An Optimal Control Problem for the Stochastic Nonlinear Schrödinger Equation in Variational Formulation

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1 Introduction and Motivation

1.1 Physical Background

The Schrödinger equation was formulated for the first time by the Austrian physicist Erwin Schrödinger in 1926. It is an evolution equation that describes, for example, how the quantum state of a physical system changes over time or characterizes the motion of a charged particle in an electric or magnetic field. Therefore, Schrödinger equations have many physical applications and often arise in the study of quantum mechanics, plasma physics, fiber optics etc.

Referring to [76], the Schrödinger equation is the essential equation of motion in quantum mechanics, which cannot be deduced from pure mathematics but rather should be introduced in physics axiomatic as a basic law of wave mechanics. The wave character of matter is the reason why the state of a physical system is modeled by a complex-valued wave function $X(t, x)$, the solution of the Schrödinger equation. Based on reasons of plausibility and analogies and due to the double-slit experiment, the relation between particles and waves is statistically interpreted such that (under some normalization conditions) $|X(t,x)|^2\, dx$ indicates the probability to find a considered particle in the line segment dx at place x at time t. Investigating a large number of similar particles, it results a distribution of intensity according to $|X(t,x)|^2$ signifying the particle density (for more details see [76, Chapter 2]).

To depict physical systems more realistic, nonlinear differential equations have to be regarded. Hence, mathematicians and physicists have been interested in nonlinear Schrödinger equations for over 30 years and this subject represents a large field of research today. With lots of applications mainly in mathematical physics, the nonlinear Schrödinger equation is a model for the propagation of waves in nonlinear dispersive media. An important physical special case is the so-called Gross-Pitaevskii equation including a cubic nonlinearity. This equation models the propagation of waves in fiber optics and the envelope of water waves (compare [86]). It is used to explain the principle of the tunnel effect and a laser beamer (see [74, 75]). Moreover, it emphasizes the concept of waveguides by focusing or guiding waves, for example electromagnetic, acoustic or optical waves, to transmit signals or power over long distances with high rates (compare [18, 31]).

The Schrödinger equation with such a cubic nonlinearity also describes the wave function of a particle in a Bose-Einstein condensate, a state of matter of a dilute gas of bosons that is cooled down near to $0\,\mathrm{K}\,(-273,15°\mathrm{C})$. In contrast to fermions, bosons are particles following the Bose-Einstein statistics, which signifies the statistical distribution of identical particles with integer spin, for example photons, gluons or the still-theoretical graviton. At very low temperature most bosons condensate in the lowest energy state that is called the ground state. In Bose-Einstein condensates the bosons become indistinguishable, which means that they all occupy the same quantum state. The probability to find a boson at a special point is equal everywhere within the condensate. Thus, this idealized state at absolute zero can be characterized by only one wave function, the solution of the Gross-Pitaevskii equation. Based on this method, one can conclude from microscopic structures to macroscopic quantum phenomena like superfluidity or superconductivity (compare [1, 6, 70]).

Finally, (see [5, 31]) one can model spontaneous emission and excitation, thermal fluctuation or general random disturbances and phenomena by stochastic processes in form of additive or multiplicative noise that leads to the theory of stochastic differential equations.

1.2 Former and Current Investigations

We focus on the nonlinear Schrödinger equation which is part of many publications in recent years (compare [51, 55, 92]). Throughout this thesis, we are especially interested in the power-type nonlinearity of the form $f(z) = |z|^{2\sigma}z$ with $z \in \mathbb{C}$ and $\sigma > 0$ (for $\sigma = 1$ we get the so-called Kerr-nonlinearity).

Deterministic Schrödinger equations including this kind of nonlinearity are already studied on bounded or unbounded domains for different types of solutions, from classical solutions in [48, 80, 97] over strong solutions in [11, 54, 96] and mild solutions in [14, 16, 40, 49] right up to generalized (weak/variational) solutions in [36, 37, 54, 89, 94]. Their results concern local or global existence and uniqueness of solutions, regularity properties, finite-time blow-up, smoothing effects etc. These properties are closely related to the values of the parameter σ and the spatial dimension n, where one distinguishes three cases: the subcritical case for $0 < \sigma < 2/n$, the critical one for $\sigma = 2/n$ and the supercritical case for $\sigma > 2/n$.

Since physical experiments are burdened with random disturbances, stochastic Schrödinger equations with power-type nonlinearities are treated in the case of additive noise in [21, 32], with respect to multiplicative noise in [8, 20, 21, 81] and referring to white noise dispersion in [24, 27]. Based on the semigroup approach, similar properties of the solutions of stochastic Schrödinger equations as in the deterministic case are obtained in these articles. Evolution equation approaches to linear and nonlinear stochastic Schrödinger equations perturbed by cylindrical Brownian motions are given in [71, 72].

Besides the vast amount of research results concerning the Schrödinger equation, there are still open problems. Here, we consider the variational (generalized) solution of stochastic Schrödinger equations with power-type nonlinearity. Such a weak solution concept is very important for solutions of stochastic partial differential equations which are not smooth enough to be a strong solution. The idea is to multiply the state equation by a sufficiently smooth test function and to transfer some differentiability to the test function through integration by parts. Observe that the existence of a variational solution implies the existence of a mild solution, but not vice versa. So far, variational solutions of stochastic Schrödinger equations are only investigated in [42] under the assumption that the nonlinear drift and diffusion terms are of bounded growth and globally or locally Lipschitz continuous. Notice that the power-type nonlinearity does not satisfy the bounded growth and locally Lipschitz continuity assumptions from [42].

This is the first work concerning variational solutions of the stochastic Schrödinger problem with power-type nonlinearities. Hence, we fill the gap of solution concepts of stochastic nonlinear Schrödinger equations by investigating existence and uniqueness of variational solutions. Sometimes, the deterministic strategy can be applied to the stochastic case as well. Thus, we enlarge the ideas of [65, pp. 131–133] concerning crucial inequalities of the power-type nonlinearity. Since other deterministic approaches of the nonlinear Schrödinger equation fail in the case of our stochastic problem, the transformation into a pathwise problem is one main idea of this thesis. This method is applied to the linear Schrödinger equation in [47] and to parabolic stochastic partial differential equations in [34], both in the context of variational solutions.

Since the Schrödinger equation cannot be classified as an elliptic, parabolic or hyperbolic partial differential equation, it is not possible to follow a given pattern (like in [66, 95]) in order to solve a corresponding optimal control problem. As far as we know, nobody has treated optimal control problems of the stochastic nonlinear Schrödinger equation until now. Modest beginnings in control theory of deterministic linear Schrödinger equations can be found in [33, 90, 102]. Besides the unique existence of an optimal control, these articles contain gradient formulas, appropriate necessary optimality conditions and some discretization schemes. Except for the discretization procedure, these results are extended to the deterministic nonlinear case in [3, 68, 69]. First studies in optimal control referring to the stochastic Schrödinger equation are suggested for the linear case in [56, 57] and for the nonlinear case in [58].

1.3 Structure of the Thesis

The first Chapter "Introduction and Motivation" contains some physical applications and backgrounds of the nonlinear Schrödinger equation and useful references to former and current works in order to classify the present thesis within the mathematical context. Now, we indicate the structure of the dissertation to outline its content in detail.

