{k}_t)^\gamma W(t,r_t)\left\{\gamma\frac{-l^*}{(1-l^*\hat{k}_t)}\dot{\hat{k}}_t+\gamma\left(\phi(t,\hat{k}_t)-\phi(t,k_t^{(0)*})\right)\right\}dt\\
&+\frac{1}{\gamma}(\tilde{X}_t^{\hat{k}})^\gamma(1-l^*\hat{k}_t)^\gamma W(t,r_t)\left\{\left(\gamma\sigma_1\hat{k}_t+\rho\sigma_2\beta(t)\right)dw_{1,t}+\sqrt{1-\rho^2}\sigma_2\beta(t)\,dw_{2,t}\right\}.
\end{aligned}$$

Because of the fact that $\hat{k}_t$ fulfills (55), it remains to show that:

$$dM_t^{\hat{k}} = \underbrace{\frac{1}{\gamma}(\tilde{X}_t^{\hat{k}})^\gamma(1-l^*\hat{k}_t)^\gamma W(t,r_t)}_{=M_t^{\hat{k}}} \cdot \left\{\left(\gamma\sigma_1\hat{k}_t + \rho\sigma_2\beta(t)\right)dw_{1,t} + \sqrt{1-\rho^2}\sigma_2\beta(t)\,dw_{2,t}\right\}$$

is a martingale. The solution of this SDE is given by

$$M_t^{\hat{k}} = \underbrace{M_0^{\hat{k}}}_{const} \cdot \exp\left(\int_0^t \left(\gamma\sigma_1\hat{k}_s + \rho\sigma_2\beta(s)\right)dw_{1,s} + \int_0^t \sqrt{1-\rho^2}\sigma_2\beta(s)\,dw_{2,s}\right.$$
$$\left. - \frac{1}{2}\int_0^t (\gamma\sigma_1\hat{k}_s + \rho\sigma_2\beta(s))^2 + (1-\rho^2)\sigma_2^2\beta^2(s)\,ds\right).$$

By Novikov's condition (see e.g. [21, Chp.3,Corollary 5.13]), the second factor is a martingale and therefore $M^{\hat{k}}$ is a martingale on $[0,T]$. It remains to show, that

$$\mathbb{E}\left(M_\infty^{\hat{k}}|\mathcal{F}_T\right) = M_T^{\hat{k}}.$$

By definition of M_∞^k and by the terminal condition of ODE (55), given by $\hat{k}_T = 0$, we have

$$\mathbb{E}\left(M_\infty^{\hat{k}}|\mathcal{F}_T\right) = \mathbb{E}\left(\underbrace{V^0(T,\tilde{X}_T^{\hat{k}},r_T)}_{\mathcal{F}_T-\text{measurable}}|\mathcal{F}_T\right) = V^0(T,\tilde{X}_T^{\hat{k}},r_T) = M_T^{\hat{k}}.$$

Thus, $M^{\hat{k}}$ is a martingale on $[0,T]\cup\{\infty\}$. By Doob's Optional Sampling Theorem (see Appendix A, Theorem A.4.7), we obtain

$$\mathbb{E}\left(M_\tau^{\hat{k}}\right) = \mathbb{E}\left(M_{\tau'}^{\hat{k}}\right)$$

for all $[0,T]\cup\{\infty\}$-valued stopping times τ,τ'. By definition, $\hat{k}^{(1)}$ is an indifference strategy for the controller vs. stopper game (54). □

Now, we use the notion of an *indifference frontier* (see for example [26, p.343]), which leads to the fact that the optimal strategy of the controller vs. stopper game (54) has to be an element of a certain class of admissible strategies:

Let $k^{(1)} \in \Pi$ be an arbitrary admissible pre-crash strategy and let $\hat{k}^{(1)}$ be the solution of ODE (55), then $M^{\hat{k}^{(1)}}$ is a martingale on $[0,T]\cup\{\infty\}$ by Lemma 2.4.2. Define $\eta := \inf\{t \geq 0 : k_t > \hat{k}_t\}$ and

$$\tilde{k}_t := \begin{cases} k_t^{(1)} & : t < \eta \\ \hat{k}_t^{(1)} & : t \geq \eta \end{cases}.$$

Then, as in [26, Lemma 4.3], we obtain by the martingale property of $M^{\hat{k}^{(1)}}$ and by continuity of $k^{(1)}$ (see condition 4 in Definition 2.1.1) that

$$\inf_{\tau\in\mathcal{C}} \mathbb{E}\left(M_\tau^{\tilde{k}}\right) \geq \inf_{\tau\in\mathcal{C}} \mathbb{E}\left(M_\tau^{k^{(1)}}\right).$$

The inequality implies that it is sufficient to consider pre-crash strategies $k^{(1)}$ for which $k_t^{(1)} \leq \hat{k}_t^{(1)}$ for all $t \in [0,T]$. The optimal strategy cannot cross the indifference frontier $\hat{k}^{(1)}$, because one could improve its performance by cutting it off at $\hat{k}^{(1)}$, and therefore it would not be optimal.

Thus, the optimal pre-crash strategy is an element of the set

$$\mathcal{A}(\hat{k}^{(1)}) := \left\{k^{(1)} \in \Pi : k_t^{(1)} \leq \hat{k}_t^{(1)}, \quad \forall\, t \in [0,T]\right\}.$$

Note that the indifference frontier $\hat{k}^{(1)}$ prevents the investor from too optimistic investment in the stock which is threatened by a market crash. The next Lemma shows that $k_t^{(1)*} = \hat{k}_t^{(1)} \wedge k_t^{(0)*}$, is optimal in the no-crash scenario, denoted by $\tau = \infty$, in the class $\mathcal{A}(\hat{k}^{(1)})$.

Lemma 2.4.4 (cf. [14]). *Let $k^{(0)*}$ be given by* (53)*, and let $\hat{k}^{(1)}$ be the uniquely determined indifference strategy as a solution of ODE* (55)*. Then, the solution of the constrained stochastic optimal control problem:*

$$\begin{aligned}
&\sup_{k_t \leq \hat{k}_t^{(1)}, t\in[0,T]} \mathbb{E}\left(\frac{1}{\gamma}(\tilde{X}_T^k)^\gamma\right), \\
w.r.t \quad & d\tilde{X}_s^k = \tilde{X}_s^k\,[r_s + \mu k_s]\,ds + \tilde{X}_s^k \sigma_1 k_s\, dw_{1,s}, \qquad \tilde{X}_0 = x^0, \\
& dr_s = a(r_M - r_s)\,ds + \sigma_2\, d\tilde{w}_s, \qquad r_0 = r^0,
\end{aligned}$$

is given by $k_t^{(1)*} = \hat{k}_t^{(1)} \wedge k_t^{(0)*}$.

Proof. Let $\tilde{V}(t,x,r)$ denote the value function of the constrained stochastic optimal control problem above. Here, we use again DPP and solve the corresponding HJB equation which is given by:

$$\begin{aligned}
\sup_{k \leq \hat{k}_t^{(1)}} \mathcal{L}^k \tilde{v}(t,x,r) = 0, \qquad & (t,x,r) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}, \\
\tilde{v}(T,x,r) = \frac{1}{\gamma} x^\gamma, \qquad & (x,r) \in \mathbb{R}_+ \times \mathbb{R}.
\end{aligned}$$

By the standard separation method $\tilde{v}(t,x,r) = \frac{1}{\gamma}x^\gamma \tilde{g}(t)\exp(\tilde{\beta(t)}r)$ with $\tilde{g}(t) > 0$ for $t \in [0,T]$, $\tilde{g}(T) = 1$ and $\tilde{\beta}(T) = 0$, we reduce the HJB equation above to:

$$\begin{aligned}
&\dot{\tilde{g}}(t) + [\dot{\tilde{\beta}}(t) - a\tilde{\beta}(t) + \gamma]\tilde{g}(t)r + [ar_M\tilde{\beta}(t) + \frac{\sigma_2^2}{2}\tilde{\beta}^2(t)]\tilde{g}(t) \\
&+ \gamma \sup_{k \leq \hat{k}_t^{(1)}} \left[\tilde{g}(t)\left((\mu + \rho\sigma_1\sigma_2\tilde{\beta}(t))k - \frac{\sigma_1^2}{2}(1-\gamma)k^2\right)\right] = 0, \qquad (t,r) \in [0,T] \times \mathbb{R}, \\
&\tilde{g}(T) = 1, \tilde{\beta}(T) = 0.
\end{aligned}$$

Now, by the first order optimality condition, we obtain a candidate for the optimal control

$$k_t^{(1)*} = \left(\frac{\mu}{(1-\gamma)\sigma_1^2} + \frac{\rho\sigma_2\tilde{\beta}(t)}{(1-\gamma)\sigma_1}\right) \wedge \hat{k}_t^{(1)}. \tag{56}$$

In order to eliminate the space variable r, we choose

$$\tilde{\beta}(t) = \frac{\gamma}{a}\left[1 - \exp(-a(T-t))\right] = \beta(t).$$

Inserting both $k_t^{(1)*}$ and $\tilde\beta(t)$ we arrive at an ODE for $\tilde g(t)$:

$$\dot{\tilde g}(t)+\tilde g(t)\Big(\gamma(\mu+\rho\sigma_1\sigma_2\tilde\beta(t))k_t^{(1)*}-\frac{\sigma_1^2}{2}\gamma(1-\gamma)(k_t^{(1)*})^2 \\ +ar_M\tilde\beta(t)+\frac{\sigma_2^2}{2}\tilde\beta^2(t)\Big)=0,\qquad \tilde g(T)=1.$$

Now, we have

$$\tilde g(t)=\exp\left(\int_t^T\tilde\alpha(s)\,ds\right),$$
$$\tilde\alpha(s):=\gamma(\mu+\rho\sigma_1\sigma_2\tilde\beta(s))k_s^{(1)*}-\frac{\sigma_1^2}{2}\gamma(1-\gamma)(k_s^{(1)*})^2+ar_M\tilde\beta(s)+\frac{\sigma_2^2}{2}\tilde\beta^2(s).$$

By (56) and by the fact that $\tilde\beta(t)=\beta(t)$ we obtain that

$$k_t^{(1)*}=\left(\frac{\mu}{(1-\gamma)\sigma_1^2}+\frac{\rho\sigma_2\beta(t)}{(1-\gamma)\sigma_1}\right)\wedge\hat k_t^{(1)}=k_t^{(0)*}\wedge\hat k_t^{(1)}$$

is a deterministic and continuous function in t, since $\hat k_t$ is deterministic and continuous. Finally, we have that

$$\tilde v(t,x,r)=\frac{1}{\gamma}x^\gamma\tilde g(t)\exp(\beta(t)r)$$

solves the HJB equation which corresponds to the constrained optimization problem. Using the same arguments for the verification result as in [24, Corollary 3.2], we conclude that $k_t^{(1)*}=\hat k_t^{(1)}\wedge k_t^{(0)*}$ is indeed the optimal control of the constrained optimization problem. □

Using Lemma 2.4.2, the idea of the indifference frontier and Lemma 2.4.4, we provide an alternative proof of part b) of Theorem 2.2.2 for $N=1$, that means, we show that $k_t^{(1)*}$ is the optimal strategy for the controller vs. stopper game (54).

Alternative Proof of Theorem 2.2.2 for $N=1$, cf.[14].
First, we define

$$t_S:=\inf\{t\in[0,T]:k_t^{(0)*}\geq\hat k_t^{(1)}\}.$$

Since $\hat k_T^{(1)}=0$ and $k_T^{(0)*}>0$ the infimum is attained at $t_S<T$, which is the point of intersection of $\hat k_t^{(1)}$ and $k_t^{(0)*}$ (if it exists).
Now, let us consider the stochastic process $M^{k^{(1)*}}$ on the interval $[t_S,T]$. For $t\in[t_S,T]$, it holds $k_t^{(1)*}=\hat k_t^{(1)}\wedge k_t^{(0)*}=\hat k_t^{(1)}$. In Lemma 2.4.2, we already proved that $M^{\hat k^{(1)}}$ is a martingale on $[0,T]\cup\{\infty\}$, and therefore, $M^{k^{(1)*}}$ is a martingale on $[t_S,T]\cup\{\infty\}$. Note that if $t_S=0$, that means $k_t^{(0)*}\geq\hat k_t^{(1)}$ for all $t\in[0,T]$, then $M^{k^{(1)*}}$ is a martingale on $[0,T]\cup\{\infty\}$. In particular, this is the case if $\gamma\rho\geq0$ (see Proposition 2.2.7).
Now, let $\gamma\rho<0$ and assume that $t_S>0$, that means there exists a (uniquely determined) intersection point of $\hat k_t^{(1)}$ and $k_t^{(0)*}$, denoted by t_S. Moreover, let us define

$$t_0:=\inf\{t\in[0,T]:k_t^{(0)*}\geq0\}.$$

If $t_0>0$, then t_0 denotes the uniquely determined root of $k_t^{(0)*}$ because it is strictly monotone increasing for $\gamma\rho<0$.
Let us consider the stochastic process $M^{k^{(1)*}}$ on the interval $[t_0,t_S]$. For $t\in[t_0,t_S]$ it holds

$k_t^{(1)*} = k_t^{(0)*}$ and we have

$$dM_t^{k^{(1)*}} = M_t^{k^{(1)*}} \cdot \left\{ -\gamma \frac{l^*}{1-l^* k_t^{(0)*}} \dot{k}_t^{(0)*} + \gamma\phi(t, k_t^{(0)*}) - \gamma\phi(t, k_t^{(0)*}) \right\} dt$$
$$+ M_t^{k^{(1)*}} \cdot \left\{ (\gamma\sigma_1 k_t^{(0)*} + \rho\sigma_2\beta(t)) dw_{1,t} + \sqrt{1-\rho^2}\sigma_2\beta(t)\, dw_{2,t} \right\}.$$

With

$$\dot{k}_t^{(0)*} = \frac{\rho\sigma_2}{\sigma_1(1-\gamma)}(-\gamma\exp(-a(T-t)))$$

we obtain

$$\begin{aligned} M_t^{k^{(1)*}} =& M_{t_0}^{k^{(1)*}} \cdot \exp\left\{ \int_{t_0}^t \gamma^2 \frac{l^*}{1-l^* k_s^{(0)*}} \cdot \frac{\rho\sigma_2}{\sigma_1(1-\gamma)} \exp(-a(T-s))\, ds \right\} \\ &\cdot \exp\Big\{ -\frac{1}{2}\int_{t_0}^t (\gamma\sigma_1 k_s^{(0)*} + \rho\sigma_2\beta(s))^2 + (1-\rho^2)\sigma_2^2\beta^2(s)\, ds \\ &\quad + \int_{t_0}^t (\gamma\sigma_1 k_s^{(0)*} + \rho\sigma_2\beta(s)) dw_{1,s} + \sqrt{1-\rho^2}\sigma_2\beta(s)\, dw_{2,s} \Big\}. \end{aligned}$$

Now, by Novikov's condition, the last factor is a martingale on $[t_0, t_S]$. As further, $M_{t_0}^{k^{(1)*}}$ is $\mathcal{F}_s$-measurable for $s \geq t_0$, we have for $t_0 \leq s \leq t \leq t_S$:

$$\begin{aligned} &\mathbb{E}\left(M_t^{k^{(1)*}} | \mathcal{F}_s \right) \\ &= M_{t_0}^{k^{(1)*}} \cdot \exp\left\{ \rho \int_{t_0}^t \gamma^2 \frac{l^*}{1-l^* k_u^{(0)*}} \cdot \frac{\sigma_2}{\sigma_1(1-\gamma)} \exp(-a(T-u))\, du \right\} \\ &\quad \cdot \exp\Big\{ -\frac{1}{2}\int_{t_0}^s (\gamma\sigma_1 k_u^{(0)*} + \rho\sigma_2\beta(u))^2 + (1-\rho^2)\sigma_2^2\beta^2(u)\, du \\ &\qquad + \int_{t_0}^s (\gamma\sigma_1 k_u^{(0)*} + \rho\sigma_2\beta(u)) dw_{1,u} + \sqrt{1-\rho^2}\sigma_2\beta(u)\, dw_{2,u} \Big\} \\ &\leq M_{t_0}^{k^{(1)*}} \cdot \exp\left\{ \rho \int_{t_0}^s \gamma^2 \frac{l^*}{1-l^* k_u^{(0)*}} \cdot \frac{\sigma_2}{\sigma_1(1-\gamma)} \exp(-a(T-u))\, du \right\} \\ &\quad \cdot \exp\Big\{ -\frac{1}{2}\int_{t_0}^s (\gamma\sigma_1 k_u^{(0)*} + \rho\sigma_2\beta(u))^2 + (1-\rho^2)\sigma_2^2\beta^2(u)\, du \\ &\qquad + \int_{t_0}^s (\gamma\sigma_1 k_u^{(0)*} + \rho\sigma_2\beta(u)) dw_{1,u} + \sqrt{1-\rho^2}\sigma_2\beta(u)\, dw_{2,u} \Big\} \\ &= M_s^{k^{(1)*}}. \end{aligned}$$

The inequality above holds because of two arguments: First, we observe that $k_u^{(0)*} \leq \hat{k}_u^{(1)} < \frac{1}{l^*}$ for $u \in [t_0, t_S]$ and therefore, the integrand of the deterministic integral is positive. Secondly, we only have to consider the cases $\gamma > 0, \rho < 0$ and $\gamma < 0, \rho > 0$ (because of $\gamma\rho < 0$) for the estimate of the deterministic integral. For both of these cases we easily obtain that:

$$M_{t_0}^{k^{(1)*}} \exp\left\{ \rho \int_{t_0}^t \ldots\, du \right\} < M_{t_0}^{k^{(1)*}} \exp\left\{ \rho \int_{t_0}^s \ldots\, du \right\},$$

for $s \le t$, because $M_{t_0}^{k^{(1)*}} > 0$ if $\gamma > 0$ and $M_{t_0}^{k^{(1)*}} < 0$ if $\gamma < 0$. The arguments above imply that $\mathbb{E}\left(M_t^{k^{(1)*}}|\mathcal{F}_s\right) \le M_s^{k^{(1)*}}$ for $t_0 \le s \le t \le t_S$. Therefore, $M^{k^{(1)*}}$ is a supermartingale on $[t_0, t_S]$. If $t_0 = 0$, we obtain, together with the martingale property on $[t_S, T]$, that $M^{k^{(1)*}}$ is a supermartingale on $[0,T] \cup \{\infty\}$.
Otherwise if $t_0 > 0$, then we have to consider $M^{k^{(1)*}}$ on the interval $[0, t_0]$. By definition of t_0 we have that $k_t^{(0)*} \le 0$, and therefore $k_t^{(1)*} = k_t^{(0)*} \le 0$ for $t \in [0, t_0]$. For $t \in [0, t_0]$, we obtain:

$$\begin{aligned} dM_t^{k^{(1)*}} &= d\left(V^0(t, \tilde{X}_t^{k^{(1)*}}, r_t)\right) \\ &= M_t^{k^{(1)*}}\left\{(\gamma\sigma_1 k_t^{(0)*} + \rho\sigma_2\beta(t))dw_{1,t} + \sqrt{1-\rho^2}\sigma_2\beta(t)\,dw_{2,t}\right\}. \end{aligned}$$

Again, by Novikov's condition, we obtain that $M^{k^{(1)*}}$ is a martingale on $[0, t_0]$.
Finally, $M^{k^{(1)*}}$ is a supermartingale on $[0,T] \cup \{\infty\}$ (if $\gamma\rho \ge 0$, it is even a martingale on $[0,T] \cup \{\infty\}$). Again, by Doob's Optional Sampling Theorem for supermartingales, see Theorem A.4.7, we have for all $\tau \in [0,T] \cup \{\infty\}$:

$$M_\tau^{k^{(1)*}} \ge \mathbb{E}\left(M_\infty^{k^{(1)*}}|\mathcal{F}_\tau\right)$$

and therefore

$$\mathbb{E}\left(M_\tau^{k^{(1)*}}\right) \ge \mathbb{E}\left(M_\infty^{k^{(1)*}}\right), \tag{57}$$

for all $[0,T] \cup \{\infty\}$-valued stopping times τ. The inequality implies that $\tau = \infty$ is a worst-case scenario for an investor who follows the strategy $k_t^{(1)*} = \hat{k}_t^{(1)} \wedge k_t^{(0)*}$.
Analogously to the Indifference Optimality Principle in [26] and [44], we obtain

$$\inf_{\tau\in\mathcal{C}} \mathbb{E}\left(M_\tau^{k^{(1)*}}\right) \overset{(57)}{\ge} \mathbb{E}\left(M_\infty^{k^{(1)*}}\right) \ge \mathbb{E}\left(M_\infty^{k}\right) \ge \inf_{\tau\in\mathcal{C}} \mathbb{E}\left(M_\tau^{k}\right). \tag{58}$$

for an arbitrary pre-crash strategy $k \in \mathcal{A}(\hat{k}^{(1)})$ The second inequality holds, because $k^{(1)*}$ is optimal in the no-crash scenario in the class $\mathcal{A}(\hat{k}^{(1)})$ (see Lemma 2.4.4). By inequality (58), $k^{(1)*}$ is the optimal strategy for the controller vs. stopper game in the class $\mathcal{A}(\hat{k}^{(1)})$. Due to the indifference frontier, that means, due to the fact that the optimal strategy is an element of $\mathcal{A}(\hat{k}^{(1)})$, we obtain that $k^{(1)*}$ is the optimal pre-crash strategy for the worst-case optimization problem (52). □

The proof above implies that we can also determine the optimal pre-crash strategy by applying the martingale approach. In the next section, we apply the three key steps of this section in a recursive way.

2.4.2. The recursive application of the martingale approach for $N > 1$. Here, we consider the general worst-case optimization problem (8) again:

$$\sup_{k\in\Pi(0,x^0,r^0)} \inf_{M\in\mathcal{N}(0,N)} \mathbb{E}\left(U(X_T^{k,M})\right).$$

After the $(N-1)$-th market crash the investor has to 'solve' a worst-case optimization problem of the form (52) with at most one market crash. From the previous section, we already know that after the $(N-1)$-th market crash it is optimal to follow the strategy $(k^{(1)*}, k^{(0)*})$, where $k_t^{(1)*} = \hat{k}_t^{(1)} \wedge k_t^{(0)*}$ is valid before the next market crash at τ_N and $k^{(0)*}$ is valid after τ_N. Now, using this optimal strategy, we additionally calculate the corresponding value function $V^1(t,x,r)$,

which reflects the worst-case optimal utility if the process $X^{k,M}$ starts in t with value x and the short rate process starts in r and at most one crash still can happen.
By definition, it holds:

$$V^1(t,x,r) = \sup_{k\in\Pi(t,x,r)} \inf_{M\in\mathcal{N}(t,1)} \mathbb{E}^{t,x,r,1}\left(U(X_T^{k,M})\right).$$

Note that V^1 was already determined in Section 2.2 as a solution of a HJB inequality system. Here, we demonstrate how to find $V^1(t,x,r)$ directly, using that we already know that $(k^{(1)*}, k^{(0)*})$ is optimal and by using the value function V^0, which is known from part a) of Theorem 2.2.2. Let $(t,x,r)\in[0,T]\times\mathbb{R}_+\times\mathbb{R}$ be arbitrary but fixed.
By Lemma 2.2.3 it holds

$$\begin{aligned} V^1(t,x,r) &= \sup_{k\in\Pi(t,x,r)} \inf_{M\in\mathcal{N}(t,1)} \mathbb{E}^{t,x,r,1}\left(U(X_T^{k,M})\right) \\ &= \sup_{k\in\Pi(t,x,r)} \inf_{(\tau,l)} \mathbb{E}^{t,x,r}\left(V^0(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right). \end{aligned}$$

Now, by the fact that $k^{(1)*}$ is the worst-case optimal strategy if one crash still can happen, we obtain

$$\begin{aligned} V^1(t,x,r) &= \inf_{(\tau,l)} \mathbb{E}^{t,x,r}\left(V^0(\tau, X_{\tau-}^{k^{(1)*},(\tau,l)}(1-lk_\tau^{(1)*}), r_\tau)\right) \\ &= \inf_{\tau} \mathbb{E}^{t,x,r}\left(V^0(\tau, X_{\tau-}^{k^{(1)*},(\tau,l)}(1-l^*(k_\tau^{(1)*})^+), r_\tau)\right) \\ &= \inf_{\tau} \mathbb{E}^{t,x,r}\left(V^0(\tau, \tilde{X}_{\tau}^{k^{(1)*}}(1-l^*(k_\tau^{(1)*})^+), r_\tau)\right) \\ &= \inf_{\tau} \mathbb{E}^{t,x,r}\left(M_\tau^{k^{(1)*}}\right). \end{aligned}$$

Here, the second equality holds, because V^0 is monotone increasing in the second component. Moreover, $\tilde{X}^{k^{(1)*}}$ denotes the wealth process in a crash-free market which starts in x at time t and is controlled by $k^{(1)*}$. Moreover, $(\tau,l)=(\tau_N,l_N)$ denotes the first intervention of the market after time t. The third equality holds by definition of M_t^k (see (54)). The alternative proof of Theorem 2.2.2 in Section 2.4.1 implies that $M^{k^{(1)*}}$ is a supermartingale on $[t,T]\cup\{\infty\}$. Together with Theorem A.4.7, we obtain for all $[t,T]\cup\{\infty\}$-valued stopping times τ that

$$\mathbb{E}\left(M_\tau^{k^{(1)*}}\right) \geq \mathbb{E}\left(M_\infty^{k^{(1)*}}\right).$$

Since it is clear that the processes $\{\tilde{X}_s^k\}_{s\in[t,T]}$ and $\{r_s\}_{s\in[t,T]}$ start in x and r at time t, respectively, we write $\mathbb{E}$ instead of $\mathbb{E}^{t,x,r}$. Now, we obtain by the inequality above and by definition of M_∞^k (see Lemma 2.4.2):

$$\begin{aligned} V^1(t,x,r) &= \inf_{\tau} \mathbb{E}\left(M_\tau^{k^{(1)*}}\right) = \mathbb{E}\left(M_\infty^{k^{(1)*}}\right) \\ &= \mathbb{E}\left(V^0(T, \tilde{X}_T^{k^{(1)*}}, r_T)\right) \\ &= \mathbb{E}\left(\frac{1}{\gamma}\left(\tilde{X}_T^{k^{(1)*}}\right)^\gamma\right). \end{aligned}$$

Using Ito's formula, it follows that

$$\begin{aligned}&\mathbb{E}\left(\frac{1}{\gamma}(\tilde{X}_T^{k^{(1)*}})^\gamma\right)\\&=\frac{1}{\gamma}x^\gamma\mathbb{E}\left(\exp\left\{\gamma\int_t^T r_s\,ds+\gamma\int_t^T \mu k_s^{(1)*}-\frac{\sigma_1^2}{2}(k_s^{(1)*})^2\,ds+\gamma\int_t^T\sigma_1 k_s^{(1)*}\,dw_{1,s}\right\}\right).\end{aligned}$$

From Proposition A.1.1 in Appendix A, we know that

$$\begin{aligned}\gamma\int_t^T r_s\,ds=&\gamma\frac{r}{a}(1-e^{-a(T-t)})+\gamma r_M\left((T-t)-\frac{1-e^{-a(T-t)}}{a}\right)\\&+\int_t^T\sigma_2\beta(s)\left(\rho dw_{1,s}+\sqrt{1-\rho^2}dw_{2,s}\right).\end{aligned}$$

Therefore

$$\begin{aligned}&\mathbb{E}\left(\frac{1}{\gamma}\left(\tilde{X}_T^{k^{(1)*}}\right)^\gamma\right)\\&=\frac{1}{\gamma}x^\gamma\exp\left\{\int_t^T\gamma\mu k_s^{(1)*}-\frac{\sigma_1^2}{2}\gamma(k_s^{(1)*})^2\,ds\right.\\&\qquad\left.+\gamma\frac{r}{a}(1-e^{-a(T-t)})+\gamma r_M\left((T-t)-\frac{1-e^{-a(T-t)}}{a}\right)\right\}\\&\quad\cdot\mathbb{E}\left(\exp\left\{\underbrace{\int_t^T\gamma\sigma_1k_s^{(1)*}+\rho\sigma_2\beta(s)\,dw_{1,s}}_{=:I_1}+\underbrace{\int_t^T\sqrt{1-\rho^2}\sigma_2\beta(s)\,dw_{2,s}}_{=:I_2}\right\}\right).\end{aligned}$$

Since w_1 and w_2 are independent Wiener processes, we obtain

$$\mathbb{E}\left(\exp\{I_1+I_2\}\right)=\mathbb{E}\left(\exp(I_1)\right)\mathbb{E}\left(\exp(I_2)\right).$$

Due to deterministic and bounded integrands of I_1 and I_2, we have that the stochastic integrals I_1 and I_2 are normally distributed random variables (note that t is assumed to be fixed). Thus

$$\begin{aligned}\mathbb{E}\left(\exp(I_1)\right)&=\exp\left(\mathbb{E}(I_1)+\frac{1}{2}Var(I_1)\right)\\&=\exp\left(\int_t^T\gamma^2\frac{\sigma_1^2}{2}(k_s^{(1)*})^2+\rho^2\frac{\sigma_2^2}{2}\beta^2(s)+\rho\sigma_1\sigma_2\gamma\beta(s)k_s^{(1)*}\,ds\right),\end{aligned}$$

and

$$\mathbb{E}\left(\exp(I_2)\right)=\exp\left(\int_t^T(1-\rho^2)\frac{\sigma_2^2}{2}\beta^2(s)\,ds\right),$$

and,

$$\begin{aligned}&\mathbb{E}\left(\frac{1}{\gamma}\left(\tilde{X}_T^{k^{(1)*}}\right)^\gamma\right)\\&=\frac{1}{\gamma}x^\gamma\exp\left(\int_t^T\gamma(\mu+\rho\sigma_1\sigma_2\beta(s))k_s^{(1)*}-\frac{\sigma_1^2}{2}\gamma(1-\gamma)(k_s^{(1)*})^2\,ds\right.\\&\qquad\left.+\int_t^T ar_M\beta(s)+\frac{\sigma_2^2}{2}\beta^2(s)\,ds\right)\exp(\beta(t)r),\end{aligned}$$

and, we obtain the value function $V^1(t,x,r)$ directly:

$$V^1(t,x,r) = \frac{1}{\gamma} x^\gamma g^{(1)}(t) \exp(\beta(t) r),$$

where $g^{(1)}(t)$ solves (15) for $n=1$. This result coincides with the result of Theorem 2.2.2. Thus, using the supermartingale property of the process $M^{k^{(1)*}}$ we are able to determine the value function $V^1(t,x,r)$ in an explicit form.
Hence, with $V^1(t,x,r)$ and the same procedure as in Section 2.4.1, we determine the worst-case optimal strategy $k^{(2)*}$, which is valid after the $(N-2)$-th market crash, denoted by (τ_{N-2}, l_{N-2}). Note that in comparison to Section 2.4.1, V^1 takes the role of V^0.

After the $(N-2)$-th market crash the investor is faced with a worst-case optimization problem with at most two market crashes. Again, let $(t,x,r) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}$ be arbitrary but fixed and assume that at time t there can still happen at most two market crashes and $X_t^{k,M} = x$ and $r_t = r$. Then, the aim is to determine the optimal strategy $k^* = (k^{(0)*}, k^{(1)*}, k^{(2)*})$ which is worst-case optimal for

$$\sup_{k \in \Pi(t,x,r)} \inf_{M \in \mathcal{N}(t,2)} \mathbb{E}^{t,x,r,2} \left(U(X_T^{k,M}) \right). \tag{59}$$

Lemma 2.2.3 implies that (59) is equal to:

$$\sup_{k \in \Pi(t,x,r)} \inf_{(\tau,l)} \mathbb{E}^{t,x,r,2} \left(V^1(\tau, X_{\tau-}^{k,(\tau,l)}(1 - l k_\tau), r_\tau) \right). \tag{60}$$

Here, $(\tau,l) = (\tau_{N-1}, l_{N-1})$ denotes again the first intervention of the market after time t. Since V^1 is monotone increasing in its second component, (60) is equal to

$$\begin{aligned} &\sup_{k^{(2)} \in \Pi(t,x,r)} \inf_{\tau} \mathbb{E}^{t,x,r,2} \left(V^1(\tau, X_{\tau-}^{k^{(2)},(\tau,l)}(1 - l^*(k_\tau^{(2)})^+), r_\tau) \right) \\ &= \sup_{k^{(2)} \in \Pi(t,x,r)} \inf_{\tau} \mathbb{E}^{t,x,r,2} \left(V^1(\tau, \tilde{X}_\tau^{k^{(2)}}(1 - l^*(k_\tau^{(2)})^+), r_\tau) \right), \end{aligned}$$

where $\tilde{X}^{k^{(2)}}$ denotes the wealth process in a crash-free market controlled by $k^{(2)}$with $\tilde{X}_t^{k^{(2)}} = x$ and $r_t = r$. Now, let us define

$${}_1M_s^k := V^1(s, \tilde{X}_s^k(1 - l^*(k_s)^+), r_s), \quad \text{for } s \in [t,T], \qquad {}_1M_\infty^k := V^1(T, \tilde{X}_T^k, r_T).$$

We reformulate the worst-case optimization problem with at most two market crashes (59) as a pre-crash problem of the form

$$\sup_{k^{(2)} \in \Pi(t,x,r)} \inf_{\tau} \mathbb{E} \left({}_1M_\tau^{k^{(2)}} \right), \tag{61}$$

where the pre-crash strategy is $k^{(2)}$. We use that the optimal post-crash strategy $(k^{(1)*}, k^{(0)*})$ and the post-crash value function $V^1(t,x,r)$ are already given. Here, the notion 'post-crash' stands for the time after the market crash (τ_{N-1}, l_{N-1}). The idea to solve the controller vs. stopper game (61) is the same as in the one-crash case in Section 2.4.1. First, we determine a strategy $\hat{k}^{(2)}$ such that the process ${}_1M^{\hat{k}^{(2)}}$ is a martingale on $[t,T] \cup \{\infty\}$.

Assume that $\hat{k}^{(2)}$ is the uniquely determined solution of

$$\dot{k}_t^{(2)} = \frac{1 - l^* k_t^{(2)}}{l^*} \left(\phi(t, k_t^{(2)}) - \phi(t, k_t^{(1)*}) \right), \qquad k_T^{(2)} = 0,$$

then, ${}_1M^{\hat{k}^{(2)}}$ is a martingale on $[t,T] \cup \{\infty\}$. This assertion is the analogue of the assertion in Lemma 2.4.2. We obtain the martingale property by the following arguments:
Using $V^1(t,x,r) = \frac{1}{\gamma} x^\gamma W^{(1)}(t,r)$, $W^{(1)}(t,r) = g^{(1)}(t)\exp(\beta(t)r)$, the fact that $\hat{k}_s^{(2)} \geq 0$ and by Ito's formula we obtain for $s \in [t,T]$:

$$\begin{aligned}
d\left({}_1M_s^{\hat{k}^{(2)}}\right) =& d\left(V^1(s, \tilde{X}_s^{\hat{k}^{(2)}}(1 - l^*\hat{k}_s^{(2)}), r_s)\right) \\
=& \frac{1}{\gamma}(\tilde{X}_s^{\hat{k}^{(2)}})^\gamma (1 - l^*\hat{k}_s^{(2)})^\gamma \\
& \quad \Bigg\{ \gamma \frac{-l^*}{(1 - l^*\hat{k}_s^{(2)})} \dot{\hat{k}}_s^{(2)}\, g^{(1)}(s)\exp(\beta(s)r_s) + \gamma\phi(t, \hat{k}_s^{(2)})\, g^{(1)}(s)\exp(\beta(s)r_s) \\
& \quad + \dot{g}^{(1)}(s)\exp(\beta(s)r_s) + \dot{\beta}(s) r g^{(1)}(s)\exp(\beta(s)r_s) \\
& \quad + \frac{\sigma_2^2}{2}\beta^2(s) g^{(1)}(s)\exp(\beta(s)r_s) + a(r_M - r_s)\beta(s) g^{(1)}(s)\exp(\beta(s)r_s) \\
& \quad + \gamma r_s g^{(1)}(s)\exp(\beta(s)r_s) \Bigg\}\, ds \\
& + \frac{1}{\gamma}(\tilde{X}_s^{\hat{k}^{(2)}})^\gamma (1 - l^*\hat{k}_s^{(2)})^\gamma W^{(1)}(s, r_s) \\
& \qquad \left\{ \left(\gamma\sigma_1\hat{k}_s^{(2)} + \rho\sigma_2\beta(s)\right) dw_{1,s} + \sqrt{1-\rho^2}\sigma_2\beta(s)\, dw_{2,s} \right\}.
\end{aligned}$$

Using that $g^{(1)}(s)$ solves (15) for $n = 1$ and using that $\beta(s)$ is a solution of

$$\dot{\beta}(s) - a\beta(s) + \gamma = 0, \qquad \beta(T) = 0,$$

we have

$$\begin{aligned}
d\left({}_1M_s^{\hat{k}^{(2)}}\right) =& \frac{1}{\gamma}(\tilde{X}_s^{\hat{k}^{(2)}})^\gamma (1 - l^*\hat{k}_s^{(2)})^\gamma W^{(1)}(s, r_s) \\
& \quad \left\{ \gamma \frac{-l^*}{(1 - l^*\hat{k}_s^{(2)})} \dot{\hat{k}}_s^{(2)} + \gamma(\phi(t, \hat{k}_s^{(2)}) - \phi(t, k_s^{(1)*}) \right\} ds \\
& + \frac{1}{\gamma}(\tilde{X}_s^{\hat{k}^{(2)}})^\gamma (1 - l^*\hat{k}_s^{(2)})^\gamma W^{(1)}(t, r_s) \\
& \quad \left\{ \left(\gamma\sigma_1\hat{k}_s^{(2)} + \rho\sigma_2\beta(s)\right) dw_{1,s} + \sqrt{1-\rho^2}\sigma_2\beta(s)\, dw_{2,s} \right\} \\
=& {}_1M_s^{\hat{k}^{(2)}} \left\{ \left(\gamma\sigma_1\hat{k}_s^{(2)} + \rho\sigma_2\beta(s)\right) dw_{1,s} + \sqrt{1-\rho^2}\sigma_2\beta(s)\, dw_{2,s} \right\}.
\end{aligned}$$

By Novikov's condition, it follows that the process ${}_1M^{\hat{k}^{(2)}}$ is a martingale on $[t,T]$. The martingale property between T and ∞ holds because $\hat{k}_T^{(2)} = 0$ and

$$\mathbb{E}\left({}_1M_\infty^{\hat{k}^{(2)}} | \mathcal{F}_T\right) = \mathbb{E}\left(\underbrace{V^1(T, \tilde{X}_T^{\hat{k}^{(2)}}, r_T)}_{\mathcal{F}_T-\text{measurable}} | \mathcal{F}_T\right) = V^1(T, \tilde{X}_T^{\hat{k}^{(2)}}, r_T) = {}_1M_T^{\hat{k}^{(2)}}.$$

This shows that ${}_1M^{\hat{k}^2}$ is a martingale on $[t,T]\cup\{\infty\}$. Doob's Optional Sampling Theorem again implies

$$\mathbb{E}\left({}_1M_{\tau}^{\hat{k}^{(2)}}\right)=\mathbb{E}\left({}_1M_{\tau'}^{\hat{k}^{(2)}}\right),$$

for all $[t,T]\cup\{\infty\}$-valued stopping times τ,τ'. By definition, $\hat{k}^{(2)}$ is an indifference strategy for the controller vs. stopper game (61). Analogously to the construction of an indifference frontier on page 34, we conclude that the optimal strategy $k^{(2)*}$ has to be an element of the set

$$\mathcal{A}(\hat{k}^{(2)})=\{k^{(2)}\in\Pi: k_s^{(2)}\le\hat{k}_s^{(2)},\forall s\in[t,T]\}.$$

Now, let us determine the optimal strategy in the no-crash scenario $\tau=\infty$ in the class $\mathcal{A}(\hat{k}^{(2)})$ for the controller vs. stopper game (61), that means

$$\sup_{k^{(2)}\in\mathcal{A}(\hat{k}^{(2)})}\mathbb{E}\left({}_1M_{\infty}^{k^{(2)}}\right)=\sup_{k^{(2)}\in\mathcal{A}(\hat{k}^{(2)})}\mathbb{E}\left(V^1(T,\tilde{X}_T^{k^{(2)}},r_T)\right)=\sup_{k^{(2)}\in\mathcal{A}(\hat{k}^{(2)})}\mathbb{E}\left(U(\tilde{X}_T^{k^{(2)}})\right).$$

The optimal strategy of this constrained optimization problem is given by $k_s^{(2)*}=\hat{k}_s^{(2)}\wedge k_s^{(0)*}$ (see Lemma 2.4.4 and replace $\hat{k}^{(1)}$ by $\hat{k}^{(2)}$). By similar arguments as in the proof on page 36, we show that $k_s^{(2)*}$ is optimal for the controller vs. stopper game (61):

Proof for optimal strategy after τ_{N-2}. Let $t\in[0,T]$ be arbitrary but fixed. Define

$$t_S^{(2)}:=\inf\{s\in[t,T]:k_s^{(0)*}\ge\hat{k}_s^{(2)}\}$$

and let us consider ${}_1M^{k^{(2)*}}$ on $[t_S^{(2)},T]\cup\{\infty\}$. On $[t_S^{(2)},T]$ we have that $k_s^{(2)*}=\hat{k}_s^{(2)}$ and therefore ${}_1M^{k^{(2)*}}$ is a martingale on $[t_S^{(2)},T]\cup\{\infty\}$. If $t_S^{(2)}=t$ we immediately have that ${}_1M^{k^{(2)*}}$ is a martingale on $[t,T]\cup\{\infty\}$. This is especially the case if $\gamma\rho\ge 0$. Thus, throughout the rest of this proof we assume that $\gamma\rho<0$ and $t_S^{(2)}>t$. Then, we define

$$t_0:=\inf\{s\in[t,T]:k_s^{(0)*}\ge 0\}$$

and we consider ${}_1M^{k^{(2)*}}$ on $[t_0,t_S^{(2)}]$. Since $k_s^{(1)*}=k_s^{(0)*}$ on $[t_0,t_S^{(1)}]$ and $t_0\le t_S^{(2)}\le t_S^{(1)}$, where

$$t_S^{(1)}:=\inf\{s\in[t,T]:k_s^{(0)*}\ge\hat{k}_s^{(1)}\},$$

we especially have that $k_s^{(2)*}=k_s^{(1)*}=k_s^{(0)*}$ on $[t_0,t_S^{(2)}]$. For $s\in[t_0,t_S^{(2)}]$, it holds

$$\begin{aligned}d\left({}_1M_s^{k^{(2)*}}\right)&={}_1M_s^{k^{(2)*}}\cdot\left\{-\gamma\frac{l^*}{1-l^*k_s^{(0)*}}\dot{k}_s^{(0)*}+\gamma\phi(s,k_s^{(0)*})-\gamma\phi(s,k_s^{(0)*})\right\}ds\\&\quad+{}_1M_s^{k^{(2)*}}\cdot\left\{(\gamma\sigma_1k_s^{(0)*}+\rho\sigma_2\beta(s))dw_{1,s}+\sqrt{1-\rho^2}\sigma_2\beta(s)\,dw_{2,s}\right\}.\end{aligned}$$

Using the same arguments as on page 36, we have that ${}_1M^{k^{(2)*}}$ is a supermartingale on $[t_0,t_S^{(2)}]$. If $t_0=t$, then it is a supermartingale on $[t,T]\cup\{\infty\}$.
Now, let us consider the case $t_0>t$. For $s\in[t,t_0]$, it holds $k_s^{(2)*}=k_s^{(0)*}\le 0$ and $k_s^{(1)*}=k_s^{(0)*}$. On $[t,t_0]$, we obtain

$$\begin{aligned}d\left({}_1M_s^{k^{(2)*}}\right)&=d\left(V^1(s,\tilde{X}_s^{k^{(2)*}},r_s)\right)\\&={}_1M_s^{k^{(2)*}}\left\{(\gamma\sigma_1k_s^{(0)*}+\rho\sigma_2\beta(s))dw_{1,s}+\sqrt{1-\rho^2}\sigma_2\beta(s)\,dw_{2,s}\right\},\end{aligned}$$

and, therefore, ${}_1M^{k^{(2)*}}$ is a martingale on $[t, t_0]$. All together, we conclude that ${}_1M^{k^{(2)*}}$ is a supermartingale on $[t, T] \cup \{\infty\}$. If $\gamma\rho \geq 0$, then it is even a martingale. By Doob's Optional Sampling Theorem (see Theorem A.4.7), we obtain

$$\mathbb{E}\left({}_1M_\tau^{k^{(2)*}}\right) \geq \mathbb{E}\left({}_1M_\infty^{k^{(2)*}}\right),$$

for all $[t, T] \cup \{\infty\}$-valued stopping times τ and

$$\inf_\tau \mathbb{E}\left({}_1M_\tau^{k^{(2)*}}\right) \geq \mathbb{E}\left({}_1M_\infty^{k^{(2)*}}\right) \geq \mathbb{E}\left({}_1M_\infty^{k^{(2)}}\right) \geq \inf_\tau \mathbb{E}\left({}_1M_\tau^{k^{(2)}}\right)$$

for all $k^{(2)} \in \mathcal{A}(\hat{k}^{(2)})$. By the fact that the optimal strategy for the controller vs. stopper game (61) has to be an element of the set $\mathcal{A}(\hat{k}^{(2)})$ (see construction of the indifference frontier), we have that $k_s^{(2)*} = \hat{k}_s^{(2)} \wedge k_s^{(0)}$ is optimal for (61).
Finally $k^* = (k^{(0)*}, k^{(1)*}, k^{(2)*})$ is the worst-case optimal strategy for

$$\sup_{k\in\Pi(t,x,r)} \inf_{M\in\mathcal{N}(t,2)} \mathbb{E}^{t,x,r,2}\left(U(X_T^{k,M})\right).$$

□

Again, one can determine the corresponding value function $V^2(t, x, r)$ and afterwards one can determine $k^{(3)*}$, valid after the $(N-3)$-th market crash using similar arguments as above. The general recursive procedure for an arbitrary $N \geq 1$, can be written in the following scheme:

Initialization (After the N-th market crash):
Solve the classical stochastic optimal control problem

$$V^0(t, x, r) = \sup_{k^{(0)}\in\Pi(t,x,r)} \mathbb{E}^{t,x,r}\left(U(\overline{X}_T)\right),$$

where $\overline{X}$ denotes the wealth process if no crash can occur anymore.