Chapter two is called "Existence and Uniqueness Results" and covers all types of stochastic Schrödinger problems we are concerned with. After introducing the necessary notations, we first formulate the stochastic nonlinear Schrödinger problem

$$dX(t) = i\Delta X(t)\,dt + i\lambda f(t, X(t))\,dt + ig(t, X(t))\,dW(t), \qquad \text{for all } t \in [0, T],$$

with an initial condition $X(0) = \varphi$ and homogeneous Neumann boundary conditions. Then we define its variational solution that is the appropriate concept of solution accompanying us throughout this work. In general, we are interested in the unique existence and some corresponding smoothness properties of the variational solution over a finite time horizon and a bounded one-dimensional domain.

At first, we deal with the nonlinear Schrödinger problem perturbed by additive or multiplicative Gaussian noise in Section 2.2. We assume that $\lambda := i\tilde{\lambda}$ with $\tilde{\lambda} > 0$, the drift function is represented by the power-type nonlinearity $f(\,\cdot\,, v) := |v|^{2\sigma}v$ for all $v \in \mathbb{C}$ with $\sigma \geq 1$ and the noise term contains the diffusion function $g(\,\cdot\,, v) := -i\tilde{g}(\,\cdot\,, v)$ for all $v \in \mathbb{C}$ with a Lipschitz continuous function $\tilde{g}(\,\cdot\,, v)$ of bounded growth and the cylindrical Wiener process W (compare [59]). The missing imaginary unit in front of the nonlinear drift term is crucial for the approach of Section 2.2 (which is applied for all $\sigma \geq 1$). However, the appearance or disappearance of the imaginary unit in front of the diffusion term does not imply major changes.

We proceed in the following way: Initially, we show the uniqueness of the variational solution. Then the Schrödinger problem is approximated by the Galerkin method and a special truncation is introduced to obtain an existence result and to state and prove some a priori estimates for the finite-dimensional solution by enlarging the ideas of the deterministic work [65] to the stochastic case. Thereafter, we deduce global existence of the solution of the stochastic nonlinear Schrödinger equation by showing that $X \in L^{2p}(\Omega; C([0, T]; L^2(0, 1))) \cap L^{2p}(\Omega \times [0, T]; H^1(0, 1))$ for all $p \geq 1$. We finish this section with possible generalizations regarding other boundary conditions, locally Lipschitz continuous noise, more general nonlinearities and an unbounded domain.

In Section 2.3, we analyze the nonlinear Schrödinger problem with linear multiplicative noise, where $\lambda > 0$, the drift function is the same power-type nonlinearity $f(\,\cdot\,, v) = |v|^{2\sigma}v$ for all $v \in \mathbb{C}$ with $\sigma \in (0, 2)$, g is a special linear function in X and W represents an infinite-dimensional Wiener process (see [60]). Due to a different approach, σ is restricted to the interval $(0, 2)$.

Referring to another stochastic process, it is possible to transfer the stochastic nonlinear Schrödinger problem into a pathwise one. Exploiting the absence of noise and using Galerkin approximations and compact embedding results, we obtain a priori estimates, existence and uniqueness of the variational solution of the pathwise nonlinear Schrödinger problem. Moreover, the sequence of Galerkin approximations converges to the solution of the pathwise problem. Then we extend the existence and uniqueness properties to the variational solution of the nonlinear Schrödinger problem with linear multiplicative noise and prove that its solution belongs to $L^2(\Omega; C([0, T]; L^2(0, 1))) \cap L^2(\Omega; L^\infty([0, T]; H^1(0, 1)))$. Finally, we state some remarks concerning further generalizations and research perspectives analogous to the end of Section 2.2 and especially indicate the necessary changes for other λ.

Section 2.4 is devoted to the two preceding cases of Schrödinger problems with respect to a Lipschitz continuous drift function f of bounded growth. Based on the concept of Wirtinger derivatives, we keep our assumptions to a minimum implying the properties of Lipschitz continuity and bounded growth that is at first proved and then illustrated by two examples. The case of Lipschitz continuous noise of bounded growth is reduced to the results in [42], but they can also be shown with the approach in Section 2.2. Moreover, the nonlinear Schrödinger problem perturbed by linear multiplicative noise is handled as in Section 2.3 by the investigation of the equivalent pathwise nonlinear Schrödinger problem.

The third Chapter "On a Problem of Optimal Control" possesses three sections and refers to two selected cases of Schrödinger problems of Chapter 2 including bilinear controls. Initially, in Section 3.1, we present the controlled Schrödinger problem and analyze the additional control term that preserves the existence and uniqueness results. After introducing the objective functional

$$J(U) := \gamma E \left\| X^U(T) - y \right\|^2 + \beta E \int_0^T \|U(t) - \Upsilon(t)\|^2 \, dt$$

for all U from the set of all admissible controls, the question of solvability of this control problem is treated just as an appropriate gradient formula to minimize the objective functional. Currently, this problem of optimal control can only be considered for stochastic nonlinear Schrödinger problems that can be reduced to a pathwise analogue and includes a control term that either depends on time or on space. Thus, we treat the optimal control problem corresponding to the Schrödinger problem with linear multiplicative noise while we refer to the power-type nonlinearity in Section 3.2 and to the Lipschitz continuous drift function of bounded growth in Section 3.3.

For both cases, we proceed in the same way: Under some stronger assumptions as in Chapter 2, we transfer the stochastic nonlinear Schrödinger problem into a pathwise one and apply the constituted estimates in form of constants depending on various parameters. At first, we investigate the difference of two variational solutions of Schrödinger problems referring to two admissible controls that differ only slightly. It results a process that is also a variational solution of a pathwise Schrödinger problem and depends continuously on the difference of the two considered controls. For this reason, we can show that there really exists a unique optimal control which minimizes the given objective functional.

Aiming to obtain a gradient formula, the variational solution of the complex conjugated adjoint Schrödinger problem is regarded. We show again the uniqueness of the variational solution and observe the corresponding Galerkin equations that possess a unique solution. Then we state suitable a priori estimates and obtain the same convergence results for the Galerkin approximations of the complex conjugated adjoint Schrödinger problem as in the case of the pathwise Schrödinger problem. Thereafter, we establish analogue variational formulations of the difference process of two variational solutions of controlled Schrödinger problems and of the complex conjugated adjoint Schödinger problem. While calculating the gradient formula in the sense of Gâteaux, we skillfully combine these two variational formulations. This procedure arises from the deterministic linear control theory and, therefore, we emphasize that the nonlinear terms of the state equation are managed by a linear Taylor approximation based on Wirtinger derivatives. At the end, we obtain a gradient formula whose structure corresponds to the linear case although the complex conjugated adjoint Schrödinger equation differs from the linear case. As a conclusion, we formulate a necessary optimality condition in form of a stochastic variational inequality and discuss further generalizations.

Finally, some auxiliary results and useful hints of stochastic and functional analysis are stated in the Appendix. There are details for deeper understanding regarding the cylindrical Wiener process and different types of solution concepts of stochastic partial differential equations. Furthermore, the frequently used norm square Itô formula, various important inequalities and relations of the power-type nonlinearity and a generalized drift function are indicated. In addition, some basic convergence results and a local martingale property are proved. The principle of Wirtinger derivatives and the derivation of the complex conjugated adjoint Schrödinger equation are explained as well.