Output: $k^{(0)*}$ and $V^0(t, x, r)$.

For n=1,...,N (After the $(N-n)$-th market crash):
(1) Apply the martingale approach to determine $k^{(n)*}$ using V^{n-1}:

a) Reformulation as controller vs. stopper game

$$\sup_{k\in\Pi(t,x,r)} \inf_{M\in\mathcal{N}(t,n)} \mathbb{E}^{t,x,r,n}\left(U(X_T^{k,M})\right) = \sup_{k^{(n)}\in\Pi(t,x,r)} \inf_\tau \mathbb{E}\left({}_{n-1}M_\tau^{k^{(n)}})\right),$$

with

$${}_nM_t^k := V^n(t, \tilde{X}_t^k(1 - l^*(k_t)^+), r_t).$$

b) Assume that $\hat{k}^{(n)}$ solves (13), then ${}_{n-1}M^{\hat{k}^{(n)}}$ is a martingale.

c) Indifference frontier: $k^{(n)*}$ is an element of

$$\mathcal{A}(\hat{k}^{(n)}) := \{k^{(n)} \in \Pi : k_s^{(n)} \leq \hat{k}_s^{(n)}, \forall s \in [t, T]\}.$$

d) $k_t^{(n)*} := \hat{k}_t^{(n)} \wedge k_t^{(0)*}$ is optimal in the no-crash scenario (see Lemma 2.4.4).

e) Show supermartingale property of ${}_{n-1}M^{k^{(n)*}}$ and it follows that $k^{(n)*}$ is optimal.

(2) Determine $V^n(t,x,r)$ by the following calculation:

$$\begin{aligned} V^n(t,x,r) &= \sup_{k\in\Pi(t,x,r)} \inf_{M\in\mathcal{N}(t,n)} \mathbb{E}^{t,x,r,n}\Big(U(X_T^{k,M})\Big) \\ &= \sup_{k\in\Pi(t,x,r)} \inf_{(\tau,l)} \mathbb{E}^{t,x,r}\Big[V^{n-1}(\tau, X_{\tau_-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\Big] \\ &= \inf_{\tau} \mathbb{E}^{t,x,r}\Big[V^{n-1}(\tau, \tilde{X}_\tau^{k^{(n)*}}(1-l^*(k_\tau^{(n)*})^+), r_\tau)\Big] \\ &= \inf_{\tau} \mathbb{E}^{t,x,r}\Big[{}_{n-1}M_\tau^{k^{(n)*}}\Big] \\ &= \mathbb{E}^{t,x,r}\Big[{}_{n-1}M_\infty^{k^{(n)*}}\Big] \\ &= \mathbb{E}^{t,x,r}\Big[V^{n-1}(T, \tilde{X}_T^{k^{(n)*}}, r_T)\Big] \\ &= \frac{1}{\gamma}x^\gamma g^{(n)}(t)\exp(\beta(t)r). \end{aligned} \tag{62}$$

Output: $k^{(n)*}$ and $V^n(t,x,r)$.
end

REMARK 2.4.5. *First, we give some arguments for the proofs in part (1) of the scheme above. The reformulation in (a) holds due to Lemma 2.2.3 and by the monotonicity of V^{n-1} in its second component. The proof of the fact that ${}_{n-1}M^{\hat{k}^{(n)}}$ is a martingale in (b) works again by applying Ito's formula and by using that $g^{(n-1)}(t)$ and $\beta(t)$ are given (see e.g. page 42). The indifference frontier of the controller vs. stopper game in c) results from the martingale property of ${}_{n-1}M^{\hat{k}^{(n)}}$ and (d) follows in the same way as in Lemma 2.4.4. In order to show the supermartingale property of ${}_{n-1}M^{k^{(n)*}}$, we can proceed as before: First one defines*

$$t_S^{(n)} = \inf\{s\in[t,T] : k_s^{(0)*} \geq \hat{k}_s^{(n)}\}.$$

Then, we obtain that ${}_{n-1}M^{k^{(n)*}}$ is a martingale on $[t_S^{(n)},T]\cup\{\infty\}$. For $\gamma\rho\geq 0$, we have that $k_s^{(n)*} = \hat{k}_s^{(n)}$ on $[0,T]\cup\{\infty\}$ (see Remark 2.2.8), and in that case $t_S^{(n)} = t$. Thus, for $\gamma\rho < 0$ and $t_S^{(n)} > t$, we again define t_0 as before. Moreover, we have to note that $t_S^{(n)} \leq t_S^{(n-1)}$ because of the following two facts. First, $k_s^{(0)*}$ is monotone increasing for $\gamma\rho<0$ and $\hat{k}_s^{(n)}$ is monotone decreasing. Second, we have that $\hat{k}_s^{(n)} \leq \hat{k}_s^{(n-1)}$. Thus, it follows that $k_s^{(n-1)*} = k_s^{(0)*}$ on $[t_0, t_S^{(n)}]$ and we can show the supermartingale property of ${}_{n-1}M^{k^{(n)*}}$ on $[t_0,t_S^{(n)}]$ using the same arguments as above. Finally, if $t_0>t$, we can also show the martingale property on $[t,t_0]$ using again that $k_s^{(n-1)*} = k_s^{(0)*}$ on this interval. All together, leads to the supermartingale property of ${}_{n-1}M^{k^{(n)*}}$ on $[t,T]\cup\{\infty\}$ and it follows that $k^{(n)*}$ is worst-case optimal by the Indifference Optimality Principle.
Moreover, we give a short explanation for (62). The first equality holds by definition of V^n and the second holds by Lemma 2.2.3. Moreover the third equality follows by the fact that $k^{(n)*}$ is worst-case optimal and V^{n-1} is monotone increasing in its second component, whereas the fourth equality holds by definition of ${}_nM^k$. Due to the fact that ${}_{n-1}M^{k^{(n)*}}$ is a supermartingale, the worst-case scenario is the no-crash scenario $\tau=\infty$ and therefore the fifth equality holds. Finally the sixth equality follows again by definition. Since

$$\mathbb{E}^{t,x,r}\Big[V^{n-1}(T,\tilde{X}_T^{k^{(n)*}}, r_T)\Big] = \mathbb{E}^{t,x,r}\left(\frac{1}{\gamma}(\tilde{X}_T^{k^{(n)*}})^\gamma\right)$$

we obtain the last equality by Ito's formula and using the same arguments as on page 40.

With the procedure above we demonstrated how we can apply the martingale approach in a recursive way to determine the optimal strategies $k^{(n)*}$ which are valid if n crashes still can occur. The main issue of this procedure is the determination of the value function V^n using the knowledge that $k^{(n)*}$ is worst-case optimal und using the supermartingale property. This allows to obtain the value function V^n in an explicit form by applying Ito's formula and Proposition A.1.1.

REMARK 2.4.6 (Log utility via martingale approach). *Here, we emphasize that the recursive application of the martingale approach works analogously for an investor with a logarithmic utility function. The solution $k^{(0)*}$ and the corresponding value function $V^0(t,x,r)$ is again obtained by solving the so-called post-crash optimization problem and it is already given in part a) of Theorem 2.3.2. After reformulating the worst-case optimization problem as a controller vs. stopper game, one can again show an analog version of Lemma 2.4.2, which ensures that the uniquely determined solution $\hat{k}^{(n)}$ of the ODE:*

$$\dot{k}_t^{(n)} = \frac{1 - l^* k_t^{(n)}}{l^*}\left(\phi(k_t^{(n)}) - \phi(\hat{k}_t^{(n-1)})\right), \qquad k_T^{(n)} = 0, \qquad \phi(k) := \mu k - \frac{\sigma_1^2}{2}k^2,$$

is an indifference strategy (that means one shows that ${}_{n-1}M^{\hat{k}^{(n)}}$ is a martingale on $[0,T]\cup\{\infty\}$). In contrast to the non-log HARA utility case, we even obtain that the process ${}_{n-1}M^{k^{(n)*}}$ is a martingale and, therefore, $k_t^{(n)*} = \hat{k}_t^{(n)}$ is the worst-case optimal strategy. Again using Ito's formula, one can determine $V^n(t,x,r) = \log(x) + W^{(n)}(t,r)$ in an explicit form, where $W^{(n)}(t,r)$ is given as in Theorem 2.3.2. Thus, the recursive application of the martingale approach is an alternative way to prove Theorem 2.3.2.

2.4.3. Discussion and numerical examples. In this section, we discuss properties of the optimal strategies for the non-log HARA utility case, which we obtained in the previous sections. By Theorem 2.2.2 we have that the optimal strategy after the N-th market crash is given by

$$k_t^{(0)*} = \frac{\mu}{(1-\gamma)\sigma_1^2} + \frac{\rho\sigma_2\beta(t)}{(1-\gamma)\sigma_1}.$$

Now, we analyse the influence of the utility preferences of the investor on this strategy. Note that $1-\gamma$ represents the investor's relative risk aversion, the higher $1-\gamma$ the higher the risk aversion. In the case of constant interest rates, we easily see, that $k^{(0)*} = \mu((1-\gamma)\sigma_1^2)^{-1}$ is constant with respect to time and monotone increasing with respect to γ. That means, the higher the investor's risk aversion, the lower the investment in the stock. This standard monotonicity behavior might vanish when considering stochastic interest rates.

PROPOSITION 2.4.7. *Let $k_{t,\gamma_1}^{(0)*}$ and $k_{t,\gamma_2}^{(0)*}$ be given by (11) for given γ_1 and γ_2, respectively. If*

$$\rho > -\frac{\mu a}{\sigma_1\sigma_2(1-e^{-aT})}, \tag{63}$$

then for $\gamma_1 < \gamma_2$ it holds: $k_{t,\gamma_1}^{(0)*} < k_{t,\gamma_2}^{(0)*}$ for all $t \in [0,T]$ (standard monotonicity behavior). On the other hand, if

$$\rho \le -\frac{\mu a}{\sigma_1\sigma_2(1-e^{-aT})}, \tag{64}$$

then, there exists a uniquely determined intersection point $S \in [0,T]$ *and it holds for* $\gamma_1 < \gamma_2$*:* $k^{(0)*}_{t,\gamma_1} \leq k^{(0)*}_{t,\gamma_2}$ *for* $t \in [S,T]$ *(standard monotonicity behavior) and* $k^{(0)*}_{t,\gamma_1} \geq k^{(0)*}_{t,\gamma_2}$ *for* $t \in [0,S]$*.*

PROOF. If the parameters fulfill the inequality (63), then for all $t \in [0,T]$ we obtain:

$$\frac{\partial}{\partial \gamma}\left(\frac{\mu}{\sigma_1^2(1-\gamma)} + \frac{\rho\sigma_2\beta(t)}{\sigma_1(1-\gamma)}\right) = \frac{a\mu + (1-e^{-a(T-t)})\rho\sigma_1\sigma_2}{a(\gamma-1)^2\sigma_1^2} > 0.$$

Thus, if the parameters fulfill (63), we observe the standard monotonicity behavior (the higher the risk aversion $1-\gamma$, the lower the investment in the stock). Now, let us consider the case where the parameters fulfill (64). Let $\tilde{\beta}(t) := \frac{1}{a}(1-e^{-a(T-t)})$. Since $\rho < 0$, (64) is equivalent to

$$\tilde{\beta}(0) \geq -\frac{\mu}{\sigma_1\sigma_2\rho} > 0.$$

Now, using the fact that $\tilde{\beta}(t)$ is monotone decreasing with $\tilde{\beta}(T) = 0$, we obtain that there exists a uniquely determined point $S \in [0,T]$ such that $\tilde{\beta}(S) = -\frac{\mu}{\sigma_1\sigma_2\rho}$, and thus,

$$k^{(0)*}_{S,\gamma_1} = \frac{\mu}{\sigma_1^2(1-\gamma_1)} + \frac{\rho\sigma_2\gamma_1\tilde{\beta}(S)}{\sigma_1(1-\gamma_1)} = \frac{\mu}{\sigma_1^2(1-\gamma_2)} + \frac{\rho\sigma_2\gamma_2\tilde{\beta}(S)}{\sigma_1(1-\gamma_2)} = k^{(0)*}_{S,\gamma_2}.$$

and the intersection point S is given by

$$S = \frac{1}{a}\left(\log\left(1 + \frac{\mu a}{\sigma_1\sigma_2\rho}\right) + aT\right).$$

Obviously, S is independent of the choice of γ_1, γ_2. One can easily show for $\gamma_1 < \gamma_2$, that $k^{(0)*}_{t,\gamma_1} \leq k^{(0)*}_{t,\gamma_2}$ for $t \in [S,T]$ (standard monotonicity behavior) and $k^{(0)*}_{t,\gamma_1} \geq k^{(0)*}_{t,\gamma_2}$ for $t \in [0,S]$. □

Figure 2.1 shows the optimal strategies $k^{(0)*}_{t,\gamma}$ for market parameters which fulfill the condition (64). For $t \in [S,T]$, we observe: the higher the risk aversion the lower the investment in the stock. For $t \in [0,S]$, we have the contrary, which means that a more risk averse investor invests more in the stock than an investor with a lower risk aversion. It is worth mentioning, that the investor has no riskless asset in the financial market model. Thus, for certain parameters (which fulfill condition (64)), it is less risky to invest in the stock, than in the bank account with stochastic instantaneous interest rates.

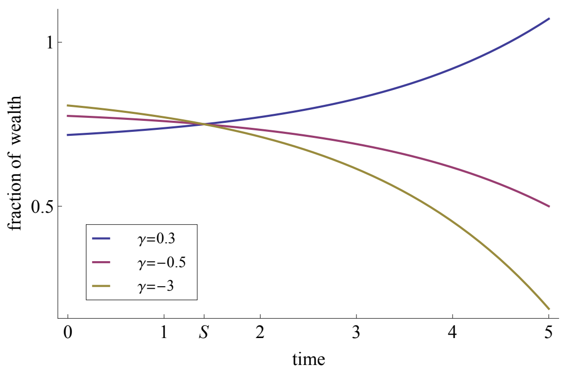

Figure 2.1. Optimal post-crash strategy $k^{(0)*}_{t,\gamma}$ for different values of γ and for parameters $\mu = 0.03$, $\sigma_1 = 0.2$, $a = 0.5$, $\sigma_2 = 0.1$, $T = 5$, $\rho = -0.9$ with $S = 1.42$.

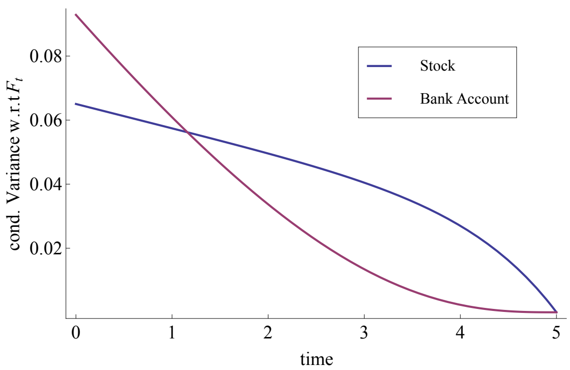

Figure 2.2. Comparison of conditional variances $Var(\log(P_T)|\mathcal{F}_t)$ and $Var(log(B_t)|\mathcal{F}_t)$ for parameters $\sigma_1 = 0.2$, $a = 0.5$, $\sigma_2 = 0.1$, $T = 5$, $\rho = -0.9$.

Another heuristic explanation for this effect might show Figure 2.2, where $Var(\log(P_T)|\mathcal{F}_t)$ and $Var(\log(B_T)|\mathcal{F}_t)$ are plotted. Therein, one can see that the variance of the savings account is higher than the variance of the stock price for small t.
If we consider a financial market model, where the condition (63) is fulfilled, then we obtain the standard behavior with respect to the investor's risk aversion. An Example is shown in Figure 2.3, where it holds: the higher the risk aversion, the lower the stock investment for all $t \in [0, T]$. Figure 2.4 shows that the variance of the stock is greater than the variance of the bank account for all $t \in [0, T]$.

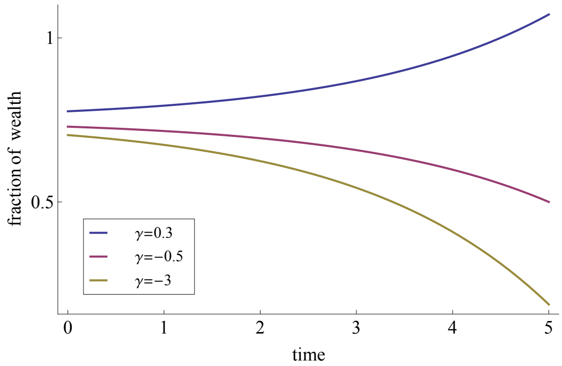

Figure 2.3. Optimal post-crash strategy $k_{t,\gamma}^{(0)*}$ for different values of γ and for parameters $\mu = 0.03$, $\sigma_1 = 0.2$, $a = 0.5$, $\sigma_2 = 0.1$, $T = 5$, $\rho = -0.75$.

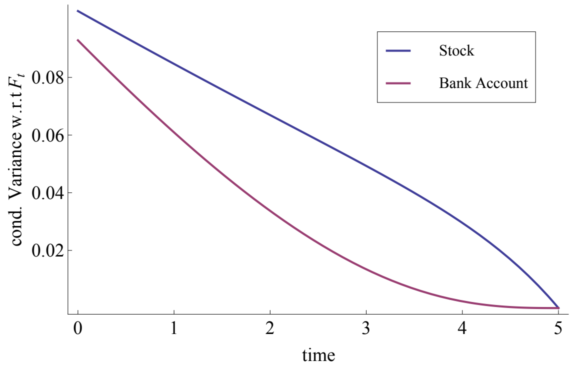

Figure 2.4. Comparison of conditional variances $Var(\log(P_T)|\mathcal{F}_t)$ and $Var(log(B_t)|\mathcal{F}_t)$ for parameters $\sigma_1 = 0.2$, $a = 0.5$, $\sigma_2 = 0.1$, $T = 5$, $\rho = -0.75$.

Analogously to the strategy $k^{(0)*}$, valid after the N-th market crash, we illustrate the strategies $k_t^{(n)*} = \hat{k}_t^{(n)} \wedge k_t^{(0)*}$ which are worst-case optimal if $n \geq 1$ market crashes still can happen. Note, that $\hat{k}_t^{(n)}$ is the uniquely determined solution of the ODE (13) which we calculate numerically.

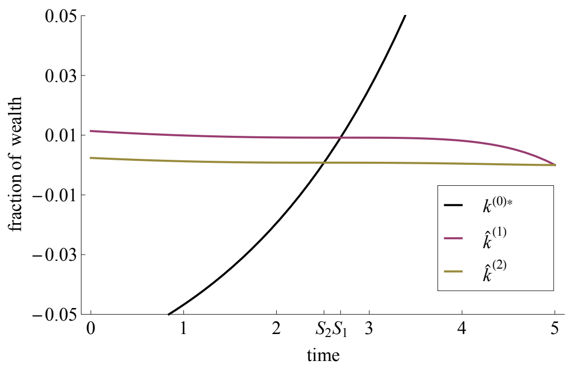

Figure 2.5. Optimal strategies for market parameters: $N = 2$, $l^* = 0.4$, $T = 5$, $\mu = 0.06$, $\sigma_1 = 0.3$, $a = 0.5$, $r_M = 0.05$, $\sigma_2 = 0.1$, $\rho = 0.7$ and risk preference: $\gamma = -2$.

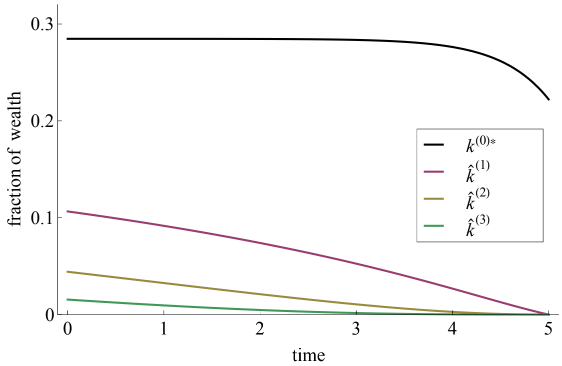

Figure 2.6. Optimal strategies for market parameters: $N = 3$, $l^* = 0.4$, $T = 5$, $\mu = 0.08$, $\sigma_1 = 0.3$, $a = 2$, $r_M = 0.05$, $\sigma_2 = 0.1$, $\rho = -0.5$ and risk preference: $\gamma = -3$.

In Figure 2.5, we consider a financial market where at most $N = 2$ market crashes can happen and we plotted $k^{(0)*}$, $\hat{k}^{(1)}$ and $\hat{k}^{(2)}$. Note, that if there still can happen two market crashes, it is worst-case optimal to follow $\hat{k}_t^{(2)} \wedge k_t^{(0)*}$. If the first crash has happened, the investor changes his strategy to $\hat{k}_t^{(1)} \wedge k_t^{(0)*}$. After the second market crash, it is optimal to follow the strategy $k_t^{(0)*}$. S_1 and S_2 refer to the uniquely determined intersection points $t_S^{(1)}$ and $t_S^{(2)}$, which were mentioned in Remark 2.4.5. These intersection points do not exist in Figure 2.6. Therein, we consider a financial market model with at most $N = 3$ market crashes. Since, $\gamma\rho \geq 0$, we have that $\hat{k}_t^{(3)} \leq \hat{k}_t^{(2)} \leq \hat{k}_t^{(1)} \leq k_t^{(0)*}$ for all $t \in [0, T]$ (see Remark 2.2.8). Thus, if there are n crashes left, it is worst-case optimal to follow the indifference strategy $\hat{k}^{(n)}$.

Analogously to the analysis in Figure 2.1 and Figure 2.3, we consider how the worst-case optimal strategies depend on the investor's risk aversion $1 - \gamma$ in Figure 2.7 and Figure 2.8. For the sake of simplicity, we consider financial markets with at most one market crash. The sensitivity of optimal strategies in the case of constant interest rates was already considered in [44, Chp.6.2]. Both figures below show the worst-case optimal investment strategies $k_t^{(1)*} = \hat{k}_t^{(1)}$ on a financial market with parameters:

$$N = 1,\ l^* = 0.4,\ T = 5,\ \mu = 0.08,\ \sigma_1 = 0.3,\ r_M = 0.05,\ \sigma_2 = 0.1, \rho = -0.8.$$

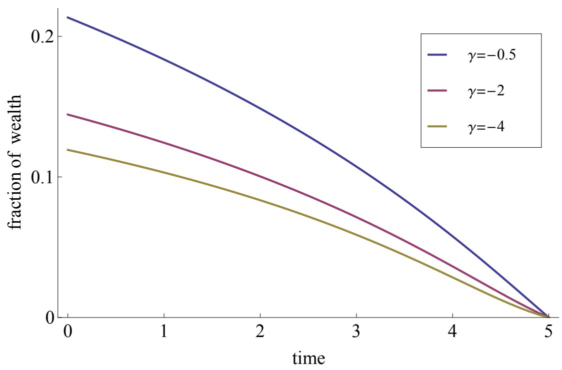

Figure 2.7. Optimal pre-crash strategy $k_t^{(1)*}$ for $a = 2$ and different values of γ.

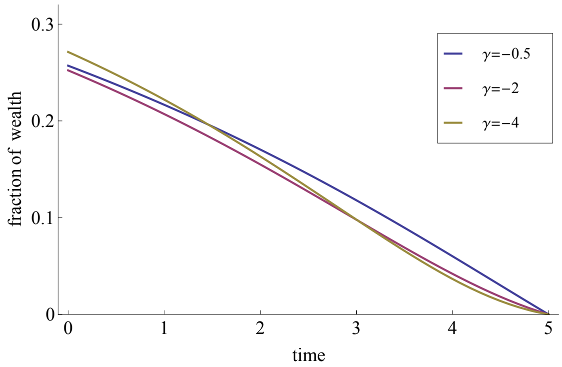

Figure 2.8. Optimal pre-crash strategy $k_t^{(1)*}$ for $a = 0.5$ and different values of γ.

Note, that in Figure 2.7 we used a speed of reversion $a = 2$ and in Figure 2.8 we used $a = 0.5$. In Figure 2.7, we observe the standard monotonicity behavior as in [44, Chp.6.2], that means, the higher the risk aversion $1 - \gamma$ the lower the investment in the stock for all $t \in [0, T]$. A contrary behavior can be seen in Figure 2.8, where the speed of reversion is lower. Here, one can see that at a certain point of time, the investor with a higher risk aversion invests more in the stock than the investor with a lower risk aversion. Compare for example the blue and the yellow line: The investor with risk level $\gamma = -0.5$ invests less in the stock than the investor with risk level $\gamma = -4$ until time $t \approx 1.4$. Again, note that in our financial market model there is no riskless asset due to the considered short rate model. If the short rate becomes more risky, for example due to a lower speed of reversion a, then for small t, it may be more risky to invest in the savings account than in the stock.

2.5. Changing market parameters and a general affine short rate model

In this section, we consider again the worst-case optimization problem (8) with a logarithmic utility function. But, in contrast to the previous sections, we assume that the short rate process is

a solution of SDE (3), such that the Cox-Ingersoll-Ross model is also covered by the considerations in this section.
Furthermore, we extend the financial market model by assuming changing market parameters at the crash time. In our previous considerations, the market crash only causes a sudden downward jump of the price process. Before and after this crash the stock price process follows the SDE given in (5). Now, we assume that the crash has one more impact on the stock price dynamics, namely that the crash might also cause a change in the market parameters μ and σ_1. This concept was already considered in the literature about worst-case optimization with constant interest rates (see e.g. [25, 33]). For the sake of simplicity we consider a financial market model where at most one market crash can occur, which is denoted by the pair (τ, l).

2.5.1. The generalized financial market model. Here, we assume that the short rate process $\{r_t\}_{t\in[0,T]}$ is given by the SDE (3), that is:

$$\begin{aligned} dr_t &= (\lambda_1 r_t + \lambda_2)\,dt + \sqrt{\xi_1 r_t + \xi_2}\,d\tilde{w}_t, \\ r_0 &= r^0 > 0, \end{aligned}$$

for suitable constants $\lambda_1, \lambda_2, \xi_1, \xi_2$ and r^0, where $\xi_2 \geq 0$. First, we have to ensure that there exists a uniquely determined solution of the SDE (3). If $\xi_1 = 0$, then there exists a uniquely determined solution of the SDE because the coefficients fulfill the classical Lipschitz and growth conditions. If $\xi_1 \neq 0$, the diffusion function $\sqrt{\xi_1 r + \xi_2}$ is, in general, not Lipschitz continuous. Moreover, we have to ensure that the process $\xi_1 r_t + \xi_2$ is nonnegative. In this chapter, our basic assumption on the parameters is that

$$\xi_1\lambda_2 - \lambda_1\xi_2 > \frac{\xi_1^2}{2}.$$

Under this assumption, Proposition A.1.2 in Appendix A provides the existence of a unique solution of SDE (3) that remains in the domain

$$\mathcal{D} := \{r \in \mathbb{R} : \xi_1 r + \xi_2 > 0\}.$$

While the short rate dynamics is not affected by the market crash, we assume that the stock price loses a fraction $l \in [0, l^*]$ of its value and the market parameters may change at the crash time. That is,

$$\begin{aligned} P_0 &= p^0, \\ dP_t &= P_t\left[(\mu^{(1)} + r_t)\,dt + \sigma_1^{(1)}\,dw_{1,t}\right], \qquad t \in (0, \tau), \\ P_\tau &= P_{\tau-}(1 - l), \\ dP_t &= P_t\left[(\mu^{(0)} + r_t)\,dt + \sigma_1^{(0)}\,dw_{1,t}\right], \qquad t \in (\tau, T]. \end{aligned}$$

In the market before the market crash the excess return and the volatility of the stock are given by the positive constants $\mu^{(1)}$ and $\sigma_1^{(1)}$, respectively. After the crash, the market conditions may change to $\mu^{(0)}$ and $\sigma_1^{(0)}$. For example, one could assume that the stock price after a crash is more volatile than before, then $\sigma_1^{(0)} > \sigma_1^{(1)}$.

Now, analogously to (7), the investor's wealth $X^{k,(\tau,l)} = \{X^{k,(\tau,l)}\}_{t\in[0,T]}$, given a strategy $k = (k^{(0)}, k^{(1)})$ and a market crash (τ, l), evolves as

$$\begin{aligned} X_0^{k,(\tau,l)} &= x^0 > 0, \\ dX_t^{k,(\tau,l)} &= X_t^{k,(\tau,l)} \left[r_t + \mu^{(1)} k_t^{(1)} \right] dt + X_t^{k,(\tau,l)} \sigma_1^{(1)} k_t^{(1)} \, dw_{1,t}, \quad t \in (0,\tau), \\ X_\tau^{k,(\tau,l)} &= (1 - l k_\tau^{(1)}) X_{\tau-}^{k,(\tau,l)}, \\ dX_t^{k,(\tau,l)} &= X_t^{k,(\tau,l)} \left[r_t + \mu^{(0)} k_t^{(0)} \right] dt + X_t^{k,(\tau,l)} \sigma_1^{(0)} k_t^{(0)} \, dw_{1,t}, \quad t \in (\tau, T]. \end{aligned} \tag{65}$$

On a market with at most one market crash and logarithmic utility function, the worst-case optimization problem (8) simplifies to:

$$\sup_{k\in\Pi(0,x^0,r^0)} \inf_{(\tau,l)\in\mathcal{C}} \mathbb{E}\left(\log(X_T^{k,(\tau,l)})\right). \tag{66}$$

We define the corresponding value function

$$V^1(t,x,r) := \sup_{k\in\Pi(t,x,r)} \inf_{(\tau,l)\in\mathcal{C}} \mathbb{E}^{t,x,r,1}\left(\log(X_T^{k,(\tau,l)})\right),$$

where $\mathcal{C}$ denotes the set of crash scenarios and Π is the set of admissible controls (see Definition 2.1.1). The following Corollary ensures that the value function above is well-defined.

COROLLARY 2.5.1. *For $(t,x,r) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}$, let $k \in \Pi(t,x,r)$ be an arbitrary admissible strategy and let $(\tau,l) \in \mathcal{C}$ be an arbitrary crash scenario on $[t,T]$. Moreover, let $\{r_t\}_{t\in[0,T]}$ and $X^{k,(\tau,l)} = \{X_t^{k,(\tau,l)}\}_{t\in[0,T]}$ be given by (3) and (65), respectively. Then*

$$\mathbb{E}^{t,x,r,1}\left(\left|\log(X_T^{k,(\tau,l)})\right|\right) < \infty.$$

PROOF. For the proof we refer to Appendix 2.6.9. □

In this section, we use the martingale approach to determine the worst-case optimal strategy $k^* = (k^{(0)*}, k^{(1)*})$. We proceed as in Section 2.4.1. First, we determine the optimal post-crash strategy $k^{(0)*}$ by DPP, then we reformulate the problem and determine the worst-case optimal pre-crash strategy $k^{(1)*}$. The main difference to Section 2.4.1 is how to determine the post-crash value function V^0 in the case of the affine model (3) and how to determine $k^{(1)*}$ under changing market parameters.

2.5.2. The optimal post-crash strategy $k^{(0)*}$. As in the proof of part a) of Theorem 2.3.2, the investor is faced with a classical stochastic optimal control problem after the crash. Here, the corresponding value function $V^0(t,x,r)$ takes the form

$$V^0(t,x,r) = \sup_{k^{(0)}\in\Pi(t,x,r)} \mathbb{E}^{t,x,r}\left(\log(\overline{X}_T)\right), \tag{67}$$

where $\overline{X}_s$ denotes the wealth at time $s \geq t$, that is:

$$\begin{aligned} d\overline{X}_s &= \overline{X}_s \left[\overline{r}_s + \mu^{(0)} k_s^{(0)} \right] ds + \overline{X}_s \sigma_1^{(0)} k_s^{(0)} \, dw_{1,s}, \qquad \overline{X}_t = x, \\ d\overline{r}_s &= (\lambda_1 \overline{r}_s + \lambda_2) \, ds + \sqrt{\xi_1 \overline{r}_s + \xi_2} \, d\tilde{w}_s, \qquad \overline{r}_t = r. \end{aligned}$$

The corresponding HJB equation is given by

$$\begin{aligned}
&v_t^0(t,x,r) + rxv_x^0(t,x,r)\\
&+ \sup_{k^{(0)}\in A}\Bigg[\mu^{(0)}k^{(0)}xv_x^0(t,x,r) + \frac{(\sigma_1^{(0)})^2}{2}(k^{(0)})^2x^2v_{xx}^0(t,x,r)\\
&\qquad + \rho\sigma_1^{(0)}\sqrt{\xi_1 r+\xi_2}k^{(0)}xv_{xr}^0(t,x,r)\Bigg]\\
&+ (\lambda_1 r+\lambda_2)v_r^0(t,x,r) + \frac{\xi_1 r+\xi_2}{2}v_{rr}^0(t,x,r) = 0, \qquad (t,x,r)\in[0,T)\times\mathbb{R}_+\times\mathbb{R},\\
&v^0(T,x,r) = \log(x), \qquad (x,r)\in\mathbb{R}_+\times\mathbb{R}.
\end{aligned}$$

Applying the standard separation ansatz for logarithmic utility functions, that is $v^0(t,x,r) = \log(x) + W^{(0)}(t,r)$ with $W^{(0)}(T,r) = 0$ for all $r\in\mathbb{R}$, implies that the optimal candidate is given by

$$k_t^{(0)*} \equiv \frac{\mu^{(0)}}{(\sigma_1^{(0)})^2}$$

and the equation above reduces to

$$\begin{aligned}
&W_t^{(0)}(t,r) + r + \frac{1}{2}\left(\frac{\mu^{(0)}}{\sigma_1^{(0)}}\right)^2 + (\lambda_1 r+\lambda_2)W_r^{(0)}(t,r) + \frac{\xi_1 r+\xi_2}{2}W_{rr}^{(0)}(t,r) = 0, \\
&\qquad\qquad (t,r)\in[0,T)\times\mathbb{R},\\
&W^{(0)}(T,r) = 0, \qquad r\in\mathbb{R}.
\end{aligned} \tag{68}$$

By the linear ansatz $W^{(0)}(t,r) = A(t)r + B(t)$, with $A(T) = B(T) = 0$, we obtain that

$$\begin{aligned}
A(t) &= \frac{1}{\lambda_1}\left(e^{\lambda_1(T-t)} - 1\right),\\
B(t) &= \frac{\lambda_2}{\lambda_1^2}\left(e^{\lambda_1(T-t)} - 1\right) - \frac{\lambda_2}{\lambda_1}(T-t) + \frac{1}{2}\left(\frac{\mu^{(0)}}{\sigma_1^{(0)}}\right)^2(T-t).
\end{aligned}$$

Finally, we obtained a solution of the HJB equation, which is given by

$$v^0(t,x,r) = \log(x) + W^{(0)}(t,r), \qquad W^{(0)}(t,r) = A(t)r + B(t).$$

Now, it remains to verify that the solution of the HJB equation is equal to the value function V^0 and that the candidate $k^{(0)*}$ is indeed the optimal post-crash strategy. We apply Corollary A.5.3 and show the requirements in Appendix 2.6.10.

2.5.3. Reformulation. Again, using the optimal post-crash strategy $k^{(0)*}$ and the post-crash value function $V^0(t,x,r) = \log(x) + A(t)r + B(t)$, which is monotone increasing in its second component, we reformulate the worst-case optimization problem (66) as a controller vs. stopper game:

$$\sup_{k^{(1)}\in\Pi(0,x^0,r^0)}\inf_{\tau\in\mathcal{C}}\mathbb{E}\left(M_\tau^{k^{(1)}}\right), \quad \text{where} \quad M_t^k := V^0(t,(1-l^*(k_t)^+)\tilde{X}_t^k, r_t) \tag{69}$$

and $\tilde{X}^k$ denotes the wealth process in a crash-free market, that means it fulfills

$$d\tilde{X}_t^k = \tilde{X}_t^k\left[r_t + \mu^{(1)}k_t\right]dt + \tilde{X}_t^k\sigma_1^{(1)}k_t\,dw_{1,t}, \qquad \tilde{X}_0^k = x^0, \tag{70}$$

and $\{r_t\}_{t\in[0,T]}$ is given by (3).
In order to solve the controller vs. stopper game above, we formulate an analogue of Lemma 2.4.2 for the generalized market with logarithmic utility. First, we obtain a sufficient condition for a pre-crash strategy to be an indifference strategy in the sense of Definition 2.4.1:

LEMMA 2.5.2. *Let $\hat{k}^{(1)}$ be the uniquely determined solution of the ODE*

$$\dot{k}_t^{(1)} = \frac{1-l^* k_t^{(1)}}{l^*}\left(\mu^{(1)}k_t^{(1)} - \frac{1}{2}(\sigma_1^{(1)})^2(k_t^{(1)})^2 - \frac{1}{2}\left(\frac{\mu^{(0)}}{\sigma_1^{(0)}}\right)^2\right), \qquad k_T^{(1)} = 0, \tag{71}$$

and let $M^k = \{M_t^k\}_{t\in[0,T]}$ be given by (69) *and $M_\infty^k := V^0(T, \tilde{X}_T^k, r_T)$. Then, $M^{\hat{k}^{(1)}}$ is a martingale on $[0,T]\cup\{\infty\}$ and $\hat{k}^{(1)}$ is an indifference strategy for the controller vs. stopper game* (69).