2 Existence and Uniqueness Results

2.1 Formulation of the Problem

To avoid ambiguity, we place first some notations widely used in this dissertation. Below, the set $\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$ consists of all positive real-valued numbers. $\mathcal{B}(X)$ denotes the σ-algebra of all Borel measurable sets of a topological space X. The capital letter C represents a generic positive constant, whose value may vary from line to line, and $C(\,\cdot\,)$ emphasizes its dependence.

Let K be a real separable Hilbert space and let $H := L^2(0,1)$ and $V := H^1(0,1)$ be spaces of complex-valued functions. Then the inner product in H is given by

$$\big(u, v\big) := \int_0^1 u(x)\,\overline{v}(x)\,dx, \qquad \text{for all } u, v \in H,$$

where $\overline{v}$ is the complex conjugate of v, while the inner product in V is constituted by

$$\big(u, v\big)_V := \int_0^1 \left[u(x)\,\overline{v}(x) + \frac{d}{dx}u(x)\,\frac{d}{dx}\overline{v}(x)\right] dx, \qquad \text{for all } u, v \in V.$$

The norms in H and V are represented by $\|\cdot\|$ and $\|\cdot\|_V$, respectively. Let V^* be the dual space of V and $\langle\,\cdot\,,\cdot\,\rangle$ denotes the duality pairing of V^* and V. Hence, the appropriate choice of H and V as separable Hilbert spaces and the identification of H with its dual space H^*, due to Riesz' representation theorem, allow to work on a triple of rigged Hilbert spaces (V, H, V^*). This triple has continuous and dense embeddings each and is also known as a Gelfand triple (see [82, p. 55]).

Moreover, we introduce the operator $A : V \to V^*$ defined by the symmetric bilinear form

$$\big\langle Au, v\big\rangle := \int_0^1 \frac{d}{dx}u(x)\,\frac{d}{dx}\overline{v}(x)\,dx, \qquad \text{for all } u, v \in V, \tag{2.1}$$

where the symmetry implies that $\langle Au, v\rangle = \overline{\langle Av, u\rangle}$. By definition it holds that

$$\big\langle Av, v\big\rangle = \left\|\frac{dv}{dx}\right\|^2 = \|v\|_V^2 - \|v\|^2 \qquad \text{and} \qquad \|Av\|_{V^*} \le \|v\|_V, \qquad \text{for all } v \in V.$$

Hence, $A : V \to V^*$ is a linear and continuous operator which we regard with respect to homogeneous Neumann boundary conditions. Requiring, in addition, that $Av \in H$ for all $v \in V$, the eigenvalue problem $Ah_k = \mu_k h_k$ for all $k \in \mathbb{N}$ is satisfied, where $(\mu_k)_{k\in\mathbb{N}}$ is the increasing sequence of eigenvalues and $(h_k)_{k\in\mathbb{N}}$ the corresponding sequence of eigenfunctions. The real-valued eigenvalues are given by $\mu_k := (k-1)^2\pi^2$ for $k = 1, 2, \ldots$ and the eigenfunctions

$$h_k(x) := \begin{cases} 1 & : k = 1, \\ \sqrt{2}\cos((k-1)\pi x) & : k = 2, 3, \ldots \end{cases}$$

form an orthonormal system in H and an orthogonal system in V since (using the Kronecker delta δ_{jk}) we get $\big(h_j, h_k\big)_V = \big(h_j, h_k\big) + \big\langle Ah_j, h_k\big\rangle = (1+\mu_j)\delta_{jk}$ for all $j, k \in \mathbb{N}$. Obviously, for all $u \in H$ and all $v \in V$ it follows that

$$u = \sum_{k=1}^{\infty} \big(u, h_k\big)\,h_k, \qquad Av = \sum_{k=1}^{\infty} \mu_k \big(v, h_k\big)\,h_k \qquad \text{and} \qquad \big\langle Av, v\big\rangle = \sum_{k=1}^{\infty} \mu_k \left|\big(v, h_k\big)\right|^2 \ge 0.$$

Next, we indicate some preliminaries for the finite-dimensional approximations. For each $n \in \mathbb{N}$, we introduce the finite-dimensional space $H_n := \operatorname{span}\{h_1, h_2, \ldots, h_n\}$ and the orthogonal projection $\pi_n : H \to H_n$ constituted by

$$\pi_n u := \sum_{k=1}^{n} \big(u, h_k\big)\, h_k, \qquad \text{for all } u \in H. \tag{2.2}$$

It especially holds for all $u \in H$ and all $h \in H_n$ that

$$\big(\pi_n u, h\big) = \big(u, h\big), \qquad \|\pi_n u\|^2 \le \|u\|^2 \qquad \text{and} \qquad \lim_{n\to\infty} \|\pi_n u - u\|^2 = 0. \tag{2.3}$$

Observe that the norms $\|\cdot\|$ and $\|\cdot\|_V$ are equivalent on H_n, which means that

$$\|u\|^2 \le \|u\|_V^2 = \|u\|^2 + \big\langle Au, u\big\rangle \le (1+\mu_n)\|u\|^2, \qquad \text{for all } u \in H_n, \tag{2.4}$$

since $\mu_n = \max\{\mu_k : k \in \{1, 2, \ldots, n\}\}$, and the operator $A : H_n \to H_n$ is linear and continuous and satisfies

$$Au = \sum_{k=1}^{n} \mu_k \big(u, h_k\big)\, h_k, \qquad \big\langle Au, u\big\rangle = \sum_{k=1}^{n} \mu_k \left|\big(u, h_k\big)\right|^2 = \left\|\frac{du}{dx}\right\|^2 \ge 0, \qquad \text{for all } u \in H_n, \tag{2.5}$$

and

$$\big(v, Au\big) = \overline{\big\langle Au, v\big\rangle} \le \left\|\frac{du}{dx}\right\| \left\|\frac{dv}{dx}\right\|, \qquad \text{for all } u, v \in H_n. \tag{2.6}$$

In the following, we consider the stochastic nonlinear Schrödinger equation

$$dX(t,x) = -\, iAX(t,x)\, dt + i\lambda f(t, X(t,x))\, dt + ig(t, X(t,x))\, dW(t)$$

with initial condition $X(0,\,\cdot\,) = \varphi(\,\cdot\,) \in V$ and homogeneous Neumann boundary conditions

$$\frac{\partial}{\partial x} X(t,x)\bigg|_{x=0} = \frac{\partial}{\partial x} X(t,x)\bigg|_{x=1} = 0, \qquad \text{for all } t \in [0,T].$$

A precise definition of the solution of this initial-boundary value problem is given in Definition 2.1.1. Here, X is the complex-valued random wave function depending on $t \in [0,T]$ and $x \in [0,1]$, i is the imaginary unit, A represents the one-dimensional negative Laplacian, which is in a formal sense defined by (2.1), and $T > 0$ is fixed. The constant $\lambda \in \mathbb{C}$ and the complex-valued nonlinear drift function f will be specified in each particular section. Furthermore, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered complete probability space and $L_2(K,H)$ the space of all Hilbert-Schmidt operators from K into H. We assume that the diffusion function $g : \Omega \times [0,T] \times H \to L_2(K,H)$ is measurable, which means that for all $s \in [0,t]$ it holds that $\{(\omega, s, x) : g(\omega, s, x) \in A\} \in \mathcal{F}_t \times \mathcal{B}([0,t] \times H)$ for all $A \in \mathcal{B}(L_2(K,H))$ and all $t \in [0,T]$. As customary, we suppose the finiteness of the Hilbert-Schmidt norm

$$\|g(t,u)\|_{L_2(K,H)}^2 := \sum_{j=1}^{\infty} \|g(t,u)e_j\|^2, \qquad \text{for all } t \in [0,T] \text{ and all } u \in H, \tag{2.7}$$

where $(e_j)_{j\in\mathbb{N}}$ is an orthonormal basis of K (see [46, pp. 12 f.] or [82, pp. 109–113]). In order to ensure the existence and uniqueness of the solution, g has to satisfy the subsequent assumptions:

- there exists a constant $c_g > 0$ such that for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $u, v \in H$

$$\|g(t,u) - g(t,v)\|_{L_2(K,H)}^2 \le c_g \|u - v\|^2, \tag{2.8}$$

- there exists a constant $k_g > 0$ such that for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $v \in V$

$$\|g(t,v)\|_{L_2(K,V)}^2 \le k_g \left(1 + \|v\|_V^2\right). \tag{2.9}$$

The representation of the Hilbert-Schmidt norm and properties (2.8) and (2.9) of g yield for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $u \in H$

$$\begin{aligned}\|g(t,0)\|^2_{L_2(K,H)} &\leq \|g(t,0)\|^2_{L_2(K,V)} \leq k_g\left(1+\|0\|^2_V\right) = k_g,\\ \|g(t,u)\|^2_{L_2(K,H)} &\leq 2\|g(t,u)-g(t,0)\|^2_{L_2(K,H)} + 2\|g(t,0)\|^2_{L_2(K,H)} \leq 2c_g\|u\|^2 + 2k_g. \qquad (2.10)\end{aligned}$$

Finally, let $(W(t))_{t\in[0,T]}$ be a K-valued cylindrical Wiener process adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Notice that a correct definition is given in Appendix A. For notational simplicity, the explicit dependences on $\omega \in \Omega$ and $x \in [0,1]$ will be neglected such that we regard the stochastic nonlinear Schrödinger problem

$$dX(t) = -iAX(t)\,dt + i\lambda f(t,X(t))\,dt + ig(t,X(t))\,dW(t), \qquad X(0) = \varphi \in V \qquad (2.11)$$

for a.e. $\omega \in \Omega$ and all $t \in [0,T]$. Here and below, the homogeneous Neumann boundary conditions are included implicitly in definition (2.1) of the operator A. The initial value problem (2.11) is defined by the integral equation

$$X(t) = \varphi - i\int_0^t AX(s)\,ds + i\lambda\int_0^t f(s,X(s))\,ds + i\int_0^t g(s,X(s))\,dW(s)$$

in V^* for a.e. $\omega \in \Omega$ and all $t \in [0,T]$, signifying the following variational formulation.

Definition 2.1.1. *A process $X \in L^2(\Omega;C([0,T];H)) \cap L^2(\Omega\times[0,T];V)$ is called a variational solution of the stochastic nonlinear Schrödinger problem* (2.11) *if it fulfills*

$$\begin{aligned}\big(X(t),v\big) = \big(\varphi,v\big) - i\int_0^t \big\langle AX(s),v\big\rangle\,ds + i\lambda\int_0^t \big(f(s,X(s)),v\big)\,ds\\ + i\bigg(\int_0^t g(s,X(s))\,dW(s),v\bigg)\end{aligned} \qquad (2.12)$$

for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $v \in V$.

To derive higher-order moment estimates in $L^{2p}(\Omega;C([0,T];H))$ and $L^{2p}(\Omega\times[0,T];V)$ for $p \geq 1$, the assumption $\varphi \in V$ on the initial condition is essential. Let $C(G)$ denote the space of all continuous and bounded functions $f : G \to \mathbb{R}$. Then it holds for a bounded open set $G \subset \mathbb{R}$ that $H^1(G)$ is continuously embedded in $C(G)$ because of Sobolev's embedding theorem (compare for example [105, p. 1029, (47) Case 5]) and $C(G)$ itself is continuously embedded in $L^q(G)$ for all $1 \leq q \leq \infty$. Thus, it follows from $\varphi \in V$ that $\varphi \in L^q(0,1)$ for all $1 \leq q \leq \infty$. Moreover, it is possible to generalize the initial condition $\varphi \in V$ to an $\mathcal{F}_0$-measurable $\varphi \in L^2(\Omega;V)$. In contrast, to obtain the same higher-order moment estimates in this case, one has to require that $\varphi \in L^{2p}(\Omega;H)$ and $\varphi \in L^{2p}(\Omega;V)$. Consequently, for the sake of simplicity and since an $\mathcal{F}_0$-measurable initial condition $\varphi \in L^2(\Omega;V)$ implies that φ is constant for a.e. $\omega \in \Omega$, we restrict ourself to the case $\varphi \in V$ in this thesis.

Being interested in other solution concepts than the variational one in Definition 2.1.1, we recommend to have a closer look at Appendix B. Here, we want to work with the variational solution and it is necessary to understand the nature of the stochastic integral in the sense of Itô. Under the above assumptions, the stochastic integral is defined for all $\xi \in L^2(\Omega\times[0,T];H)$ as an H-valued Gaussian random variable with zero mean and it is given by

$$\int_0^T g(s,\xi(s))\,dW(s) := \sum_{j=1}^\infty \int_0^T g(s,\xi(s))e_j\,d\beta_j(s)$$

with an orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of K and a sequence of mutually independent real-valued Wiener processes $\big((\beta_j(t))_{t\in[0,T]}\big)_{j\in\mathbb{N}}$ (see Appendix A). This series converges in $L^2(\Omega;H)$ and one can prove the Itô isometry

$$E\left\|\int_0^T g(s,\xi(s))\,dW(s)\right\|^2 = E\int_0^T \|g(s,\xi(s))\|^2_{L_2(K,H)}\,ds$$

and the Burkholder-Davis-Gundy inequality (compare [87, p. 44, Theorem 7]), which especially states that for all stopping times $\tau \in [0,T]$ we have

$$E \sup_{t\in[0,\tau]} \left\| \int_0^t g(s,\xi(s))\,dW(s) \right\| \le 3E\left[\int_0^\tau \|g(s,\xi(s))\|^2_{L_2(K,H)}\,ds \right]^{\frac{1}{2}}. \tag{2.13}$$

Thus, the stochastic integral in Definition 2.1.1 obeys

$$\left(\int_0^t g(s,X(s))\,dW(s), v \right) = \sum_{j=1}^\infty \int_0^t \big(g(s,X(s))e_j, v \big)\,d\beta_j(s), \qquad \text{for all } t \in [0,T].$$