REMARK 2.5.3. *Here, $\hat{k}^{(1)}$ is admissible in the sense of Definition 2.1.1. With similar arguments as in the proof of Proposition 2.2.4, there exists a uniquely determined solution $\hat{k}^{(1)}$ of ODE* (71) *with $\hat{k}_t^{(1)} \in [0, \frac{1}{l^*})$ for all $t \in [0,T]$.*

PROOF OF LEMMA 2.5.2. We show that $M^{\hat{k}^{(1)}}$ is a martingale on $[0,T]\cup\{\infty\}$. Throughout the proof we abbreviate $\hat{k}^{(1)}$ by $\hat{k}$. By $V^0(t,x,r) = \log(x) + W^{(0)}(t,r)$ and by applying Ito's formula we obtain

$$\begin{aligned} dM_t^{\hat{k}} =& d\left(V^0(t,(1-l^*\hat{k}_t)\tilde{X}_t^{\hat{k}}, r_t)\right) \\ =& \Bigg\{ -\frac{l^*}{1-l^*\hat{k}_t}\dot{\hat{k}}_t + \mu^{(1)}\hat{k}_t - \frac{1}{2}(\sigma_1^{(1)})^2\hat{k}_t^2 \\ & + W_t^{(0)}(t,r_t) + r_t + (\lambda_1 r_t + \lambda_2)W_r^{(0)}(t,r_t) + \frac{\xi_1 r_t + \xi_2}{2}W_{rr}^{(0)}(t,r_t)\Bigg\}\, dt \\ & + \{\sigma_1^{(1)}\hat{k}_t + \sqrt{\xi_1 r_t + \xi_2}\rho W_r^{(0)}(t,r_t)\}\, dw_{1,t} \\ & + \sqrt{\xi_1 r_t + \xi_2}\sqrt{1-\rho^2}W_r^{(0)}(t,r_t)\, dw_{2,t}. \end{aligned}$$

Since $W^{(0)}(t,r)$ is a solution of PDE (68), we obtain

$$W_t^{(0)}(t,r_t) + r_t + (\lambda_1 r_t + \lambda_2)W_r^{(0)}(t,r_t) + \frac{\xi_1 r_t + \xi_2}{2}W_{rr}^{(0)}(t,r_t) = -\frac{1}{2}\left(\frac{\mu^{(0)}}{\sigma_1^{(0)}}\right)^2.$$

By assumption, $\hat{k}_t$ fulfills (71) such that the dt-coefficient vanishes and it remains to show that the solution of

$$\begin{aligned} dM_t^{\hat{k}} =& \{\sigma_1^{(1)}\hat{k}_t + \sqrt{\xi_1 r_t + \xi_2}\rho W_r^{(0)}(t,r_t)\}\, dw_{1,t} \\ & + \sqrt{\xi_1 r_t + \xi_2}\sqrt{1-\rho^2}W_r^{(0)}(t,r_t)\, dw_{2,t} \end{aligned}$$

is a martingale on $[0,T]$. Since $\mathbb{E}(r_t)$ is given in Proposition A.1.2 and since $W_r^{(0)}(t,r_t) = A(t)$, we immediately obtain that

$$\begin{aligned} &\mathbb{E}\left(\int_0^T (\sigma_1^{(1)}\hat{k}_t + \sqrt{\xi_1 r_t + \xi_2}\rho A(t))^2\, dt\right) \\ &\le \int_0^T 2(\sigma_1^{(1)}\hat{k}_t)^2 + 2(\xi_1\mathbb{E}(r_t) + \xi_2)\rho^2 A^2(t)\, dt < \infty, \end{aligned}$$

and

$$\mathbb{E}\left(\int_0^T (\xi_1 r_t + \xi_2)(1-\rho^2)A(t)^2\,dt\right) = \int_0^T (\xi_1 \mathbb{E}(r_t) + \xi_2)(1-\rho^2)A(t)^2\,dt < \infty,$$

and it follows that $M^{\hat{k}}$ is a martingale on $[0,T]$. By definition of $M^{\hat{k}}_\infty$ and by $\hat{k}_T = 0$, we also have

$$\mathbb{E}\left(M^{\hat{k}}_\infty|\mathcal{F}_T\right) = \mathbb{E}\left(V^0(T,\tilde{X}^{\hat{k}}_T, r_T)|\mathcal{F}_T\right) = V^0(T,\tilde{X}^{\hat{k}}_T, r_T) = M^{\hat{k}}_T.$$

Thus, $M^{\hat{k}}$ is a martingale on $[0,T]\cup\{\infty\}$.
By Doob's Optional Sampling Theorem, we obtain

$$\mathbb{E}\left(M^{\hat{k}^{(1)}}_\tau\right) = \mathbb{E}\left(M^{\hat{k}^{(1)}}_{\tau'}\right),$$

for all $[0,T]\cup\{\infty\}$-valued stopping times τ,τ'. By definition, $\hat{k}^{(1)}$ is an indifference strategy. □

Let $k^M := \frac{\mu^{(1)}}{(\sigma_1^{(1)})^2}$. Then, we can easily see, that k^M is the classical optimal investment strategy in a crash-free market.
By applying the invariance argument, we can show that $\hat{k}^{(1)}_t \in [0,k^M]$ for all $t\in[0,T]$ if condition (72) in the following Proposition is fulfilled.

PROPOSITION 2.5.4. *Let $\hat{k}^{(1)}$ be the uniquely determined solution of* (71) *and let* $k^M = \frac{\mu^{(1)}}{(\sigma_1^{(1)})^2}$. *Moreover, assume that*

$$\frac{\mu^{(1)}}{\sigma_1^{(1)}} - \frac{\mu^{(0)}}{\sigma_1^{(0)}} \geq 0. \tag{72}$$

Then $\hat{k}^{(1)}_t \in \left[0,k^M\right]$ *for all* $t\in[0,T]$.

PROOF. We refer to Appendix 2.6.11 for the proof. □

REMARK 2.5.5. *If the inequality* (72) *is not fulfilled, there might exist an intersection point* t_S *of* $\hat{k}^{(1)}_t$ *and* k^M *on* $[0,T]$. *Since* $\hat{k}^{(1)}_t$, *as a solution of an autonomous first order ODE, is monotone decreasing on* $[0,T]$ *and since* $\hat{k}^{(1)}_T = 0 \leq k^M$, *we have that* t_S *is uniquely determined and*

$$\begin{aligned}\hat{k}^{(1)}_t &\geq k^M \quad \forall\, t\in[0,t_S],\\ \hat{k}^{(1)}_t &\leq k^M \quad \forall\, t\in[t_S,T].\end{aligned}$$

Now, we can apply the notion of an indifference frontier, which was already explained in Section 2.4. As in [26],[44], we obtain by the martingale property of $M^{\hat{k}^{(1)}}$, that

$$\inf_{\tau\in\mathcal{C}} \mathbb{E}(M^{\tilde{k}}_\tau) \geq \inf_{\tau\in\mathcal{C}} \mathbb{E}(M^{k}_\tau),$$

where k is an arbitrary admissible pre-crash strategy and $\tilde{k}$ is defined, as before, by

$$\tilde{k}_t := \begin{cases} k_t & : t<\eta \\ \hat{k}^{(1)}_t & : t\geq\eta \end{cases}, \qquad \eta := \inf\{t\geq 0 : k_t > \hat{k}_t\}.$$

We conclude that the worst-case optimal pre-crash strategy is an element of the set

$$\mathcal{A}(\hat{k}^{(1)}) := \left\{k\in\Pi : k^{(1)}_t \leq \hat{k}^{(1)}_t, \quad \forall\, t\in[0,T]\right\}.$$

For a detailed explanation of the idea of the indifference frontier we refer to the literature [26, 44] or to Section 2.4.1. In order to apply the Indifference Optimality Principle below, we first determine the optimal pre-crash strategy in the no-crash scenario in the class $\mathcal{A}(\hat{k}_t^{(1)})$. This leads to the constrained optimization problem

$$\sup_{k_t \le \hat{k}_t^{(1)}, t\in[0,T]} \mathbb{E}\left(\log(\tilde{X}_T^k)\right) \qquad \text{w.r.t.} \quad (3), (70).$$

By DPP and similar ideas as in the post-crash problem, we obtain that the strategy $k_t^{(1)*} = \hat{k}_t^{(1)} \wedge k^M$ is optimal in the no-crash scenario $\tau = \infty$. Now, we show that $k^{(1)*}$ is worst-case optimal for the controller vs. stopper game (69).

2.5.4. The worst-case optimal pre-crash strategy.

THEOREM 2.5.6. *Suppose that* $\hat{k}^{(1)}$ *is the uniquely determined solution of* (71). *Let*

$$k_t^{(1)*} := \hat{k}_t^{(1)} \wedge k^M.$$

Then $k^{(1)*}$ *is the optimal pre-crash strategy for the worst-case optimization problem. Moreover,* $k_t^{(0)*} \equiv \frac{\mu^{(0)}}{(\sigma_1^{(0)})^2}$ *is the optimal post-cash strategy.*

PROOF. Since $k^M > 0$ is constant, the proof is similar to the proof of [44, Thm.5.1]. After showing the supermartingale property of $M^{k^{(1)*}}$, we apply the indifference optimality principle to obtain the assertion.
The proof works with similar arguments as the proof on page 36. The key assumption, that $M_t^{\hat{k}^{(1)}}$ is a martingale, is fulfilled. For the sake of completeness, we give some details of the proof. Let us define

$$t_S := \inf\{t \in [0,T] : k^M \ge \hat{k}_t^{(1)}\}.$$

Again, t_S denotes the uniquely determined point of intersection of k^M and $\hat{k}_t^{(1)}$ if it exists. By Lemma 2.5.2, $M^{k^{(1)*}}$ is a martingale on $[t_S, T] \cup \{\infty\}$ because $k_t^{(1)*} = \hat{k}_t^{(1)}$ on $[t_S, T] \cup \{\infty\}$. If (72) is fulfilled, we immediately obtain that $t_S = 0$ and $M^{k^{(1)*}}$ is a martingale on $[0,T] \cup \{\infty\}$. Now, let us assume that

$$\frac{\mu^{(1)}}{\sigma_1^{(1)}} - \frac{\mu^{(0)}}{\sigma_1^{(0)}} < 0,$$

and that $t_S > 0$. Then, on $[0, t_S]$, we have that $k_t^{(1)*} = k^M$ and by Ito's formula:

$$\begin{aligned} dM_t^{k^{(1)*}} &= \left\{ \frac{1}{2}\left(\frac{\mu^{(1)}}{\sigma_1^{(1)}}\right)^2 - \frac{1}{2}\left(\frac{\mu^{(0)}}{\sigma_1^{(0)}}\right)^2 \right\} dt \\ &\quad + \left\{ \sigma_1^{(1)} k^M + \sqrt{\xi_1 r_t + \xi_2}\rho W_r^{(0)}(t, r_t) \right\} dw_{1,t} \\ &\quad + \sqrt{\xi_1 r_t + \xi_2}\sqrt{1-\rho^2} W_r^{(0)}(t, r_t)\, dw_{2,t}. \end{aligned}$$

Due to the fact that the stochastic integral above is a martingale on $[0, t_S]$, we have for $0 \leq s \leq t \leq t_S$:

$$\begin{aligned}
&\mathbb{E}\left(M_t^{k^{(1)*}} | \mathcal{F}_s\right) \\
&= M_0^{k^{(1)*}} + \frac{t}{2}\left(\left(\frac{\mu^{(1)}}{\sigma_1^{(1)}}\right)^2 - \left(\frac{\mu^{(0)}}{\sigma_1^{(0)}}\right)^2\right) \\
&\quad + \mathbb{E}\bigg(\int_0^t \left\{\sigma_1^{(1)} k^M + \sqrt{\xi_1 r_u + \xi_2}\rho W_r^{(0)}(u, r_u)\right\} dw_{1,u} \\
&\qquad + \sqrt{\xi_1 r_t + \xi_2}\sqrt{1-\rho^2} W_r^{(0)}(u, r_u)\, dw_{2,u} | \mathcal{F}_s\bigg) \\
&\leq M_0^{k^{(1)*}} + \frac{s}{2}\left(\left(\frac{\mu^{(1)}}{\sigma_1^{(1)}}\right)^2 - \left(\frac{\mu^{(0)}}{\sigma_1^{(0)}}\right)^2\right) \\
&\quad + \int_0^s \left\{\sigma_1^{(1)} k^M + \sqrt{\xi_1 r_u + \xi_2}\rho W_r^{(0)}(u, r_u)\right\} dw_{1,u} \\
&\quad + \sqrt{\xi_1 r_t + \xi_2}\sqrt{1-\rho^2} W_r^{(0)}(u, r_u)\, dw_{2,u} = M_s^{k^{(1)*}}.
\end{aligned}$$

Therefore, $M^{k^{(1)*}}$ is a supermartingale on $[0, T] \cup \{\infty\}$ and by Doob's Optional Sampling Theorem for supermartingales (see Appendix A, Theorem A.4.7), we have for all $[0, T] \cup \{\infty\}$-valued stopping times τ:

$$\mathbb{E}\left(M_\tau^{k^{(1)*}}\right) \geq \mathbb{E}\left(M_\infty^{k^{(1)*}}\right).$$

Analogously to the Indifference Optimality Principle in [26] and [44], we obtain

$$\inf_{\tau \in \mathcal{C}} \mathbb{E}\left(M_\tau^{k^{(1)*}}\right) \geq \mathbb{E}\left(M_\infty^{k^{(1)*}}\right) \geq \mathbb{E}\left(M_\infty^{k}\right) \geq \inf_{\tau \in \mathcal{C}} \mathbb{E}\left(M_\tau^{k}\right).$$

for an arbitrary pre-crash strategy $k \in \mathcal{A}(\hat{k}^{(1)})$. The second inequality holds, because $k^{(1)*}$ is optimal in the no-crash scenario in the class $\mathcal{A}(\hat{k}^{(1)})$. Since the optimal strategy for the controller vs. stopper game is an element of the class $\mathcal{A}(\hat{k}^{(1)})$, we have that $k^{(1)*}$ is the optimal strategy, and it is the worst-case optimal pre-crash strategy for the problem (66). □

2.5.5. Discussion and numerical examples. We have shown that $k_t^{(1)*} = \hat{k}_t^{(1)} \wedge k^M$, where $\hat{k}^{(1)}$ is the uniquely determined solution of (71) and $k^M = \frac{\mu^{(1)}}{(\sigma_1^{(1)})^2}$, is the worst-case optimal pre-crash strategy and $k^{(0)*} = \frac{\mu^{(0)}}{(\sigma_1^{(0)})^2}$ is the optimal post-crash strategy for (66).

REMARK 2.5.7. *Comparing the result for the worst-case optimization problem* (66) *with the worst-case optimal strategies from the literature, where constant interest rates are used, we obtain the following points:*

- *The worst-case optimal strategy $k^* = (k^{(0)*}, k^{(1)*})$ for the problem with stochastic interest rates equals the strategy in the case of constant interest rates (see e.g. [33]). The explanation is the same as the one already mentioned in Section 2.3.1: Due to the logarithmic utility function we can additively separate the control variable k from the short rate r_t, which means that the maximization of the goal function does not depend on the short rate, neither through $r_t(\omega)$ itself nor through the parameters which determine* *the short rate equation. Again, the logarithmic utility function eliminates the stochastic interest rate risk. Further, if the short rate would somehow be affected by the market crash, this effect might vanish.*

- *Nevertheless, we had to show the martingale property in Lemma 2.5.2 and the super-martingale property in Theorem 2.5.6 taking into account that the short rate is stochastic and that it follows a general affine model of the form* (3)*.*
- *If* $\mu^{(1)} = \mu^{(0)}$ *and* $\sigma_1^{(1)} = \sigma_1^{(0)}$*, then ODE* (71) *coincides with ODE* (48) *for* $n = 1$ *and* $k_t^{(1)*} = \hat{k}_t^{(1)}$*.*

For the logarithmic utility case, the optimal strategies do not differ from the ones with constant interest rates. Nevertheless, we give a short illustration of them. Similar plots can be found for example in [25].

Example 2.5.8. *Here, we use ODE* (71) *to calculate the indifference strategy* $\hat{k}^{(1)}$ *and the corresponding optimal pre-crash strategy* $k^{(1)*} = \hat{k}_t^{(1)} \wedge k^M$ *numerically . In Figure 2.9 and Figure 2.10 we plotted the optimal strategy in a crash-free market* k^M *(blue dashed line), the optimal pre-crash strategy* $k^{(1)*}$ *(pink solid line) and the optimal post-crash strategy* $k^{(0)*}$ *(yellow solid line) for* $T = 10$*, a maximum crash size* $l^* = 0.4$ *and for different choices of market parameters.*

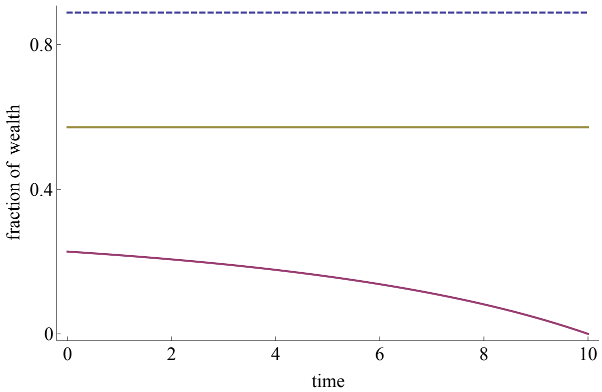

Figure 2.9. Optimal strategies for market parameters $\mu^{(1)} = 0.08$, $\sigma_1^{(1)} = 0.3$, $\mu^{(0)} = 0.07$, $\sigma_1^{(0)} = 0.35$.

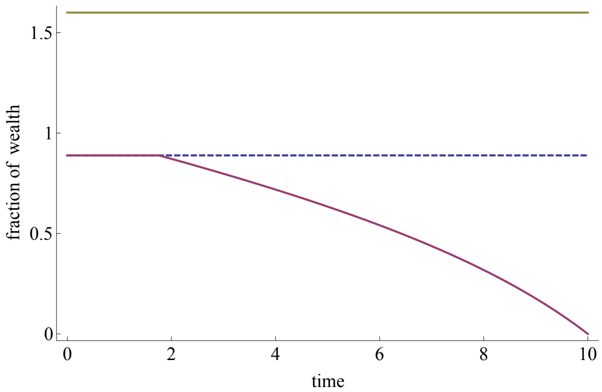

Figure 2.10. Optimal strategies for market parameters $\mu^{(1)} = 0.08$, $\sigma_1^{(1)} = 0.3$, $\mu^{(0)} = 0.1$, $\sigma_1^{(0)} = 0.25$.

In Figure 2.9 we assumed that the market after the crash is worse than before (lower excess return, higher volatility). The market parameters fulfill (72) *and therefore* $\hat{k}_t^{(1)} \in [0, k^M]$ *for all* $t \in [0, T]$*. On the other hand, in Figure 2.10, we assumed that the market after the crash is better than before (higher excess return, lower volatility). There exists an intersection point of* $\hat{k}^{(1)}$ *and* k^M*, such that the optimal pre-crash strategy is to follow* k^M *for* $t \leq t_S \approx 2$ *and to follow* $\hat{k}_t^{(1)}$ *for* $t \geq t_S$*. Once the market crash has happened, the investor changes to* $k^{(0)*}$*.*

2.6. Appendix

2.6.1. Proof of Lemma 2.2.3.

For the readers convenience we repeat the assertion of Lemma 2.2.3:
Let $V^n(t,x,r)$ be given by (10) and let (τ,l) be the first intervention after time t. Then, we have

$$\begin{aligned}
V^n(t,x,r) &= \sup_{k\in\Pi}\inf_{M\in\mathcal{N}(t,n)} \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right]\\
&= \inf_{M\in\mathcal{N}(t,n)}\sup_{k\in\Pi} \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right]\\
&= \sup_{k\in\Pi}\inf_{(\tau,l)} \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right]\\
&= \inf_{(\tau,l)}\sup_{k\in\Pi} \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right]
\end{aligned}$$

Proof. Since the short rate dynamics is not affected by the market crash, the proof works in the same manner as in [28, Lemma 3]. In contrast to this literature the infimum above is taken over the pairs (τ,l), which denote the first intervention after time t.
Let $\varepsilon > 0$ and let (τ,l) be a given first intervention. Now, we choose a strategy k^*, which is $\frac{\varepsilon}{4}$-optimal until time τ in the sense that

$$\begin{aligned}
&\sup_k \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right]\\
&\quad\le \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_{\tau-}^{k^*,(\tau,l)}(1-lk^*_\tau), r_\tau)\right] + \frac{\varepsilon}{4}. \qquad (73)
\end{aligned}$$

Moreover, we choose a strategy k^{**}, which is arbitrary until τ and $\frac{\varepsilon}{4}$-optimal after the intervention (τ,l), that means

$$\begin{aligned}
&\sup_k \inf_{M\in\mathcal{N}(\tau,n-1)} \mathbb{E}^{\tau,X_\tau^{k,M},r_\tau,n-1}\left[U(X_T^{k,M})\right]\\
&\quad\le \inf_{M\in\mathcal{N}(\tau,n-1)} \mathbb{E}^{\tau,X_\tau^{k^{**},M},r_\tau,n-1}\left[U(X_T^{k^{**},M})\right] + \frac{\varepsilon}{4}. \qquad (74)
\end{aligned}$$

Now, let k be a given portfolio strategy. Then, we define an $\frac{\varepsilon}{4}$-optimal first intervention strategy (τ_*,l_*) in the following sense

$$\begin{aligned}
&\inf_{(\tau,l)} \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right]\\
&\quad\ge \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau_*, X_{\tau_*-}^{k,(\tau_*,l_*)}(1-l_*k_{\tau_*}), r_{\tau_*})\right] - \frac{\varepsilon}{4}, \qquad (75)
\end{aligned}$$

and for an arbitrary but fixed first intervention strategy (τ,l), we define a strategy $M^*\in\mathcal{N}(\tau,n-1)$, which is $\frac{\varepsilon}{4}$-optimal after (τ,l) in the following sense

$$\begin{aligned}
&\inf_{M\in\mathcal{N}(\tau,n-1)} \mathbb{E}^{\tau,X_\tau^{k,M},r_\tau,n-1}\left[U(X_T^{k,M})\right]\\
&\qquad\ge \mathbb{E}^{\tau,X_\tau^{k,M^*},r_\tau,n-1}\left[U(X_T^{k,M^*})\right] - \frac{\varepsilon}{4}. \qquad (76)
\end{aligned}$$

With these definitions we obtain the following inequalities:

$$\begin{aligned}
&\sup_k \inf_{M\in\mathcal{N}(t,n)} \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right] \\
&\geq \inf_{M\in\mathcal{N}(t,n)} \mathbb{E}^{t,x,r,n}\left[U(X_T^{k^{**},M})\right] \\
&\geq \inf_{M\in\mathcal{N}(t,n)} \mathbb{E}^{t,x,r,n}\left[\mathbb{E}^{\tau,X_\tau^{k^{**},M},r_\tau,n-1}\left[U(X_T^{k^{**},M})\right]\right] \\
&\geq \inf_{(\tau,l)} \mathbb{E}^{t,x,r,n}\left[\inf_{M\in\mathcal{N}(\tau,n-1)} \mathbb{E}^{\tau,X_\tau^{k^{**},M},r_\tau,n-1}\left[U(X_T^{k^{**},M})\right]\right] \\
&\overset{(74)}{\geq} \inf_{(\tau,l)} \mathbb{E}^{t,x,r,n}\left[\sup_k \inf_{M\in\mathcal{N}(\tau,n-1)} \mathbb{E}^{\tau,X_\tau^{k,M},r_\tau,n-1}\left[U(X_T^{k,M})\right]\right] - \frac{\varepsilon}{4} \\
&= \inf_{(\tau,l)} \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_\tau^{k,(\tau,l)}, r_\tau)\right] - \frac{\varepsilon}{4} \\
&= \inf_{(\tau,l)} \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right] - \frac{\varepsilon}{4} \\
&\overset{(75)}{\geq} \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau_*, X_{\tau_*-}^{k,(\tau_*,l_*)}(1-l_*k_{\tau_*}), r_{\tau_*})\right] - \frac{\varepsilon}{2}.
\end{aligned}$$

The second inequality follows by the tower property of the conditional expectation. Thus, we have

$$\begin{aligned}
&\sup_k \inf_{M\in\mathcal{N}(t,n)} \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right] \\
&\geq \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau_*, X_{\tau_*-}^{k,(\tau_*,l_*)}(1-l_*k_{\tau_*}), r_{\tau_*})\right] - \frac{\varepsilon}{2}.
\end{aligned}$$

Taking the supremum on both sides leads to

$$\begin{aligned}
&\sup_k \inf_{M\in\mathcal{N}(t,n)} \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right] \\
&\geq \sup_k \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau_*, X_{\tau_*-}^{k,(\tau_*,l_*)}(1-l_*k_{\tau_*}), r_{\tau_*})\right] - \frac{\varepsilon}{2}. \qquad (77)
\end{aligned}$$

Moreover, for an arbitrary but fixed first intervention (τ, l) after t, it holds

$$\begin{aligned}
&\inf_{M\in\mathcal{N}(t,n)} \sup_k \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right] \\
&\leq \sup_k \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M^*})\right] \\
&\leq \sup_k \mathbb{E}^{t,x,r,n}\left[\mathbb{E}^{\tau,X_\tau^{k,M^*},r_\tau,n-1}\left[U(X_T^{k,M^*})\right]\right] \\
&\overset{(76)}{\leq} \sup_k \mathbb{E}^{t,x,r,n}\left[\inf_{M\in\mathcal{N}(\tau,n-1)} \mathbb{E}^{\tau,X_\tau^{k,M},r_\tau,n-1}\left[U(X_T^{k,M})\right]\right] + \frac{\varepsilon}{4} \\
&\leq \sup_k \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_\tau^{k,(\tau,l)}, r_\tau)\right] + \frac{\varepsilon}{4} \\
&= \sup_k \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right] + \frac{\varepsilon}{4}.
\end{aligned}$$

Thus,

$$\inf_{M\in\mathcal{N}(t,n)} \sup_k \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right]$$
$$\le \sup_k \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right] + \frac{\varepsilon}{4}.$$

Taking the infimum over the first intervention (τ, l) on both sides leads to

$$\inf_{M\in\mathcal{N}(t,n)} \sup_k \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right]$$
$$\le \inf_{(\tau,l)} \sup_k \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right] + \frac{\varepsilon}{4}. \tag{78}$$

Now, we conclude that

$$\sup_k \inf_{M\in\mathcal{N}(t,n)} \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right]$$
$$\overset{(77)}{\ge} \sup_k \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau_*, X_{\tau_*-}^{k,(\tau_*,l_*)}(1-l_*k_{\tau_*}), r_{\tau_*})\right] - \frac{\varepsilon}{2}$$
$$\ge \inf_{(\tau,l)} \sup_k \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right] - \frac{\varepsilon}{2}$$
$$\ge \inf_{(\tau,l)} \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_{\tau-}^{k^*,(\tau,l)}(1-lk_\tau^*)), r_\tau)\right] - \frac{\varepsilon}{2}$$
$$\overset{(73)}{\ge} \inf_{(\tau,l)} \sup_k \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)})(1-lk_\tau), r_\tau)\right] - \frac{3\varepsilon}{4}$$
$$\overset{(78)}{\ge} \inf_{M\in\mathcal{N}(t,n)} \sup_k \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right] - \varepsilon$$
$$\ge \sup_k \inf_{M\in\mathcal{N}(t,n)} \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right] - \varepsilon,$$

and analogously

$$\inf_{M\in\mathcal{N}(t,n)} \sup_k \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right]$$
$$\overset{(78)}{\le} \inf_{(\tau,l)} \sup_k \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right] + \frac{\varepsilon}{4}$$
$$\overset{(73)}{\le} \inf_{(\tau,l)} \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_{\tau-}^{k^*,(\tau,l)}(1-lk_\tau^*), r_\tau)\right] + \frac{\varepsilon}{2}$$
$$\le \sup_k \inf_{(\tau,l)} \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right] + \frac{\varepsilon}{2}$$
$$\le \sup_k \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau_*, X_{\tau_*-}^{k,(\tau_*,l_*)}(1-l_*k_{\tau_*}), r_{\tau_*})\right] + \frac{\varepsilon}{2}$$
$$\overset{(77)}{\le} \sup_k \inf_{M\in\mathcal{N}(t,n)} \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right] + \varepsilon$$
$$\le \inf_{M\in\mathcal{N}(t,n)} \sup_k \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right] + \varepsilon.$$

Thus, the inequalities

$$\begin{aligned}
&\sup_k \inf_{M\in\mathcal{N}(t,n)} \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right] \\
&\quad\geq \inf_{(\tau,l)} \sup_k \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right] - \frac{\varepsilon}{2} \\
&\quad\geq \sup_k \inf_{M\in\mathcal{N}(t,n)} \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right] - \varepsilon
\end{aligned}$$

and

$$\begin{aligned}
&\inf_{M\in\mathcal{N}(t,n)} \sup_k \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right] \\
&\quad\leq \sup_k \inf_{(\tau,l)} \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right] + \frac{\varepsilon}{2} \\
&\quad\leq \inf_{M\in\mathcal{N}(t,n)} \sup_k \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right] + \varepsilon
\end{aligned}$$

hold for any $\varepsilon > 0$ and we have that

$$\begin{aligned}
\sup_k \inf_{M\in\mathcal{N}(t,n)} \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right] &= \inf_{(\tau,l)} \sup_k \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right], \\
\inf_{M\in\mathcal{N}(t,n)} \sup_k \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right] &= \sup_k \inf_{(\tau,l)} \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right].
\end{aligned}$$

Moreover, by the fact that

$$\begin{aligned}
&\sup_k \inf_{M\in\mathcal{N}(t,n)} \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right] \\
&\quad\leq \inf_{M\in\mathcal{N}(t,n)} \sup_k \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right] \\
&\quad\leq \sup_k \inf_{M\in\mathcal{N}(t,n)} \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right] + \varepsilon,
\end{aligned}$$

for any $\varepsilon > 0$, we finally obtain

$$\begin{aligned}
V^n(t,x,r) &\overset{(10)}{=} \sup_k \inf_{M\in\mathcal{N}(t,n)} \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right] \\
&= \inf_{M\in\mathcal{N}(t,n)} \sup_k \mathbb{E}^{t,x,r,n}\left[U(X_T^{k,M})\right] \\
&= \sup_k \inf_{(\tau,l)} \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right] \\
&= \inf_{(\tau,l)} \sup_k \mathbb{E}^{t,x,r,n}\left[V^{n-1}(\tau, X_{\tau-}^{k,(\tau,l)}(1-lk_\tau), r_\tau)\right].
\end{aligned}$$

□

2.6.2. Proof of Proposition 2.2.4.

We prove the assertion for the corresponding forward ODE (16) via induction. For the readers convenience we repeat the definition of $f^{(n)}(t, h^{(n)})$:

$$\begin{aligned}
f^{(n)}(t,h^{(n)}) := -\frac{1-l^*h^{(n)}}{l^*}\Bigg[&(\mu + \rho\sigma_1\sigma_2\beta(T-t))\left(h^{(n)} - k_{T-t}^{(n-1)*}\right) \\
&- \frac{\sigma_1^2}{2}(1-\gamma)\left((h^{(n)})^2 - (k_{T-t}^{(n-1)*})^2\right)\Bigg].
\end{aligned}$$

For $n = 1$ we have to consider the following ODE

$$\dot{h}_t^{(1)} = f^{(1)}(t, h_t^{(1)}), \qquad h_0^{(1)} = 0. \tag{79}$$

By definition we have that $k_{T-t}^{(0)*}$ is continuous in t, and therefore $f^{(1)} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous in t and $h^{(1)}$ and continuously differentiable in $h^{(n)}$. Thus, $f^{(1)}$ is locally Lipschitz continuous in $h^{(1)}$. Thus, there exists a uniquely determined local solution of (79) (see e.g. [42, Thm. 2.2.2]). Now, we show that the solution exists for all $t \geq 0$. Let $[0, t^+)$ be the maximal interval of existence and let $h^{(1)}$ be a maximal solution. Assume that $t^+ < \infty$. Choose $J_1 := [0, t^+]$, then we have

$$f^{(1)}(t, h^{(1)})h^{(1)} = -\frac{\sigma_1^2}{2}(1-\gamma)h^{(1)4} + \sum_{j=0}^{3} s_j(t)(h^{(1)})^j \leq C_{J_1} \qquad \forall t \in J_1, h^{(1)} \in \mathbb{R},$$

because $-\frac{\sigma_1^2}{2}(1-\gamma) < 0$ and $s_j(t)$ are continuous functions in t and therefore we can choose C_{J_1} such that the above inequality holds. Then, we easily obtain with $\varphi_t := (h_t^{(1)})^2$ for all $t \in [0, t_1]$, $t_1 < t^+$ arbitrary but fixed, that

$$\dot{\varphi}_t \leq 2C_{J_1}$$

and therefore

$$\varphi_t \leq 2C_{J_1}(t^+ - t_0), \qquad \forall t \in [0, t_1].$$

Thus, $\lim_{t \to t^+} |h_t^{(1)}| \neq \infty$, which is a contradiction to the assumption that $t^+ < \infty$. Therefore, $t^+ = \infty$ and there exists a uniquely determined global solution $h^{(1)}$ of (79). Thus, the assertion holds for $n = 1$. Now, assume that the assertion holds for $(n-1)$, that means

$$\dot{h}_t^{(n-1)} = -\frac{1 - l^* h_t^{(n-1)}}{l^*} \Big[(\mu + \rho\sigma_1\sigma_2\beta(T-t)) \left(h_t^{(n-1)} - k_{T-t}^{(n-2)*} \right) - \frac{\sigma_1^2}{2}(1-\gamma) \left((h_t^{(n-1)})^2 - (k_{T-t}^{(n-2)*})^2 \right) \Big], \qquad h_0^{(n-1)} = 0,$$

has a uniquely determined solution $h_t^{(n-1)}$. Under this assumption, we want to show that

$$\dot{h}_t^{(n)} = -\frac{1 - l^* h_t^{(n)}}{l^*} \Big[(\mu + \rho\sigma_1\sigma_2\beta(T-t)) \left(h_t^{(n)} - k_{T-t}^{(n-1)*} \right) - \frac{\sigma_1^2}{2}(1-\gamma) \left((h_t^{(n)})^2 - (k_{T-t}^{(n-1)*})^2 \right) \Big], \qquad h_0^{(n)} = 0. \tag{80}$$

has a uniquely determined solution $h_t^{(n)}$. The right hand side $f^{(n)} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous in t and $h^{(n)}$ (because, by assumption, $k_{T-t}^{(n-1)*} = h_t^{(n-1)} \wedge k_{T-t}^{(0)*}$ is continuous in t) and continuously differentiable in $h^{(n)}$. Therefore $f^{(n)}(t, h^{(n)})$ is locally Lipschitz continuous in $h^{(n)}$. Using the same arguments as above, one can easily see that there exists a uniquely determined solution of (80) on the maximal interval of existence $[0, \infty)$. By time reversion, we obtain the existence and uniqueness of a solution $\hat{k}_t^{(n)}$ of (13) on $(-\infty, T]$.

In the following two steps, we show that $\hat{k}_t^{(n)} \in [0, \frac{1}{l^*})$. Step 1 shows that $\hat{k}_t^{(n)} \in [0, \frac{1}{l^*}]$ for all $t \in [0,T]$ via induction over n, while Step 2 shows that $\hat{k}_t^{(n)} < \frac{1}{l^*}$ by contradiction.

Step 1: By an invariance argument in the sense of qualitative theory of ODE's, we show that $h_t^{(n)} \in [0, \frac{1}{l^*}]$ for all $t \geq 0$, $n \in \{1, \dots, N\}$. For a detailed definition of an invariant set and for the theorem which is used for the proof, we refer to Appendix A.2.
Let $D := [0, \frac{1}{l^*}]$. We show that D is positively invariant for (16), that means $h_t^{(n)} \in D$ provided that $h_0^{(n)} \in D$, for $n = 1, \dots, N$. We prove this via induction.
Let $n = 1$. Then, $f^{(1)}(t, \frac{1}{l^*}) \cdot y \leq 0$ for all $t \in \mathbb{R}$, $y \in \mathcal{N}_D(\frac{1}{l^*}) = (0, \infty)$, where $\mathcal{N}_D(h)$ denotes the set of outer normals on D. Moreover,

$$f^{(1)}(t,0) \cdot y = \frac{1}{l^*}\frac{\sigma_1^2}{2}(1-\gamma)\left(-k_{T-t}^{(0)*}\right)^2 \cdot y \leq 0, \qquad \forall\, t \in \mathbb{R},\, y \in \mathcal{N}_D(0) = (-\infty, 0).$$

By Theorem A.2.3, we obtain that D is positively invariant for (16) with $n = 1$ and therefore, $h_t^{(1)} \in D$ for all $t \geq 0$. Thus the assertion holds for $n = 1$. Now, assume that the assertion holds for $n-1$, that means $h_t^{(n-1)} \in D$. The aim is to show that D is positively invariant for

$$\dot{h}_t^{(n)} = f^{(n)}(t, h_t^{(n)}), \qquad h_0^{(n)} = 0.$$

Obviously, $f^{(n)}(t, \frac{1}{l^*}) = 0$ for all $t \in \mathbb{R}$ and therefore $f^{(n)}(t, \frac{1}{l^*}) \cdot y \leq 0$ for all $t \in \mathbb{R}$ and $y \in \mathcal{N}_D(\frac{1}{l^*})$.
Moreover,

$$f^{(n)}(t,0) = \frac{1}{l^*}\left[(\mu + \rho\sigma_1\sigma_2\beta(T-t))k_{T-t}^{(n-1)*} - \frac{\sigma_1^2}{2}(1-\gamma)(k_{T-t}^{(n-1)*})^2\right].$$

Since $\frac{1}{l^*} > 0$, it remains to show that

$$(\mu + \rho\sigma_1\sigma_2\beta(T-t))k_{T-t}^{(n-1)*} - \frac{\sigma_1^2}{2}(1-\gamma)(k_{T-t}^{(n-1)*})^2 \geq 0, \quad \forall\, t \in \mathbb{R}.$$

In order to show this inequality, we differentiate between two cases:
Case 1. For $t \in \mathbb{R}$ it holds that $h_t^{(n-1)} \geq k_{T-t}^{(0)*}$ and therefore $k_{T-t}^{(n-1)*} = k_{T-t}^{(0)*}$.
Then, we obtain

$$\begin{aligned}
&(\mu + \rho\sigma_1\sigma_2\beta(T-t))k_{T-t}^{(n-1)*} - \frac{\sigma_1^2}{2}(1-\gamma)(k_{T-t}^{(n-1)*})^2 \\
&= (\mu + \rho\sigma_1\sigma_2\beta(T-t))k_{T-t}^{(0)*} - \frac{\sigma_1^2}{2}(1-\gamma)(k_{T-t}^{(0)*})^2 \\
&= \frac{1}{2}\frac{(\mu + \rho\sigma_1\sigma_2\beta(T-t))^2}{\sigma_1^2(1-\gamma)} \geq 0.
\end{aligned}$$

Case 2. For $t \in \mathbb{R}$ it holds that $h_t^{(n-1)} \leq k_{T-t}^{(0)*}$ and therefore $k_{T-t}^{(n-1)*} = h_t^{(n-1)}$.
Since $h_t^{(n-1)} \geq 0$, we have that $k_{T-t}^{(0)*} \geq 0$ which is equivalent to

$$\mu + \rho\sigma_1\sigma_2\beta(T-t) \geq 0.$$

Thus, we obtain

$$\begin{aligned}
&(\mu+\rho\sigma_1\sigma_2\beta(T-t))k_{T-t}^{(n-1)*}-\frac{\sigma_1^2}{2}(1-\gamma)(k_{T-t}^{(n-1)})^2\\
&=h_t^{(n-1)}\left[\mu+\rho\sigma_1\sigma_2\beta(T-t)-\frac{\sigma_1^2}{2}(1-\gamma)h_t^{(n-1)}\right]\\
&\geq h_t^{(n-1)}\left[\mu+\rho\sigma_1\sigma_2\beta(T-t)-\frac{\sigma_1^2}{2}(1-\gamma)k_{T-t}^{(0)*}\right]\\
&=h_t^{(n-1)}\left[\frac{1}{2}(\mu+\rho\sigma_1\sigma_2\beta(T-t))\right]\geq 0.
\end{aligned}$$

Case 1 and Case 2 together yield:

$$(\mu+\rho\sigma_1\sigma_2\beta(T-t))k_{T-t}^{(n-1)*}-\frac{\sigma_1^2}{2}(1-\gamma)(k_{T-t}^{(n-1)*})^2\geq 0,\quad \forall\, t\in\mathbb{R},$$

and therefore

$$f^{(n)}(t,0)\cdot y\leq 0,\quad \forall t\in\mathbb{R}, y\in\mathcal{N}_D(0).$$

Finally, we have

$$f^{(n)}(t,h^{(n)})\cdot y\leq 0,\qquad \forall\, t\in\mathbb{R}, h^{(n)}\in\partial D, y\in\mathcal{N}_D(h^{(n)}),$$

and therefore, by Theorem A.2.3, we obtain that D is positively invariant for (16), and finally $h_t^{(n)}\in[0,\frac{1}{l^*}]$ for all $t\geq 0$. By time reversion we obtain that $\hat{k}_t^{(n)}\in[0,\frac{1}{l^*}]$ for all $t\in(-\infty,T]$.

Step 2: Here, we show that $\hat{k}_t^{(n)}<\frac{1}{l^*}$ for all $t\in[0,T]$ for arbitrary but fixed $n\in\{1,\ldots,N\}$. Let $\tilde{t}:=\inf\{t\in[0,T]:\hat{k}_t^{(n)}\leq\frac{1}{l^*}-\delta,\hat{k}_s^{(n)}\leq\frac{1}{l^*}-\delta,\,\forall s\in[t,T]\}$ for some $\delta>0$. Since $\hat{k}_T^{(n)}=0$ and by continuity of $\hat{k}_t^{(n)}$, the infimum is attained at some $\tilde{t}$. First, we show that $\hat{k}_{\tilde{t}}^{(n)}\leq\frac{1}{l^*}-2\delta$ for some $\delta>0$ if $\hat{k}_s^{(n)}\leq\frac{1}{l^*}-2\delta$ for $s\in[\tilde{t},T]$, and in a second step we deduce that $\tilde{t}=0$ by contradiction.