Notice that this kind of noise includes additive as well as multiplicative Gaussian noise. However, we think of additive Gaussian noise with $g \in L_2(K,H)$ for the time being. Therefore, let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of K consisting of the eigenvectors of the covariance operator Q with corresponding eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$ such that $Qe_k = \lambda_k e_k$ for all $k \in \mathbb{N}$ with $\lambda_k \ge 0$, and zero is the only accumulation point of the sequence $(\lambda_k)_{k\in\mathbb{N}}$ (compare Appendix A). Letting $(h_k)_{k\in\mathbb{N}}$ be an orthonormal system of H, we consider a Hilbert-Schmidt operator g from K into H (see [46, pp. 12 f.]) which is especially constituted by

$$gu := \sum_{k=1}^\infty \sqrt{\lambda_k}\big(u, e_k\big)_K\, h_k, \qquad \text{for all } u \in K.$$

Due to the finiteness of the Hilbert-Schmidt norm, one obtains the condition

$$\|g\|^2_{L_2(K,H)} = \sum_{j=1}^\infty \|ge_j\|^2 = \sum_{j=1}^\infty \left\| \sum_{k=1}^\infty \sqrt{\lambda_k}\big(e_j, e_k\big)_K\, h_k \right\|^2 = \sum_{j=1}^\infty \left\| \sqrt{\lambda_j} h_j \right\|^2 = \sum_{j=1}^\infty \lambda_j < \infty.$$

Hence, we get an H-valued Q-Wiener process $(\tilde{W}(t))_{t\in[0,T]}$ which is represented by

$$\tilde{W}(t) := gW(t) = \sum_{j=1}^\infty g e_j \beta_j(t) = \sum_{j=1}^\infty \sqrt{\lambda_j} h_j \beta_j(t), \qquad \text{for all } t \in [0,T], \tag{2.14}$$

where $(\beta_j(t))_{t\in[0,T]}$ with $j \in \{n \in \mathbb{N} : \lambda_n > 0\}$ are independent real-valued Wiener processes (compare Appendix A). The series (2.14) even converges in $L^2(\Omega; C([0,T];H))$, and thus always has a P-a.s. continuous modification (see [19, pp. 86–89] or [82, p. 13, Proposition 2.1.10]). To ensure that the trajectories $(X(t))_{t\in[0,T]}$ of the stochastic nonlinear Schrödinger problem

$$\big(X(t), v\big) = \big(\varphi, v\big) - i\int_0^t \big\langle AX(s), v \big\rangle\,ds + i\lambda \int_0^t \big(f(s,X(s)), v\big)\,ds + i\left(\int_0^t g\,dW(s), v \right) \tag{2.15}$$

for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $v \in V$ take values in the Sobolev space V, the assumption

$$\sum_{j=1}^\infty \lambda_j \mu_j^2 < \infty$$

has to be fulfilled as well, where $(\lambda_j)_{j\in\mathbb{N}}$ are the eigenvalues of the covariance operator Q and $(\mu_j)_{j\in\mathbb{N}}$ are those of the negative Laplacian A with homogeneous Neumann boundary conditions. Accordingly, due to the representation (2.14) of an H-valued Q-Wiener process, the stochastic integral in (2.15) in the case of additive noise with $\tilde{W}(t) = gW(t)$ and $g \in L_2(K,H)$ suffices

$$\left(\int_0^t d\tilde{W}(s), v \right) = \left(\int_0^t g\,dW(s), v \right) = \sum_{j=1}^\infty \sqrt{\lambda_j} \int_0^t \big(h_j, v\big)\,d\beta_j(s), \qquad \text{for all } t \in [0,T].$$

In [56, 57] the controlled stochastic linear Schrödinger equation

$$dX(t) = -\,iAX(t)\,dt + iU(t)X(t)\,dt + i\,d\tilde{W}(t), \qquad \text{for all } t \in [0,T],$$

with homogeneous initial and Neumann boundary conditions is analyzed. The equation contains an admissible control U and is perturbed by an additive H-valued Q-Wiener process $(\tilde{W}(t))_{t\in[0,T]}$ (for more details have a closer look at [56, Chapter 2]). With the help of the Galerkin method, the norm square Itô formula (C.2) and the Burkholder-Davis-Gundy inequality (2.13), it is shown that there exists a unique variational solution $X \in L^2(\Omega; C([0,T];H)) \cap L^2(\Omega \times [0,T];V)$, which is especially true for the uncontrolled problem. This result can be simply expanded, for example, by choosing a non-homogeneous initial condition $\varphi \in V$, any kind of additive noise including deterministic functions g in front of the Wiener process or even multiplicative noise as the following approach suggests. Keeping these results in mind, we are interested in the unique existence of the variational solution of the stochastic nonlinear Schrödinger problem.

2.2 Study of Lipschitz Continuous Noise

Referring to [59], we initially consider the stochastic nonlinear Schrödinger problem

$$dX(t) = -\,iAX(t)\,dt - \lambda f(X(t))\,dt + g(t,X(t))\,dW(t), \qquad X(0) = \varphi \in V \tag{2.16}$$

for a.e. $\omega \in \Omega$ and all $t \in [0,T]$. Additionally to the notations in Section 2.1, we presume $\lambda \in \mathbb{R}_+$ and the nonlinear drift function $f : V \to H$ to be defined by $f(v) := |v|^{2\sigma} v$ for all $v \in V$, where $\sigma \geq 1$ is fixed. A more general form of the nonlinearity f is discussed in Subsection 2.2.3. In comparison with (2.11), notice that the sign and the missing imaginary unit in front of the drift term are necessary due to our approach, while the sign and the missing imaginary unit in front of the diffusion term do not include major changes and are, therefore, only adjusted to the problem. Indeed, the Schrödinger problem (2.16) is equivalent to problem (2.11) which can be seen by choosing $\lambda := i\tilde{\lambda}$ and $g(t,X(t)) := -i\,\tilde{g}(t,X(t))$ in (2.11) with $\tilde{\lambda} > 0$ and the same properties for $\tilde{g}$ as for g. For the sake of simplicity, we write λ and g instead of $\tilde{\lambda}$ and $\tilde{g}$. Accordingly, a process $X \in L^2(\Omega; C([0,T];H)) \cap L^2(\Omega \times [0,T];V)$ is called a variational solution of the stochastic nonlinear Schrödinger problem (2.16) if it fulfills

$$\big(X(t), v\big) = \big(\varphi, v\big) - i\int_0^t \big\langle AX(s), v\big\rangle\,ds - \lambda\int_0^t \big(f(X(s)), v\big)\,ds + \bigg(\int_0^t g(s,X(s))\,dW(s), v\bigg) \tag{2.17}$$

for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $v \in V$. Using the Galerkin method and a special truncation function, we investigate the existence and uniqueness of the variational solution of (2.17).