(1) Here we show that $\hat{k}_{\tilde{t}}^{(n)}\leq\frac{1}{l^*}-2\delta$:

By definition, we have for $t\in[\tilde{t},T]$ that $\hat{k}_t^{(n)}<\frac{1}{l^*}$, and therefore, we obtain together with (13):

$$\frac{d}{dt}\log(1-l^*\hat{k}_t^{(n)})=-F(t,\hat{k}_t^{(n)}),\qquad F(t,\hat{k}^{(n)}):=-\left(\phi(t,\hat{k}^{(n)})-\phi(t,\hat{k}_t^{(n-1)*})\right).$$

Integrating on both sides and using that $\hat{k}_s^{(n)}<\frac{1}{l^*}$ for $s\in[\tilde{t},T]$ leads to

$$\begin{aligned}
&\int_{\tilde{t}}^{T}\frac{d}{ds}\log(1-l^*\hat{k}_s^{(n)})\,ds=-\int_{\tilde{t}}^{T}F(s,\hat{k}_s^{(n)})\,ds\\
\Leftrightarrow\ &\log(1-l^*\hat{k}_T^{(n)})-\log(1-l^*\hat{k}_{\tilde{t}}^{(n)})=-\int_{\tilde{t}}^{T}F(s,\hat{k}_s^{(n)})\,ds\\
&\Leftrightarrow\ \log(1-l^*\hat{k}_{\tilde{t}}^{(n)})=\int_{\tilde{t}}^{T}F(s,\hat{k}_s^{(n)})\,ds.
\end{aligned}$$

Moreover, by Step 1, we know that $\hat{k}_t^{(n)} \in [0, \frac{1}{l^*}]$ for all $t \in [0,T]$. Since $F(s,k)$ is a continuous function in k, we have $|F(s, \hat{k}_s^{(n)})| \leq M$ for all $s \in [0,T]$, and we obtain

$$\begin{aligned} 1 - l^* \hat{k}_{\tilde{t}}^{(n)} &= \exp\left\{ \int_{\tilde{t}}^T F(s, \hat{k}_s^{(n)})\, ds \right\} \geq \exp\left\{ \int_{\tilde{t}}^T -M\, ds \right\} \\ &= \exp\{-M(T-\tilde{t})\} \geq e^{-MT} \end{aligned}$$

Thus, with $\tilde{\delta} := \frac{1}{2} e^{-MT}$, we have

$$1 - l^* \hat{k}_{\tilde{t}}^{(n)} \geq 2\tilde{\delta} \Leftrightarrow \hat{k}_{\tilde{t}}^{(n)} \leq \frac{1}{l^*} - 2\delta, \qquad \delta := \frac{\tilde{\delta}}{l^*} > 0 \tag{81}$$

(2) We show that $\tilde{t} = 0$ by contradiction:
Assume that $\tilde{t} > 0$, then inequality (81) implies

$$\hat{k}_{\tilde{t}}^{(n)} \leq \frac{1}{l^*} - 2\delta,$$

because $\hat{k}_s^{(n)} < \frac{1}{l^*}$ for $s \in [\tilde{t}, T]$. By continuity, there exists $t' < \tilde{t}$, such that $\hat{k}_{t'}^{(n)} \leq \frac{1}{l^*} - \delta$ which is a contradiction to the definition of $\tilde{t}$. Thus, $\tilde{t} = 0$ and therefore, $\hat{k}_t^{(n)} < \frac{1}{l^*}$ for all $t \in [0,T]$.

2.6.3. Proof of Proposition 2.2.6.

Here, we prove the assertion:
Let $u_t^{(n)} := h_t^{(n)} - h_t^{(n-1)}$, where $h^{(n)}$ is the solution of the corresponding equation (16). Then, $u_t^{(n)} \leq 0$ for all $t \in [0,T]$, $n \in \{2, 3, \ldots, N\}$.

Proof. We prove the assertion via induction. First, by definition, we have for arbitrary n:

$$h_t^{(n)} = u_t^{(n)} + h_t^{(n-1)}, \tag{82}$$

$$h_t^{(n)} - k_{T-t}^{(n-1)*} = h_t^{(n)} - h_t^{(n-1)} + h_t^{(n-1)} - k_{T-t}^{(n-1)*} = u_t^{(n)} + h_t^{(n-1)} - k_{T-t}^{(n-1)*}, \tag{83}$$

$$\begin{aligned} (h_t^{(n)})^2 - (k_{T-t}^{(n-1)*})^2 &= (h_t^{(n)})^2 - (h_t^{(n-1)})^2 + (h_t^{(n-1)})^2 - (k_{T-t}^{(n-1)*})^2, \\ &= u_t^{(n)}(h_t^{(n)} + h_t^{(n-1)}) + ((h_t^{(n-1)})^2 - (k_{T-t}^{(n-1)*})^2), \end{aligned} \tag{84}$$

and therefore, by (16) and (82)-(84) and with $\nu(T-t) := \mu + \rho\sigma_1\sigma_2\beta(T-t)$ it holds for $n \geq 2$:

$$\begin{aligned} \dot{u}_t^{(n)} &= f^{(n)}(t, h_t^{(n)}) - f^{(n-1)}(t, h_t^{(n-1)}) \\ &= -\frac{1 - l^*(u_t^{(n)} + h_t^{(n-1)})}{l^*} \\ &\quad \cdot \Bigg[\nu(T-t) \left(u_t^{(n)} + h_t^{(n-1)} - k_{T-t}^{(n-1)*} \right) \\ &\qquad - \frac{\sigma_1^2}{2}(1-\gamma) \left(u_t^{(n)} \left(h_t^{(n)} + h_t^{(n-1)} \right) + \left((h_t^{(n-1)})^2 - (k_{T-t}^{(n-1)*})^2 \right) \right) \Bigg] \\ &\quad + \frac{1 - l^* h_t^{(n-1)}}{l^*} \\ &\quad \cdot \left[\nu(T-t) \left(h_t^{(n-1)} - k_{T-t}^{(n-2)*} \right) - \frac{\sigma_1^2}{2}(1-\gamma) \left((h_t^{(n-1)})^2 - (k_{T-t}^{(n-2)*})^2 \right) \right] \\ &=: f_d^{(n)}(t, u_t^{(n)}), \\ u_0^{(n)} &= h_0^{(n)} - h_0^{(n-1)} = 0. \end{aligned}$$

First, we show that $u_t^{(2)} \le 0$ by showing that $E := (-\infty, 0]$ is positively invariant for

$$\dot{u}_t^{(2)} = f_d^{(2)}(t, u_t^{(2)}), \qquad u_0^{(2)} = 0. \tag{85}$$

Here, we have

$$\begin{aligned} f_d^{(2)}(t,0) =& -\frac{1-l^*h_t^{(1)}}{l^*} \cdot \left[\nu(T-t)\left(h_t^{(1)} - k_{T-t}^{(1)*}\right) - \frac{\sigma_1^2}{2}(1-\gamma)\left((h_t^{(1)})^2 - (k_{T-t}^{(1)*})^2\right)\right] \\ &+ \frac{1-l^*h_t^{(1)}}{l^*} \cdot \left[\nu(T-t)\left(h_t^{(1)} - k_{T-t}^{(0)*}\right) - \frac{\sigma_1^2}{2}(1-\gamma)\left((h_t^{(1)})^2 - (k_{T-t}^{(0)*})^2\right)\right] \\ =&\frac{1-l^*h_t^{(1)}}{l^*}\left[\sigma_1^2(1-\gamma)k_{T-t}^{(0)*} \cdot k_{T-t}^{(1)*} - \frac{\sigma_1^2}{2}(1-\gamma)(k_{T-t}^{(1)*})^2 - \frac{\sigma_1^2}{2}(1-\gamma)(k_{T-t}^{(0)*})^2\right] \\ =&\frac{1-l^*h_t^{(1)}}{l^*}\left[-\frac{\sigma_1^2}{2}(1-\gamma)\left(k_{T-t}^{(1)*} - k_{T-t}^{(0)*}\right)^2\right] \le 0. \end{aligned}$$

The last inequality holds because $h_t^{(1)} \in [0, \frac{1}{l^*})$ and $1-\gamma > 0$. We obtain that E is positively invariant for (85), because

$$f_d^{(2)}(t,0) \cdot y \le 0, \quad \forall t \in \mathbb{R}, y \in \mathcal{N}_E(0) = (0, \infty),$$

and it follows that $u_t^{(2)} = h_t^{(2)} - h_t^{(1)} \le 0$ for all $t \in [0,T]$, and the assertion holds for $n = 2$. Now, we assume that the assertion holds for $n-1$, that is $u_t^{(n-1)} = h_t^{(n-1)} - h_t^{(n-2)} \le 0$. Then, the aim is to show that E is positively invariant for

$$\dot{u}_t^{(n)} = f_d^{(n)}(t, u_t^{(n)}), \qquad u_0^{(n)} = 0.$$

It holds

$$\begin{aligned} f_d^{(n)}(t,0) =& -\frac{1-l^*h_t^{(n-1)}}{l^*} \\ &\cdot \left[\nu(T-t)\left(h_t^{(n-1)} - k_{T-t}^{(n-1)*}\right) - \frac{\sigma_1^2}{2}(1-\gamma)\left((h_t^{(n-1)})^2 - (k_{T-t}^{(n-1)*})^2\right)\right] \\ &+ \frac{1-l^*h_t^{(n-1)}}{l^*} \\ &\cdot \left[\nu(T-t)\left(h_t^{(n-1)} - k_{T-t}^{(n-2)*}\right) - \frac{\sigma_1^2}{2}(1-\gamma)\left((h_t^{(n-1)})^2 - (k_{T-t}^{(n-2)*})^2\right)\right] \\ =&\frac{1-l^*h_t^{(n-1)}}{l^*} \\ &\cdot \left[\nu(T-t)\left(k_{T-t}^{(n-1)*} - k_{T-t}^{(n-2)*}\right) - \frac{\sigma_1^2}{2}(1-\gamma)\left((k_{T-t}^{(n-1)*})^2 - (k_{T-t}^{(n-2)*})^2\right)\right] \\ =&\frac{1-l^*h_t^{(n-1)}}{l^*}\left(k_{T-t}^{(n-1)*} - k_{T-t}^{(n-2)*}\right) \\ &\cdot \left[\nu(T-t) - \frac{\sigma_1^2}{2}(1-\gamma)\left(k_{T-t}^{(n-1)*} + k_{T-t}^{(n-2)*}\right)\right]. \end{aligned}$$

By definition and by the assumption $h_t^{(n-1)} \le h_t^{(n-2)}$, we conclude that

$$k_{T-t}^{(n-1)*} = h_t^{(n-1)} \wedge k_{T-t}^{(0)*}, \qquad k_{T-t}^{(n-2)*} = h_t^{(n-2)} \wedge k_{T-t}^{(0)*}$$

and hence that

$$k_{T-t}^{(n-1)*} \leq k_{T-t}^{(n-2)*}. \tag{86}$$

Moreover, by definition, it holds

$$k_{T-t}^{(n-1)*} \leq k_{T-t}^{(0)*}, \quad k_{T-t}^{(n-2)*} \leq k_{T-t}^{(0)*}.$$

Thus,

$$\nu(T-t) - \frac{\sigma_1^2}{2}(1-\gamma)\left(k_{T-t}^{(n-1)*} + k_{T-t}^{(n-2)*}\right) \geq \nu(T-t) - \frac{\sigma_1^2}{2}(1-\gamma)2 \cdot k_{T-t}^{(0)*} = 0, \tag{87}$$

and by $h_t^{(n-1)} < \frac{1}{l^*}$, by (86) and (87), we obtain the following inequality:

$$\begin{aligned} f_d^{(n)}(t,0) =& \frac{1 - l^* h_t^{(n-1)}}{l^*}\left(k_{T-t}^{(n-1)*} - k_{T-t}^{(n-2)*}\right) \\ & \cdot \left[\nu(T-t) - \frac{\sigma_1^2}{2}(1-\gamma)\left(k_{T-t}^{(n-1)*} + k_{T-t}^{(n-2)*}\right)\right] \leq 0. \end{aligned}$$

Finally, we conclude that

$$f_d^{(n)}(t,0) \cdot y \leq 0, \quad \forall t \in \mathbb{R}, y \in \mathcal{N}_E(0)$$

and it follows that $u_t^{(n)} = h_t^{(n)} - h_t^{(n-1)} \leq 0$ for all $t \in [0,T]$. □

2.6.4. Proof of Proposition 2.2.7.

Here we prove the assertion:
Let $\gamma\rho \geq 0$. Then $u_t^{(1)} := h_t^{(1)} - h_t^{(0)} \leq 0$ for all $t \in [0,T]$, where $h_t^{(0)} := k_{T-t}^{(0)*}$ and $h_t^{(1)}$ is the uniquely determined solution of (16) with $n = 1$.

Proof. By definition we have

$$\begin{aligned} \dot{h}_t^{(0)} &= \frac{\rho\sigma_2}{\sigma_1(1-\gamma)}\gamma\exp(-at), \qquad h_0^{(0)} = k_T^{(0)*} = \frac{\mu}{\sigma_1^2(1-\gamma)} > 0, \\ \dot{h}_t^{(1)} &= \frac{1 - l^* h_t^{(1)}}{l^*} \cdot \frac{\sigma_1^2}{2}(1-\gamma)\left(h_t^{(1)} - k_{T-t}^{(0)*}\right)^2, \qquad h_0^{(1)} = 0. \end{aligned}$$

With $u_t^{(1)} := h_t^{(1)} - h_t^{(0)}$, we have $u_0^{(1)} = -\frac{\mu}{\sigma_1^2(1-\gamma)}$ and

$$\begin{aligned} \dot{u}_t^{(1)} &= \dot{h}_t^{(1)} - \dot{h}_t^{(0)} \\ &= \frac{1 - l^*\left(u_t^{(1)} + h_t^{(0)}\right)}{l^*} \cdot \frac{\sigma_1^2}{2}(1-\gamma)\left(u_t^{(1)}\right)^2 - \frac{\rho\sigma_2}{\sigma_1(1-\gamma)}\gamma\exp(-at) \\ &=: f_d^{(1)}(t, u_t^{(1)}) \end{aligned} \tag{88}$$

In order to show that $u_t^{(1)} \leq 0$, we use again the invariance argument. Let $E := (-\infty, 0]$. Then, E is positively invariant for (88), because $\gamma\rho \geq 0$ implies

$$f_d^{(1)}(t,0) \cdot y = -\frac{\sigma_2}{\sigma_1(1-\gamma)}\exp(-at)\underbrace{\gamma \cdot \rho}_{\geq 0} \cdot y \leq 0, \quad \forall t \in \mathbb{R}, \forall y \in \mathcal{N}_E(0) = (0,\infty).$$

By Theorem A.2.3, we have that E is positively invariant for (88), that means $u_t^{(1)} \leq 0$ for all $t \in [0,\infty)$. □

2.6.5. Verification argument for the post-crash problem.

Here we apply the Verification Theorem A.5.2, which was formulated in [24, Corollary 3.2] for a more general stochastic optimal control problem. First, we have to prove several conditions for the stochastic optimal control problem (17) on the relevant interval $[t_0, T]$ with $0 \leq t_0 < T$.

i) $k^{(0)*}$ is progressively measurable.

ii) for all $n \in \mathbb{N}$ the integrability condition
$$\mathbb{E}\left(\int_{t_0}^{T} |k_t^{(0)*}|^n \, dt\right) < \infty$$
is satisfied.

iii) The corresponding state process $\{\overline{X}_t^*\}_{t\in[t_0,T]}$, controlled by $k^{(0)*}$, satisfies
$$\mathbb{E}^{t_0,x_0}\left(\sup_{t\in[t_0,T]} |\overline{X}_t^*|^n\right) < \infty.$$

iv) $v^0(t,x,r) = \frac{1}{\gamma}x^\gamma g^{(0)}(t)\exp(\beta(t)r)$ is an element of $C^{1,2,2}([t_0,T]\times\mathbb{R}_+\times\mathbb{R})$ and it is a solution of the HJB equation.

v) For all $(t,x) \in [t_0,T) \times \mathbb{R}$ and for all $k^{(0)} \in \Pi(t,x,r)$, there exists a $q > 1$, such that
$$\mathbb{E}\left(\sup_{s\in[t,T]} \left|v^0\left(s, \overline{X}_s, \overline{r}_s\right)\right|^q\right) < \infty.$$

vi) $k_s^{(0)*} \in \arg\max_{k\in A} \mathcal{L}^k v^0(s, \overline{X}_s^*, \overline{r}_s)$ for all $s \in [t_0,T]$.

Remark 2.6.1. *As in the model of [24], we can treat the post-crash problem as if the state process consists only of $\overline{X}_t$, because $\overline{r}_t$, as a solution of* (2)*, has a uniquely determined solution and it holds $\mathbb{E}\left(\max_{t_0\leq s\leq T}|\overline{r}_s|^q\right) < \infty$ for $q \in \mathbb{N}$ (see for example [32, Chp. 5.2, Corollary 12]).*

All the steps work in the same way as for the model of [24], but for the sake of completeness we show it here.

Proof of i): $k_t^{(0)*} = \frac{\mu}{(1-\gamma)\sigma_1^2} + \frac{\rho\sigma_2\beta(t)}{\sigma_1(1-\gamma)}$ is a deterministic and continuous function, and therefore it is progressively measurable.

Proof of ii): Due to that fact that $k_t^{(0)*}$ is bounded on $[0,T]$, we immediately obtain the integrability condition for all $n \in \mathbb{N}$.

Proof of iii): Here, we consider the wealth equation
$$d\overline{X}_t^* = \overline{X}_t^*\left[\overline{r}_t + \mu\, k_t^{(0)*}\right] dt + \overline{X}_t\sigma_1 k_t^{(0)*}\, dw_{1,t}, \qquad \overline{X}_{t_0}^* = x.$$
By Corollary A.5.1, there exists a uniquely determined solution given by
$$\overline{X}_t^* = x_0 \cdot \exp\left(\int_{t_0}^{t} \overline{r}_s + \mu k_s^{(0)*} - \frac{\sigma_1^2}{2}(k_s^{(0)*})^2\, ds + \int_{t_0}^{t} \sigma_1 k_s^{(0)*}\, dw_{1,s}\right),$$
because $k_t^{(0)*}$ is bounded on $[0,T]$ and $\int_{t_0}^{t} |\overline{r}_s|\, ds < +\infty$, $\mathbb{P}$- a.s., for all $t \in [t_0,T]$. Obviously, $\overline{X}_t^* > 0$ $\mathbb{P}$- a.s for $t \in [t_0,T]$. Now, let $n \in \mathbb{N}$ be arbitrary but fixed. Then, we have the following

estimate with a universal constant $K > 0$:

$$\begin{aligned}\overline{X}_t^{*n} &= x_0^n \cdot \exp\left(n\int_{t_0}^t \overline{r}_s + \mu k_s^{(0)*} - \frac{\sigma_1^2}{2}(k_s^{(0)*})^2\, ds + n\int_{t_0}^t \sigma_1 k_s^{(0)*}\, dw_{1,s}\right)\\ &\le K\cdot \exp\left(n\int_{t_0}^t \overline{r}_s\, ds + n\int_{t_0}^t \sigma_1 k_s^{(0)*}\, dw_{1,s}\right)\\ &\le K\exp\left(2n\int_{t_0}^t \overline{r}_s\, ds\right) + K\exp\left(2n\int_{t_0}^t \sigma_1 k_s^{(0)*}\, dw_{1,s}\right).\end{aligned}$$

By Proposition A.1.1 in Appendix A, it holds

$$\int_{t_0}^t \overline{r}_s\, ds = \frac{r_0}{a}(1-e^{-a(t-t_0)}) + r_M\left((t-t_0) - \frac{1-e^{-a(t-t_0)}}{a}\right) + \sigma_2\int_{t_0}^t \frac{1-e^{-a(t-s)}}{a}\, d\tilde{w}_s.$$

Thus,

$$\overline{X}_t^{*n} \le K\exp\left(\int_{t_0}^t h_1(s)\, d\tilde{w}_s\right) + K\exp\left(\int_{t_0}^t h_2(s)\, dw_{1,s}\right),$$

where $h_1(s) := 2n\frac{\sigma_2}{a}\left(1-e^{a(t-s)}\right)$ and $h_2(s) := 2n\sigma_1 k_s^{(0)*}$ are deterministic and bounded functions on $[0,T]$. With

$$\exp\left(\int_{t_0}^t h_1(s)\, d\tilde{w}_s\right) = \exp\left(\int_{t_0}^t \frac{1}{2}h_1^2(s)\, ds\right)\cdot\exp\left(-\int_{t_0}^t \frac{1}{2}h_1^2(s)\, ds + \int_{t_0}^t h_1(s)\, d\tilde{w}_s\right)$$

and with

$$Z_{1,t} := \exp\left(-\int_{t_0}^t \frac{1}{2}h_1^2(s)\, ds + \int_{t_0}^t h_1(s)\, d\tilde{w}_s\right),$$

which is a solution of

$$dZ_{1,t} = Z_{1,t}h_1(t)\, d\tilde{w}_t, \qquad Z_{1,t_0} = 1,$$

we have by [32, Chp. 5.2, Cor. 12]:

$$\mathbb{E}\left(\max_{t\in[t_0,T]} Z_{1,t}\right) < \infty.$$

Analogously, with

$$Z_{2,t} := \exp\left(-\int_{t_0}^t \frac{1}{2}h_2^2(s)\, ds + \int_{t_0}^t h_2(s)\, dw_{1,s}\right),$$

we obtain $\mathbb{E}\left(\max_{t\in[t_0,T]} Z_{2,t}\right) < \infty$. Now, we can conclude that

$$\begin{aligned}\overline{X}_t^{*n} &\le K\exp\left(\int_{t_0}^t \frac{1}{2}h_1^2(s)\, ds\right)\cdot Z_{1,t} + K\exp\left(\int_{t_0}^t \frac{1}{2}h_2^2(s)\, ds\right)\cdot Z_{2,t}\\ &\le K\cdot\max_{t\in[t_0,T]}(Z_{1,t}) + K\max_{t\in[t_0,T]}(Z_{2,t}),\end{aligned}$$

because $h_1(t)$ and $h_2(t)$ are deterministic and bounded functions. Taking the supremum on the left hand side and the expectations on both sides leads to

$$\mathbb{E}^{t_0,x_0}\left(\sup_{t\in[t_0,T]} \overline{X}_t^{*n}\right) \le K\mathbb{E}^{t_0,x_0}\left(\sup_{t\in[t_0,T]}(Z_{1,t})\right) + K\mathbb{E}^{t_0,x_0}\left(\sup_{t\in[t_0,T]}(Z_{2,t})\right) < +\infty,$$

and iii) follows.

Proof of iv): We have already shown that $v^0(t,x,r) = \frac{1}{\gamma}x^\gamma g^{(0)}(t)\exp(\beta(t)r)$ solves the HJB equation (18).

Proof of v): The candidate of the value function is given by

$$v^0(t,x,r) = \frac{1}{\gamma}x^\gamma g^{(0)}(t)\exp(\beta(t)r),$$

where $g^{(0)}(t) = \exp(\int_t^T \alpha^{(0)}(s)\,ds)$ and $\beta(t) = \frac{\gamma}{a}\left[1-\exp(-a(T-t))\right]$ are deterministic and bounded functions. Let $k^{(0)} \in \Pi(t,x,r)$ be arbitrary but fixed. Then, we have that $|k_s^{(0)}| \leq K$ for some $K < \infty$. Let $(t',x',r') \in [t_0,T]\times\mathbb{R}_+\times\mathbb{R}$ be arbitrary but fixed. Then, we have with constants $K_i > 0$:

$$\begin{aligned}
\left|v^0(t,\overline{X}_t,\overline{r}_t)\right| &= |\gamma^{-1}|\cdot\overline{X}_t^\gamma\exp\left(\int_t^T \alpha^{(0)}(s)\,ds + \frac{\gamma}{a}\left[1-\exp(-a(T-t))\right]\overline{r}_t\right)\\
&\leq K_1\cdot\exp\left(\gamma\int_{t'}^t\left[\overline{r}_s + \mu k_s^{(0)} - \frac{\sigma_1^2}{2}(k_s^{(0)})^2\right]ds + \gamma\int_{t'}^t \sigma_1 k_s^{(0)}\,dw_{1,s}\right)\\
&\quad\cdot\exp\left(\frac{\gamma}{a}\left[1-\exp(-a(T-t))\right]\overline{r}_t\right)\\
&\leq K_2\cdot\exp\left(\gamma\int_{t'}^t \overline{r}_s\,ds + \gamma\int_{t'}^t\sigma_1 k_s^{(0)}\,dw_{1,s}\right)\cdot\exp\left(\frac{\gamma}{a}\overline{r}_t\right)\\
&\quad\cdot\exp\left(-\frac{\gamma}{a}\exp(-a(T-t))\overline{r}_t\right).
\end{aligned}$$

Again, by Proposition A.1.1, we obtain

$$\begin{aligned}
\exp(-a(T-t))\cdot\overline{r}_t =& \exp\left(-a(T-t')\right)r' + \int_{t'}^t a r_M\exp\left(-a(T-s)\right)\,ds\\
&+ \int_{t'}^t \sigma_2\exp\left(-a(T-s)\right)\,d\tilde{w}_s.
\end{aligned}$$

Now, we obtain

$$\begin{aligned}
&\left|v^0(t,\overline{X}_t,\overline{r}_t)\right|\\
&\leq K_3\cdot\exp\left(\gamma\int_{t'}^t\overline{r}_s\,ds + \gamma\int_{t'}^t\sigma_1 k_s^{(0)}\,dw_{1,s} + \frac{\gamma}{a}\left(r' + \int_{t'}^t a(r_M-\overline{r}_s)\,ds + \int_{t'}^t\sigma_2\,d\tilde{w}_s\right)\right)\\
&\qquad\cdot\exp\left(\int_{t'}^t -\frac{\gamma}{a}\sigma_2\exp\left(-a(T-s)\right)\,d\tilde{w}_s\right)\\
&\leq K_4\cdot\exp\left(\gamma\int_{t'}^t\sigma_1 k_s^{(0)}\,dw_{1,s} + \frac{\gamma}{a}\int_{t'}^t\sigma_2\,d\tilde{w}_s\right)\cdot\exp\left(\int_{t'}^t -\frac{\gamma}{a}\sigma_2\exp\left(-a(T-s)\right)\,d\tilde{w}_s\right)\\
&= K_4\cdot\exp\left(\int_{t'}^t \underbrace{\gamma\sigma_1 k_s^{(0)}}_{:=h_3(s)}\,dw_{1,s}\right)\cdot\exp\left(\int_{t'}^t\underbrace{\frac{\gamma}{a}\sigma_2\left(1-\exp\left(-a(T-s)\right)\right)}_{:=h_4(s)}\,d\tilde{w}_s\right)\\
&\leq K_5\cdot Z_{3,t}
\end{aligned}$$

where

$$Z_{3,t} := \exp\Bigg(\int_{t'}^{t} h_3(s) + \rho h_4(s)\, dw_{1,s} + \int_{t'}^{t} \sqrt{1-\rho^2} h_4(s)\, dw_{2,s} - \frac{1}{2}\int_{t'}^{t} \Big[(h_3(s) + \rho h_4(s))^2 + (\sqrt{1-\rho^2} h_4(s))^2 \Big]\, ds \Bigg).$$

Z_3 is a solution of the SDE

$$dZ_{3,t} = Z_{3,t}(h_3(t) + \rho h_4(t))\, dw_{1,t} + Z_{3,t}\sqrt{1-\rho^2} h_4(t)\, dw_{2,t}, \qquad Z_{3,t'} = 1,$$

and now, by [32, Chp.5.2,Cor.12], we conclude that

$$\mathbb{E}\left(\sup_{t\in[t',T]} |Z_{3,t}|^2 \right) < \infty.$$

With

$$\left| v^0(t, \overline{X}_t, \overline{r}_t) \right|^2 \le K_5^2 \cdot Z_{3,t}^2 \le K_5^2 \sup_{t\in[t',T]} Z_{3,t}^2, \qquad \forall t \in [t', T]$$

we have

$$\mathbb{E}\left(\sup_{t\in[t',T]} \left| v^0(t, \overline{X}_t, \overline{r}_t) \right|^2 \right) \le K_5^2 \mathbb{E}\left(\sup_{t\in[t',T]} Z_{3,t}^2 \right) < \infty,$$

and therefore v) follows for $q = 2$.

Proof of vi):
This condition is fulfilled (see page 16).
Obviously, the function $U(x) = \frac{1}{\gamma}x^\gamma$ with $\gamma < 1, \gamma \neq 0$, does not fulfill the growth condition (158), which is required for applying Theorem A.5.2. Nevertheless, we can replace this condition if we can show that the functional

$$J(t,x,r;k) := \mathbb{E}^{t,x,r}\left(\frac{1}{\gamma} \overline{X}_T^\gamma \right)$$

is well-defined for all admissible controls $k \in \Pi(t,x,r)$. For further details about models in which the utility function does not satisfy the growth conditions, we refer to Kraft [31, p.18]. In our case, we can show that $J(t,x,r;k)$ is well-defined using the explicit expression of the solution of the wealth equation and the short rate equation and the conditions of an admissible control strategy (see Definition 2.1.1).
By proving the conditions i)- vi), we now apply Theorem A.5.2. Note that our control problem does not include a running utility such that the function L in Theorem A.5.2 is equal to zero. Moreover, in our case we have that $Q \triangleq [t_0, T) \times \mathcal{O}$ and $\mathcal{O} \triangleq (0,\infty) \times \mathbb{R}$. Since $\overline{X}_t > 0$ $\mathbb{P}$-a.s. for every $t \in [0,T]$ and $k \in \Pi(t,x,r)$, the process $(\overline{X}_t, \overline{r}_t)$ never leaves the set $\mathcal{O}$ and therefore $\eta \triangleq T$, where η is defined in (156). Now, we obtain that $v^0(t,x,r)$ is indeed equal to the post-crash value function and $k_t^{(0)*}$ is the optimal post-crash strategy.

2.6.6. Heuristic characterization of $v^n(t,x,r)$- The non-log utility case. Here, we want to characterize a candidate for the value function $V^n \in C^{1,2,2}$ which solves the system:

$$0 \leq \sup_{k\in\mathcal{A}''_n(t,x,r)} \left[\mathcal{L}^k v^n(t,x,r)\right], \tag{89}$$

$$0 \leq \sup_{k\in\mathcal{A}'_n(t,x,r)} \left[v^{n-1}\left(t,x(1-l^*k^+),r\right) - v^n(t,x,r)\right], \tag{90}$$

$$0 = \sup_{k\in\mathcal{A}''_n(t,x,r)} \left[\mathcal{L}^k v^n(t,x,r)\right] \sup_{k\in\mathcal{A}'_n(t,x,r)} \left[v^{n-1}\left(t,x(1-l^*k^+),r\right) - v^n(t,x,r)\right], \tag{91}$$

$$v^n(T,x,r) = \frac{1}{\gamma}x^\gamma. \tag{92}$$

We can do this using analogous arguments as in [28, Section 4]. Let $(t,x,r) \in [0,T]\times\mathbb{R}_+\times\mathbb{R}$ and $n \in \{1,\dots,N\}$ be arbitrary but fixed. Let us start with the inequality (90). In contrast to [28, Section 4], this inequality has the positive part k^+ instead of k. Assuming that the candidate v^n is strictly monotone increasing in x we easily have, for arbitrary but fixed (t,x,r), that $v^{n-1}\left(t,x(1-l^*k^+),r\right) - v^n(t,x,r)$ is constant for $k \leq 0$ and strictly monotone decreasing in k for $k \geq 0$. Therefore, the supremum in (90) is taken for the smallest k for which $k \in \mathcal{A}'_n(t,x,r)$, that is, for which the inequality

$$\begin{aligned} &v^n_t(t,x,r) + a(r_M - r)v^n_r(t,x,r) + \frac{\sigma_2^2}{2}v^n_{rr}(t,x,r) \\ &\quad \geq -x(\mu k + r)v^n_x(t,x,r) - \frac{\sigma_1^2}{2}k^2x^2v^n_{xx}(t,x,r) - \rho\sigma_1\sigma_2 kxv^n_{xr}(t,x,r) \end{aligned} \tag{93}$$

holds. With a second assumption that $v^n(t,x,r)$ is concave in x, we have that $v^n_{xx}(t,x,r) < 0$. Thus, the right hand side of the inequality, as function of k, is a parabola opens upward. Note that if the inequality holds for $k \in [k',0]$ with $k' \leq 0$, then the supremum is attained for all $k \in [k',0]$ due to the positive part. Especially, the supremum in (90) is attained for the smallest value k for which (93) holds as an equality. Analogously to the literature, we can separate the (t,x,r) space into the set $\mathcal{Y}^{(n)}$, where the right hand side of inequality (90) is strictly positive, and its complement. That means

$$\mathcal{Y}^{(n)} := \left\{(t,x,r) : \sup_{k\in\mathcal{A}'_n(t,x,r)} \left[v^{n-1}\left(t,x(1-l^*k^+),r\right) - v^n(t,x,r)\right] > 0\right\}.$$

For $(t,x,r) \notin \mathcal{Y}^{(n)}$, k and v^n are determined by the following equalities:

$$\begin{aligned} &v^{n-1}(t,x(1-l^*k^+),r) = v^n(t,x,r), \\ &v^n_t(t,x,r) + a(r_M - r)v^n_r(t,x,r) + \frac{\sigma_2^2}{2}v^n_{rr}(t,x,r) \\ &\quad = -x(\mu k + r)v^n_x(t,x,r) - \frac{\sigma_1^2}{2}k^2x^2v^n_{xx}(t,x,r) - \rho\sigma_1\sigma_2 kxv^n_{xr}(t,x,r). \end{aligned} \tag{94}$$

By the complementarity condition (91), we have to require for $(t,x,r) \in \mathcal{Y}^{(n)}$, that

$$\sup_{k\in\mathcal{A}''_n(t,x,r)} \left[\mathcal{L}^k v^n(t,x,r)\right] = 0. \tag{95}$$

Ignoring the condition $k \in \mathcal{A}''_n(t,x,r)$ and using the first order optimality condition, we obtain a candidate

$$k = -\frac{v^n_x(t,x,r)\mu + \rho\sigma_1\sigma_2 v^n_{xr}(t,x,r)}{\sigma_1^2 x v^n_{xx}(t,x,r)}. \tag{96}$$

If k, given by (96), fulfills the condition $k \in \mathcal{A}''_n(t,x,r)$, that means, if

$$v^n(t,x,r) \leq v^{n-1}(t,x(1-l^*k^+),r),$$

then the supremum in (95) is attained in (96). Otherwise, since $\mathcal{L}^k v^n(t,x,r)$ is monotone increasing in k for

$$k < -\frac{\mu v^n_x(t,x,r) + \rho\sigma_1\sigma_2 v^n_{xr}(t,x,r)}{\sigma_1^2 x v^n_{xx}(t,x,r)},$$

and since $v^{n-1}(t,x(1-l^*k^+),r)$ is monotone decreasing in k, it follows that the supremum in (95) is attained for k, for which

$$v^n(t,x,r) = v^{n-1}(t,x(1-l^*k^+),r).$$

In this case k and v^n are determined by the following equalities:

$$\begin{aligned} v^n(t,x,r) &= v^{n-1}(t,x(1-l^*k^+),r), \\ \mathcal{L}^k v^n(t,x,r) &= 0. \end{aligned}$$

Summarizing, for $(t,x,r) \in \mathcal{Y}^{(n)}$, k and v^n are either determined by

$$\begin{aligned} k &= -\frac{\mu v^n_x(t,x,r) + \rho\sigma_1\sigma_2 v^n_{xr}(t,x,r)}{\sigma_1^2 x v^n_{xx}(t,x,r)}, \\ 0 &= \mathcal{L}^k v^n(t,x,r), \end{aligned} \tag{97}$$

or by

$$\begin{aligned} v^n(t,x,r) &= v^{n-1}(t,x(1-l^*k^+),r), \\ 0 &= \mathcal{L}^k v^n(t,x,r). \end{aligned} \tag{98}$$

But, the last equalities also determine k and v^n for $(t,x,r) \notin \mathcal{Y}^{(n)}$ (see (94)). As in the literature [28], we can now separate the (t,x,r) space into the set

$$\mathcal{Z}^{(n)} = \left\{(t,x,r) : k(t,x,r) = -\frac{\mu v^n_x(t,x,r) + \rho\sigma_1\sigma_2 v^n_{xr}(t,x,r)}{\sigma_1^2 x v^n_{xx}(t,x,r)}, 0 = \mathcal{L}^k v^n(t,x,r)\right\}$$

and its complement, where k and v^n are determined by

$$\begin{aligned} v^n(t,x,r) &= v^{n-1}(t,x(1-l^*k^+),r), \\ 0 &= \mathcal{L}^k v^n(t,x,r). \end{aligned}$$

By the proof of part a) in Theorem 2.2.2, we know that $v^0(t,x,r)$ given by (12) and $k^{(0)}$ given by (11) is the usual solution of the HJB equation

$$0 = \mathcal{L}^{k^{(0)}} v^0(t,x,r), \quad v^0(T,x,r) = \frac{1}{\gamma}x^\gamma, \quad (t,x,r) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R},$$

$$k^{(0)} = \arg\sup_{k\in A}\left[\mathcal{L}^k v^0(t,x,r)\right] = -\frac{\mu v^0_x(t,x,r) + \rho\sigma_1\sigma_2 v^0_{xr}(t,x,r)}{\sigma_1^2 x v^0_{xx}(t,x,r)}.$$

Thus, for $n = 0$ we have that $\mathcal{Z}^{(0)}$ is the whole (t, x, r) space. For $n > 0$, we demonstrate how we derived $v^n(t, x, r)$ heuristically and afterwards we obtain that v^n, given by (14), is a solution of the system of inequalities (89)-(91). Analogously to the case of $n = 0$, we assume that the solution takes the form

$$v^n(t, x, r) = \frac{1}{\gamma} x^\gamma g^{(n)}(t) \exp(\beta(t) r),$$

with $g^{(n)}(T) = 1$ and $\beta(T) = 0$. By inserting $v^n(t, x, r)$ and its derivatives in (97) for $(t, x, r) \in \mathcal{Z}^{(n)}$ and in (98) for $(t, x, r) \notin \mathcal{Z}^{(n)}$, and by assuming that $k^{(n)*}(t, x, r) \geq 0$ for $(t, x, r) \notin \mathcal{Z}^{(n)}$ (this will be shown later), we obtain:

$$k_t^{(n)*} = \begin{cases} \frac{\mu + \rho\sigma_1\sigma_2\beta(t)}{\sigma_1^2(1-\gamma)} & : (t, x, r) \in \mathcal{Z}^{(n)}, \\ \hat{k}_t^{(n)} := \frac{1}{l^*}\left[1 - \left(\frac{g^{(n)}(t)}{g^{(n-1)}(t)}\right)^{\frac{1}{\gamma}}\right] & : (t, x, r) \notin \mathcal{Z}^{(n)}. \end{cases} \tag{99}$$

Moreover, for both cases, we have to require that $0 = \mathcal{L}^{k_t^{(n)*}} v^n(t, x, r)$. This is equivalent to

$$v_t^n(t, x, r) + x(\mu k_t^{(n)*} + r) v_x^n(t, x, r) + \frac{\sigma_1^2}{2}(k_t^{(n)*})^2 x^2 v_{xx}^n(t, x, r)$$
$$+ \rho\sigma_1\sigma_2 k_t^{(n)*} x v_{xr}^n(t, x, r) + a(r_M - r) v_r^n(t, x, r) + \frac{\sigma_2^2}{2} v_{rr}^n(t, x, r) = 0.$$

Now, by inserting $v^n(t, x, r)$ and its derivatives and by dividing by $\frac{1}{\gamma} x^\gamma \exp(\beta(t) r) \neq 0$, we obtain

$$\dot{g}^{(n)}(t) + g^{(n)}(t)\left(\gamma(\mu + \rho\sigma_1\sigma_2\beta(t))k_t^{(n)*} - \frac{\sigma_1^2}{2}\gamma(1-\gamma)(k_t^{(n)*})^2 + a r_M \beta(t) + \frac{\sigma_2^2}{2}\beta^2(t)\right)$$
$$+ g^{(n)}(t) r\left(\dot{\beta}(t) - a\beta(t) + \gamma\right) = 0, \qquad \beta(T) = 0, g^{(n)}(T) = 1.$$

In order to eliminate r, we choose $\beta(t)$ as given in (11) and therefore $\dot{\beta}(t) - a\beta(t) + \gamma = 0, \beta(T) = 0$ and the equation above reduces to the ordinary differential equation for $g^{(n)}(t)$ given in (15). Thus, we obtain that $0 = \mathcal{L}^{k_t^{(n)*}} v^n(t, x, r)$. Now, for $(t, x, r) \notin \mathcal{Z}^{(n)}$, we have that

$$k^{(n)*}(t, x, r) = \hat{k}_t^{(n)} := \frac{1}{l^*}\left[1 - \left(\frac{g^{(n)}(t)}{g^{(n-1)}(t)}\right)^{\frac{1}{\gamma}}\right], \tag{100}$$

where $g^{(n)}(t)$ fulfills (15). Using (15) and (100), we have for $\hat{k}_t^{(n)}$ for $(t, x, r) \notin \mathcal{Z}^{(n)}$:

$$\begin{aligned} \dot{\hat{k}}_t^{(n)} &= -\frac{1}{l^*}\frac{1}{\gamma}\left(\frac{g^{(n)}(t)}{g^{(n-1)}(t)}\right)^{\frac{1}{\gamma}-1}\left[\frac{\dot{g}^{(n)}(t) g^{(n-1)}(t) - g^{(n)}(t)\dot{g}^{(n-1)}(t)}{g^{(n-1)}(t)^2}\right] \\ &= -\frac{1}{l^*}\left(1 - l^*\hat{k}_t^{(n)}\right) \cdot \underbrace{\left(\frac{1}{\gamma}\frac{g^{(n-1)}(t)}{g^{(n)}(t)} \cdot \left[\frac{\dot{g}^{(n)}(t) g^{(n-1)}(t) - g^{(n)}(t)\dot{g}^{(n-1)}(t)}{g^{(n-1)}(t)^2}\right]\right)}_{:=(A)}, \end{aligned}$$

where

$$(A) = -(\mu + \rho\sigma_1\sigma_2\beta(t))\hat{k}_t^{(n)} + \frac{\sigma_1^2}{2}(1-\gamma)(\hat{k}_t^{(n)*})^2 - \frac{1}{\gamma} a r_M \beta(t) - \frac{1}{\gamma}\frac{\sigma_2^2}{2}\beta^2(t)$$
$$+ (\mu + \rho\sigma_1\sigma_2\beta(t))k_t^{(n-1)*} - \frac{\sigma_1^2}{2}(1-\gamma)(k_t^{(n-1)*})^2 + \frac{1}{\gamma} a r_M \beta(t) + \frac{1}{\gamma}\frac{\sigma_2^2}{2}\beta^2(t).$$

Thus, $\hat{k}_t^{(n)}$ has to fulfill the equation:

$$\dot{\hat{k}}_t^{(n)} = \frac{1 - l^* \hat{k}_t^{(n)}}{l^*} \left(\phi(t, \hat{k}_t^{(n)}) - \phi(t, k_t^{(n-1)*}) \right), \qquad \hat{k}_T^{(n)} = 0.$$

Note that the terminal condition $\hat{k}_T^{(n)} = 0$ follows by $g^{(n)}(T) = 1$ and therefore

$$\hat{k}_T^{(n)} = \frac{1}{l^*} \left[1 - \left(\frac{g^{(n)}(T)}{g^{(n-1)}(T)} \right)^{\frac{1}{\gamma}} \right] = 0$$

By definition of $k_t^{(0)*}$ in (11) and by (99) we finally obtain that

$$k_t^{(n)*} = \begin{cases} k_t^{(0)*} & : (t, x, r) \in \mathcal{Z}^{(n)} \\ \hat{k}_t^{(n)} & : (t, x, r) \notin \mathcal{Z}^{(n)} \end{cases}.$$

In order to show that $v^n(t, x, r) \in C^{1,2,2}$, given by (14), is a solution of the system (22)-(24), it remains to show that $k_t^{(n)*}$ is indeed equal to

$$\hat{k}_t^{(n)} \wedge k_t^{(0)*}.$$

We can see this by the following arguments. Let $n \in \{1, \dots, N\}$ be fixed. First, one can easily show that $\hat{k}_t^{(n)}$ is strictly monotone decreasing with $\hat{k}_T^{(n)} = 0$. Now, we show that

$$\mathcal{Z}^{(n)} = \{(t, x, r) : \hat{k}_t^{(n)} \geq k_t^{(0)*}\}.$$

Let $(t', x', r') \in \{(t, x, r) : \hat{k}_t^{(n)} \geq k_t^{(0)*}\}$. Then, we have by the fact that $\hat{k}_{t'}^{(n)} \geq 0$ (see Proposition 2.2.4), by the fact that $v^n(t', x'(1 - l^* k), r')$ is monotone decreasing in k:

$$\begin{aligned}
&\hat{k}_{t'}^{(n)} \geq k_{t'}^{(0)*} \\
&\Rightarrow (\hat{k}_{t'}^{(n)})^+ \geq (k_{t'}^{(0)*})^+ \\
&\Rightarrow v^{n-1}(t', x'(1 - l^*(k_{t'}^{(0)*})^+, r') \geq v^{n-1}(t', x'(1 - l^*(\hat{k}_{t'}^{(n)})^+, r') = v^n(t', x', r') \\
&\Rightarrow k_{t'}^{(0)*} \in \mathcal{A}_n''(t', x', r').
\end{aligned}$$

Since

$$k_{t'}^{(0)*} = \frac{\mu + \rho\sigma_1\sigma_2\beta(t')}{\sigma_1^2(1-\gamma)} = -\frac{\mu v_x^n(t', x', r') + \rho\sigma_1\sigma_2 v_{xr}^n(t', x', r')}{\sigma_1^2 x' v_{xx}^n(t', x', r')}$$

and $\mathcal{L}^{k^{(0)*}} v^n(t', x', r') = 0$, it follows that $(t', x', r') \in \mathcal{Z}^{(n)}$.
Now, let $(t', x', r') \in \mathcal{Z}^{(n)}$. Then, by definition, it holds

$$k(t', x', r') = -\frac{v_x^n(t', x', r')\mu + \rho\sigma_1\sigma_2 v_{xr}^n(t', x', r')}{\sigma_1^2 x' v_{xx}^n(t', x', r')} = k_{t'}^{(0)*} \in \mathcal{A}_n''(t', x', r')$$

and $\mathcal{L}^k v^n(t', x', r') = 0$. $k_{t'}^{(0)*} \in \mathcal{A}_n''(t', x', r')$ implies

$$v^{n-1}(t', x'(1 - l^*(k_{t'}^{(0)*})^+), r') \geq v^n(t', x', r') = v^{n-1}(t', x'(1 - l^*(\hat{k}_{t'}^{(n)})^+), r'). \tag{101}$$

Now, assume that $\hat{k}_{t'}^{(n)} < k_{t'}^{(0)*}$, then $(\hat{k}_{t'}^{(n)})^+ < (k_{t'}^{(0)*})^+$. Since $v^{n-1}(t, x, r)$ is strictly monotone increasing in x, we obtain

$$v^{n-1}(t', x'(1 - l^*(k_{t'}^{(0)*})^+), r') < v^{n-1}(t', x'(1 - l^*(\hat{k}_{t'}^{(n)})^+), r'),$$

which is a contradiction to (101). Thus, $\hat{k}_{t'}^{(n)} \geq k_{t'}^{(0)*}$ and therefore $(t', x', r') \in \{(t, x, r) : \hat{k}_t^{(n)} \geq k_t^{(0)*}\}$.
Summing up, we obtained that $v^n(t, x, r) \in C^{1,2,2}$, given by (14), is a solution of the system (22)-(24).

REMARK 2.6.2.

1. *By Proposition 2.2.4, we have that $\hat{k}_t^{(n)} \geq 0$, which justifies the assumption that $k_t^{(n)*} \geq 0$ for $(t, x, r) \notin \mathcal{Z}^{(n)}$ (see assumption before* (99)*)*
2. *If $\gamma\rho \geq 0$, then $k_t^{(n)*} = \hat{k}_t^{(n)}$ for all $t \in [0, T]$, $n = 1, \ldots, N$, and therefore $\mathcal{Z}^{(n)} = \emptyset$ for $n = 1, \ldots, N$. (see Remark 2.2.8)*
3. *Korn and Steffensen [28] characterized the solution of the system of inequalities and note that -in their model with a constant interest rate-, the set $\mathcal{Z}^{(n)}$ is the whole (t, x, r) space for $n = 0$ and for $n > 0$ the set $\mathcal{Z}^{(n)}$ is empty. The authors also mentioned that 'examples where neither $\mathcal{Z}^{(n)}$ nor its complement are empty for a given n may require a generalized model, such as, e.g., the case of crashed coefficients where the diffusion coefficients react on crashes' ([28, p.2020]).*
   Here, we characterized a solution of a model where neither $\mathcal{Z}^{(n)}$ nor its complement are empty for a given n. This effect is due to the short rate model. (An example for the case where neither $\mathcal{Z}^{(n)}$ nor its complement are empty is illustrated in Figure 2.5).

2.6.7. Characterization of $v^n(t, x, r)$ - The Log utility case. Analogously to Section 2.6.6, we can separate the (t, x, r) space into the set $\mathcal{Z}^{(n)}$, given by

$$\mathcal{Z}^{(n)} = \left\{(t, x, r) : k(t, x, r) = -\frac{v_x^n(t, x, r)\mu + \rho\sigma_1\sigma_2 v_{xr}^n(t, x, r)}{\sigma_1^2 x v_{xx}^n(t, x, r)}, 0 = \mathcal{L}^k v^n(t, x, r)\right\}$$

and its complement, where k and v^n are determined by

$$\begin{aligned} v^n(t, x, r) &= v^{n-1}(t, x(1 - l^* k^+), r), \\ 0 &= \mathcal{L}^k v^n(t, x, r). \end{aligned} \tag{102}$$

Inspired by the case $n = 0$, we try the ansatz of the form

$$v^n(t, x, r) = \log(x) + W^{(n)}(t, r) \qquad \text{with} \quad W^{(n)}(T, r) = 0, \quad r \in \mathbb{R}.$$

Then, for $(t, x, r) \in \mathcal{Z}^{(n)}$ we obtain that $k^{(n)*}(t, x, r) = \frac{\mu}{\sigma_1^2} = k_t^{(0)*}$. For $(t, x, r) \notin \mathcal{Z}^{(n)}$, $k^{(n)*}$ is determined by the condition $v^n(t, x, r) = v^{n-1}(t, x(1 - l^* k^+), r)$, and therefore, we have

$$k^{(n)*}(t, x, r) = \begin{cases} k_t^{(0)*} & : (t, x, r) \in \mathcal{Z}^{(n)} \\ \frac{1}{l^*}\left(1 - \exp\left(W^{(n)}(t, r) - W^{(n-1)}(t, r)\right)\right) & : (t, x, r) \notin \mathcal{Z}^{(n)} \end{cases}.$$

In contrast to the case of non-log HARA utility functions, we assume that $\mathcal{Z}^{(n)}$ is empty for $n > 0$. Now, by inserting $v^n(t, x, r)$ and $k^{(n)*}$ in condition (102), we obtain the following PDE

$$\begin{aligned} &W_t^{(n)} + \frac{\sigma_2^2}{2} W_{rr}^{(n)} + a(r_M - r)W_r^{(n)} + r + \mu k_t^{(n)*} - \frac{\sigma_1^2}{2}(k_t^{(n)*})^2 = 0, \quad (t, r) \in [0, T] \times \mathbb{R}, \\ &W(T, r) = 0, \quad r \in \mathbb{R}. \end{aligned}$$

Since the short rate dynamics is not affected by the market crash, we try the following separation ansatz for $W^{(n)}(t,r)$ of the form

$$W^{(n)}(t,r) = g^{(n)}(t) + h(t,r), \qquad g^{(n)}(T) = 0,\, h(T,r) = 0, \qquad r \in \mathbb{R},$$

where the function $h(t,r)$ does not depend on the remaining number of crashes n. Thus, for $(t,x,r) \notin \mathcal{Z}^{(n)}$, we obtain

$$k^{(n)*}(t,x,r) = k_t^{(n)*} = \frac{1}{l^*}\left(1 - \exp\left(g^{(n)}(t) - g^{(n-1)}(t)\right)\right),$$

and $g^{(n)}(t)$ is given by

$$\dot{g}^{(n)}(t) = -\mu k_t^{(n)*} + \frac{\sigma_1^2}{2}(k_t^{(n)*})^2, \quad g^{(n)}(T) = 0,$$

and $h(t,r)$ has to fulfill the following PDE

$$\begin{aligned} 0 &= h_t(t,r) + \frac{\sigma_2^2}{2}h_{rr}(t,r) + a(r_M - r)h_r(t,r) + r, \quad (t,r) \in [0,T] \times \mathbb{R}, \\ 0 &= h(T,r), \quad r \in \mathbb{R}. \end{aligned}$$

By applying Feynman-Kac Theorem (see Appendix Theorem A.4.1), we obtain that

$$h(t,r) = \mathbb{E}^{t,r}\left(\int_t^T \overline{r}_s\, ds\right) = r_M(T-t) + \frac{1}{a}(r - r_M + e^{-a(T-t)}(r_M - r))$$

is a solution of the PDE above. Thus, $v^n(t,x,r) = \log(x) + W^{(n)}(t,r)$ with $W^{(n)}(t,r) = g^{(n)}(t) + h(t,r)$ fulfills (102). Using these calculations, we obtain that $k_t^{(n)*}$ is determined by the following ODE

$$\begin{aligned} \dot{k}_t^{(n)*} &= -\frac{1}{l^*}\exp\left(g^{(n)}(t) - g^{(n-1)}(t)\right)(\dot{g}^{(n)}(t) - \dot{g}^{(n-1)}(t)) \\ &= \frac{1 - l^* k_t^{(n)*}}{l^*}\left(\phi(k_t^{(n)*}) - \phi(k_t^{(n-1)*})\right), \quad k_T^{(n)*} = 0, \end{aligned}$$

where $\phi(k) := \mu k - \frac{\sigma_1^2}{2}k^2$.

2.6.8. Proof of (31) **and** (41)**.** Here, the aim is to show that the expectation of stochastic integrals, which appear in the proof of the verification theorem 2.2.2, vanish, that means we show that (31) and (41) hold. First, we prove that (41) holds:

$$\mathbb{E}^{t,x,r,1}\left[\int_t^\theta f^{(k)}(s)\, dw_s\right] = 0.$$

$f^{(k)}(s) := (f_1^{(k)}(s), f_2^{(k)}(s)), dw_s := (dw_{1,s}, dw_{2,s})^T$ with

$$\begin{aligned} f_1^{(k)}(s) &:= v^1(s, X_s^{k,(\theta,\tilde{l})}, r_s)\,(\gamma\sigma_1 k_s + \beta(s)\rho\sigma_2), \\ f_2^{(k)}(s) &:= v^1(s, X_s^{k,(\theta,\tilde{l})}, r_s)\sqrt{1-\rho^2}\sigma_2\beta(s), \end{aligned}$$

where $\theta = \theta^1(t,x,r)$ (for fixed (t,x,r)) and $\tilde{l} = l^*\mathbb{1}_{k_\theta \geq 0}$ and $k \in \Pi(t,x,r)$ arbitrary but fixed. Since θ is given by (26), we can interpret θ as a first exit time of the process

$$\left(v^0(s, X_s^{k,(\theta,\tilde{l})}(1 - l^* k_s^+), r_s) - v^1(s, X_s^{k,(\theta,\tilde{l})}, r_s)\right)_{s \geq t}$$

from the open set $\mathbb{R}_+$. By [46, Chp.1,Example 3.3] we know that θ is a stopping time. Thus, we have to consider a stochastic integral, where the upper limit is a stopping time. Using [17, Chp.4,Thm. 4.3], it remains to show that

$$\mathbb{E}\left[\int_t^T |f_1^{(k)}(s)|^2\,ds\right] < \infty \qquad \text{and} \qquad \mathbb{E}\left[\int_t^T |f_2^{(k)}(s)|^2\,ds\right] < \infty.$$

Let $k \in \Pi$ be arbitrary but fixed, then we have

$$\begin{aligned}
\mathbb{E}\left[\int_t^T |f_1^{(k)}(s)|^2\,ds\right] &= \mathbb{E}\left[\int_t^T |v^1(s, X_s^{k,(\theta,\tilde{l})}, r_s)\,(\gamma\sigma_1 k_s + \beta(s)\rho\sigma_2)\,|^2\,ds\right] \\
&= \mathbb{E}\left[\int_t^T |v^1(s, X_s^{k,(\theta,\tilde{l})}, r_s)|^2 \cdot \underbrace{|\gamma\sigma_1 k_s + \beta(s)\rho\sigma_2|^2}_{\leq K_1}\,ds\right] \\
&\leq K_1 \cdot \mathbb{E}\left[\int_t^T |v^1(s, X_s^{k,(\theta,\tilde{l})}, r_s)|^2\,ds\right] \\
&\leq K_1 \cdot \mathbb{E}\left[\int_t^T \sup_{s\in[t,T]} |v^1(s, X_s^{k,(\theta,\tilde{l})}, r_s)|^2\,ds\right]. \qquad (103)
\end{aligned}$$

Now, we use that we already have the explicit structure of $v^n(t,x,r)$, that is

$$v^n(t,x,r) = \frac{1}{\gamma} x^\gamma g^{(n)}(t)\exp(\beta(t)r),$$

where $g^{(n)}(t) \leq K^{(n)}$ for all $t \in [0,T]$ and some positive constant $K^{(n)}$. We show that

$$\mathbb{E}\left[\sup_{s\in[t,T]} |v^1(s, X_s^{k,(\theta,\tilde{l})}, r_s)|^2\right] < \infty. \qquad (104)$$

For some constant $K_2 > 0$ it holds

$$\begin{aligned}
|v^1(s, X_s^{k,(\theta,\tilde{l})}, r_s)| &= |\gamma^{-1}| \left(X_s^{k,(\theta,\tilde{l})}\right)^\gamma g^{(1)}(s)\exp(\beta(s)r_s) \\
&\leq K_2 \left(X_s^{k,(\theta,\tilde{l})}\right)^\gamma \exp(\beta(s)r_s).
\end{aligned}$$

Now, we use the same methods as on page 70 and obtain (104) by using the solution $X_s^{k,(\theta,\tilde{l})}$ and r_s of the corresponding SDE's. Together with (103) we have

$$\mathbb{E}\left[\int_t^T |f_1^{(k)}(s)|^2\,ds\right] < \infty.$$

Analogously, we prove that

$$\mathbb{E}\left[\int_t^T |f_2^{(k)}(s)|^2\,ds\right] < \infty$$

and finally obtain that

$$\mathbb{E}^{t,x,r,1}\left[\int_t^\theta f^{(k)}(s)dw_s\right] = 0$$

by [17, Chp.4,Thm. 4.3]. Moreover, for an arbitrary but fixed strategy (τ, l), we can prove (31), that is

$$\mathbb{E}^{t,x,r,1}\left[-\int_t^\tau f^{(\tau,l)}(s)dw_s\right] = 0,$$

where $f^{(\tau,l)}(s) := (f_1^{(\tau,l)}(s), f_2^{(\tau,l)}(s)), dw_s := (dw_{1,s}, dw_{2,s})^T$ with

$$f_1^{(\tau,l)}(s) := v^1(s, X_s^{k^{(1)*},(\tau,l)}, r_s)\left(\gamma\sigma_1 k_s^{(1)*} + \beta(s)\rho\sigma_2\right),$$

$$f_2^{(\tau,l)}(s) := v^1(s, X_s^{k^{(1)*},(\tau,l)}, r_s)\sqrt{1-\rho^2}\sigma_2\beta(s).$$

This can be done in the same manner as above. Thus we omit it here.
Note that the stochastic integrals in (47) also vanish with the same arguments as above.

2.6.9. Proof of Corollary 2.5.1. Here, the aim is to show for arbitrary but fixed $(t,x,r) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}$ and $k \in \Pi(t,x,r)$ and $(\tau,l) \in \mathcal{C}$:

$$\mathbb{E}^{t,x,r,1}\left(\left|\log(X_T^{k,(\tau,l)})\right|\right) < \infty,$$

where $\{r_t\}_{t\in[0,T]}$ and $X^{k,(\tau,l)} = \{X_t^{k,(\tau,l)}\}_{t\in[0,T]}$ are given by (3) and (65), respectively.

Proof. By Proposition A.1.2 in Appendix A, we have that $\mathbb{E}^{t,r}\left(r_s^2\right) < \infty$ for all $s \geq t$, and therefore,

$$\mathbb{E}^{t,r}\left(\int_t^T |r_s|\, ds\right) \leq \frac{1}{2}\int_t^T \mathbb{E}^{t,r}\left(r_s^2\right) ds + \frac{1}{2}(T-t) < \infty$$

$$\Rightarrow \int_t^T |r_s|\, ds < \infty \qquad \mathbb{P} - a.s.$$

By Corollary A.5.1 the SDE (65) has a uniquely determined solution and with $X_\tau^{k,(\tau,l)} = (1 - lk_\tau^{(1)})X_{\tau-}^{k,(\tau,l)}$, we obtain for a $[t,T]$-valued stopping time τ and a universal constant $K > 0$:

$$\begin{aligned}
X_T^{k,(\tau,l)} &= X_\tau^{k,(\tau,l)} \exp\left(\int_\tau^T (\mu^{(0)}k_u^{(0)} - \frac{(\sigma_1^{(0)})^2}{2}(k_u^{(0)})^2 + r_u)\, du + \int_\tau^T \sigma_1^{(0)}k_u^{(0)}\, dw_{1,u}\right)\\
&= x(1-lk_\tau^{(1)})\\
&\quad\cdot \exp\Bigg(\int_t^\tau \mu^{(1)}k_u^{(1)} - \frac{(\sigma_1^{(1)})^2}{2}(k_u^{(1)})^2\, du + \int_t^\tau \sigma_1^{(1)}k_u^{(1)}\, dw_{1,u}\\
&\qquad + \int_\tau^T \mu^{(0)}k_u^{(0)} - \frac{(\sigma_1^{(0)})^2}{2}(k_u^{(0)})^2\, du + \int_\tau^T \sigma_1^{(0)}k_u^{(0)}\, dw_{1,u} + \int_t^T r_u\, du\Bigg).
\end{aligned}$$

Since $k = (k^{(0)}, k^{(1)}) \in \Pi(t,x,r)$, we obtain

$$\begin{aligned}
\left|\log(X_T^{k,(\tau,l)})\right| &\leq K + \left|\int_0^\tau \sigma_1^{(1)}k_u^{(1)}\, dw_{1,u}\right| + \left|\int_\tau^T \sigma_1^{(0)}k_u^{(0)}\, dw_{1,u}\right| + \int_t^T |r_u|\, du\\
&\leq K + \frac{1}{2}\left|\int_0^\tau \sigma_1^{(1)}k_u^{(1)}\, dw_{1,u}\right|^2 + \frac{1}{2}\left|\int_\tau^T \sigma_1^{(0)}k_u^{(0)}\, dw_{1,u}\right|^2 + \frac{1}{2}\int_t^T r_u^2\, du.
\end{aligned}$$

Taking the expectation on both sides leads to

$$\begin{aligned}
\mathbb{E}^{t,x,r,1}\left(\left|\log(X_T^{k,(\tau,l)})\right|\right) &\leq K + \frac{1}{2}\mathbb{E}\left(\int_t^T (\sigma_1^{(1)}k_u^{(1)})^2\, du\right)\\
&\quad + \frac{1}{2}\mathbb{E}\left(\int_t^T (\sigma_1^{(0)}k_u^{(0)})^2\, du\right) + \frac{1}{2}\int_t^T \mathbb{E}(r_u^2)\, du < \infty.
\end{aligned}$$

Here, we used that the Ito Isometry also holds for stochastic integrals where the limits of the integration are stopping times (see for example [17, Thm 4.2,Chp.4]). In the no-crash scenario

$\tau = \infty$ we have

$$X_T^{k,(\tau,l)} = x \cdot \exp\left(\int_t^T (\mu^{(1)} k_u^{(1)} - \frac{(\sigma_1^{(1)})^2}{2} (k_u^{(1)})^2 + r_u)\, du + \int_t^T \sigma_1^{(1)} k_u^{(1)}\, dw_{1,u} \right)$$

and the assertion follows by the same arguments as above. □

2.6.10. Verification argument for post-crash problem (67)**.** Here we show, that we are allowed to apply Corollary A.5.3 in Appendix A for our post-crash optimization problem. In order to apply this, it remains to show that the candidate for the optimal control $k_t^{(0)*} \equiv \frac{\mu^{(0)}}{(\sigma_1^{(0)})^2}$ and the candidate for the value function $v^0(t,x,r) = \log(x) + A(t)r + B(t)$ fulfill the following assumptions on the relevant time interval $[t_0, T]$:

i) $k^{(0)*}$ is progressively measurable: This is obviously met.
ii) For all initial conditions $r_0 > 0$ and $x_0 > 0$, the corresponding state processes $\{\overline{r}_s\}$ and $\{\overline{X}_s\}$ with $r_{t_0} = r_0$ and $\overline{X}_{t_0} = x_0$ have a pathwise unique solution $\{r_s\}_{s\in[t_0,T]}$ and $\{\overline{X}_s\}_{s\in[t_0,T]}$: At the beginning of Section 2.5 we already mentioned that the short rate equation (3) has a pathwise unique solution. Since $\int_0^T |\overline{r}_s|\, ds < \infty$ $\mathbb{P}$-a.s. the requirements of Corollary A.5.1 are fulfilled, and therefore, the wealth equation has a unique solution for every $k^{(0)} \in \Pi(t_0, x_0, r_0)$.
iii) Obviously $\mathbb{E}\left(\int_{t_0}^T |k_s^{(0)}|^4\, ds\right) < \infty$ for all $k^{(0)} \in \Pi(t_0, x_0, r_0)$.
iv) The utility functional

$$J(t_0, x_0, r_0; k) := \mathbb{E}^{t_0,x_0,r_0}\left(\log(\overline{X}_T)\right)$$

is well defined for each initial value (t_0, x_0, r_0) and each $k \in \Pi(t_0, x_0, r_0)$: This can be shown by the same arguments as in the proof of Corollary 2.5.1.
v) Let $\mathcal{O}_p := \mathcal{O} \cap \{y = (x,r) \in \mathbb{R}^2 : |y| < p, dist(y, \partial\mathcal{O}) > p^{-1}\}$ for $p \in \mathbb{N}$, where $\mathcal{O} = \mathbb{R}_+ \times \mathbb{R}$, and let θ_p be the first exit time of $(s, \overline{X}_s, \overline{r}_s)$ from $Q_p := [t_0, T - p^{-1}) \times \mathcal{O}_p$. Note that Q_p is not empty for $p \in \mathbb{N}$ with $p > \tilde{p} := (T - t_0)^{-1}$. Moreover, $\theta_p \to T$ $\mathbb{P}$-a.s. for $p \to \infty$. Now, we have to show that $\{v^0(\theta_p, \overline{X}^*_{\theta_p}, \overline{r}_{\theta_p})\}_{p\in\mathbb{N}}$ is uniformly integrable, given that $v^0(t,x,r) = \log(x) + A(t)r + B(t)$ and $\overline{X}^*$ denotes the wealth process controlled by $k^{(0)*}$. We prove that

$$\sup_{p>\tilde{p}} \mathbb{E}\left(|v^0(\theta_p, \overline{X}^*_{\theta_p}, \overline{r}_{\theta_p})|^2\right) < \infty.$$

Let $p > \tilde{p}$ be arbitrary but fixed. Using the explicit solution of the state equation $\overline{X}^*$, using the inequality $(a+b+c+d)^2 \le 4(a^2+b^2+c^2+d^2)$, by the fact that $\theta_p \in [t_0, T)$ and by

Cauchy-Schwarz inequality, we obtain that

$$
\begin{aligned}
&|v^0(\theta_p, \overline{X}^*_{\theta_p}, \overline{r}_{\theta_p})|^2 \\
&= \Bigg| \log(x_0) + (\theta_p - t_0)\frac{1}{2}\left(\frac{\mu^{(0)}}{\sigma_1^{(0)}}\right)^2 + B(\theta_p) \\
&\quad + \int_{t_0}^{\theta_p} \overline{r}_s \, ds + \int_{t_0}^{\theta_p} \sigma_1^{(0)} k^{(0)*} \, dw_{1,s} + A(\theta_p)\overline{r}_{\theta_p} \Bigg|^2 \\
&\leq 4 \left| \log(x_0) + (\theta_p - t_0)\frac{1}{2}\left(\frac{\mu^{(0)}}{\sigma_1^{(0)}}\right)^2 + B(\theta_p) \right|^2 + 4 \left| \int_{t_0}^{\theta_p} \overline{r}_s \, ds \right|^2 + 4 \left| \int_{t_0}^{\theta_p} \sigma_1^{(0)} k^{(0)*} \, dw_{1,s} \right|^2 \\
&\quad + 4 \left| A(\theta_p)\overline{r}_{\theta_p} \right|^2 \\
&\leq K_1 + 4(T - t_0) \int_{t_0}^{T} |\overline{r}_s|^2 \, ds + 4 \sup_{t \in [t_0, T]} \left| \int_{t_0}^{t} \sigma_1^{(0)} k^{(0)*} \, dw_{1,s} \right|^2 + K_2 |\overline{r}_{\theta_p}|^2,
\end{aligned}
$$

where

$$
\begin{aligned}
K_1 &:= 4 \sup_{t \in [t_0, T]} \left| \log(x_0) + (t - t_0)\frac{1}{2}\left(\frac{\mu^{(0)}}{\sigma_1^{(0)}}\right)^2 + B(t) \right|^2, \\
K_2 &:= 4 \sup_{t \in [t_0, T]} |A(t)|^2.
\end{aligned}
$$

Now, taking the expectation on both sides, leads to

$$
\begin{aligned}
&\mathbb{E}\left(|v^0(\theta_p, \overline{X}^*_{\theta_p}, \overline{r}_{\theta_p})|^2 \right) \\
&\leq K_1 + 4(T - t_0) \int_{t_0}^{T} \mathbb{E}|\overline{r}_s|^2 \, ds + 4\mathbb{E}\left(\sup_{t \in [t_0, T]} \left| \int_{t_0}^{t} \sigma_1^{(0)} k^{(0)*} \, dw_{1,s} \right|^2 \right) + K_2 \mathbb{E}|\overline{r}_{\theta_p}|^2 \\
&\leq K_1 + 4(T - t_0) \int_{t_0}^{T} \mathbb{E}|\overline{r}_s|^2 \, ds + 4\mathbb{E}\left(\sup_{t \in [t_0, T]} \left| \int_{t_0}^{t} \sigma_1^{(0)} k^{(0)*} \, dw_{1,s} \right|^2 \right) \\
&\quad + K_2 \sup_{t \in [t_0, T]} \left(\mathbb{E}|\overline{r}_t|^2 \right) \\
&=: K_3 < \infty.
\end{aligned}
$$

The, inequality above holds for all $p > \tilde{p}$. Hence, we obtain:

$$
\sup_{p > \tilde{p}} \mathbb{E}\left(|v^0(\theta_p, \overline{X}^*_{\theta_p}, \overline{r}_{\theta_p})|^2 \right) < \infty.
$$

This implies uniformly integrability of the sequence of random variables $\{v^0(\theta_p, \overline{X}^*_{\theta_p}, \overline{r}_{\theta_p})\}_p$. Note that the uniformly integrability is a key tool to prove Corollary A.5.3. For further details we refer to the literature [31].

By showing conditions i)-v), we obtain that $k^{(0)*}$ is a weak admissible control (for Definition we refer to Appendix A.5). Thus, we can apply Corollary A.5.3 and obtain that $k^{(0)*}$ is the optimal strategy after the market crash and the solution of the HJB equation v^0 coincides with the post-crash value function V^0.

2.6.11. Proof of Proposition 2.5.4. Here, we prove the following assertion: Let $\hat{k}^{(1)}$ be the uniquely determined solution of (71) and let $k^M = \frac{\mu^{(1)}}{(\sigma_1^{(1)})^2}$. Moreover, assume that

$$\frac{\mu^{(1)}}{\sigma_1^{(1)}} - \frac{\mu^{(0)}}{\sigma_1^{(0)}} \geq 0. \tag{105}$$

Then $\hat{k}_t^{(1)} \in \big[0, k^M\big]$ for all $t \in [0,T]$.

PROOF. If $k^M \geq \frac{1}{l^*}$, then the assertion follows by the fact that $\hat{k}_t^{(1)} \in [0, \frac{1}{l^*})$ for all $t \in [0,T]$. Now, let $k^M < \frac{1}{l^*}$. In this case, we prove the assertion by showing that the solution $h_t^{(1)}$ of the corresponding forward equation of (71), given by

$$\dot{h}_t^{(1)} = \tilde{f}^{(1)}(h_t^{(1)}), \qquad h_0^{(1)} = 0,$$

where

$$\tilde{f}^{(1)}(h) := -\frac{1-l^*h}{l^*}\left(\mu^{(1)}h - \frac{1}{2}(\sigma_1^{(1)})^2h^2 - \frac{1}{2}\left(\frac{\mu^{(0)}}{\sigma_1^{(0)}}\right)^2\right),$$

fulfills $h_t^{(1)} \in [0, k^M]$ for $t \in [0,T]$. This will be done by the invariance argument. Let $\tilde{D} := [0, k^M]$. On the one hand, we have that

$$\tilde{f}^{(1)}(0) \cdot y = \frac{1}{2l^*}\left(\frac{\mu^{(0)}}{\sigma_1^{(0)}}\right)^2 \cdot y \leq 0, \quad \forall y \in \mathcal{N}_{\tilde{D}}(0) = (-\infty, 0),$$

and, on the other hand,

$$\tilde{f}^{(1)}(k^M) \cdot y = -\frac{1-l^*k^M}{2l^*}\left(\left(\frac{\mu^{(1)}}{\sigma_1^{(1)}}\right)^2 - \left(\frac{\mu^{(0)}}{\sigma_1^{(0)}}\right)^2\right) y \leq 0, \quad \forall y \in \mathcal{N}_{\tilde{D}}(k^M) = (0, \infty).$$

By Theorem A.2.3 in Appendix A, we obtain that $h_t^{(1)} \in [0, k^M]$ for all $t \in [0,T]$ if (105) holds. By time reversion the assertion follows.

□

CHAPTER 3

Worst-Case Optimal Investment and Consumption with an Infinite Time Horizon for Log utility Function

3.1. The financial market model

In this chapter, we consider an infinite horizon financial market model where the investor is again allowed to invest in a savings account and in a stock, but in contrast to Chapter 2, he additionally can consume a fraction of his wealth. Here, we assume that at most one market crash can happen which is modeled as an uncertain event. This market crash causes a sudden downward jump in the stock price evolution. The short rate dynamics of the savings account evolves as a stochastic process with continuous paths which is not affected by the market crash. The investor is acting on an infinite time interval and he aims to maximize his expected discounted utility of consumption in the worst-case crash scenario by choosing an investment and consumption strategy. Using the same notation as in Chapter 2, we specify the short rate models, which we use in this chapter, in Section 3.1.1. In Section 3.1.2 we define the stock price process and in Section 3.1.3 we present the investor's wealth equation and the corresponding worst-case investment and consumption problem.

3.1.1. The short rate models. The value of the savings account $\{B_t\}_{t\geq 0}$ is assumed to follow the differential equation (1). In this chapter, we consider two different short rate models. In Section 3.2, we assume that the short rate follows a slightly more general process than the Vasicek process which we already used in the previous chapter. That is, we assume the process $\{r_t\}_{t\geq 0}$ to be a solution of the SDE:

$$dr_t = f(r_t)\,dt + \sigma_2\,d\tilde{w}_t, \qquad r_0 = r^0, \tag{106}$$

where $f \in C^1(\mathbb{R})$ and $c_2 \leq f_r(r) \leq c_1$, where c_1, c_2 are constants. Again, σ_2 denotes the volatility. This type of short rate model was already considered in [15] and [39] in the context of a similar investment consumption model, but without the possibility of a market crash. Fleming and Pang [15] and Pang [39] refer to the above short rate model as *generalized Vasicek model.* Note that for $f(r) = a(r_M - r)$ we obtain the classical Vasicek model with a speed of reversion a to the long term mean level r_M.

In Section 3.3, we assume that the short rate follows an affine model. Analogously to Section 2.5, we assume that the process $\{r_t\}_{t\geq 0}$ is a solution of the SDE (3):

$$dr_t = (\lambda_1 r_t + \lambda_2)\,dt + \sqrt{\xi_1 r_t + \xi_2}\,d\tilde{w}_t, \qquad r_0 = r^0,$$

where $\lambda_1, \lambda_2, \xi_1, \xi_2, r^0$ are given constants. Within this model, we also cover the well-known Cox-Ingersoll-Ross process.

3.1.2. The stock price process. In order to model the stock price evolution, we make two basic assumptions which are valid for the rest of this chapter. First, as in [10], we assume that there can happen at most one market crash on the infinite time interval. Thus, in this chapter the market crash is a once-in-a-lifetime event. The crash is represented by the pair $(\tau, l) \in \mathcal{C}'$, where

$$\begin{aligned} \mathcal{C}' := \{(\tau, l) : \tau \in [0, \infty],\ &\text{stopping time}, \\ l \in [0, l^*]\, \mathcal{F}_\tau\text{- }&\text{measurable random variable}\}. \end{aligned} \tag{107}$$

As in the previous chapter, we emphasize that $\tau = \infty$ describes the case if no crash occurs at all and the random variable $l \in [0, l^*]$ denotes the crash size, where the maximum crash size $l^* < 1$ is given. Additionally, we assume that the drift and the volatility of the price process will change at time τ. The concept of changing market parameters was already applied and motivated in Section 2.5. Now, we obtain the equations which describe the evolution of the stock price process $\{P_t\}_{t\geq 0}$:

$$\begin{aligned} P_0 &= p^0, \\ dP_t &= P_t \left[(\mu^{(1)} + r_t)\, dt + \sigma_1^{(1)}\, dw_{1,t} \right], \qquad t \in (0, \tau), \\ P_\tau &= P_{\tau-}(1 - l), \\ dP_t &= P_t \left[(\mu^{(0)} + r_t)\, dt + \sigma_1^{(0)}\, dw_{1,t} \right], \qquad t \in (\tau, \infty], \end{aligned}$$

where $\mu^{(1)} > 0$ and $\sigma_1^{(1)} > 0$ denote the market parameters valid before the crash, and $\mu^{(0)} > 0$ and $\sigma_1^{(0)} > 0$ denote the market parameters valid after the crash. Again, the Wiener processes w_1 and $\tilde{w}$ may be correlated with correlation coefficient $\rho \in [-1, 1]$.

3.1.3. Admissible controls and the worst-case optimization problem. In contrast to the previous chapter, the investor's behavior is described by the portfolio process $k = \{k_t\}_{t\geq 0}$ and by the consumption process $c = \{c_t\}_{t\geq 0}$. k_t denotes the fraction of wealth invested in the stock and c_t denotes the rate at which the investor consumes. Accordingly, $1 - k_t$ is the fraction of wealth invested in the savings account. We denote the investment and consumption strategy valid for $t \in [0, \tau]$ by $(k^{(1)}, c^{(1)})$ and the strategy valid for $t \in (\tau, \infty]$ by $(k^0, c^{(0)})$. Thus, we call $(k^{(1)}, c^{(1)})$ and $(k^0, c^{(0)})$ pre- and post-crash strategy, respectively. Below, we define the admissible control space.

Definition 3.1.1 (Admissible control). *A process $(k, c) = (k^{(0)}, c^{(0)}, k^{(1)}, c^{(1)})$, where the post-crash strategy $(k^{(0)}, c^{(0)})$ is valid for $t \in (\tau, \infty]$ and the pre-crash strategy $(k^{(1)}, c^{(1)})$ is valid for $t \in [0, \tau]$, is called admissible control if $k = (k^{(0)}, k^{(1)})$ is nonnegative and is admissible in the sense of Definition 2.1.1 and $c = (c^{(0)}, c^{(1)})$ fulfills the following conditions:*

(1) c is a $\mathbb{F}$-adapted process,
(2) $0 \leq c_t \leq C$ for all $t \geq 0$ for a sufficiently large constant C,

The set of admissible controls is denoted by Π.

Remark 3.1.2. *Note that the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is again generated by the processes w_1, $\tilde{w}$ and $\tilde{N}$, where $\tilde{N}$ is again the counting process defined in* (6).

Given a market crash (τ, l) and a self-financing investment consumption strategy (k, c), we denote the investor's wealth at time t by X_t and obtain the corresponding wealth equation:

$$\begin{aligned} X_0 &= x^0 > 0, \\ dX_t &= X_t \left[r_t + \mu^{(1)} k_t^{(1)} - c_t^{(1)} \right] dt + X_t \sigma_1^{(1)} k_t^{(1)} \, dw_{1,t}, \quad t \in (0, \tau), \\ X_\tau &= (1 - l k_\tau^{(1)}) X_{\tau-}, \\ dX_t &= X_t \left[r_t + \mu^{(0)} k_t^{(0)} - c_t^{(0)} \right] dt + X_t \sigma_1^{(0)} k_t^{(0)} \, dw_{1,t}, \quad t \in (\tau, \infty]. \end{aligned}$$

Again, the short rate process $\{r_t\}_{t \geq 0}$ is not affected by the market crash. Note that the wealth equation of this chapter slightly differs from the one in the previous chapter because consumption reduces the investor's wealth. Using the dynamics above, we formulate the worst-case optimization problem. Here, the investor wants to maximize his expected discounted logarithmic utility of consumption over an infinite time interval in the worst-case crash scenario. Thus, we formulate the following worst-case optimization problem:

$$\sup_{(k,c) \in \Pi(x^0, r^0)} \inf_{(\tau,l) \in \mathcal{C}'} \mathbb{E} \left(\int_0^\infty e^{-\varepsilon t} \log(c_t X_t) \, dt \right), \tag{108}$$

where $\varepsilon > 0$ denotes the discount factor. $\Pi(x^0, r^0)$ denotes the set of admissible controls, corresponding to the condition that $X_0 = x^0$ and $r_0 = r^0$. The worst-case optimization problem above provides a generalization of the problem considered in [10] where constant interest rates are used in a similar infinite time horizon model. Note that Desmettre et al. [10] extended the martingale approach of Seifried [44] for the infinite time horizon problem and interpreted it as a controller vs. stopper game. In this thesis, we already used the martingale approach to solve the finite time horizon problem considered in Chapter 2. In this chapter, we apply the method again for problem (108).

In the next section, we investigate the worst-case optimal investment and consumption behavior if the short rate r_t follows the generalized Vasicek process specified in (106).

3.2. The generalized Vasicek Model

Here, we assume that the short rate r_t will change according to (106), that is

$$dr_t = f(r_t) \, dt + \sigma_2 \, d\tilde{w}_t, \quad r_0 = r^0,$$

where $\sigma_2 > 0$ and

$$f \in C^1(\mathbb{R}), \quad c_2 \leq f_r(r) \leq c_1, \quad \forall r \in \mathbb{R}. \tag{109}$$

We conclude from condition (109) that $f(r)$ is globally Lipschitz continuous and fulfills the growth condition $|f(r)| \leq C(1 + |r|)$ for some constant $C > 0$. By the classical existence and uniqueness theorem for SDEs (see e.g. [38, Theorem 5.2.1]), there exists a uniquely determined solution $\{r_t\}_{t \geq 0}$ of equation (106).

The aim of this section is to solve the worst-case optimization problem (108) under the generalized Vasicek model. As in [10] and [44], and as in Section 2.4, we apply the following steps. First, we solve the post-crash optimization problem which is a classical infinite horizon stochastic optimal control problem. Afterwards, we reformulate the problem (108) as a pre-crash problem which is interpreted as a controller vs. stopper game. Using the martingale approach we finally obtain the worst-case optimal pre-crash strategy.