At first, we point out some important properties of the nonlinear function f. Due to Lemma D.2, it follows for all $v \in V$ that

$$\|f(v)\|^2 = \int_0^1 |v(x)|^{2(2\sigma+1)}\,dx \leq \sup_{x\in[0,1]} |v(x)|^{2(2\sigma+1)} \leq 2^{2\sigma+1}\|v\|_V^{2(2\sigma+1)}. \tag{2.18}$$

Hence, $f : V \to H$ is well-defined and no function from H into H. Moreover, f is not locally Lipschitz continuous in the classical sense as in [42] because Lemma D.4 (b) and Lemma D.2 yield

$$\begin{aligned}\|f(u) - f(v)\|^2 &= \int_0^1 \big||u(x)|^{2\sigma}u(x) - |v(x)|^{2\sigma}v(x)\big|^2\,dx \\ &\leq 4\sigma^2 \int_0^1 \big(|u(x)|^{2\sigma} + |v(x)|^{2\sigma}\big)^2 |u(x) - v(x)|^2\,dx \\ &\leq 8\sigma^2 \sup_{x\in[0,1]} \big(|u(x)|^{4\sigma} + |v(x)|^{4\sigma}\big)\,\|u - v\|^2 \\ &\leq 2^{2\sigma+3}\sigma^2 \big(\|u\|_V^{4\sigma} + \|v\|_V^{4\sigma}\big)\,\|u - v\|^2, \qquad \text{for all } u, v \in V.\end{aligned} \tag{2.19}$$

However, we can make up for this fact by applying Lemma D.5 that results in

$$\operatorname{Re}\big(f(u) - f(v), u - v\big) \geq 0, \qquad \text{for all } u, v \in V, \tag{2.20}$$

and, choosing $v \equiv 0$, it especially holds that

$$\operatorname{Re}\big(f(u), u\big) \geq 0, \qquad \text{for all } u \in V. \tag{2.21}$$

2.2.1 Uniqueness and A Priori Estimates

While the existence of a variational solution of the stochastic nonlinear Schrödinger problem (2.17) is shown in Subsection 2.2.2, we first investigate its uniqueness.

Theorem 2.2.1. *If* $X \in L^2(\Omega; C([0,T]; H)) \cap L^2(\Omega \times [0,T]; V)$ *is a variational solution of the Schrödinger problem* (2.17)*, then it is unique.*

Proof. Assume that $X, \hat{X} \in L^2(\Omega; C([0,T]; H)) \cap L^2(\Omega \times [0,T]; V)$ are two variational solutions of the stochastic nonlinear Schrödinger problem (2.17). By denoting $\delta X := X - \hat{X}$, we get

$$\begin{aligned}\big(\delta X(t), v\big) = &- i \int_0^t \big\langle A\delta X(s), v\big\rangle \, ds - \lambda \int_0^t \big(f(X(s)) - f(\hat{X}(s)), v\big) \, ds \\ &+ \sum_{j=1}^{\infty} \int_0^t \Big(\Big[g(s, X(s)) - g(s, \hat{X}(s))\Big] \, e_j, v\Big) \, d\beta_j(s)\end{aligned}$$

for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $v \in V$. Applying the norm square Itô formula (C.2), which is also known as the stochastic energy equality, we obtain

$$\begin{aligned}\|\delta X(t)\|^2 = &\,2 \operatorname{Im} \int_0^t \big\langle A\delta X(s), \delta X(s)\big\rangle \, ds - 2\lambda \operatorname{Re} \int_0^t \big(f(X(s)) - f(\hat{X}(s)), \delta X(s)\big) \, ds \\ &+ 2 \operatorname{Re} \sum_{j=1}^{\infty} \int_0^t \Big(\Big[g(s, X(s)) - g(s, \hat{X}(s))\Big] \, e_j, \delta X(s)\Big) \, d\beta_j(s) \\ &+ \int_0^t \Big\|g(s, X(s)) - g(s, \hat{X}(s))\Big\|_{L_2(K,H)}^2 \, ds\end{aligned} \tag{2.22}$$

for a.e. $\omega \in \Omega$ and all $t \in [0,T]$. The first addend on the right-hand side vanishes immediately since $\big\langle Av, v\big\rangle \geq 0$ implies $\operatorname{Im}\big\langle Av, v\big\rangle = 0$ for all $v \in V$, and the second one is less than or equal to zero because of relation (2.20). Hence, we only have to regard the terms induced by noise. With the help of the Burkholder-Davis-Gundy inequality (2.13), the definition of the Hilbert-Schmidt norm (2.7) and the Lipschitz continuity (2.8) of g, we estimate the Itô integral in (2.22) by

$$\begin{aligned}&E \sup_{t \in [0,T]} 2 \operatorname{Re} \sum_{j=1}^{\infty} \int_0^t \Big(\Big[g(s, X(s)) - g(s, \hat{X}(s))\Big] \, e_j, \delta X(s)\Big) \, d\beta_j(s) \\ &\leq 2E \sup_{t \in [0,T]} \left| \sum_{j=1}^{\infty} \int_0^t \Big(\Big[g(s, X(s)) - g(s, \hat{X}(s))\Big] \, e_j, \delta X(s)\Big) \, d\beta_j(s) \right| \\ &\leq 6E \left[\int_0^T \sum_{j=1}^{\infty} \left| \Big(\Big[g(s, X(s)) - g(s, \hat{X}(s))\Big] \, e_j, \delta X(s)\Big) \right|^2 ds \right]^{\frac{1}{2}} \\ &\leq 6E \left[\int_0^T \Big\|g(s, X(s)) - g(s, \hat{X}(s))\Big\|_{L_2(K,H)}^2 \, \|\delta X(s)\|^2 \, ds \right]^{\frac{1}{2}} \\ &\leq E \left[\left(\sup_{t \in [0,T]} \|\delta X(t)\|^2 \right)^{\frac{1}{2}} \left(36 \int_0^T \Big\|g(s, X(s)) - g(s, \hat{X}(s))\Big\|_{L_2(K,H)}^2 \, ds \right)^{\frac{1}{2}} \right] \\ &\leq \frac{1}{2} E \sup_{t \in [0,T]} \|\delta X(t)\|^2 + 18 c_g E \int_0^T \|\delta X(s)\|^2 \, ds.\end{aligned}$$

The Hilbert-Schmidt norm in (2.22) is treated analogously with the Lipschitz continuity (2.8) of g such that

$$E \sup_{t \in [0,T]} \int_0^t \Big\|g(s, X(s)) - g(s, \hat{X}(s))\Big\|_{L_2(K,H)}^2 \, ds \leq c_g E \int_0^T \|\delta X(s)\|^2 \, ds.$$

Since for all $u \in L^2(\Omega; C([0,T]; H))$ it holds the relation

$$E \int_0^T \|u(s)\|^2 \, ds = \int_0^T E\|u(s)\|^2 \, ds \le \int_0^T E \sup_{s\in[0,t]} \|u(s)\|^2 \, dt, \tag{2.23}$$

it follows from equation (2.22) that

$$E \sup_{t\in[0,T]} \|\delta X(t)\|^2 \le 38 c_g E \int_0^T \|\delta X(s)\|^2 \, ds \le 38 c_g \int_0^T E \sup_{s\in[0,t]} \|\delta X(s)\|^2 \, dt.$$

Consequently, we deduce by Gronwall's lemma that

$$E \, \|\delta X(t)\|^2 \le E \sup_{t\in[0,T]} \|\delta X(t)\|^2 = 0, \qquad \text{for all } t \in [0,T],$$

which entails that $X(t) = \hat{X}(t)$ for a.e. $\omega \in \Omega$ and all $t \in [0,T]$. □