3.2.1. The post-crash optimization problem. The aim of this section is to find the optimal post-crash strategy $(k^{(0)*}, c^{(0)*})$ by using stochastic optimal control theory. After the market crash, the investor has to 'solve' a classical stochastic optimal control problem with initial values x and r over an infinite time interval. Let us define the *post-crash value function* by

$$V^0(x,r) = \sup_{(k^{(0)},c^{(0)})\in\Pi(x,r)} \mathbb{E}\left(\int_0^\infty e^{-\varepsilon t}\log(c_t^{(0)}\overline{X}_t)\,dt\right), \tag{110}$$

where $\overline{X}_t$ denotes the wealth at time t if no crash can occur anymore, that means $\overline{X}_t$ is assumed to solve the following wealth equation controlled by $(k^{(0)}, c^{(0)})$, starting in x and the short rate process starts with value r:

$$\begin{aligned} d\overline{X}_t &= \overline{X}_t\left[\overline{r}_t + \mu^{(0)}k_t^{(0)} - c_t^{(0)}\right]dt + \overline{X}_t\sigma_1^{(0)}k_t^{(0)}\,dw_{1,t}, \qquad &\overline{X}_0 = x, \\ d\overline{r}_t &= f(\overline{r}_t)\,dt + \sigma_2(\rho\,dw_{1,t} + \sqrt{1-\rho^2}\,dw_{2,t}), &\overline{r}_0 = r. \end{aligned} \tag{111}$$

REMARK 3.2.1. *The post-crash value function $V^0(x,r)$ depends on the initial values of the post-crash dynamics, given by arbitrary $x \in \mathbb{R}_+$ and $r \in \mathbb{R}$, that will later represent the wealth and the short rate at the crash time, respectively.*

The corresponding HJB equation to the post-crash problem (110) is given by

$$\begin{aligned} &\sup_{k^{(0)}\in A}\left[\mu^{(0)}k^{(0)}xv_x^0(x,r) + \frac{(\sigma_1^{(0)})^2}{2}(k^{(0)})^2x^2v_{xx}^0(x,r) + \rho\sigma_1^{(0)}\sigma_2k^{(0)}xv_{xr}^0(x,r)\right] \\ &+ \sup_{c^{(0)}\geq 0}\left[\log(c^{(0)}x) - c^{(0)}xv_x^0(x,r)\right] \\ &+ rxv_x^0(x,r) + f(r)v_r^0(x,r) + \frac{\sigma_2^2}{2}v_{rr}^0(x,r) - \varepsilon v^0(x,r) = 0, \qquad (t,x)\in\mathbb{R}_+\times\mathbb{R}. \end{aligned} \tag{112}$$

In order to find a solution of the HJB equation above, we apply the separation ansatz $v^0(x,r) = B\log(x) + W(r)$, where $B \in \mathbb{R}, W \in C^2(\mathbb{R})$, and obtain the reduced HJB equation:

$$\begin{aligned} &\sup_{k^{(0)}\in A}\left[\mu^{(0)}k^{(0)}B - \frac{(\sigma_1^{(0)})^2}{2}(k^{(0)})^2B\right] + \sup_{c^{(0)}\geq 0}\left[\log(c^{(0)}) - c^{(0)}B\right] \\ &+ rB + f(r)W_r(r) + \frac{\sigma_2^2}{2}W_{rr}(r) - \varepsilon\left(B\log(x) + W(r)\right) + \log(x) = 0, \qquad r\in\mathbb{R}. \end{aligned}$$

By choosing $B = \frac{1}{\varepsilon}$, we eliminate the state variable x. Furthermore, we obtain the following candidates for the optimal post-crash strategy:

$$k^{(0)*} = \frac{\mu^{(0)}}{(\sigma_1^{(0)})^2}, \qquad c^{(0)*} = \varepsilon. \tag{113}$$

Inserting these candidates, leads to an ODE for W of the form

$$\frac{\sigma_2^2}{2}W_{rr}(r) + f(r)W_r(r) - \varepsilon W(r) + Q(r) = 0, \qquad r\in\mathbb{R}, \tag{114}$$

where

$$Q(r) := \frac{1}{2\varepsilon}\left(\frac{\mu^{(0)}}{\sigma_1^{(0)}}\right)^2 + \frac{r}{\varepsilon} + \log(\varepsilon) - 1. \tag{115}$$

This kind of ODE was already investigated in [15] and [39] in the context of a similar model by means of a sub- and supersolution method. For the definition of a subsolution and a supersolution we refer to Appendix A.3. We apply this method to get an existence result of a function $W \in C^2(\mathbb{R})$ such that (114) is fulfilled.
Let $L : C^2(\mathbb{R}) \to C(\mathbb{R})$ with

$$LW := \frac{\sigma_2^2}{2} W_{rr} + f(r) W_r,$$

and

$$h(r, W) := Q(r) - \varepsilon W.$$

Then, we rewrite the differential equation (114) briefly as:

$$-LW = h(r, W). \tag{116}$$

By Definition A.3.1 in Appendix A.3, we obtain that $\underline{W}$ is a subsolution of (116) if $-L\underline{W} \leq h(r, \underline{W})$ and $\overline{W}$ is a supersolution if $-L\overline{W} \geq h(r, \overline{W})$ for all $r \in \mathbb{R}$. This is used to prove the following two Lemma.

LEMMA 3.2.2. *Suppose that* $\varepsilon - 2c_1 > 0$. *Then, there exist constants* $\alpha_1, \alpha_2 < 0$ *such that*

$$\underline{W}(r) := \alpha_2 r^2 + \alpha_1 \tag{117}$$

is a subsolution of (116).

PROOF. By the mean value theorem, by (109) and by $\alpha_2 < 0$, we have

$$\begin{aligned} -L\underline{W} &= -\sigma_2^2 \alpha_2 - 2f(r)\alpha_2 r \\ &= -\sigma_2^2 \alpha_2 - 2\alpha_2 r \left(f_r(\xi) r + f(0)\right) \\ &\leq -\sigma_2^2 \alpha_2 - 2\alpha_2 r^2 c_1 - 2\alpha_2 r f(0). \end{aligned}$$

Moreover, it holds

$$h(r, \underline{W}) = -\varepsilon \alpha_2 r^2 + \frac{r}{\varepsilon} + \frac{1}{2\varepsilon}\left(\frac{\mu^{(0)}}{\sigma_1^{(0)}}\right)^2 + \log(\varepsilon) - 1 - \varepsilon\alpha_1.$$

Thus, condition $-L\underline{W} \leq h(r, \underline{W})$ holds, if we have that:

$$\alpha_2 (\varepsilon - 2c_1) r^2 - \left(2\alpha_2 f(0) + \frac{1}{\varepsilon}\right) r - \sigma_2^2 \alpha_2 - \frac{1}{2\varepsilon}\left(\frac{\mu^{(0)}}{\sigma_1^{(0)}}\right)^2 - \log(\varepsilon) + 1 + \varepsilon\alpha_1 \leq 0.$$

Since $\varepsilon - 2c_1 > 0$ and $\alpha_2 < 0$, we obtain the inequality above by choosing $\alpha_1 < 0$ sufficiently small. That means $\underline{W}$ given by (117) is a subsolution of (116). □

Analogously, we determine a supersolution.

LEMMA 3.2.3. *Suppose that* $\varepsilon - 2c_1 > 0$. *Then, there exist constants* $\beta_1, \beta_2 > 0$ *such that*

$$\overline{W}(r) := \beta_2 r^2 + \beta_1 \tag{118}$$

is a supersolution of (116).

PROOF. Analogously to the Lemma before, we use the mean value theorem, the condition (109) and the fact that $\beta_2 > 0$ and obtain

$$-L\overline{W} \geq -\sigma_2^2\beta_2 - 2\beta_2 r^2 c_1 - 2\beta_2 r f(0).$$

Now, $-L\overline{W} \geq h(r, \overline{W})$ holds, if we have that

$$\beta_2 (\varepsilon - 2c_1) r^2 - \left(2\beta_2 f(0) + \frac{1}{\varepsilon}\right) r - \sigma_2^2\beta_2 - \frac{1}{2\varepsilon}\left(\frac{\mu^{(0)}}{\sigma_1^{(0)}}\right)^2 - \log(\varepsilon) + 1 + \varepsilon\beta_1 \geq 0.$$

Since $\varepsilon - 2c_1 > 0$ and $\beta_2 > 0$ we can choose $\beta_1 > 0$ sufficiently large such that the inequality holds. □

REMARK 3.2.4. *Since $\alpha_2, \alpha_1 < 0$ and $\beta_2, \beta_1 > 0$, we conclude that $\underline{W}(r) \leq \overline{W}(r)$ for all $r \in \mathbb{R}$. Thus, $(\underline{W}, \overline{W})$ is an ordered pair of sub- and supersolution (see Definition A.3.1).*

Since we were able to find a sub- and supersolution, we can apply the Theorem A.3.2 by Fleming and Pang [15] to show the following existence result.

THEOREM 3.2.5. *Let $\varepsilon - 2c_1 > 0$. Then, the ODE* (114) *has a classical solution $\tilde{W} \in C^2(\mathbb{R})$ such that*

$$\underline{W}(r) \leq \tilde{W}(r) \leq \overline{W}(r), \qquad \forall r \in \mathbb{R}, \tag{119}$$

where $\underline{W}(r)$ and $\overline{W}(r)$ are given by (117) *and* (118), *respectively.*

PROOF. First, we define

$$\bar{H}(r, w, p) := \frac{2}{\sigma_2^2}\left[-f(r)p + \varepsilon w - Q(r)\right].$$

Then, ODE (114) can be rewritten as

$$W_{rr} = \bar{H}(r, W, W_r), \qquad r \in \mathbb{R}.$$

Obviously, $\bar{H}(r, w, p)$ is strictly increasing with respect to w, because $\varepsilon > 0$. Let $I_m := [-m, m]$ for $m \in \mathbb{N}$. Moreover, define

$$M \equiv \max\left\{\sup_{r\in I_m} |\overline{W}(r)|, \sup_{r\in I_m} |\underline{W}(r)|\right\}.$$

Let $m \in \mathbb{N}$ be fixed. We have that $|f(r)| \leq \tilde{C}_1(m)$ and $|Q(r)| \leq \tilde{C}_2(m)$ for all $r \in I_m$. Then, for $r \in I_m$ and $|w| \leq 3M$, it holds

$$\begin{aligned}
|\bar{H}(r, w, p)| &\leq \frac{2}{\sigma_2^2}\left(|f(r)||p| + \varepsilon|w| + |Q(r)|\right) \\
&\leq \frac{2}{\sigma_2^2}\tilde{C}_1(m)|p| + \frac{2}{\sigma_2^2}(3\varepsilon M + \tilde{C}_2(m)) \\
&\leq \frac{1}{\sigma_2^2}p^2 + \tilde{C}_3(m),
\end{aligned}$$

where $\tilde{C}_3(m) := \frac{2}{\sigma_2^2}(3\varepsilon M + \tilde{C}_2(m) + \frac{(\tilde{C}_1(m))^2}{2})$. Defining $C_1 := \frac{1}{\sigma_2^2}$ and $C_2(m) := \frac{\tilde{C}_3(m)}{C_1}$, we obtain

$$|\bar{H}(r, w, p)| \leq C_1(p^2 + C_2(m)), \qquad \text{for} \quad r \in I_m, |w| \leq 3M.$$

By the fact that $(\underline{W}, \overline{W})$, given by (117) and (118), is an ordered pair of sub- and supersolution of (116), the assertion follows by Theorem A.3.2. □

Now, using Theorem 3.2.5, we conclude that $v^0(x,r) = \frac{1}{\varepsilon}\log(x) + \tilde{W}(r)$ is a solution of the HJB equation (112). Now, it remains to verify that $v^0(x,r)$ is indeed equal to the value function $V^0(x,r)$ and that the candidates given in (113) are the optimal post-crash strategies. In order to prove such verification result, we first need an analogue estimate for $\tilde{W}_r(r)$ as in [39]. This estimate is given in the following Lemma.

LEMMA 3.2.6. *Let* $\varepsilon - 2c_1 > 0$ *and let* $\tilde{W}(r)$ *be a classical solution of* (114) *such that* $\underline{W}(r) \leq \tilde{W}(r) \leq \overline{W}(r)$ *for all* $r \in \mathbb{R}$, *where* $(\underline{W}, \overline{W})$ *is given by* (117) *and* (118), *respectively. Then,*

$$\tilde{W}_r^2(R) \leq \sum_{i=0}^{3} \nu_{2i} R^{2i}, \quad \forall R \in \mathbb{R},$$

where $\nu_6 > 0$ *and* $\nu_{2i} \geq 0$ $(i = 0, 1, 2)$ *are constants.*

PROOF. The idea of the proof is similar to that in [39, Lemma 1.44] and we refer to Appendix 3.5.1 for the proof. □

THEOREM 3.2.7 (Verification Theorem). *Let* $\varepsilon - 2c_1 > 0$ *and let* $\tilde{W}(r)$ *be a classical solution of* (114) *such that* (119) *holds. Moreover, let*

$$v^0(x,r) = \frac{1}{\varepsilon}\log(x) + \tilde{W}(r).$$

Then

(1) For every strategy $(k^{(0)}, c^{(0)}) \in \Pi$, *with*

$$\mathbb{E}^{x,r}\left(\int_0^\infty e^{-\varepsilon t} |\log(c_t^{(0)}\overline{X}_t)|\, dt\right) < \infty \tag{120}$$

it holds

$$v^0(x,r) \geq \mathbb{E}^{x,r}\left(\int_0^\infty e^{-\varepsilon t} \log(c_t^{(0)}\overline{X}_t)\, dt\right).$$

(2) If $k^{(0)*}(\overline{X}_t, \overline{r}_t) \equiv \frac{\mu^{(0)}}{(\sigma_1^{(0)})^2}$ *and* $c^{(0)*}(\overline{X}_t, \overline{r}_t) \equiv \varepsilon$, *then* $(k^{(0)*}, c^{(0)*}) \in \Pi$ *and*

$$v^0(x,r) = \mathbb{E}^{x,r}\left(\int_0^\infty e^{-\varepsilon t} \log(c_t^{(0)*}\overline{X}_t^*)\, dt\right),$$

where $\overline{X}^* = \{\overline{X}_t^*\}_{t\geq 0}$ *solves SDE* (111) *which is controlled by* $(k^{(0)*}, c^{(0)*})$. *Thus,* $v^0(x,r) = V^0(x,r)$, *where* $V^0(x,r)$ *is the post-crash value function* (110).

PROOF. The proof is technical but standard. Thus, we refer to Appendix 3.5.2 for the proof. □

In the next section, we use the explicit structure of the post-crash value function in order to reformulate the worst-case optimization problem.

3.2.2. Reformulation of the Worst-Case Problem. Analogously to Section 2.4.1, we reformulate the worst-case optimization problem (108) as a pre-crash problem. Let $\tilde{X} = \{\tilde{X}_t\}_{t\geq 0}$

be the wealth process in a crash-free market controlled by an arbitrary admissible pre-crash strategy $(k^{(1)}, c^{(1)})$. That means, $\tilde{X}$ solves

$$d\tilde{X}_t = \tilde{X}_t \left[r_t + \mu^{(1)} k_t^{(1)} - c_t^{(1)} \right] dt + \tilde{X}_t \sigma_1^{(1)} k_t^{(1)} \, dw_{1,t}, \qquad \tilde{X}_0 = x^0, \tag{121}$$
$$dr_t = f(r_t)\, dt + \sigma_2 \, d\tilde{w}_t, \qquad r_0 = r^0.$$

At the crash time τ the investor's wealth is given by $x = (1 - l k_\tau^{(1)}) \tilde{X}_\tau$ and the short rate is given by $r = r_\tau$. The performance of the optimal post-crash strategy at the crash time is then given by $V^0((1 - l k_\tau^{(1)}) \tilde{X}_\tau, r_\tau)$, where $V^0(x, r) = \frac{1}{\varepsilon} \log(x) + \tilde{W}(r)$. Obviously, $V^0(x, r)$ is monotone increasing in x. Thus,

$$V^0((1 - l k_\tau^{(1)}) \tilde{X}_\tau, r_\tau) \geq V^0((1 - l^* k_\tau^{(1)}) \tilde{X}_\tau, r_\tau), \qquad \forall l \in [0, l^*].$$

Since $k_t^{(1)} \geq 0$ for all $t \geq 0$, the worst-case crash size is given by $l = l^*$. Using the inequality above, we reformulate the worst-case optimization problem (108) as a pre-crash problem of the form:

$$\sup_{(k^{(1)}, c^{(1)}) \in \Pi(x^0, r^0)} \inf_{\tau \in \mathcal{C}'} \mathbb{E} \left(\int_0^\tau e^{-\varepsilon t} \log(c_t^{(1)} \tilde{X}_t) \, dt + e^{-\varepsilon \tau} V^0((1 - l^* k_\tau^{(1)}) \tilde{X}_\tau, r_\tau) \right). \tag{122}$$

The pre-crash problem (122) can be interpreted as a controller vs. stopper game, where the investor chooses $(k^{(1)}, c^{(1)})$ and the market chooses the crash time τ. As in Section 2.4, this problem will be solved by a martingale approach developed in [44] for the finite time horizon and in [10] for the infinite time horizon.

In what follows, we write (k, c) instead of $(k^{(1)}, c^{(1)})$ to denote the pre-crash strategy. Moreover, we define the process $M^{k,c} = \{M_t^{k,c}\}_{t \geq 0}$ by:

$$M_t^{k,c} := \int_0^t e^{-\varepsilon s} \log(c_s \tilde{X}_s) \, ds + e^{-\varepsilon t} V^0((1 - l^* k_t) \tilde{X}_t, r_t), \qquad t \geq 0,$$

such that (122) is given by

$$\sup_{(k,c) \in \Pi(x^0, r^0)} \inf_{\tau \in \mathcal{C}'} \mathbb{E} \left(M_\tau^{k,c} \right). \tag{123}$$

3.2.3. The worst-case optimal pre-crash strategy. As in [10] and [44], we use the concept of indifference and the Indifference Optimality Principle to determine the worst-case optimal pre-crash strategy. Here, a pre-crash strategy (k, c) is called indifference strategy if

$$\mathbb{E} \left(M_\tau^{k,c} \right) = \mathbb{E} \left(M_{\tau'}^{k,c} \right)$$

for two stopping times τ, τ'. This definition is similar to the literature [10, 44] where constant interest rates are used and is already applied in Section 2.4.
Moreover, it implies that a pre-crash strategy, which is an indifference strategy, makes the investor indifferent with respect to the crash time τ, because he reaches the same performance for two different (arbitrary) stopping times.
Due to the infinite time horizon and the time-independent market coefficients, we can assume that the worst-case optimal pre-crash strategy (k, c) does not depend on time t. Moreover we assume that it will not depend on the short rate r_t. Below, we will see that the optimal pre-crash strategy fulfills these assumptions.

Now, we formulate a sufficient condition for a strategy to be such indifference strategy.

LEMMA 3.2.8. *Let $\varepsilon - 3c_1 > 0$ and let $(\hat{k}, \hat{c})$ be a constant admissible pre-crash strategy such that $H(\hat{k}, \hat{c}) = 0$, where*

$$H(k,c) := \log\left(\frac{c}{1-l^*k}\right) + \frac{1}{\varepsilon}\left(\mu^{(1)}k - \frac{(\sigma_1^{(1)})^2}{2}k^2\right) - \frac{c}{\varepsilon} - \frac{1}{2\varepsilon}\left(\frac{\mu^{(0)}}{\sigma_1^{(0)}}\right)^2 - \log(\varepsilon) + 1. \tag{124}$$

Then, $M^{\hat{k},\hat{c}}$ is a uniformly integrable martingale and $(\hat{k}, \hat{c}) \in \Pi$ is an indifference strategy for the controller vs. stopper game (123)*.*

REMARK 3.2.9. *Note that for an arbitrary but fixed $\hat{c} > 0$ it holds $\lim_{k \nearrow \frac{1}{l^*}} H(k, \hat{c}) = \infty$ and*

$$H(0,\hat{c}) = \log\left(\frac{\hat{c}}{\varepsilon}\right) - \frac{\hat{c}}{\varepsilon} - \frac{1}{2\varepsilon}\left(\frac{\mu^{(0)}}{\sigma_1^{(0)}}\right)^2 + 1 \leq -\frac{1}{2\varepsilon}\left(\frac{\mu^{(0)}}{\sigma_1^{(0)}}\right)^2 \leq 0.$$

Thus, for each $\hat{c} > 0$ there exists $\hat{k} \in [0, \frac{1}{l^*})$ such that $H(\hat{k}, \hat{c}) = 0$.

PROOF OF LEMMA 3.2.8. Let $(\hat{k}, \hat{c})$ be a solution of $H(\hat{k}, \hat{c}) = 0$. By the definition of $M^{k,c}$, we obtain:

$$dM_t^{\hat{k},\hat{c}} = e^{-\varepsilon t}\log(\hat{c}\tilde{X}_t)\,dt + d\left(e^{-\varepsilon t}V^0((1-l^*\hat{k})\tilde{X}_t, r_t)\right),$$

where $\tilde{X}$ is the wealth process in a crash-free market controlled by $(\hat{k}, \hat{c})$ (see (121)). Using $V^0(x,r) = \frac{1}{\varepsilon}\log(x) + \tilde{W}(r)$ and using that $\tilde{W} \in C^2(\mathbb{R})$, we apply Ito's formula and obtain

$$\begin{aligned} dM_t^{\hat{k},\hat{c}} =& e^{-\varepsilon t}\Bigg\{\log\left(\frac{\hat{c}}{1-l^*\hat{k}}\right) + \frac{1}{\varepsilon}\left(\mu^{(1)}\hat{k} - \frac{(\sigma_1^{(1)})^2}{2}\hat{k}^2\right) - \frac{\hat{c}}{\varepsilon} + \frac{r_t}{\varepsilon} \\ & + f(r_t)\tilde{W}_r(r_t) + \frac{\sigma_2^2}{2}\tilde{W}_{rr}(r_t) - \varepsilon\tilde{W}(r_t)\Bigg\}\,dt \\ & + e^{-\varepsilon t}\left(\frac{\sigma_1^{(1)}}{\varepsilon}\hat{k} + \sigma_2\rho\tilde{W}_r(r_t)\right)\,dw_{1,t} + e^{-\varepsilon t}\sigma_2\sqrt{1-\rho^2}\tilde{W}_r(r_t)\,dw_{2,t}. \end{aligned}$$

Now, we use that $\tilde{W}(r)$ is a solution of the ODE (114) and, therefore, we have

$$\frac{\sigma_2^2}{2}\tilde{W}_{rr}(r_t) + f(r_t)\tilde{W}_r(r_t) - \varepsilon\tilde{W}(r_t) + \frac{r_t}{\varepsilon} = -\frac{1}{2\varepsilon}\left(\frac{\mu^{(0)}}{\sigma_1^{(0)}}\right)^2 - \log(\varepsilon) + 1.$$

Thus,

$$dM_t^{\hat{k},\hat{c}} = e^{-\varepsilon t}\left(\frac{\sigma_1^{(1)}}{\varepsilon}\hat{k} + \sigma_2\rho\tilde{W}_r(r_t)\right)\,dw_{1,t} + e^{-\varepsilon t}\sigma_2\sqrt{1-\rho^2}\tilde{W}_r(r_t)\,dw_{2,t}.$$

Now, we define

$$\begin{aligned} {}_1M_t^{\hat{k},\hat{c}} &:= \int_0^t e^{-\varepsilon s}\left(\frac{\sigma_1^{(1)}}{\varepsilon}\hat{k} + \sigma_2\rho\tilde{W}_r(r_s)\right)\,dw_{1,s}, \\ {}_2M_t^{\hat{k},\hat{c}} &:= \int_0^t e^{-\varepsilon s}\sigma_2\sqrt{1-\rho^2}\tilde{W}_r(r_s)\,dw_{2,s}, \end{aligned}$$

such that $M_t^{\hat{k},\hat{c}} = M_0^{\hat{k},\hat{c}} + {}_1M_t^{\hat{k},\hat{c}} + {}_2M_t^{\hat{k},\hat{c}}$. Let us consider the quadratic variation process of ${}_1M^{\hat{k},\hat{c}}$, which is given by

$$\langle {}_1M^{\hat{k},\hat{c}} \rangle_t = \int_0^t e^{-2\varepsilon s} \left(\frac{\sigma_1^{(1)}}{\varepsilon} \hat{k} + \rho\sigma_2 \tilde{W}_r(r_s) \right)^2 ds$$

and we define

$$\langle {}_1M^{\hat{k},\hat{c}} \rangle_\infty := \int_0^\infty e^{-2\varepsilon s} \left(\frac{\sigma_1^{(1)}}{\varepsilon} \hat{k} + \rho\sigma_2 \tilde{W}_r(r_s) \right)^2 ds.$$

Lemma 3.2.6 implies

$$\begin{aligned} \mathbb{E}\left(\langle {}_1M^{\hat{k},\hat{c}} \rangle_\infty \right) &\leq \frac{1}{\varepsilon^3} (\sigma_1^{(1)} \hat{k})^2 + 2\rho^2\sigma_2^2 \mathbb{E}\left(\int_0^\infty e^{-2\varepsilon s} \tilde{W}_r^2(r_s)\, ds \right) \\ &\leq \frac{1}{\varepsilon^3} (\sigma_1^{(1)} \hat{k})^2 + 2\rho^2\sigma_2^2 \int_0^\infty e^{-2\varepsilon s} \sum_{i=0}^3 \nu_{2i} \mathbb{E}\left(r_s^{2i} \right) ds. \end{aligned}$$

By Lemma A.1.3 in Appendix A, it holds for any integer $m > 0$ and any $\tilde{\epsilon} > 0$ that $\mathbb{E}\left(r_s^{2m}\right) \leq \Lambda_1^{(m)}$ if $c_1 < 0$ and $\mathbb{E}\left(r_s^{2m}\right) \leq \Lambda_2^{(m)} e^{2m(c_1+\tilde{\epsilon})s}$ if $c_1 \geq 0$, where $\Lambda_1^{(m)}, \Lambda_2^{(m)}$ are positive constants independent of time s. Thus, for $c_1 < 0$ we immediately obtain that

$$\mathbb{E}\left(\langle {}_1M^{\hat{k},\hat{c}} \rangle_\infty \right) \leq \frac{1}{\varepsilon^3} (\sigma_1^{(1)} \hat{k})^2 + 2\rho^2\sigma_2^2 \int_0^\infty e^{-2\varepsilon s} \bar{\Lambda}_1\, ds < \infty,$$

where $\bar{\Lambda}_1$ is a positive constant. We assumed that $\varepsilon - 3c_1 > 0$. Thus, if $c_1 > 0$ there exists $\tilde{\epsilon} > 0$ such that $\varepsilon - 3(c_1 + \tilde{\epsilon}) > 0$ and we obtain

$$\mathbb{E}\left(\langle {}_1M^{\hat{k},\hat{c}} \rangle_\infty \right) \leq \frac{1}{\varepsilon^3} (\sigma_1^{(1)} \hat{k})^2 + 2\rho^2\sigma_2^2 \int_0^\infty e^{-2\varepsilon s} \sum_{i=0}^3 \nu_{2i} \Lambda_2^{(i)} e^{2i(c_1+\tilde{\epsilon})s}\, ds < \infty.$$

Analogously to the definition of $\langle {}_1M^{\hat{k},\hat{c}} \rangle_\infty$, we define $\langle {}_2M^{\hat{k},\hat{c}} \rangle_\infty$ and by similar arguments we obtain that

$$\mathbb{E}\left(\langle {}_2M^{\hat{k},\hat{c}} \rangle_\infty \right) < \infty.$$

Now, we apply the Burkholder-Davis-Gundy inequality (see e.g. [43, Chp.IV]) and we conclude that there exist constants C^i $(i = 1, 2)$ such that

$$\mathbb{E}\left(\left(\sup_{t\geq 0} |{}_iM_t^{\hat{k},\hat{c}}| \right)^2 \right) \leq C^i \mathbb{E}\left(\langle {}_iM^{\hat{k},\hat{c}} \rangle_\infty \right) < \infty. \qquad i = 1, 2.$$

Since $M_t^{\hat{k},\hat{c}} = M_0^{\hat{k},\hat{c}} + {}_1M_t^{\hat{k},\hat{c}} + {}_2M_t^{\hat{k},\hat{c}}$, we obtain that the process $M^{\hat{k},\hat{c}}$ is dominated by an integrable random variable $\bar{M}$, that is for all $t \geq 0$ it holds

$$|M_t^{\hat{k},\hat{c}}| \leq |M_0^{\hat{k},\hat{c}}| + \sup_{t\geq 0} |{}_1M_t^{\hat{k},\hat{c}}| + \sup_{t\geq 0} |{}_2M_t^{\hat{k},\hat{c}}| =: \bar{M},$$

with $\mathbb{E}\left(\bar{M}\right) < \infty$. Therefore, $M^{\hat{k},\hat{c}}$ is a uniformly integrable martingale which implies that it is closed by a random variable $M_\infty^{\hat{k},\hat{c}} = \lim_{t\to\infty} M_t^{\hat{k},\hat{c}}$ a.s.(see Theorem A.4.5). We apply Doob's Optional Sampling Theorem (see Theorem A.4.6) and we get:

$$\mathbb{E}\left(M_\tau^{\hat{k},\hat{c}} \right) = \mathbb{E}\left(M_{\tau'}^{\hat{k},\hat{c}} \right).$$

By definition, it follows that $(\hat{k}, \hat{c})$ is an indifference strategy. □

REMARK 3.2.10. *In Section 2.4 we obtained that there exists a uniquely determined indifference strategy which is a solution of an ODE. For the case of maximizing the lifetime consumption, an indifference strategy* $(\hat{k}, \hat{c})$ *has to fulfill* $H(\hat{k}, \hat{c}) = 0$ *and therefore, there exist infinitely many strategies* $(\hat{k}, \hat{c})$ *which are indifference strategies. Thus, the main difference to Section 2.4 is the sufficient condition for a pre-crash strategy to be an indifference strategy: In Section 2.4,* $\hat{k}$ *is the uniquely determined solution of an ODE and in this Chapter,* $(\hat{k}, \hat{c})$ *is an indifference strategy if* $H(\hat{k}, \hat{c}) = 0$.

Now, having a sufficient condition for a pre-crash strategy to be an indifference strategy, we use the notion of an *indifference frontier* which was defined [10] for the infinite time horizon. Note, that we already applied the concept of an indifference frontier and the Indifference Optimality Principle in Section 2.4 and 2.5 for the finite time horizon model. Analogously to [10, Lemma 4.2], let us consider the indifference frontier:
Let $(\hat{k}, \hat{c})$ be an indifference strategy and $(k, c) \in \Pi$ be an arbitrary admissible pre-crash strategy and let $\eta := \inf\{t \geq 0 : k_t > \hat{k}\}$ and let

$$\tilde{k}_t = \begin{cases} k_t & : t < \eta \\ \hat{k} & : t \geq \eta \end{cases}, \qquad \tilde{c}_t = \begin{cases} c_t & : t < \eta \\ \hat{c} & : t \geq \eta \end{cases}.$$

Then, by the same arguments as in the proof of [10, Lemma 4.2], one can show that

$$\inf_{\tau} \mathbb{E}(M_{\tau}^{\tilde{k},\tilde{c}}) \geq \inf_{\tau} \mathbb{E}(M_{\tau}^{k,c}). \tag{125}$$

The proof of the inequality above works in the same way as in the literature because it needs the right continuity of the pre-crash strategy, the fact that $M^{\hat{k},\hat{c}}$ is a uniformly integrable martingale and that $V^0(x, r)$ is monotone increasing in x. By Definition 3.1.1, Theorem 3.2.7 and Lemma 3.2.8 the requirements for the proof are given. As in [10, 44], an indifference strategy $(\hat{k}, \hat{c})$ can be interpreted as a frontier that prevents too bold investment decisions (cf. [10, p.13]).
By (125), we can restrict our considerations on strategies that are dominated by an indifference strategy because all other strategies would provide worse performances. In order to do this, we define the set of such strategies by

$$\mathcal{A}(\hat{k}) := \left\{ (k, c) \in \Pi : k_t \leq \hat{k}, \quad \forall t \geq 0 \right\}.$$

Next, we apply the Indifference Optimality Principle (see [10, Proposition 5.1]) to identify the worst-case optimal pre-crash strategy. This principle provides a sufficient condition for a pre-crash strategy (k, c) to be optimal in the worst-case scenario: An indifference strategy $(\hat{k}, \hat{c}) = (k^*, c^*)$ is the worst-case optimal investment consumption strategy for (108), if it is optimal in the no-crash scenario $\tau = \infty$ in the class of all strategies respecting the associated indifference frontier, that means:

$$\mathbb{E}(M_{\infty}^{k,c}) \leq \mathbb{E}(M_{\infty}^{k^*,c^*}), \qquad \forall\, (k, c) \in \mathcal{A}(k^*).$$

The following Theorem provides the worst-case optimal pre-crash strategy for the controller vs. stopper game (122) by using the Indifference Optimality principle.

THEOREM 3.2.11. *Let $\varepsilon - 3c_1 > 0$ and let $H(k,c)$ be given by* (124). *Moreover, we define*

$$m := \min\left\{\frac{1}{l^*}, k^M\right\} \quad \text{with} \quad k^M := \frac{\mu^{(1)}}{(\sigma_1^{(1)})^2}.$$

(1) *Suppose that there exists $\kappa \in [0,m]$ such that $H(\kappa,\varepsilon) = 0$. Then, κ is uniquely determined and $(k^*,c^*) = (\kappa,\varepsilon)$ is a worst-case optimal pre-crash strategy for* (108).

(2) *If there is no $\kappa \in [0,m]$ such that $H(\kappa,\varepsilon) = 0$, then $(k^*,c^*) = (k^M,\varepsilon)$ is the worst-case optimal pre-crash strategy for* (108).

PROOF. First, by Remark 3.2.9, there exists a $\kappa \in [0, \frac{1}{l^*})$ such that $H(\kappa,\varepsilon) = 0$.
Now, we prove (1):
Assume that there exists a $\kappa \in [0,m]$ such that $H(\kappa,\varepsilon) = 0$. For $k \in [0,m]$ it holds

$$\frac{\partial}{\partial k} H(k,\varepsilon) = \frac{l^*}{1-l^*k} + \frac{1}{\varepsilon}\left(\mu^{(1)} - (\sigma_1^{(1)})^2 k\right) > 0.$$

Therefore, if there exists a root $\kappa \in [0,m]$ of $H(k,\varepsilon)$, then κ is uniquely determined on $[0,m]$. By Lemma 3.2.8, (κ,ε) is an indifference strategy. Now, we show that (κ,ε) is an optimal strategy in the no-crash scenario $\tau = \infty$ in the class $\mathcal{A}(\kappa)$. In order to do this, we consider the following constrained stochastic optimal control problem:

$$\sup_{(k,c)\in\mathcal{A}(\kappa)} \mathbb{E}\left(M_\infty^{k,c}\right) = \sup_{(k,c)\in\mathcal{A}(\kappa)} \mathbb{E}\left(\int_0^\infty e^{-\varepsilon t}\log(c_t\tilde{X}_t)\,dt\right). \tag{126}$$

Similar to [27], where the authors showed that investment and consumption decisions can be separated for general optimal control problems with logarithmic utility function, we obtain for an arbitrary but fixed admissible pre-crash strategy $(k,c) \in \mathcal{A}(\kappa)$:

$$\begin{aligned}
&\mathbb{E}\left(\int_0^\infty e^{-\varepsilon t}\log(c_t\tilde{X}_t)\,dt\right)\\
&= \mathbb{E}\left(\int_0^\infty e^{-\varepsilon t}\left(\log(c_t) - \int_0^t c_s\,ds\right)\,dt\right)\\
&\quad + \mathbb{E}\left(\int_0^\infty e^{-\varepsilon t}\left(\log(x^0) + \int_0^t r_s + \psi(k_s)\,ds + \int_0^t \sigma_1^{(1)} k_s\,dw_{1,s}\right)\,dt\right),
\end{aligned}$$

where $\psi(k) := \mu^{(1)}k - \frac{(\sigma_1^{(1)})^2}{2}k^2$. Thus, maximizing the expectation in (126) is equivalent to maximize the first and the second summand above to obtain the optimal consumption and investment decision, respectively. Considering the first summand above, the optimal consumption strategy is given by $c = \varepsilon$ (see e.g. [27, Thm.2]). Since ψ is concave and strictly monotone increasing for $k < k^M$ and since, by assumption, $\kappa \le k^M$, we obtain for all strategies $k \in \mathcal{A}(\kappa)$ that $\psi(k_s) \le \psi(\kappa)$ for all $s \ge 0$ and

$$\begin{aligned}
&\mathbb{E}\left(\int_0^t \psi(k_s)\,ds + \int_0^t \sigma_1^{(1)} k_s\,dw_{1,s}\right) = \mathbb{E}\left(\int_0^t \psi(k_s)\,ds\right)\\
&\le \mathbb{E}\left(\int_0^t \psi(\kappa)\,ds\right) = \mathbb{E}\left(\int_0^t \psi(\kappa)\,ds + \int_0^t \sigma_1^{(1)}\kappa\,dw_{1,s}\right).
\end{aligned}$$

This implies that the second summand fulfills the inequality

$$\begin{aligned}&\mathbb{E}\left(\int_0^\infty e^{-\varepsilon t}\left(\log(x^0)+\int_0^t r_s+\psi(k_s)\,ds+\int_0^t \sigma_1^{(1)}k_s\,dw_{1,s}\right)\,dt\right)\\&\leq \mathbb{E}\left(\int_0^\infty e^{-\varepsilon t}\left(\log(x^0)+\int_0^t r_s+\psi(\kappa)\,ds+\int_0^t \sigma_1^{(1)}\kappa\,dw_{1,s}\right)\,dt\right)\end{aligned}$$

for all $k\in\mathcal{A}(\kappa)$ and therefore (κ,ε) is the optimal strategy for the constrained problem (126). Now, we apply the Indifference Optimality Principle (see e.g. [10, Prop.5.1]) and obtain that $(k^*,c^*)=(\kappa,\varepsilon)$ is the optimal strategy for the controller vs. stopper game (122), because

$$\inf_{\tau\in\mathcal{C}'}\mathbb{E}\left(M_\tau^{k,c}\right)\leq\mathbb{E}\left(M_\infty^{k,c}\right)\leq\mathbb{E}\left(M_\infty^{\kappa,\varepsilon}\right)=\inf_{\tau\in\mathcal{C}'}\mathbb{E}\left(M_\tau^{\kappa,\varepsilon}\right),\quad\forall\,(k,c)\in\mathcal{A}(\kappa).$$

Note that the second inequality holds because (κ,ε) is optimal in the no-crash scenario and the equality above is true because (κ,ε) is an indifference strategy.
Thus, (κ,ε) is a worst-case optimal pre-crash strategy for problem (108).

Now, we prove (2):
Suppose that there is no $\kappa\in[0,m]$ such that $H(\kappa,\varepsilon)=0$. Since $H(0,\varepsilon)<0$, it follows that $H(k^M,\varepsilon)<0$. By Ito's formula, we obtain

$$M_t^{k^M,\varepsilon}=M_0^{k^M,\varepsilon}+\frac{1}{\varepsilon}H(k^M,\varepsilon)(1-e^{-\varepsilon t})+{}_1M_t^{k^M,\varepsilon}+{}_2M_t^{k^M,\varepsilon},$$

where

$$\begin{aligned}{}_1M_t^{k^M,\varepsilon}&:=\int_0^t e^{-\varepsilon s}\left(\frac{\sigma_1^{(1)}}{\varepsilon}k^M+\sigma_2\rho\tilde{W}_r(r_s)\right)\,dw_{1,s},\\{}_2M_t^{k^M,\varepsilon}&:=\int_0^t e^{-\varepsilon s}\sigma_2\sqrt{1-\rho^2}\tilde{W}_r(r_s)\,dw_{2,s}.\end{aligned}$$

In order to apply Doob's Optional Sampling Theorem, we show that $M^{k^M,\varepsilon}=\{M_t^{k^M,\varepsilon}\}_{t\geq0}$ is a supermartingale which is closed by a random variable (we refer to Definition A.4.3 for details). Since (k^M,ε) is time independent, one can apply the same arguments as in the proof of Lemma 3.2.8 in order to show that the processes ${}_1M^{k^M,\varepsilon}$ and ${}_1M^{k^M,\varepsilon}$ are uniformly integrable martingales with

$$\mathbb{E}\left(\sup_{t\geq0}|{}_iM_t^{k^M,\varepsilon}|\right)<\infty,\qquad i=1,2.$$

Using the martingale property of ${}_1M^{k^M,\varepsilon}$ and ${}_2M^{k^M,\varepsilon}$ and $H(k^M,\varepsilon)<0$ we obtain

$$\mathbb{E}\left(M_t^{k^M,\varepsilon}|\mathcal{F}_s\right)\leq M_0^{k^M,\varepsilon}+\frac{1}{\varepsilon}H(k^M,\varepsilon)(1-e^{-\varepsilon s})+{}_1M_s^{k^M,\varepsilon}+{}_2M_s^{k^M,\varepsilon}=M_s^{k^M,\varepsilon},$$

for $s\leq t$, which implies that $M^{k^M,\varepsilon}$ is a supermartingale. Since ${}_1M^{k^M,\varepsilon}$ and ${}_2M^{k^M,\varepsilon}$ are uniformly integrable martingales, we obtain by Theorem A.4.5 that ${}_1M^{k^M,\varepsilon}:=\lim_{t\to\infty}{}_1M_t^{k^M,\varepsilon}$ and ${}_2M^{k^M,\varepsilon}:=\lim_{t\to\infty}{}_2M_t^{k^M,\varepsilon}$ a.s. exist and $\mathbb{E}|{}_1M_\infty^{k^M,\varepsilon}|<\infty$ and $\mathbb{E}|{}_2M_\infty^{k^M,\varepsilon}|<\infty$, and therefore,

$$M_\infty^{k^M,\varepsilon}:=\lim_{t\to\infty}M_t^{k^M,\varepsilon}=M_0^{k^M,\varepsilon}+\frac{1}{\varepsilon}H(k^M,\varepsilon)+{}_1M_\infty^{k^M,\varepsilon}+{}_2M_\infty^{k^M,\varepsilon}$$

a.s. exists and $\mathbb{E}|M_\infty^{k^M,\varepsilon}| < \infty$. Moreover, for each $t \geq 0$ it holds

$$\begin{aligned} M_t^{k^M,\varepsilon} &> M_0^{k^M,\varepsilon} + \frac{1}{\varepsilon} H(k^M,\varepsilon) + {}_1M_t^{k^M,\varepsilon} + {}_2M_t^{k^M,\varepsilon} \\ &= M_0^{k^M,\varepsilon} + \frac{1}{\varepsilon} H(k^M,\varepsilon) + \mathbb{E}\left({}_1M_\infty^{k^M,\varepsilon}|\mathcal{F}_t\right) + \mathbb{E}\left({}_2M_\infty^{k^M,\varepsilon}|\mathcal{F}_t\right) \quad = \mathbb{E}\left(M_\infty^{k^M,\varepsilon}|\mathcal{F}_t\right). \end{aligned}$$

Definition A.4.3 implies that $M^{k^M,\varepsilon}$ is a supermartingale which is closed by the random variable $M_\infty^{k^M,\varepsilon}$. By Doob's Optional Sampling Theorem (see e.g. Theorem A.4.6), we obtain

$$\mathbb{E}\left(M_\tau^{k^M,\varepsilon}\right) \geq \mathbb{E}\left(M_\infty^{k^M,\varepsilon}\right) \tag{127}$$

for all stopping times τ. This inequality implies that $\tau = \infty$ is a worst-case scenario for an investor who follows the strategy (k^M, ε) before the market crash.
Moreover, we have that (k^M, ε) is optimal in the no-crash scenario, that is, (k^M, ε) is the optimal control of the problem

$$\sup_{(k,c)\in\Pi(x^0,r^0)} \mathbb{E}\left(M_\infty^{k,c}\right) = \sup_{(k,c)\in\Pi(x^0,r^0)} \mathbb{E}\left(\int_0^\infty e^{-\varepsilon t}\log(c_t\tilde{X}_t)\,dt\right).$$

Finally, we obtain

$$\inf_{\tau\in\mathcal{C}'} \mathbb{E}\left(M_\tau^{k,c}\right) \leq \mathbb{E}\left(M_\infty^{k,c}\right) \leq \mathbb{E}\left(M_\infty^{k^M,\varepsilon}\right) \leq \inf_{\tau\in\mathcal{C}'} \mathbb{E}\left(M_\tau^{k^M,\varepsilon}\right), \quad \forall\,(k,c)\in\Pi.$$

Note, that the second inequality holds because (k^M, ε) is optimal in the no-crash scenario and (127) implies the third inequality. Thus, (k^M, ε) is the optimal strategy for the controller vs. stopper game (122) and therefore, it is the worst-case optimal strategy for problem (108). ☐

EXAMPLE 3.2.12. *Let us consider a market which becomes worse after the market crash has happened. If for example* $\mu^{(1)} \geq \mu^{(0)}$ *and* $\sigma_1^{(1)} \leq \sigma_1^{(0)}$ *then there exists* $\kappa \in [0, m]$ *such that* $H(\kappa, \varepsilon) = 0$*, because* $H(0, \varepsilon) < 0$*,* $H(k^M, \varepsilon) > 0$ *and* $H(k, \varepsilon)$ *is strictly monotone increasing for* $k \in [0, m]$*. Thus, we can apply part (1) of the Theorem above and obtain that* (κ, ε) *is the worst-case optimal pre-crash strategy for problem* (108)*.*

3.3. The general affine short rate model

In Section 2.5, we investigated the finite time horizon worst-case optimization problem under an affine short rate model. Here again, we consider this short rate model for the infinite horizon worst-case optimization problem (108). We assume that the short rate process $\{r_t\}_{t\geq 0}$ is a solution of the SDE

$$dr_t = (\lambda_1 r_t + \lambda_2)\,dt + \sqrt{\xi_1 r_t + \xi_2}\,d\tilde{w}_t, \qquad r_0 = r^0, \tag{128}$$

where the constants $\lambda_1, \lambda_2, \xi_1, \xi_2$ fulfill condition (149) in Proposition A.1.2 in Appendix A such that there exists a uniquely determined solution of the SDE above. Again, we will determine the optimal pre- and post-crash strategy of the worst-case optimization problem (108) by means of the same steps as in the previous section. First, let us consider the post-crash optimization problem.

3.3.1. The post-crash optimization problem.

The aim of this section is to find the optimal post-crash Strategy $(k^{(0)*}, c^{(0)*})$ under the affine short rate model. As in section 3.2.1, we use standard stochastic optimal control theory. Analogously, for $(x, r) \in \mathbb{R}_+ \times \mathbb{R}$ we define the post-crash value function as follows:

$$V^0(x,r) = \sup_{(k^{(0)},c^{(0)})\in\Pi(x,r)} \mathbb{E}\left(\int_0^\infty e^{-\varepsilon t}\log(c_t^{(0)}\overline{X}_t)\,dt\right) \tag{129}$$

with respect to the post-crash wealth and short rate dynamics:

$$\begin{aligned} d\overline{X}_t &= \overline{X}_t\left[\overline{r}_t + \mu^{(0)}k_t^{(0)} - c_t^{(0)}\right]dt + \overline{X}_t\sigma_1^{(0)}k_t^{(0)}\,dw_{1,t}, & \overline{X}_0 &= x,\\ d\overline{r}_t &= (\lambda_1 r_t + \lambda_2)\,dt + \sqrt{\xi_1 r_t + \xi_2}\,(\rho\,dw_{1,t} + \sqrt{1-\rho^2}\,dw_{2,t}), & \overline{r}_0 &= r. \end{aligned} \tag{130}$$

The corresponding HJB equation to the post-crash problem above is given by:

$$\begin{aligned} &\sup_{k^{(0)}\in A}\left[\mu^{(0)}k^{(0)}xv_x^0(x,r) + \frac{(\sigma_1^{(0)})^2}{2}(k^{(0)})^2x^2v_{xx}^0(x,r) + \rho\sigma_1^{(0)}\sqrt{\xi_1 r+\xi_2}\,k^{(0)}xv_{xr}^0(x,r)\right]\\ &+\sup_{c^{(0)}\geq 0}\left[\log(c^{(0)}x) - c^{(0)}xv_x^0(x,r)\right] + rxv_x^0(x,r)\\ &+(\lambda_1 r+\lambda_2)v_r^0(x,r) + \frac{\xi_1 r+\xi_2}{2}v_{rr}^0(x,r) - \varepsilon v^0(x,r) = 0, \qquad (t,x)\in\mathbb{R}_+\times\mathbb{R}. \end{aligned}$$

By the standard ansatz for the Log utility case, we set $v^0(x,r) = B\log(x) + W(r)$, where $B\in\mathbb{R}, W\in C^2(\mathbb{R})$. Then, we obtain:

$$\begin{aligned} &\sup_{k^{(0)}\in A}\left[\mu^{(0)}k^{(0)}B - \frac{(\sigma_1^{(0)})^2}{2}(k^{(0)})^2B\right] + \sup_{c^{(0)}\geq 0}\left[\log(c^{(0)}) - c^{(0)}B\right] + rB\\ &+(\lambda_1 r+\lambda_2)W_r(r) + \frac{\xi_1 r+\xi_2}{2}W_{rr}(r) - \varepsilon\left(B\log(x) + W(r)\right) + \log(x) = 0, \quad r\in\mathbb{R}. \end{aligned}$$

As in the previous section, we eliminate x by choosing $B = \frac{1}{\varepsilon}$ and obtain the candidates

$$k^{(0)*} = \frac{\mu^{(0)}}{(\sigma_1^{(0)})^2}, \qquad c^{(0)*} = \varepsilon.$$

Due to the fact that the stochastic control $k^{(0)}$ is not coupled with short rate in the wealth equation (130), we obtain the same candidates for the optimal post-crash strategy as in Section 3.2.1. Inserting the candidates leads again to an ODE for W:

$$\frac{\xi_1 r+\xi_2}{2}W_{rr}(r) + (\lambda_1 r+\lambda_2)W_r(r) - \varepsilon W(r) + Q(r) = 0, \qquad r\in\mathbb{R}, \tag{131}$$

where $Q(r)$ is given by (115). In comparison to (114), the coefficient of the second derivative in (131) depends on r, whereas the coefficient of the first derivative is explicitly given as a linear function in r. Under the assumption that $\varepsilon \neq \lambda_1$, we assume that the solution of (131) is a linear function in r, that is

$$\tilde{W}(r) = a_1 r + a_0. \tag{132}$$

Inserting $\tilde{W}$ and its derivatives in (131) and comparing the coefficients, leads to

$$a_1 = \frac{1}{\varepsilon(\varepsilon-\lambda_1)}, \qquad a_0 = \frac{1}{\varepsilon}\left[\frac{\lambda_2}{\varepsilon(\varepsilon-\lambda_1)} + \frac{1}{2\varepsilon}\left(\frac{\mu^{(0)}}{\sigma_1^{(0)}}\right)^2 + \log(\varepsilon) - 1\right].$$

Thus, we find an explicit solution of the HJB equation given by $v^0(x,r) = \frac{1}{\varepsilon}\log(x) + \tilde{W}(r)$. The following theorem implies that v^0 is indeed equal to the post-crash value function and that the candidates $k^{(0)*}$ and $c^{(0)*}$ are the optimal post-crash strategies.

THEOREM 3.3.1. *Let $\varepsilon - \lambda_1 > 0$ and let $\tilde{W}$ be given by* (132)*. Moreover, let $v^0(x,r) = \frac{1}{\varepsilon}\log(x) + \tilde{W}(r)$. Then, the assertion (1) and (2) in Theorem 3.2.7 hold. Thus, $v^0(x,r) = V^0(x,r)$, where $V^0(x,r)$ is the post-crash value function given by* (129)*.*

PROOF. The method of the proof of Theorem 3.2.7 carries over to this proof. We have used only the fact that $\mathbb{E}(r_s)$ and $\mathbb{E}\left(r_s^2\right)$ are given in Proposition A.1.2 in Appendix A. □

3.3.2. Reformulation of the worst-case problem. For the sake of brevity, we write (k,c) instead of $(k^{(1)}, c^{(1)})$ for the pre-crash strategies and we define for $t \geq 0$:

$$M_t^{k,c} := \int_0^t e^{-\varepsilon s}\log(c_s\tilde{X}_s)\,ds + e^{-\varepsilon t}V^0((1-l^*k_t)\tilde{X}_t, r_t), \qquad t \geq 0,$$

where the wealth and the short rate evolve as

$$d\tilde{X}_t = \tilde{X}_t\left[r_t + \mu^{(1)}k_t - c_t\right]dt + \tilde{X}_t\sigma_1^{(1)}k_t\,dw_{1,t}, \qquad X_0 = x^0 > 0,$$
$$dr_t = (\lambda_1 r_t + \lambda_2)\,dt + \sqrt{\xi_1 r_t + \xi_2}(\rho\,dw_{1,t} + \sqrt{1-\rho^2}\,dw_{2,t}), \qquad r_0 = r^0 > 0.$$

Since $V^0(x,r)$ is strictly monotone increasing in x, and using the same arguments as in Section 3.2.2, we reformulate the worst-case optimization problem (108) as a controller vs. stopper game of the form

$$\sup_{(k,c)\in\Pi(x^0,r^0)}\inf_{\tau\in\mathcal{C}'}\mathbb{E}\left(M_\tau^{k,c}\right). \tag{133}$$

3.3.3. The worst-case optimal pre-crash strategy. We obtain a sufficient condition for a pre-crash strategy to be an indifference strategy.

LEMMA 3.3.2. *Let $\varepsilon - \lambda_1 > 0$, and let $(\hat{k},\hat{c})$ be a constant pre-crash strategy such that $H(\hat{k},\hat{c}) = 0$, where H is given by* (124)*. Then, $M^{\hat{k},\hat{c}} = \{M^{\hat{k},\hat{c}}\}_{t\geq 0}$ is a uniformly integrable martingale and $(\hat{k},\hat{c}) \in \Pi$ is an indifference strategy for the controller vs. stopper game* (133)*.*

PROOF. The proof works with the same ideas as in Lemma 3.2.8. Using that $V^0(x,r) = \frac{1}{\varepsilon}\log(x) + \tilde{W}(r)$, where $\tilde{W}(r) = a_1 r + a_0$, and applying Ito's formula leads to

$$\begin{aligned} dM_t^{\hat{k},\hat{c}} =& e^{-\varepsilon t}\Bigg[\log\left(\frac{\hat{c}}{1-l^*\hat{k}}\right) + \frac{1}{\varepsilon}\left(\mu^{(1)}\hat{k} - \frac{(\sigma_1^{(1)})^2}{2}\hat{k}^2\right) - \frac{\hat{c}}{\varepsilon} + \frac{r_t}{\varepsilon} \\ & \qquad + (\lambda_1 r_t + \lambda_2)\tilde{W}_r(r_t) + \frac{\xi_1 r_t + \xi_2}{2}\tilde{W}_{rr}(r_t) - \varepsilon\tilde{W}(r_t)\Bigg]dt \\ & + e^{-\varepsilon t}\left(\frac{\sigma_1^{(1)}}{\varepsilon}\hat{k} + \sqrt{\xi_1 r_t + \xi_2}\rho\tilde{W}_r(r_t)\right)dw_{1,t} \\ & + e^{-\varepsilon t}\sqrt{\xi_1 r_t + \xi_2}\sqrt{1-\rho^2}\tilde{W}_r(r_t)\,dw_{2,t}. \end{aligned}$$

Now, under the condition that $\varepsilon \neq \lambda_1$, we have that $\tilde{W}(r) = a_1 r + a_0$ is a solution of the ODE (131) and therefore, since $H(\hat{k},\hat{c}) = 0$, the dt coefficient vanishes and it remains to show that

$$M_t^{\hat{k},\hat{c}} = M_0^{\hat{k},\hat{c}} + {}_1M_t^{\hat{k},\hat{c}} + {}_2M_t^{\hat{k},\hat{c}},$$

where

$$ {}_1M_t^{\hat{k},\hat{c}} := \int_0^t e^{-\varepsilon s} \left(\frac{\sigma_1^{(1)}}{\varepsilon} \hat{k} + \rho a_1 \sqrt{\xi_1 r_t + \xi_2} \right) dw_{1,s}, $$

$$ {}_2M_t^{\hat{k},\hat{c}} := \int_0^t e^{-\varepsilon s} \sqrt{1-\rho^2} a_1 \sqrt{\xi_1 r_s + \xi_2}\, dw_{2,s}, $$

is a uniformly integrable martingale. Note, that the difference to the proof of Lemma 3.2.8 is that the square root of r_t occurs in the stochastic integrals. Again, we consider the quadratic variation process of ${}_1M^{\hat{k},\hat{c}}$ and ${}_2M^{\hat{k},\hat{c}}$ and obtain

$$ \langle {}_1M^{\hat{k},\hat{c}} \rangle_\infty = \int_0^\infty e^{-2\varepsilon s} \left(\frac{\sigma_1^{(1)}}{\varepsilon} \hat{k} + \rho a_1 \sqrt{\xi_1 r_t + \xi_2} \right)^2 ds, $$

and therefore,

$$ \mathbb{E}\left(\langle {}_1M^{\hat{k},\hat{c}} \rangle_\infty \right) \le \frac{1}{\varepsilon^3} (\sigma_1^{(1)} \hat{k})^2 + \frac{1}{\varepsilon} (\rho a_1)^2 \xi_2 + 2(\rho a_1)^2 \xi_1 \mathbb{E}\left(\int_0^\infty e^{-2\varepsilon s} r_s \, ds \right). $$

Proposition A.1.2 in Appendix A implies that

$$ \mathbb{E}(r_s) = r^0 e^{\lambda_1 s} + \frac{\lambda_2}{\lambda_1} \left(e^{\lambda_1 s} - 1 \right). $$

Under the assumption that $\varepsilon - \lambda_1 > 0$, we obtain that

$$ \mathbb{E}\left(\langle {}_1M^{\hat{k},\hat{c}} \rangle_\infty \right) < \infty. $$

Analogously, we obtain $\mathbb{E}\left(\langle {}_2M^{\hat{k},\hat{c}} \rangle_\infty \right) < \infty$. Again, by the Burkholder-Davis-Gundy inequality it follows

$$ \mathbb{E}\left(\sup_{t\ge 0} |{}_1M_t^{\hat{k},\hat{c}}| \right) < \infty, \qquad \mathbb{E}\left(\sup_{t\ge 0} |{}_2M_t^{\hat{k},\hat{c}}| \right) < \infty, $$

such that $M^{\hat{k},\hat{c}}$ is dominated by an integrable random variable. It follows that $M^{\hat{k},\hat{c}}$ is a uniformly integrable martingale. Theorem A.4.5 implies that it is closed by a random variable $M_\infty^{\hat{k},\hat{c}} := \lim_{t\to\infty} M_t^{\hat{k},\hat{c}}$. Doob's Optional Sampling Theorem (see Theorem A.4.6) implies

$$ \mathbb{E}\left(M_\tau^{\hat{k},\hat{c}} \right) = \mathbb{E}\left(M_{\tau'}^{\hat{k},\hat{c}} \right) $$

for two stopping times τ, τ'. Therefore, the pre-crash strategy $(\hat{k}, \hat{c})$ is an indifference strategy. □

REMARK 3.3.3. *Lemma 3.3.2 provides a sufficient condition for a pre-crash strategy to be an indifference strategy. This condition does not differ from the indifference condition in Section 3.2. That means the short rate model has no influence on the indifference condition. By means of the indifference frontier, which we already explained in Section 3.2, we know that the optimal pre-crash strategy (k^*, c^*) has to be an element of the set*

$$ \mathcal{A}(\hat{k}) := \left\{ (k,c) \in \Pi : k_t \le \hat{k}, \quad \forall t \ge 0 \right\}. $$

THEOREM 3.3.4. *Let $\varepsilon - \lambda_1 > 0$ and let $H(k,c)$ be given by* (124). *Then, the worst-case optimal pre-crash strategy for problem* (108) *under the affine short rate model is determined by statement (1) and (2) of Theorem 3.2.11.*

PROOF.
(1): By Lemma 3.3.2, we have that $M^{\kappa,\varepsilon}$ is a uniformly integrable martingale. Then, the proof follows by the same steps as in the proof of Theorem 3.2.11.
(2): Since ${}_1M^{k^M,\varepsilon}$ and ${}_2M^{k^M,\varepsilon}$ are uniformly integrable martingales (we refer to the proof of Lemma 3.3.2), we use the same steps as in Theorem 3.2.11 to show that $M^{k^M,\varepsilon}$ is a supermartingale which is closed by a random variable. The remaining steps of the proof are equal to the previous section. □

3.3.4. Discussion and numerical examples. Theorem 3.2.11 and Theorem 3.3.4 provide the optimal pre-crash strategy $(k^{(1)*}, c^{(1)*})$ for the generalized Vasicek model and for the affine short rate model, respectively. For both models, it is optimal to invest a constant fraction of wealth κ (in case (1)) or k^M (in case (2)) in the stock and to consume at a rate $c^{(1)*} = \varepsilon$ before the market crash. After the crash has happened, it is optimal to invest a fraction $k^{(0)*} = \frac{\mu^{(0)}}{(\sigma_1^{(0)})^2}$ and continue consuming at a rate $c^{(0)*} = \varepsilon$.

REMARK 3.3.5. *The following remarks are valid for the worst-case optimization problem* (108) *both under the generalized Vasicek model from Section 3.2 and the affine short rate model of this section.*

a) As in Section 2.5 the optimal strategies neither depend on the short rate $r_t(\omega)$ itself nor on the parameters which determine the short rate equation. This is due to the logarithmic utility function which eliminates the stochastic interest rate risk.
b) An investor with logarithmic utility function can separate the consumption decision from the investment decision (cf. [27]) such that it is optimal to consume at a rate ε before and after the market crash.

In the following example, we give a short illustration of the optimal strategies. In Figure 3.1 and Figure 3.2, we assume that the maximum crash size is given by $l^* = 0.4$ and the discount factor is given by $\varepsilon = 0.1$. Therein, we calculate the optimal investment strategy in a crash-free market k^M, the optimal pre-crash strategy $k^{(1)*}$, the optimal post-crash strategy $k^{(0)*}$ and the optimal consumption strategy $c^{(0)*} = c^{(1)*}$. In Figure 3.1 we assume that the market after the crash is worse than before (higher volatility). In this case, there exists $\kappa \in [0, m]$ such that $H(\kappa, \varepsilon) = 0$ (see Example 3.2.12). Part (1) of Theorem 3.2.11 implies that it is optimal to invest and consume along the indifference strategy (κ, ε) before the market crash. Contrary, in Figure 3.2, we assume that the volatility after the crash is lower than before. Here, we obtain that $\kappa > m$ such that we apply part (2) of Theorem 3.2.11, which implies that it is optimal to invest a fraction k^M of wealth in the stock before the crash.

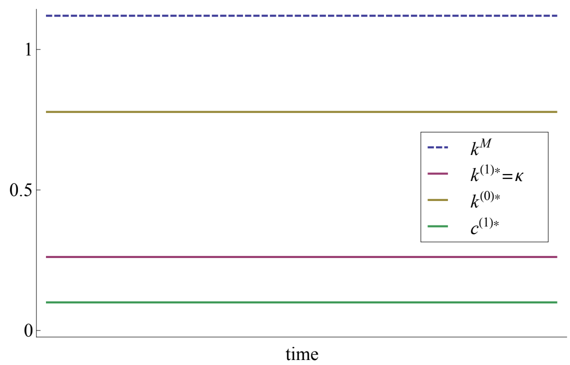

Figure 3.1. Optimal strategies with market parameters $\mu^{(1)} = 0.07$, $\sigma_1^{(1)} = 0.25$, $\mu^{(0)} = 0.07$, $\sigma_1^{(0)} = 0.3$.

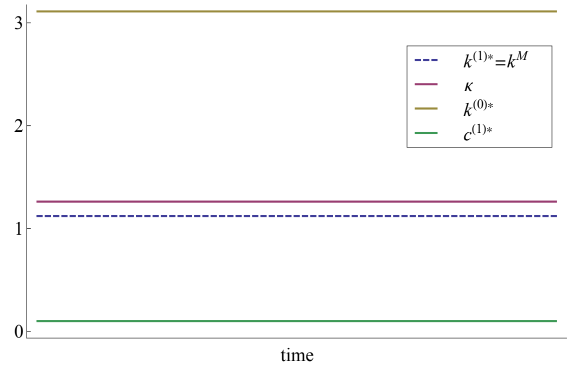

Figure 3.2. Optimal strategies with market parameters $\mu^{(1)} = 0.07$, $\sigma_1^{(1)} = 0.25$, $\mu^{(0)} = 0.07$, $\sigma_1^{(0)} = 0.15$.

3.4. Uncertain post-crash parameters

Here, we extend the worst-case optimization problem of Section 3.3. Therein, we assumed that the post-crash parameters $\mu^{(0)}$ and $\sigma_1^{(0)}$ are given quantities. That means, we assumed that the investor has full information about the market, especially about the drift and the volatility of the asset price process, after a significant market crash, that is modeled as a 'once in a lifetime' event. Thus, it is self-evident to assume that the drift and the volatility are also uncertain parameters. The modeling of uncertain post-crash parameters and the corresponding worst-case optimization problem will be subject of this section.

The stock price process $P = \{P_t\}_{t\geq 0}$ is again given as in Section 3.1. But now, we assume that the post-crash parameters $(\mu^{(0)}, \sigma_1^{(0)})$ are $\mathcal{F}_\tau$- measurable random variables on a given interval. That is, we assume that

$$(\mu^{(0)}, \sigma_1^{(0)}) \in [\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}] =: \mathcal{P},$$

where $0 \leq \underline{\mu} \leq \overline{\mu}$ and $0 < \underline{\sigma} \leq \overline{\sigma}$ are given.
As before, we assume that the risky asset loses a fraction $l \in [0, l^*]$ of its value at the crash time τ and we assume that pre-crash parameters $(\mu^{(1)}, \sigma_1^{(1)})$ will change at the crash time to $(\mu^{(0)}, \sigma_1^{(0)})$. In order to allow that the market parameters do not change, we assume that

$$(\mu^{(1)}, \sigma_1^{(1)}) \in \mathcal{P}. \tag{134}$$

The investor takes a cautious attitude towards the uncertainty about the market crash (τ, l) and towards the uncertainty about the post-crash parameters $(\mu^{(0)}, \sigma_1^{(0)})$. He wants to maximize his expected discounted utility of consumption over an infinite time interval in the worst-case scenario with respect to $(\tau, l) \in \mathcal{C}'$ and $(\mu^{(0)}, \sigma_1^{(0)}) \in \mathcal{P}$. Then, the corresponding worst-case optimization problem is given by:

$$\sup_{(k,c)\in\Pi(x^0,r^0)} \inf_{\substack{(\tau,l)\in\mathcal{C}' \\ (\mu^{(0)},\sigma_1^{(0)})\in\mathcal{P}}} \mathbb{E}\left(\int_0^\infty e^{-\varepsilon t} \log(c_t X_t)\, dt\right), \tag{135}$$

where

$$
\begin{aligned}
X_0 &= x^0 > 0, \\
dX_t &= X_t \left[r_t + \mu^{(1)} k_t^{(1)} - c_t^{(1)} \right] dt + X_t \sigma_1^{(1)} k_t^{(1)} \, dw_{1,t}, \quad t \in (0, \tau), \\
X_\tau &= (1 - l k_\tau^{(1)}) X_{\tau-}, \\
dX_t &= X_t \left[r_t + \mu^{(0)} k_t^{(0)} - c_t^{(0)} \right] dt + X_t \sigma_1^{(0)} k_t^{(0)} \, dw_{1,t}, \quad t \in (\tau, \infty],
\end{aligned}
$$

and the short rate process $\{r_t\}_{t \geq 0}$ is assumed to be a solution of SDE (128). The set of admissible controls Π is given by Definition 3.1.1 and the set of market crash scenarios $\mathcal{C}'$ is given by (107).

The Post-Crash Optimization Problem.
Suppose that the investor has wealth $x > 0$ and interest rate r at the crash time τ. Again, he is faced with a stochastic optimal control problem over an infinite time interval starting with initial wealth x and interest rate r. The post-crash wealth dynamics depends on the post-crash parameters $(\mu^{(0)}, \sigma_1^{(0)})$. Thus, the post-crash value function now depends on x and r and on $(\mu^{(0)}, \sigma_1^{(0)})$ and we define

$$
V^0(x, r, \mu^{(0)}, \sigma_1^{(0)}) := \sup_{(k^{(0)}, c^{(0)}) \in \Pi(x,r)} \mathbb{E} \left(\int_0^\infty e^{-\varepsilon t} \log(c_t^{(0)} \overline{X}_t) \, dt \right)
$$

with respect to the post-crash dynamics:

$$
\begin{aligned}
d\overline{X}_t &= \overline{X}_t \left[\overline{r}_t + \mu^{(0)} k_t^{(0)} - c_t^{(0)} \right] dt + \overline{X}_t \sigma_1^{(0)} k_t^{(0)} \, dw_{1,t}, & \overline{X}_0 &= x, \\
d\overline{r}_t &= (\lambda_1 r_t + \lambda_2) \, dt + \sqrt{\xi_1 r_t + \xi_2} (\rho \, dw_{1,t} + \sqrt{1 - \rho^2} \, dw_{2,t}), & \overline{r}_0 &= r.
\end{aligned}
$$

For given post-crash parameters $(\mu^{(0)}, \sigma_1^{(0)})$, which are available at the crash time, the stochastic optimal control problem can be solved as in Section 3.3.1. Assuming that $\varepsilon - \lambda_1 > 0$, we obtain that the optimal post-crash strategy is given by

$$
k^{(0)*}(\mu^{(0)}, \sigma_1^{(0)}) = \frac{\mu^{(0)}}{(\sigma_1^{(0)})^2}, \qquad c^{(0)*}(\mu^{(0)}, \sigma_1^{(0)}) = \varepsilon,
$$

and the post-crash value function is given by

$$
V^0(x, r, \mu^{(0)}, \sigma_1^{(0)}) = \frac{1}{\varepsilon} \log(x) + \tilde{W}(r, \mu^{(0)}, \sigma_1^{(0)}),
$$

where $\tilde{W}(r, \mu^{(0)}, \sigma_1^{(0)})$ solves ODE (131) and is given by

$$
\tilde{W}(r, \mu^{(0)}, \sigma_1^{(0)}) = \frac{1}{\varepsilon(\varepsilon - \lambda_1)} r + \frac{1}{\varepsilon} \left[\frac{\lambda_2}{\varepsilon(\varepsilon - \lambda_1)} + \frac{1}{2\varepsilon} \left(\frac{\mu^{(0)}}{\sigma_1^{(0)}} \right)^2 + \log(\varepsilon) - 1 \right].
$$

Thus, the investor's optimal post-crash strategy is the classical Merton strategy depending on the relevant post-crash parameters $(\mu^{(0)}, \sigma_1^{(0)})$. Based on the post-crash value function, we reformulate the worst-case optimization problem (135) by identifying the worst-case scenario of the crash size $l \in [0, l^*]$ and the worst-case scenario with respect to the post-crash parameters $(\mu^{(0)}, \sigma_1^{(0)}) \in \mathcal{P}$.

Reformulation.
At the crash time, the investor has wealth $x = (1 - lk_\tau^{(1)})\tilde{X}_\tau$ and he is faced with a short rate $r = r_\tau$. Note, that $\tilde{X} = \{\tilde{X}_t\}_{t\geq 0}$ denotes the wealth process in a crash-free market (see e.g. (121)). We reformulate the problem (135) as a pre-crash problem of the form:

$$\sup_{(k^{(1)},c^{(1)})\in\Pi(x^0,r^0)} \inf_{\substack{(\tau,l)\in\mathcal{C}' \\ (\mu^{(0)},\sigma_1^{(0)})\in\mathcal{P}}} \mathbb{E}\Bigg(\int_0^\tau e^{-\varepsilon t}\log(c_t^{(1)}\tilde{X}_t)\,dt + e^{-\varepsilon\tau}V^0((1-lk_\tau^{(1)})\tilde{X}_\tau, r_\tau, \mu^{(0)}, \sigma_1^{(0)})\Bigg).$$

Since $V^0(x, r, \mu^{(0)}, \sigma_1^{(0)})$ is monotone increasing in x, we obtain that $l = l^*$ is the worst-case scenario with respect to the crash size. The worst-case post-crash parameters are given by the pair $(\underline{\mu}, \overline{\sigma})$, because they minimize the function $\tilde{W}(r, \mu^{(0)}, \sigma_1^{(0)})$ for any arbitrary but fixed $r \in \mathbb{R}$. For the sake of brevity, we write (k, c) instead of $(k^{(1)}, c^{(1)})$ for the pre-crash strategies and the pre-crash problem above reads as the following controller vs. stopper game:

$$\sup_{(k,c)\in\Pi(x^0,r^0)} \inf_{\tau\in\mathcal{C}'} \mathbb{E}\left(M_\tau^{k,c}\right),$$

where

$$M_t^{k,c} := \int_0^t e^{-\varepsilon s}\log(c_s\tilde{X}_s)\,ds + e^{-\varepsilon t}V^0((1-l^*k_t)\tilde{X}_t, r_t, \underline{\mu}, \overline{\sigma}), \quad t \geq 0.$$

The worst-case optimal pre-crash strategy
Let $H(k, c)$ be defined as in (124), where $\mu^{(0)}$ and $\sigma_1^{(0)}$ are replaced by $\underline{\mu}$ and $\overline{\sigma}$, respectively. Then, we obtain the following sufficient condition for a pre-crash strategy $(\hat{k}, \hat{c})$ to be an indifference strategy.

Corollary 3.4.1. *Let $\varepsilon - \lambda_1 > 0$, and let $(\hat{k}, \hat{c})$ be a constant pre-crash strategy such that $H(\hat{k}, \hat{c}) = 0$, where H is given by*

$$H(k, c) = \log\left(\frac{c}{1-l^*k}\right) + \frac{1}{\varepsilon}\left(\mu^{(1)}k - \frac{(\sigma_1^{(1)})^2}{2}k^2\right) - \frac{c}{\varepsilon} - \frac{1}{2\varepsilon}\left(\frac{\underline{\underline{\mu}}}{\overline{\sigma}}\right)^2 - \log(\varepsilon) + 1. \quad (136)$$

Then, $M^{\hat{k},\hat{c}} = \{M^{\hat{k},\hat{c}}\}_{t\geq 0}$ is a uniformly integrable martingale and $(\hat{k}, \hat{c}) \in \Pi$ is an indifference strategy for the controller vs. stopper game.

Proof. By replacing $\mu^{(0)}$ and $\sigma_1^{(0)}$ by $\underline{\mu}$ and $\overline{\sigma}$ in the proof of Lemma 3.3.2, we immediately obtain the assertion. □

Again, by the concept of an indifference frontier (see Section 3.2 and Section 3.3) and by the same arguments as in Theorem 3.2.11, we identify the worst-case optimal pre-crash strategy.

Corollary 3.4.2. *Let $\varepsilon - \lambda_1 > 0$ and let $H(k, c)$ be given by (136). Moreover, let*

$$m := \min\left\{\frac{1}{l^*}, k^M\right\}, \qquad k^M := \frac{\mu^{(1)}}{(\sigma_1^{(1)})^2}.$$

Then, $(k^*, c^*) = (\kappa, \varepsilon)$ is a worst-case optimal pre-crash strategy, where $\kappa \in [0, m]$ is the uniquely determined solution of $H(k, \varepsilon) = 0$.

PROOF. Obviously, we have that $H(k,\varepsilon)$ is strictly monotone increasing in k for $0 \le k \le m$, $H(0,\varepsilon) < 0$ and $\lim_{k \nearrow \frac{1}{l^*}} H(k,\varepsilon) = \infty$. If $k^M < \frac{1}{l^*}$, we additionally obtain by assumption (134):

$$H(k^M,\varepsilon) = -\log\left(1 - l^* k^M\right) + \frac{1}{2\varepsilon}\left(\left(\frac{\mu^{(1)}}{\sigma_1^{(1)}}\right)^2 - \left(\frac{\underline{\mu}}{\overline{\sigma}}\right)^2\right) > 0.$$

Thus, there exists a uniquely determined $\kappa \in [0,m]$ such that $H(\kappa,\varepsilon) = 0$ and therefore, $(\hat{k},\hat{c}) = (\kappa,\varepsilon)$ is an indifference strategy. Moreover, (κ,ε) is optimal in the no-crash scenario in the class $\mathcal{A}(\kappa)$. The indifference optimality principle implies that (κ,ε) is a worst-case optimal strategy. □

The result of this section is, that the investor's optimal investment strategy before the market crash is given by $k_t^{(1)*} \equiv \kappa$, where $\kappa \in [0,m]$ fulfills $H(\kappa,\varepsilon) = 0$. Note, that $k^{(1)*}$ not only depends on the worst-case scenario with respect to the crash size, given by l^*, but also on the worst-case scenario with respect to the post-crash parameters, given by $(\underline{\mu},\overline{\sigma})$. Furthermore, the optimal pre-crash consumption strategy is given by $c_t^{(1)*} \equiv \varepsilon$. After the market crash, the optimal investment strategy is the classical Merton strategy with the relevant market parameters:

$$k_t^{(0)*} \equiv \frac{\mu^{(0)}}{(\sigma_1^{(0)})^2}.$$

Due to the logarithmic utility function the optimal post-crash consumption strategy is again given by $c_t^{(0)*} \equiv \varepsilon$.

3.5. Appendix

3.5.1. Proof of Lemma 3.2.6.

PROOF. Let $\tilde{W}(r)$ be a classical solution of (114) with

$$\underline{W}(r) \le \tilde{W}(r) \le \overline{W}(r). \tag{137}$$

Let $R > 0$ be fixed. Since $\tilde{W}(r)$ is a solution of

$$0 = \frac{\sigma_2^2}{2} W_{rr}(r) + f(r) W_r(r) - \varepsilon W(r) + Q(r), \quad \forall\, r \in \mathbb{R},$$

we can integrate over $[0,R]$ and obtain

$$\tilde{W}_r(R) = \tilde{W}_r(0) + \frac{2}{\sigma_2^2}\left[\varepsilon \int_0^R \tilde{W}(r)\,dr - \int_0^R Q(r)\,dr - \int_0^R f(r) W_r(r)\,dr\right].$$

Analogously, for fixed $R < 0$ we integrate over $[R,0]$ and obtain

$$\begin{aligned}\tilde{W}_r(R) &= \tilde{W}_r(0) - \frac{2}{\sigma_2^2}\left[\varepsilon \int_R^0 \tilde{W}(r)\,dr - \int_R^0 Q(r)\,dr - \int_R^0 f(r) W_r(r)\,dr\right] \\ &= \tilde{W}_r(0) + \frac{2}{\sigma_2^2}\left[\varepsilon \int_0^R \tilde{W}(r)\,dr - \int_0^R Q(r)\,dr - \int_0^R f(r) W_r(r)\,dr\right].\end{aligned}$$

Then, for arbitrary but fixed $R \in \mathbb{R}$, it holds

$$\begin{aligned}
&\left|\tilde{W}_r(R)\right|^2 \\
&\leq 2\tilde{W}_r^2(0) + \frac{8}{\sigma_2^4}\left|\varepsilon \int_0^R \tilde{W}(r)\,dr - \int_0^R Q(r)\,dr - \int_0^R f(r)W_r(r)\,dr\right|^2 \\
&\leq 2\tilde{W}_r^2(0) + \frac{8}{\sigma_2^4}\left[3\varepsilon^2\left|\int_0^R \tilde{W}(r)\,dr\right|^2 + 3\left|\int_0^R Q(r)\,dr\right|^2\right. \\
&\qquad\qquad\qquad \left. + 3\left|\int_0^R f(r)\tilde{W}_r(r)\,dr\right|^2\right]. \qquad (138)
\end{aligned}$$

By

$$Q(r) = \frac{1}{2\varepsilon}\left(\frac{\mu^{(0)}}{\sigma_1^{(0)}}\right)^2 + \frac{r}{\varepsilon} + \log(\varepsilon) - 1,$$

it holds

$$\left|\int_0^R Q(r)\,dr\right|^2 \leq R\int_0^R |m_1 r + m_2|^2\,dr \leq \frac{2}{3}m_1^2 R^4 + 2m_2^2 R^2, \qquad (139)$$

where $m_1 := \frac{1}{\varepsilon}$ and $m_2 := \frac{1}{2\varepsilon}\left(\frac{\mu^{(0)}}{\sigma_1^{(0)}}\right)^2 + \log(\varepsilon) - 1$. Moreover, by (137) we have

$$|\tilde{W}(r)| \leq |\underline{W}(r)| + |\overline{W}(r)|, \quad \forall r \in \mathbb{R}. \qquad (140)$$

Thus,

$$\begin{aligned}
\left|\int_0^R \tilde{W}(r)\,dr\right|^2 &\leq R\int_0^R (|\underline{W}(r)| + |\overline{W}(r)|)^2\,dr, \\
&\leq 2R\int_0^R \gamma_2^2 r^4 + \gamma_1^2\,dr = \frac{2}{5}\gamma_2^2 R^6 + 2\gamma_1^2 R^2, \qquad (141)
\end{aligned}$$

where $\gamma_2 := |\alpha_2| + |\beta_2|$ and $\gamma_1 := |\alpha_1| + |\beta_1|$.
Moreover, integrating by parts, and using assumption (109) leads to

$$\begin{aligned}
\left|\int_0^R f(r)\tilde{W}_r(r)\,dr\right| &= \left|f(R)\tilde{W}(R) - f(0)\tilde{W}(0) - \int_0^R f_r(r)\tilde{W}(r)\,dr\right| \\
&\leq |f(R)||\tilde{W}(R)| + |f(0)\tilde{W}(0)| + \left|\int_0^R f_r(r)\tilde{W}(r)\,dr\right| \\
&\leq (\tilde{c}|R| + |f(0)|)(\gamma_2 R^2 + \gamma_1) + |f(0)\tilde{W}(0)| + \left|\int_0^R f_r(r)\tilde{W}(r)\,dr\right|,
\end{aligned}$$

where $\tilde{c} := \max\{|c_1|, |c_2|\}$. Moreover, using (140), we have

$$\begin{aligned}
&\left|\int_0^R f(r)\tilde{W}_r(r)\,dr\right|^2 \\
&\quad\leq 3\left[(\tilde{c}|R| + |f(0)|)^2\,(\gamma_2 R^2+\gamma_1)^2 + |f(0)\tilde{W}(0)|^2 + \left|\int_0^R f_r(r)\tilde{W}(r)\,dr\right|^2\right] \\
&\quad\leq 3\left[4(\tilde{c}^2R^2 + |f(0)|^2)(\gamma_2^2R^4+\gamma_1^2) + |f(0)\tilde{W}(0)|^2 + 2\tilde{c}R\int_0^R \gamma_2^2 r^4 + \gamma_1^2\,dr\right] \\
&\quad\leq \sum_{j=0}^{3} \tilde{\nu}_{2j} R^{2j}. \qquad (142)
\end{aligned}$$

Finally, by (138),(139),(141) and (142), we obtain:

$$\tilde{W}_r^2(R) \leq \sum_{i=0}^{3} \nu_{2i} R^{2i}.$$

□

3.5.2. Proof of Theorem 3.2.7.

Proof. The proof works with the same arguments as in [39, Thm.4.2]. Therein, the control variable k is coupled with the stochastic interest rate in the wealth equation. This is not the case for our wealth equation (111). This fact makes the proof a bit easier. For the readers convenience, we note that the candidate for the value function is given by

$$v^0(x,r) = \frac{1}{\varepsilon}\log(x) + \tilde{W}(r),$$

where $\tilde{W}$ is a classical solution of (114) with $\underline{W}(r) \leq \tilde{W}(r) \leq \overline{W}(r)$ and $(\underline{W}, \overline{W})$ are given by (117) and (118), respectively.
Let $(k^{(0)}, c^{(0)}) \in \Pi$ be an arbitrary admissible control, then we can apply Ito's formula and obtain:

$$d\left(e^{-\varepsilon t}v^0(\overline{X}_t,\overline{r}_t)\right) = e^{-\varepsilon t}\,dv^0(\overline{X}_t,\overline{r}_t) - \varepsilon e^{-\varepsilon t}v^0(\overline{X}_t,\overline{r}_t)\,dt \qquad (143)$$

and using that $v^0(x,r)$ solves the HJB equation (112), we obtain

$$\begin{aligned}
dv^0(\overline{X}_t,\overline{r}_t) =& \Big[\mu^{(0)}k_t^{(0)}\overline{X}_t v_x^0 + \frac{(\sigma_1^{(0)})^2}{2}(k_t^{(0)})^2\overline{X}_t^2 v_{xx}^0 + \rho\sigma_1^{(0)}\sigma_2 k_t^{(0)}\overline{X}_t v_{xr}^0 + \overline{r}_t\overline{X}_t v_x^0 \\
&+ f(\overline{r}_t)v_r^0 + \frac{\sigma_2^2}{2}v_{rr}^0 - c_t^{(0)}\overline{X}_t v_x^0\Big]\,dt + \sigma_1^{(0)}k_t^{(0)}\overline{X}_t v_x^0\,dw_{1,t} + \sigma_2 v_r^0\,d\tilde{w}_t \\
\overset{(112)}{\leq}& \varepsilon v^0\,dt - \log(c_t^{(0)}\overline{X}_t)\,dt + \sigma_1^{(0)}k_t^{(0)}\overline{X}_t v_x^0\,dw_{1,t} + \sigma_2 v_r^0\,d\tilde{w}_t. \qquad (144)
\end{aligned}$$

Let $M^A := \{M_T^A\}_{T\geq 0}$ be defined by

$$M_T^A := \int_0^T e^{-\varepsilon t}\sigma_1^{(0)}k_t^{(0)}\overline{X}_t v_x^0(\overline{X}_t,\overline{r}_t)\,dw_{1,t} = \frac{\sigma_1^{(0)}}{\varepsilon}\int_0^T e^{-\varepsilon t}k_t^{(0)}\,dw_{1,t},$$

then, M^A is a martingale because $k^{(0)} \in \Pi$. Moreover, let $M^B := \{M_T^B\}_{T\geq 0}$ be defined by

$$M_T^B := \int_0^T e^{-\varepsilon t}\sigma_2 v_r^0(\overline{X}_t, \overline{r}_t)\, d\tilde{w}_t = \sigma_2 \int_0^T e^{-\varepsilon t}\tilde{W}_r(\overline{r}_t)\, d\tilde{w}_t.$$

Using Lemma 3.2.6 and Lemma A.1.3, we obtain

$$\begin{aligned}&\mathbb{E}\int_0^T e^{-2\varepsilon t}\tilde{W}_r^2(\overline{r}_t)\, dt\\ &= \int_0^T e^{-2\varepsilon t}\mathbb{E}\left(\tilde{W}_r^2(\overline{r}_t)\right) dt \leq \int_0^T e^{-2\varepsilon t}\sum_{i=0}^3 \nu_{2i}\mathbb{E}\left(\overline{r}_t^{2i}\right)\, dt < \infty, \qquad \forall T \geq 0,\end{aligned}$$

and therefore M^B is a martingale. Thus, $\mathbb{E}(M_T^A) = 0$ and $\mathbb{E}(M_T^B) = 0$ for all $T \geq 0$. Multiplying (144) by $e^{-\varepsilon t}$, integrating over $[0, T]$ and taking the expectation leads to:

$$\begin{aligned}&\mathbb{E}\left(\int_0^T e^{-\varepsilon t}\, dv^0(\overline{X}_t, \overline{r}_t)\right) - \mathbb{E}\left(\int_0^T \varepsilon e^{-\varepsilon t}v^0(\overline{X}_t, \overline{r}_t)\, dt\right)\\ &\leq -\mathbb{E}\left(\int_0^T e^{-\varepsilon t}\log(c_t^{(0)}\overline{X}_t)\, dt\right).\end{aligned}$$

Together with (143) this yields

$$v^0(x, r) \geq \mathbb{E}\left(\int_0^T e^{-\varepsilon t}\log(c_t^{(0)}\overline{X}_t)\, dt\right) + \mathbb{E}\left(e^{-\varepsilon T}v^0(\overline{X}_T, \overline{r}_T)\right), \qquad \forall T \geq 0. \tag{145}$$

In the next step, we show that

$$\limsup_{T\to\infty} e^{-\varepsilon T}\mathbb{E}\left(v^0(\overline{X}_T, \overline{r}_T)\right) = \limsup_{T\to\infty} e^{-\varepsilon T}\mathbb{E}\left(\frac{1}{\varepsilon}\log\left(\overline{X}_T\right) + \tilde{W}\left(\overline{r}_T\right)\right) \geq 0.$$

Since $\tilde{W}(\overline{r}_T) \geq \underline{W}(\overline{r}_T) = \alpha_2\overline{r}_T^2 + \alpha_1$, where $\alpha_1, \alpha_2 < 0$, we verify that

$$\limsup_{T\to\infty} e^{-\varepsilon T}\mathbb{E}(\tilde{W}(\overline{r}_T)) \geq \limsup_{T\to\infty} e^{-\varepsilon T}\left(\alpha_2\mathbb{E}(\overline{r}_T^2) + \alpha_1\right). \tag{146}$$

By Lemma A.1.3 in Appendix A, we have

$$\mathbb{E}(\overline{r}_T^2) \leq \begin{cases}\Lambda_1 & : c_1 \leq 0\\ \Lambda_2 e^{2(c_1+\tilde{\epsilon})T} & : c_1 > 0\end{cases}, \tag{147}$$

for any $\tilde{\epsilon} > 0$. Together with (146) and if $c_1 \leq 0$, we immediately conclude that

$$\limsup_{T\to\infty} e^{-\varepsilon T}\mathbb{E}(\tilde{W}(\overline{r}_T)) \geq \limsup_{T\to\infty} e^{-\varepsilon T}\left(\alpha_2\Lambda_1 + \alpha_1\right) = 0,$$

because $\alpha_2 < 0$. Since we assumed that $\varepsilon - 2c_1 > 0$, there exists $\tilde{\epsilon} > 0$, such that $\varepsilon - 2(c_1 + \tilde{\epsilon}) > 0$. Thus, if $c_1 > 0$, it holds

$$\limsup_{T\to\infty} e^{-\varepsilon T}\mathbb{E}(\tilde{W}(\overline{r}_T)) \geq \limsup_{T\to\infty}\left[\alpha_2\Lambda_2 e^{-\varepsilon T}e^{2(c_1+\tilde{\epsilon})T} + e^{-\varepsilon T}\alpha_1\right] = 0.$$

Thus we have shown that

$$\limsup_{T\to\infty} e^{-\varepsilon T}\mathbb{E}(\tilde{W}(\overline{r}_T)) \geq 0. \tag{148}$$

Using the same steps as in [39, p.11-12], one can show that

$$\limsup_{T\to\infty} e^{-\varepsilon T}\mathbb{E}\left(\log(\overline{X}_T)\right) \geq 0.$$

Together with (148), we obtain

$$\limsup_{T\to\infty} e^{-\varepsilon T}\mathbb{E}\left(v^0(\overline{X}_T,\overline{r}_T)\right) = \limsup_{T\to\infty} e^{-\varepsilon T}\mathbb{E}\left(\frac{1}{\varepsilon}\log(\overline{X}_T) + \tilde{W}(\overline{r}_T)\right) \geq 0.$$

Now letting $T\to\infty$ and taking lim sup in (145) leads to

$$v^0(x,r) \geq \mathbb{E}\left(\int_0^\infty e^{-\varepsilon t}\log(c_t^{(0)}\overline{X}_t)\,dt\right).$$

Thus, part (1) of Theorem 3.2.7 holds. Now we will show part (2).