Hereafter, let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis of K and $K_n := \operatorname{span}\{e_1, e_2, \ldots, e_n\}$. Then we use the notations $\varphi_n := \pi_n \varphi$, $f_n(u) := \pi_n f(u)$ and $g_n(\,\cdot\,, u)w := \pi_n\{g(\,\cdot\,, u)w\}$ for all $u \in H_n$ and all $w \in K_n$ to denote the orthogonal projections of the initial condition, the drift term and the diffusion term on H_n (see (2.2)). The finite-dimensional Wiener process in K_n is represented by

$$W_n(s) := \sum_{j=1}^{n} e_j \, \beta_j(s).$$

To proceed with the existence of the variational solution of the stochastic nonlinear Schrödinger problem (2.17), we adapt the approach introduced in [42, Section 3.2]. Therefore, we extend the Galerkin method for deterministic nonlinear Schrödinger equations (compare [36, Section 2] or [65, pp. 131-133]) to the case of problem (2.17). For each $n \in \mathbb{N}$, we use the Galerkin approximations of $X(t)$ given by

$$X_n(t) := \sum_{k=1}^{n} c_{nk}(t) \, h_k \in H_n, \qquad \text{for all } t \in [0,T] \text{ and all } n \in \mathbb{N},$$

where $c_{nk}(t) := \big(X_n(t), h_k\big)$ for all $k = 1, 2, \ldots, n$ are unknown complex-valued random functions, and consider the finite-dimensional Galerkin equations

$$\begin{aligned} \big(X_n(t), h_k\big) = \big(\varphi_n, h_k\big) - i \int_0^t \big\langle A X_n(s), h_k \big\rangle \, ds - \lambda \int_0^t \big(f_n(X_n(s)), h_k\big) \, ds \\ + \left(\int_0^t g_n(s, X_n(s)) \, dW_n(s), h_k \right) \end{aligned} \tag{2.24}$$

for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $k \in \{1, 2, \ldots, n\}$. Furthermore, we introduce for fixed $M \in \mathbb{N}$ the Lipschitz continuous real-valued truncation function $\psi^M : [0,\infty) \to [0,\infty)$ by

$$\psi^M(r) := \begin{cases} 1 & : 0 \le r \le M, \\ M + 1 - r & : M < r < M + 1, \\ 0 & : r \ge M + 1 \end{cases}$$

and choose $f_n^M : H_n \to H_n$ defined by $f_n^M(u) := \psi^M(\|u\|) f_n(u)$ for each $u \in H_n$. Now, we deal with the system of finite-dimensional equations

$$\begin{aligned} \big(X_n^M(t), h_k\big) = \big(\varphi_n, h_k\big) - i \int_0^t \big\langle A X_n^M(s), h_k \big\rangle \, ds - \lambda \int_0^t \big(f_n^M(X_n^M(s)), h_k\big) \, ds \\ + \left(\int_0^t g_n(s, X_n^M(s)) \, dW_n(s), h_k \right) \end{aligned} \tag{2.25}$$

for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $k \in \{1, 2, \ldots, n\}$.

Referring to the equivalence of norms (2.4) and the properties (2.18) and (2.19) of f, one can show that the nonlinear truncated function $f_n^M : H_n \to H_n$ is Lipschitz continuous and of bounded growth on H_n for fixed $M, n \in \mathbb{N}$. Because of the second property in (2.3), the Lipschitz continuity (2.8) of g and estimate (2.10), the noise term g_n satisfies similar properties given by

$$\begin{aligned}\|g_n(t,u) - g_n(t,v)\|^2_{L_2(K_n,H_n)} &= \sum_{j=1}^{n} \|[g_n(t,u) - g_n(t,v)]\, e_j\|^2 = \sum_{j=1}^{n} \|\pi_n \{[g(t,u) - g(t,v)]\, e_j\}\|^2 \\ &\le \sum_{j=1}^{n} \|[g(t,u) - g(t,v)]\, e_j\|^2 \le \sum_{j=1}^{\infty} \|[g(t,u) - g(t,v)]\, e_j\|^2 \\ &= \|g(t,u) - g(t,v)\|^2_{L_2(K,H)} \le c_g \|u - v\|^2\end{aligned}$$

and

$$\begin{aligned}\|g_n(t,u)\|^2_{L_2(K_n,H_n)} &= \sum_{j=1}^{n} \|g_n(t,u)e_j\|^2 = \sum_{j=1}^{n} \|\pi_n \{g(t,u)e_j\}\|^2 \le \sum_{j=1}^{n} \|g(t,u)e_j\|^2 \\ &\le \sum_{j=1}^{\infty} \|g(t,u)e_j\|^2 = \|g(t,u)\|^2_{L_2(K,H)} \le 2c_g\|u\|^2 + 2k_g\end{aligned} \tag{2.26}$$

for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $u, v \in H_n$. Hence, we know from the theory of finite-dimensional stochastic differential equations with Lipschitz continuous mappings that the system (2.25) possesses a unique solution $X_n^M \in L^2(\Omega; C([0,T]; H_n))$ (see [61, pp. 127–141, Theorem 4.5.3 and Exercise 4.5.5]). Due to the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_V$ on H_n (given in (2.4)), the approximation X_n^M is also continuous in V and we further get $X_n^M \in L^2(\Omega \times [0,T]; V)$. These results and the fact that the equations (2.25) also hold for all $v \in V$ (since it follows that $\big(u, \pi_n v\big) = \big(u, v\big)$ for all $u \in H_n$ due to the first property in (2.3)) especially imply that the solution is a variational solution.

Now, we can apply the norm square Itô formula (C.2) to get the subsequent theorems stating uniform a priori estimates of X_n^M in the spaces $L^{2p}(\Omega; C([0,T]; H))$ and $L^{2p}(\Omega; C([0,T]; V))$ for $p \ge 1$. At first, we prove the results for $p = 1$.

Theorem 2.2.2. *Let $M, n \in \mathbb{N}$ be arbitrarily fixed. Then there exists a positive constant C depending on c_g, k_g and T such that*

$$E \sup_{t\in[0,T]} \big\|X_n^M(t)\big\|^2 \le C(c_g, k_g, T)\Big[1 + \|\varphi\|^2\Big].$$

Proof. For the sake of simplicity, we use the notation $Y(t) := X_n^M(t)$, apply the stochastic energy equality (C.2) to (2.25) and get

$$\begin{aligned}\big|\big(Y(t), h_k\big)\big|^2 &= \big|\big(\varphi_n, h_k\big)\big|^2 + 2\,\mathrm{Im}\int_0^t \big\langle AY(s), \big(Y(s), h_k\big)\, h_k\big\rangle\, ds \\ &\quad - 2\lambda\,\mathrm{Re}\int_0^t \psi^M(\|Y(s)\|)\, \big(f_n(Y(s)), \big(Y(s), h_k\big)\, h_k\big)\, ds \\ &\quad + 2\,\mathrm{Re}\sum_{j=1}^{n}\int_0^t \big(g_n(s, Y(s))e_j, \big(Y(s), h_k\big)\, h_k\big)\, d\beta_j(s) \\ &\quad + \int_0^t \sum_{j=1}^{n} \big|\big(g_n(s, Y(s))e_j, h_k\big)\big|^2\, ds\end{aligned} \tag{2.27}$$

for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $k \in \{1, 2, \dots, n\}$. By summation over all k from 1 until n, we obtain