Obviously, the strategies

$$k^{(0)*}(\overline{X}_t,\overline{r}_t) \equiv \frac{\mu^{(0)}}{(\sigma_1^{(0)})^2}, \qquad c^{(0)*}(\overline{X}_t,\overline{r}_t) \equiv \varepsilon$$

are admissible in the sense of Definition 3.1.1 and fulfill condition (120). Moreover, we have

$$\begin{aligned} k^{(0)*} &\in \arg\max_{k^{(0)}}\left[\mu^{(0)}k^{(0)}xv_x^0 + \frac{(\sigma_1^{(0)})^2}{2}(k^{(0)})^2x^2v_{xx}^0 + \rho\sigma_1^{(0)}\sigma_2 k^{(0)}xv_{xr}^0\right], \\ c^{(0)*} &\in \arg\max_{c^{(0)}\geq 0}\left[\log(c^{(0)}x) - c^{(0)}xv_x^0\right]. \end{aligned}$$

Now, denote by $\overline{X}_t^*$ the wealth process controlled by $(k^{(0)*},c^{(0)*})$. Then, inequality (144) becomes an equality of the form

$$dv^0(\overline{X}_t^*,\overline{r}_t) = \varepsilon v^0\,dt - \log(c_t^{(0)*}\overline{X}_t^*)\,dt + \sigma_1^{(0)}k_t^{(0)*}\overline{X}_t^*v_x^0\,dw_{1,t} + \sigma_2 v_r^0\,d\tilde{w}_t.$$

Due of the fact that $(k^{(0)*},c^{(0)*})\in\Pi$, we know that the last two summands are martingales. With the same arguments as before we obtain:

$$v^0(x,r) = \mathbb{E}\left(\int_0^T e^{-\varepsilon t}\log(c_t^{(0)*}\overline{X}_t^*)\,dt\right) + \mathbb{E}\left(e^{-\varepsilon T}v^0(\overline{X}_T^*,\overline{r}_T)\right), \qquad \forall\, T\geq 0.$$

Let us show that

$$\liminf_{T\to\infty} e^{-\varepsilon T}\mathbb{E}\left(v^0(\overline{X}_T^*,\overline{r}_T)\right) = \liminf_{T\to\infty} e^{-\varepsilon T}\mathbb{E}\left(\frac{1}{\varepsilon}\log(\overline{X}_t^*) + \tilde{W}(\overline{r}_t)\right) \leq 0.$$

By $\tilde{W}(\overline{r}_T) \leq \overline{W}(\overline{r}_T) = \beta_2\overline{r}_T^2 + \beta_1$, where $\beta_1,\beta_2>0$, we get

$$\begin{aligned} \liminf_{T\to\infty} e^{-\varepsilon T}\mathbb{E}(\tilde{W}(\overline{r}_T)) &\leq \liminf_{T\to\infty} e^{-\varepsilon T}\mathbb{E}\left(\beta_2\overline{r}_T^2+\beta_1\right) \\ &= \liminf_{T\to\infty} e^{-\varepsilon T}\left(\beta_2\mathbb{E}(\overline{r}_T^2)+\beta_1\right). \end{aligned}$$

If $c_1\leq 0$, then by (147), we immediately conclude that

$$\liminf_{T\to\infty} e^{-\varepsilon T}\mathbb{E}(\tilde{W}(\overline{r}_T)) \leq \liminf_{T\to\infty} e^{-\varepsilon T}\left(\beta_2\Lambda_1+\beta_1\right) = 0,$$

because $\beta_2>0$. We know, by the assumption $\varepsilon - 2c_1 > 0$, that there exists a $\tilde{\epsilon}>0$ such that $\varepsilon - 2(c_1+\tilde{\epsilon})>0$. Thus, for the case $c_1>0$ we obtain

$$\liminf_{T\to\infty} e^{-\varepsilon T}\mathbb{E}(\tilde{W}(\overline{r}_T)) \leq \liminf_{T\to\infty}\left[\beta_2\Lambda_2 e^{-\varepsilon T}e^{2(c_1+\tilde{\epsilon})T} + e^{-\varepsilon T}\beta_1\right] = 0.$$

Moreover, we have

$$
\begin{aligned}
&\mathbb{E}\left(\log(\overline{X}_T^*)\right) \\
&= \log(x) + \mathbb{E}\left(\int_0^T \left[\overline{r}_t + \mu^{(0)} k_t^{(0)*} - \frac{(\sigma_1^{(0)})^2}{2}(k_t^{(0)*})^2\right] dt\right) - \mathbb{E}\left(\int_0^T c_t^{(0)*}\, dt\right) \\
&= \log(x) + \mathbb{E}\left(\int_0^T \overline{r}_t\, dt\right) + \left(\frac{1}{2}\left(\frac{(\mu^{(0)})}{\sigma_1^{(0)}}\right)^2 - \varepsilon\right) T \\
&\leq \log(x) + \frac{1}{2}\mathbb{E}\left(\int_0^T \overline{r}_t^2\, dt\right) + \left(\frac{1}{2} + \frac{1}{2}\left(\frac{(\mu^{(0)})}{\sigma_1^{(0)}}\right)^2 - \varepsilon\right) T
\end{aligned}
$$

Thus,

$$
\liminf_{T\to\infty} e^{-\varepsilon T}\mathbb{E}\left(\frac{1}{\varepsilon}\log(\overline{X}_T^*)\right) \leq 0.
$$

Finally, we conclude that

$$
\liminf_{T\to\infty} e^{-\varepsilon T}\mathbb{E}\left(v^0(\overline{X}_T^*, \overline{r}_T)\right) = \liminf_{T\to\infty} e^{-\varepsilon T}\mathbb{E}\left(\frac{1}{\varepsilon}\log(\overline{X}_t^*) + \tilde{W}(\overline{r}_t)\right) \leq 0,
$$

which yields

$$
v^0(x,r) \leq \mathbb{E}\left(\int_0^\infty e^{-\varepsilon t}\log(c_t^{(0)*}\overline{X}_t^*)\, dt\right).
$$

Combined with part (1) we have

$$
v^0(x,r) = V^0(x,r).
$$

□

CHAPTER 4

Conclusions

The aim of this thesis was to investigate how an investor has to invest and consume optimally in a financial market in which the stock price is threatened by market crashes, which are modeled as uncertain events with an unknown probability distribution of crash times and crash sizes. The investor takes a cautious attitude towards this uncertainty which leads to the worst-case optimization approach. The main contribution to existing research in the field of worst-case portfolio optimization is the extension of the financial market model by a stochastic interest rate risk. In previous work in this field a constant interest rate is used for the savings account. For both the finite time horizon and the infinite time horizon model, we considered different short rate models: The Vasicek model, the affine short rate model and a generalized Vasicek model.
In Chapter 2, the investor is acting on a finite time interval and he maximizes his expected utility of terminal wealth in the worst-case crash scenario. We determined the solution of the corresponding worst-case optimization problem

$$\sup_{k\in\Pi(0,x^0,r^0)} \inf_{M\in\mathcal{N}(0,N)} \mathbb{E}\left(U(X_T^{k,M})\right)$$

for both utility functions $U(x) = \frac{1}{\gamma}x^\gamma$, $\gamma \neq 0, \gamma < 1$, and $U(x) = \log(x)$. Under the Vasicek short rate model, we applied two methods, the variational inequality approach and the martingale approach and obtained explicit optimal strategies. After the N-th market crash it is optimal to invest a fraction of wealth

$$k_t^{(0)*} = \frac{\mu}{(1-\gamma)\sigma_1^2} + \frac{\rho\sigma_2\beta(t)}{(1-\gamma)\sigma_1}$$

in the stock, whereas, if $n \leq N$ crashes still can occur, then it is optimal to invest the fraction

$$k_t^{(n)*} = \hat{k}_t^{(n)} \wedge k_t^{(0)*}$$

in the stock, where $\hat{k}^{(n)}$ is a uniquely determined solution of a nonlinear non-autonomous ODE. If the investor has a non-log HARA utility function and if the short rate and the stock are correlated, the worst-case optimal strategies do not depend on the short rate $r_t(\omega)$ itself, but on the speed of reversion a and the volatility σ_2 which determine the Vasicek process. Numerical experiments have shown that an investor with higher risk aversion may invest more in the stock than an investor with lower risk aversion if, for example, the parameter a is sufficiently small. This is due to the fact that there exists no riskless asset in the financial market model. If the interest rate becomes too risky, e.g. through a low speed of reversion, the risk averse investor invests more in the stock. This behaviour of course differs from the results from previous work with a constant interest rate, where the savings account is a riskless asset. If the investor's risk preferences are represented by a logarithmic utility function or if the short rate and the stock price are uncorrelated, then the optimal strategies are equal to the optimal strategies obtained

for a constant interest rate. Thus, the strategies do not depend on the short rate, neither through $r_t(\omega)$ itself nor through short rate parameters.
In Chapter 3, we determined an investment and consumption strategy, which is optimal for the infinite horizon worst-case optimization problem

$$\sup_{(k,c)\in\Pi(x^0,r^0)} \inf_{(\tau,l)\in\mathcal{C}'} \mathbb{E}\left(\int_0^\infty e^{-\varepsilon t}\log(c_t X_t)\,dt\right),$$

where market parameters of the stock price equation may change at the crash time. For both the generalized Vasicek model and the affine short rate model, it is optimal to consume at a rate ε before and after the market crash and it is optimal to invest at the constant rate $k^{(1)*} = \kappa \wedge k^M$ before the crash and $k^{(0)*} = \mu^{(0)}(\sigma_1^{(0)})^{-2}$ after the crash. Due to the logarithmic utility function, the worst-case optimal strategies do not depend on the short rate itself or on the parameters which determine the short rate model, and they are equal to the optimal strategies from previous work with a constant interest rate. Thus, the logarithmic utility function eliminates the stochastic interest rate risk and it allows to separate the investment decision from the consumption decision such that it is optimal to consume at a rate ε over the whole time interval.
Based on the research in this thesis, we mention two examples for further research in the field of worst-case optimization under stochastic interest rate risk. First, one could consider the finite time horizon non-log HARA utility model from Chapter 2 and replace the Vasicek model by an affine short rate model. For the logarithmic utility function, this was already done in Section 2.5. For the non-log HARA utility case one could apply the martingale approach. The first step of the method requires to solve the post-crash optimization problem. If the supremum in the corresponding HJB equation is attained, then it reduces to a nonlinear second order PDE in (t, x, r). If the stock price and the affine short rate are uncorrelated, then one obtains a classical solution $v^0 \in C^{1,2,2}([0,T]\times\mathbb{R}_+\times\mathbb{R})$. Otherwise, if they are correlated, it needs, from the present point of view, further intensive research to show existence of a classical solution. Possibly one can only show existence of a generalized solution, for example a viscosity solution, which is not necessarily in $C^{1,2,2}$. In that case, $v^0(t, x, r)$ would not be smooth enough to apply Ito's formula in the proof of the indifference condition.
A second direction is to consider the infinite time horizon model with a non-log HARA utility function and with a Vasicek short rate model. As above, one first has to solve a classical stochastic control problem, which is the first step of the martingale approach. The resulting HJB equation is again a nonlinear PDE in (x, r) which needs an existence result of either a classical solution or a generalized solution. The latter case prohibits the application of Ito's formula in subsequent steps of the martingale approach.

APPENDIX A

Basic Essentials

A.1. Stochastic interest rate models

PROPOSITION A.1.1. *Let a, r_M, σ_2 be given positive constants and let $\tilde{w} = \{\tilde{w}_t\}_{t\in[0,T]}$ be a Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the uniquely determined solution r_s of the SDE:*

$$\begin{aligned} dr_s &= a(r_M - r_s)\, ds + \sigma_2\, d\tilde{w}_s, \qquad s \geq t \\ r_t &= r > 0, \end{aligned}$$

is given by

$$r_s = e^{-a(s-t)} r_t + r_M \left(1 - e^{-a(s-t)}\right) + \sigma_2 \int_t^s e^{-a(s-u)}\, d\tilde{w}_u.$$

Moreover, we have the following properties:

i)

$$\mathbb{E}\left(r_s\right) = r_0 e^{-as} + r_M(1 - e^{-as}),$$

ii)

$$\begin{aligned} \int_t^T r_s\, ds =& \frac{r_t}{a}(1 - e^{-a(T-t)}) + r_M \left((T-t) - \frac{1 - e^{-a(T-t)}}{a}\right) \\ &+ \sigma_2 \int_t^T \frac{1 - e^{-a(T-u)}}{a}\, d\tilde{w}_u, \end{aligned}$$

iii)

$$\mathbb{E}^{t,r}\left(\int_t^T r_s\, ds\right) = \frac{1}{a}\left(r - r_M + e^{-a(T-t)}(r_M - r)\right) + r_M(T-t).$$

where $\mathbb{E}^{t,r}$ denotes the conditional expectation given that $r_t = r$.

PROOF. The solution given above is the uniquely determined solution due to Lipschitz continuity of the coefficients of the SDE. Assertion i) immediately follows. The second assertion follows by the following calculations

$$\begin{aligned} \int_t^T r_s\, ds &= \int_t^T r_t e^{-a(s-t)}\, ds + \int_t^T r_M(1 - e^{-a(s-t)})\, ds + \sigma_2 \int_t^T \int_t^s e^{-a(s-u)}\, d\tilde{w}_u\, ds \\ &= \int_t^T r_t e^{-a(s-t)}\, ds + \int_t^T r_M(1 - e^{-a(s-t)})\, ds + \sigma_2 \int_t^T \int_u^T e^{-a(s-u)}\, ds\, d\tilde{w}_u \\ &= \frac{r_t}{a}(1 - e^{-a(T-t)}) + r_M\left((T-t) - \frac{1 - e^{-a(T-t)}}{a}\right) \\ &+ \sigma_2 \int_t^T \frac{1 - e^{-a(T-u)}}{a}\, d\tilde{w}_u \end{aligned}$$

The second equality holds by Fubini's Theorem for stochastic integrals. The third assertion follows by the fact that the expectation of the stochastic integral vanishes. □

PROPOSITION A.1.2. *Let $\tilde{w} = \{\tilde{w}_t\}_{t\geq 0}$ be a Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\lambda_1, \lambda_2, \xi_1, \xi_2$ be constants, such that*

$$\xi_1\lambda_2 - \lambda_1\xi_2 > \frac{\xi_1^2}{2}. \tag{149}$$

Then, the SDE

$$\begin{aligned} dr_s &= (\lambda_1 r_s + \lambda_2)\, ds + \sqrt{\xi_1 r_s + \xi_2}\, d\tilde{w}_t, \qquad s \geq t \\ r_t &= r > 0, \end{aligned}$$

has a uniquely determined solution in $\mathcal{D} = \{r \in \mathbb{R} : \xi_1 r + \xi_2 > 0\}$ and for this solution it holds $\xi_1 r_t + \xi_2 > 0$ for all t almost surely. Moreover

$$\begin{aligned} \mathbb{E}^{t,r}\left(r_s\right) &= re^{\lambda_1(s-t)} + \frac{\lambda_2}{\lambda_1}\left(e^{\lambda_1(s-t)} - 1\right), \\ \mathbb{E}^{t,r}\left(r_s^2\right) &= r^2 e^{2\lambda_1(s-t)} + e^{2\lambda_1(s-t)}\int_t^s e^{-2\lambda_1(u-t)}\left((2\lambda_2 + \xi_1)\mathbb{E}(r_u) + \xi_2\right) du. \end{aligned}$$

PROOF. For $\xi_1 = 0$ the assertion follows by similar arguments as in Proposition (A.1.1). Thus, throughout the proof we assume that $\xi_1 \neq 0$. Here, we apply the main theorem from [13, Chp.4]. Therein, the assertion holds if for all r with $\xi_1 r + \xi_2 = 0$ it holds $\xi_1(\lambda_1 r + \lambda_2) > \frac{\xi_1^2}{2}$. By assumption, we have for $\xi_1 \neq 0$:

$$\xi_1\left(-\lambda_1\frac{\xi_2}{\xi_1} + \lambda_2\right) > \frac{\xi_1^2}{2}.$$

By the theorem in [13, Chp.4], we obtain that there is a uniquely determined solution of SDE (3) and $\xi_1 r_t + \xi_2 > 0$ for all t almost surely.
Since

$$r_s = r + \int_t^s (\lambda_1 r_u + \lambda_2)\, du + \int_t^s \sqrt{\xi_1 r_s + \xi_2}\, d\tilde{w}_u,$$

we calculate the first and the second moment by the following arguments. Assuming that $\mathbb{E}^{t,r}(r_s) < \infty$ for all $s \geq t$, we obtain that the expectation of the stochastic integral vanishes. Therefore, by taking the expectation and by defining $m_1(s) := \mathbb{E}^{t,r}(r_s)$, we obtain

$$\dot{m}_1(s) = \lambda_1 m_1(s) + \lambda_2, \qquad m_1(t) = r.$$

This yields

$$m_1(s) = re^{\lambda_1(s-t)} + \frac{\lambda_2}{\lambda_1}\left(e^{\lambda_1(s-t)} - 1\right).$$

By applying Ito's formula, we obtain

$$r_s^2 = r^2 + \int_t^s [(\lambda_1 r_u + \lambda_2)2r_u + \xi_1 r_u + \xi_2]\, du + \int_t^s 2r_u\sqrt{\xi_1 r_u + \xi_2}\, d\tilde{w}_u.$$

Now, we define $m_2(s) := \mathbb{E}^{t,r}(r_s^2)$ and obtain

$$\dot{m}_2(s) = 2\lambda_1 m_2(s) + (2\lambda_2 + \xi_1)m_1(s) + \xi_2, \qquad m_2(t) = r^2,$$

and therefore,

$$m_2(s) = r^2 e^{2\lambda_1(s-t)} + e^{2\lambda_1(s-t)} \int_t^s e^{-2\lambda_1(u-t)}((2\lambda_2 + \xi_1)m_1(u) + \xi_2)\, du.$$

□

LEMMA A.1.3 (Cf. [39, Lemma 3.1]). *Let $\tilde{w}$ be a Wiener process and assume that $\sigma_2 > 0$ and $f(r) \in C^1(\mathbb{R})$ with $c_2 \leq f_r(r) \leq c_1$, where c_1, c_2 are constants. Then, the SDE*

$$dr_t = f(r_t)\, dt + \sigma_2\, d\tilde{w}_t, \qquad r_0 = r^0,$$

possesses a unique solution. In addition, for any $\tilde{\epsilon} > 0$ and any integer $m > 0$, it holds

$$\mathbb{E}|r_t|^{2m} \leq \Lambda_1, \qquad \text{if} \quad c_1 \leq 0,$$
$$\mathbb{E}|r_t|^{2m} \leq \Lambda_2 e^{2m(c_1+\tilde{\epsilon})t}, \qquad \text{if} \quad c_1 > 0,$$

where Λ_1, Λ_2 are positive constants which are independent of t.

A.2. The concept of an invariant set

We introduce the concept of invariance in the sense of qualitative theory of ODE's. Here, we only give results which we use in this thesis. For the whole theory of invariance we refer to the literature [42, Chapter 7]. The following definitions and theorems can be found in [42, Chp. 7.1-7.3]. Therein the authors consider the following initial value problem for an open set $G \subset \mathbb{R}^n$ and a continuous function $f : \mathbb{R} \times G \to \mathbb{R}^n$:

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0, \tag{150}$$

where $x_0 \in G$ and $t_0 \in \mathbb{R}$. Let $x(t; t_0, x_0)$ be the unique solution of problem (150) on the maximal interval of existence $J_+(t_0, x_0) := [t_0, t_+(t_0, x_0))$.

DEFINITION A.2.1 (Cf. [42, Def.7.1.1.]). *Let $D \subset G$. D is called positively invariant for* (150), *if $x(t; t_0, x_0) \in D$ for all $t \in J_+(t_0, x_0)$ provided that $x_0 \in D$. Accordingly, D is called negatively invariant, if the solution is uniquely determined to the left. D is called invariant, if D is both positive and negative invariant.*

Moreover, we need the following definition:

DEFINITION A.2.2 (Cf. [42, Def.7.1.2.]). *Let $D \subset \mathbb{R}^n$ be closed and $x \in \partial D$. A vector $y \in \mathbb{R}^n$, $y \neq 0$, is called outer normal to the set D in x, if $B_{|y|_2}(x + y) \cap D = \emptyset$. The set of outer normals in x is denoted by $\mathcal{N}_D(x)$. ($B_{|y|_2}(x + y)$ denotes the open ball with radius $|y|_2$ and centre $x + y$.)*

In this, these we only consider invariance in the context of convex and closed sets. We repeatedly apply the following theorem:

THEOREM A.2.3. *Let $D \subset G$ be closed and convex. Then,*

(1) D is positively invariant for (150),
(2) $(f(t, x)|y) \leq 0$ for all $t \in \mathbb{R}$, $x \in \partial D$, $y \in \mathcal{N}_D(x)$

are equivalent.

A.3. The subsolution-supersolution method

For an overview of the theory of nonlinear parabolic and elliptic equations and the subsolution-supersolution method we refer to [40]. In this thesis, we apply a result from [15], where a second order differential equation of the form

$$W_{rr} = \bar{H}(r, W, W_r), \qquad r \in \mathbb{R}, \tag{151}$$

is considered.

Definition A.3.1 (Subsolution and supersolution, cf. [15, Def.3.1]). *A function $\underline{W}$ is said to be a subsolution of* (151) *on the whole real line if*

$$\underline{W}_{rr} \geq \bar{H}(r, \underline{W}, \underline{W}_r).$$

$\overline{W}$ is a supersolution if

$$\overline{W}_{rr} \leq \bar{H}(r, \overline{W}, \overline{W}_r).$$

In addition, $(\underline{W}, \overline{W})$ is said to be an ordered pair of subsolution/supersolution of (151) *if they also satisfy*

$$\underline{W}(r) \leq \overline{W}(r), \qquad \forall r \in \mathbb{R}.$$

Theorem A.3.2 (cf. [15, Thm.3.8]). *Suppose $H(r, w, p)$ is strictly increasing with respect to w and suppose that for each $m \in \mathbb{N}$ there exist constants $C_1(m) > 0$ and $C_2(m) \geq 0$ such that*

$$|\bar{H}(r, w, p)| \leq C_1(m)(p^2 + C_2(m)),$$

for all $r \in I_m := [-m, m]$ and

$$|w| \leq 3 \max\{\sup_{r \in I_m} |\underline{W}(r)|, \sup_{r \in I_m} |\overline{W}(r)|\},$$

where $(\underline{W}, \overline{W})$ is an ordered pair of subsolution/supersolution of (151) *on $\mathbb{R}$. Then,* (151) *has a solution $W(r)$ such that*

$$\underline{W}(r) \leq W(r) \leq \overline{W}(r).$$

A.4. Results from stochastic analysis

Theorem A.4.1 (Feynman-Kac Formula, see e.g. [46, Chp.7, Thm.4.1]). *Let $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$, $c, h : [0, T] \times \mathbb{R}^n \to \mathbb{R}$, and $g : \mathbb{R} \to \mathbb{R}$ be uniformly continuous maps. Moreover, assume that c is bounded and that there exists a constant $L > 0$ such that for $\varphi(t, r) = b(t, r), \sigma(t, r), h(t, r), g(t, r)$:*

$$\begin{aligned} |\varphi(t, r) - \varphi(t, r')| &\leq L(r - r'), \qquad && \forall t \in [0, T], r, r' \in \mathbb{R}, \\ |\varphi(t, 0)| &\leq L, && \forall t \in [0, T]. \end{aligned}$$

Then, the PDE

$$\begin{aligned} & w_t + \frac{1}{2} tr(\sigma(t, r)\sigma(t, r)^T w_{rr}) + \langle b(t, r), w_r \rangle + c(t, r) w + h(t, r) = 0, \\ & \qquad (t, r) \in [0, T] \times \mathbb{R}^n, \\ & w|_{t=T} = g(r), \qquad r \in \mathbb{R}, \end{aligned} \tag{152}$$

admits a unique viscosity solution w and it has the following representation

$$w(t,r) = \mathbb{E}\left(\int_t^T h(s,\overline{r}_s)e^{-\int_t^s c(\mu,\overline{r}_\mu)\,d\mu}\,ds + g(\overline{r}_T)e^{-\int_t^T c(\mu,\overline{r}_\mu)\,d\mu}\right), \quad (t,r) \in [0,T] \times \mathbb{R}^n, \tag{153}$$

where $\overline{r}_s$ is the (unique) strong solution of the following SDE:

$$d\overline{r}_s = b(s,\overline{r}_s)\,ds + \sigma(s,\overline{r}_s)dW_s, \quad s \in [t,T],$$
$$\overline{r}_t = r,$$

with $(t,r) \in [0,T) \times \mathbb{R}^n$ and W an m- dimensional standard Brownian motion. In addition, if (152) *admits a classical solution, then* (153) *gives that classical solution.*

For the definition of the notion of *viscosity solution* and for the proof of the Theorem above we refer to the literature, e.g. [46].

Definition A.4.2 (Cf. [41]). *A martingale Y is said to be closed by a random variable Y_∞ if $\mathbb{E}(|Y_\infty|) < \infty$ and $Y_t = \mathbb{E}\left(Y_\infty|\mathcal{F}_t\right)$, $0 \le t < \infty$.*

Definition A.4.3 (Cf. [41]). *A supermartingale Y is closed by a random variable Y_∞ if $\mathbb{E}(|Y_\infty|) < \infty$ and $Y_t \ge \mathbb{E}\left(Y_\infty|\mathcal{F}_t\right)$ for each $t \ge 0$.*

Definition A.4.4 (Uniformly Integrability). *A process $Y = \{Y_t\}_{t\in I}$, is called uniformly integrable if $\mathbb{E}\left(|Y_t|\mathbb{1}_{|Y_t|>n}\right)$ converges to zero as $n \to \infty$ uniformly in t, that is,*

$$\lim_{n\to\infty}\sup_t \mathbb{E}\left(|Y_t|\mathbb{1}_{|Y_t|>n}\right) = 0,$$

where the supremum is over $[0,T]$ in the case of a finite time interval $I = [0,T]$, and over $[0,\infty)$ if the process is considered on $0 \le t < \infty$.

Theorem A.4.5 (Cf. [41, Thm. I.12]). *Let Y be a right continuous martingale which is uniformly integrable. Then $\overline{Y} = \lim_{t\to\infty} Y_t$ a.s. exists, $\mathbb{E}\left(|\overline{Y}|\right) < \infty$, and $\overline{Y}$ closes Y as a martingale.*

Theorem A.4.6 (Doob's Optional Sampling Theorem, cf. [41, Thm. I.16]). *Let Y be a right continuous martingale (respectively a supermartingale), which is closed by a random variable Y_∞. Let S and T be two stopping times such that $S \le T$ a.s.. Then Y_S and Y_T are integrable and*

$$Y_S = (\ge)\mathbb{E}\left(Y_T|\mathcal{F}_S\right), \quad a.s.$$

Theorem A.4.7 ([41, Thm. I.17]). *Let Y be a right continuous supermartingale (resp. martingale), and S and T be two bounded stopping times such that $S \le T$ a.s. Then Y_S and Y_T are integrable and*

$$Y_S \ge \mathbb{E}\left(Y_T|\mathcal{F}_S\right), \quad a.s.\,(resp. =).$$

A.5. Technical results for post-crash optimization problems

In this subsection we give the setting and a verification theorem of [24] which is applied for our post-crash optimization problems. Throughout this section we look at a state process given by a general controlled SDE of the form

$$dY_t = \Lambda(t,Y_t,k_t)\,dt + \Sigma(t,Y_t,k_t)\,dW_t, \tag{154}$$

with initial value of $Y_{t_0} = y_0$ and a d-dimensional control process $k = \{k_t\}_{t\in[t_0,T]}$, where $[t_0,T]$ is the relevant time interval. A control k is a progressively measurable process with $k_t \in U \subset \mathbb{R}^d$ for all $t \in [t_0,T]$. Let $Q_0 := [t_0,T) \times \mathbb{R}^n$, $n \in \mathbb{N}$. Then, the coefficient functions

$$\Lambda : \overline{Q}_0 \times U \to \mathbb{R}^n,$$
$$\Sigma : \overline{Q}_0 \times U \to \mathbb{R}^{(n,m)},$$

are assumed to be continuous and for all $v \in U$, let $\Lambda(\cdot,\cdot,v)$ and $\Sigma(\cdot,\cdot,v)$ be in $C(\overline{Q}_0)$. Moreover, a control k is called admissible control if

(1) for all $y_0 \in \mathbb{R}^n$ the corresponding controlled SDE (154) with initial condition $Y_{t_0} = y_0$ admits a pathwise unique solution $\{Y_t^k\}_{t\in[t_0,T]}$,
(2) for all $q \in \mathbb{N}$ the condition
$$\mathbb{E}\left(\int_{t_0}^T |k_s|^q \, ds\right) < \infty \tag{155}$$
is satisfied,
(3) the corresponding state process Y^k satisfies
$$\mathbb{E}^{t_0,y_0}\left(\sup_{t\in[t_0,T]} |Y_t^k|^q\right) < \infty.$$

The set of admissible controls is denoted by $\mathcal{A}(t_0,y_0)$.

In our post-crash optimization problems (see for example (17)), we have to handle a controlled SDE where the coefficients does not satisfy the usual Lipschitz and growth conditions. Thus, we cannot apply standard existence and uniqueness theorems. Nevertheless, it is a linear SDE with a stochastic coefficient, where we can apply the following Corollary, which was proved in [24, Corollary 3.1]. Here, we state the version for bounded admissible controls k.

Corollary A.5.1 (Variation of constants by [24]). *Let $(t_0,y_0) \in [0,T) \times \mathbb{R}^n$, and let $A_1^{(j)}$, $j = 1,\dots,d$, A_2, $B_1^{(i,j)}$, $i = 1,\dots,m$, $j = 1,\dots,d$ and $B_2^{(i)}$, $i = 1,\dots,m$ be progressively measurable real-valued processes satisfying the integrability conditions*

$$\int_{t_0}^T \left(\sum_{j=1}^d |A_{1,s}^{(j)}| + |A_{2,s}|\right) ds < \infty, \quad \mathbb{P}-a.s.$$
$$\int_{t_0}^T \left(\sum_{i=1}^m \sum_{j=1}^d (B_{1,s}^{(i,j)})^2 + \sum_{i=1}^m (B_{2,s}^{(i)})^2\right) ds < \infty, \quad \mathbb{P}-a.s.$$

Further, let k be a control with property (155)*. Then the linear controlled SDE*

$$dY_t^k = Y_t^k \left[(A_{1,t}' k_t + A_{2,t})\, dt + (B_{1,t} k_t + B_{2,t})' dW_t\right]$$

admits the uniquely determined solution

$$Y_t^k = y_0 \exp\Bigg(\int_{t_0}^t \left(A_{1,s}' k_s + A_{2,s} - \frac{1}{2} |B_{1,s} k_s + B_{2,s}|^2 \right) ds + \int_{t_0}^t (B_{1,s} k_s + B_{2,s})' \, dW_s \Bigg).$$

In order to formulate the verification theorem by Korn and Kraft [24], we have to introduce the setting of a more general stochastic optimal control problem: Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$. In the case of $\mathcal{O} \neq \mathbb{R}^n$, they assume that the boundary $\partial\mathcal{O}$ is a compact $(n-1)$ dimensional C^3-manifold. Moreover, let $Q := [t_0, T) \times \mathcal{O}$ and let

$$\eta := \inf\{t \in [t_0, T] : (t, Y_t) \notin Q\}. \tag{156}$$

Let L and Ψ be continuous, real valued functions satisfying the polynomial growth conditions

$$|L(t, y, v)| \leq C(1 + |y|^q + |v|^q), \tag{157}$$

$$|\Psi(t, y)| \leq C(1 + |y|^q), \tag{158}$$

on $\overline{Q} \times U$ and $\overline{Q}$, for some constants $q \in \mathbb{N}$ and $C > 0$. Then, the utility functional for the general stochastic optimal control problem, is defined by

$$J(t_0, y_0; k) := \mathbb{E}^{t_0, y_0}\left(\int_{t_0}^{\eta} L(s, Y_s^k, k_s)\, ds + \Psi(\eta, Y_\eta^k)\right),$$

which will be maximized by choosing an admissible control k^*. The value function is then defined by

$$V(t, y) := \sup_{k \in \mathcal{A}(t,y)} J(t, y; k), \qquad (t, y) \in Q.$$

With $\Sigma^* := \Sigma\Sigma'$, the general differential operator is defined by

$$A^k G(t, y) := G_t(t, y) + \frac{1}{2}\sum_{i,j=1}^{n} \Sigma^*_{i,j}(t, y, k) G_{y_i y_j}(t, y) + \sum_{i=1}^{n} \Lambda_i(t, y, k) G_{y_i}(t, y).$$

Using this general setting, we now formulate the verification theorem, which we applied for our special setting (see e.g. Section 2.1).

THEOREM A.5.2 (Verification Theorem by [24]). *Consider a linear controlled SDE with coefficients satisfying the assumptions of Corollary A.5.1. Assume, further, that the functions L and* Ψ *satisfy the growth conditions* (157) *and* (158). *Let* $G \in C^{1,2}(Q) \cap C(\overline{Q})$ *be a solution of the following HJB equation:*

$$\sup_{k \in U}\left\{A^k G(t, y) + L(t, y)\right\} = 0, \qquad (t, y) \in Q, \tag{159}$$

$$G(t, y) = \Psi(t, y), \qquad (t, y) \in ([t_0, T) \times \partial\mathcal{O}) \cup (\{T\} \times \overline{\mathcal{O}}).$$

Assume that for all $(t, y) \in Q$ *and all admissible controls* $k \in \mathcal{A}(t, y)$ *there exists a* $q > 0$, *such that*

$$\mathbb{E}\left(\sup_{s \in [t_0, T]} |G(s, Y_s)|^q\right) < \infty.$$

Then, it holds:

i) $G(t, y) \geq J(t, y; k)$ *for all* $(t, y) \in Q$ *and* $k \in \mathcal{A}(t, y)$.
ii) If for $(t, y) \in Q$ *there exists a control* $k^* \in \mathcal{A}(t, y)$ *with*

$$k_s^* \in \arg\max_{v \in U}\left(A^v G(s, Y_s^{k^*}) + L(s, Y_s^{k^*}, v)\right)$$

for all $s \in [t, \eta]$*, then* $G(t,y) = V(t,y) = J(t,y;k^*)$*, that means* k^* *is an optimal control and* G *is equal to the value function* V*.*

For $L \equiv 0$, Kraft [31] has proven a variant of the above Theorem. In order to formulate this verification result, we need the definition of a weakly admissible control:
Given a candidate G for the value function, a control k is called weakly admissible if it has the following properties (see [31, p.18]):

i) k is progressively measurable,
ii) for all initial conditions $y_0 > 0$ the corresponding state process Y^k with $Y_{t_0} = y_0$ has a pathwise unique solution $\{Y_t^k\}_{t\in[t_0,T]}$,
iii) $\mathbb{E}\left(\int_{t_0}^T |k_s|^4\, ds\right) < \infty$,
iv) the utility functional $J(t_0, y_0; k)$ is well-defined.
v) $\{G(\theta_p, Y^k_{\theta_p})\}_p$ is uniformly integrable, where for $p \in \mathbb{N}$: $\theta_p := \min\{\theta, \eta_p\}$ and η_p denotes the first exit time of (s, Y_s) from Q_p, where

$$\begin{aligned} Q_p &:= [t_0, T - p^{-1}) \times \mathcal{O}_p, \\ \mathcal{O}_p &:= \mathcal{O} \cap \{x \in \mathbb{R}^n : |x| < p, dist(x, \partial\mathcal{O}) > p^{-1}\}. \end{aligned}$$

The set of weakly admissible controls is denoted by $\tilde{\mathcal{A}}(t_0, y_0)$. Note that the increasing sequence of bounded sets $\mathcal{O}_p$ is used to approximate the set $\mathcal{O}$ by letting $p \to \infty$. Moreover, for $p \to \infty$ one has $\eta_p \to T$ $\mathbb{P}$-a.s. For further details, we refer to [31, p.11ff]. Now, we can formulate a variant of the above Theorem.

Corollary A.5.3 (Verification Theorem by [31]). *Assume that* $L \equiv 0$ *and consider a linear SDE whose coefficients meet*

$$\mathbb{E}\left(\int_{t_0}^T \left(\sum_{i=1}^m (B_{2,s}^{(i)})^2 + \sum_{i=1}^m \sum_{j=1}^d (B_{1,s}^{(i,j)})^4\right) ds\right) < \infty,$$

and the requirements of Corollary A.5.1. Besides, assume that there exists a function $G \in C^{1,2}(Q) \cap C(\overline{Q})$ *that solves the HJB equation* (159)*. Further, suppose that for* $(t,y) \in Q$ *there exists a weakly admissible control* $k^* \in \tilde{\mathcal{A}}(t,y)$ *with*

$$k_s^* \in \arg\max_{v\in U} \left(A^v G(s, Y_s^{k^*})\right),$$

for all $s \in [t, \eta]$*. Then, the following results are valid:*

i) $G(t,y) \geq J(t,y;k)$ *for all* $(t,y) \in Q$ *and* $k \in \tilde{\mathcal{A}}(t,y)$*.*
ii) Besides, k^* *is an optimal control among all weak admissible controls and* G *corresponds to the value function of the optimization problem over all weak admissible controls.*

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Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die Arbeit selbstständig und ohne fremde Hilfe verfasst, keine anderen als die von mir angegebenen Quellen und Hilfsmittel benutzt und die den benutzten Werken wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht habe.

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