$$\begin{aligned}\|Y(t)\|^2 = \|\varphi_n\|^2 + 2\operatorname{Im}\int_0^t \big\langle AY(s), Y(s)\big\rangle\, ds - 2\lambda \operatorname{Re}\int_0^t \psi^M(\|Y(s)\|)\big(f_n(Y(s)), Y(s)\big)\, ds \\ + 2\operatorname{Re}\sum_{j=1}^n \int_0^t \big(g_n(s,Y(s))e_j, Y(s)\big)\, d\beta_j(s) + \int_0^t \|g_n(s,Y(s))\|^2_{L_2(K_n,H_n)}\, ds\end{aligned} \tag{2.28}$$

for a.e. $\omega \in \Omega$ and all $t \in [0,T]$. Due to the second property in (2.5), $\operatorname{Im}\big\langle AY(s), Y(s)\big\rangle = 0$ such that the second term on the right-hand side equals zero. Moreover, $\psi^M(\|Y(s)\|) \geq 0$ by definition and $\operatorname{Re}\big(f_n(Y(s)), Y(s)\big) = \operatorname{Re}\big(f(Y(s)), Y(s)\big) \geq 0$ because of the first property in (2.3) and relation (2.21). Thus, it only remains

$$\|Y(t)\|^2 \leq \|\varphi_n\|^2 + 2\operatorname{Re}\sum_{j=1}^n \int_0^t \big(g_n(s,Y(s))e_j, Y(s)\big)\, d\beta_j(s) + \int_0^t \|g_n(s,Y(s))\|^2_{L_2(K_n,H_n)}\, ds$$

for a.e. $\omega \in \Omega$ and all $t \in [0,T]$. Analogously to the proof of uniqueness, the Burkholder-Davis-Gundy inequality (2.13) yields

$$\begin{aligned}&E \sup_{t\in[0,T]} 2\operatorname{Re}\sum_{j=1}^n \int_0^t \big(g_n(s,Y(s))e_j, Y(s)\big)\, d\beta_j(s) \\ \leq &6E\left[\int_0^T \|g_n(s,Y(s))\|^2_{L_2(K_n,H_n)} \|Y(s)\|^2\, ds\right]^{\frac{1}{2}} \\ \leq &E\left[\left(\sup_{t\in[0,T]}\|Y(t)\|^2\right)^{\frac{1}{2}}\left(36\int_0^T \|g_n(s,Y(s))\|^2_{L_2(K_n,H_n)}\, ds\right)^{\frac{1}{2}}\right] \\ \leq &\frac{1}{2}E\sup_{t\in[0,T]}\|Y(t)\|^2 + 18E\int_0^T \|g_n(s,Y(s))\|^2_{L_2(K_n,H_n)}\, ds\end{aligned}$$

such that

$$E\sup_{t\in[0,T]}\|Y(t)\|^2 \leq 2\|\varphi_n\|^2 + 38E\int_0^T \|g_n(s,Y(s))\|^2_{L_2(K_n,H_n)}\, ds.$$

We deduce by inequality (2.26), the estimate $\|\varphi_n\|^2 \leq \|\varphi\|^2$ (see (2.3)) and relation (2.23) that

$$\begin{aligned}E\sup_{t\in[0,T]}\|Y(t)\|^2 &\leq 2\|\varphi_n\|^2 + 76k_gT + 76c_gE\int_0^T \|Y(s)\|^2\, ds \\ &\leq 2\|\varphi\|^2 + 76k_gT + 76c_g\int_0^T E\sup_{s\in[0,t]}\|Y(s)\|^2\, dt.\end{aligned}$$

Finally, the application of Gronwall's lemma results in

$$E\sup_{t\in[0,T]}\|Y(t)\|^2 \leq C(c_g,k_g,T)\left[1+\|\varphi\|^2\right],$$

and the assertion follows with $Y(t) = X_n^M(t)$. □

Theorem 2.2.3. *Let $M, n \in \mathbb{N}$ be arbitrarily fixed. Then there exists a positive constant C depending on c_g, k_g and T such that*

$$E\sup_{t\in[0,T]}\left\|X_n^M(t)\right\|_V^2 \leq C(c_g,k_g,T)\left[1+\|\varphi\|_V^2\right].$$

Proof. We denote again $Y(t) := X_n^M(t)$, consider (2.25), apply the norm square Itô formula (C.2) and receive equation (2.27) for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $k \in \{1,2,\ldots,n\}$. Multiplication with the real-valued eigenvalues μ_k of A and summing up from $k = 1,2,\ldots,n$ leads to

$$\begin{aligned}\sum_{k=1}^{n} \mu_k \left|\big(Y(t),h_k\big)\right|^2 &= \sum_{k=1}^{n} \mu_k \left|\big(\varphi_n,h_k\big)\right|^2 + 2\operatorname{Im}\int_0^t \left\langle AY(s), \sum_{k=1}^{n} \mu_k \big(Y(s),h_k\big)\, h_k \right\rangle ds \\ &\quad - 2\lambda \operatorname{Re}\int_0^t \psi^M(\|Y(s)\|) \left(f_n(Y(s)), \sum_{k=1}^{n} \mu_k \big(Y(s),h_k\big)\, h_k \right) ds \\ &\quad + 2\operatorname{Re}\sum_{j=1}^{n}\int_0^t \left(g_n(s,Y(s))e_j, \sum_{k=1}^{n} \mu_k \big(Y(s),h_k\big)\, h_k \right) d\beta_j(s) \\ &\quad + \int_0^t \sum_{j=1}^{n}\sum_{k=1}^{n} \mu_k \left|\big(g_n(s,Y(s))e_j,h_k\big)\right|^2 ds\end{aligned}$$

for a.e. $\omega \in \Omega$ and all $t \in [0,T]$. Observing the first relation in (2.5), the second term on the right-hand side vanishes once more and, because of the first property in (2.3), it holds that $\big(f_n(Y(s)), \big(Y(s),h_k\big)\, h_k\big) = \big(f(Y(s)), \big(Y(s),h_k\big)\, h_k\big)$. Thus, by relations (2.5) of the operator A, we write

$$\begin{aligned}\left\|\frac{\partial}{\partial x} Y(t)\right\|^2 &= \left\|\frac{d}{dx}\varphi_n\right\|^2 - 2\lambda \operatorname{Re}\int_0^t \psi^M(\|Y(s)\|)\, \big(f(Y(s)), AY(s)\big)\, ds \\ &\quad + 2\operatorname{Re}\sum_{j=1}^{n}\int_0^t \big(g_n(s,Y(s))e_j, AY(s)\big)\, d\beta_j(s) \\ &\quad + \int_0^t \sum_{j=1}^{n} \left\|\frac{\partial}{\partial x}\left[g_n(s,Y(s))e_j\right]\right\|^2 ds\end{aligned} \tag{2.29}$$

for a.e. $\omega \in \Omega$ and all $t \in [0,T]$. Lemma D.6 implies that the second term on the right-hand side is less than or equal to zero. Due to the definition of the Hilbert-Schmidt norm (2.7), it only remains

$$\begin{aligned}\left\|\frac{\partial}{\partial x} Y(t)\right\|^2 &\le \left\|\frac{d}{dx}\varphi_n\right\|^2 + 2\operatorname{Re}\sum_{j=1}^{n}\int_0^t \big(g_n(s,Y(s))e_j, AY(s)\big)\, d\beta_j(s) \\ &\quad + \int_0^t \|g(s,Y(s))\|^2_{L_2(K,V)}\, ds\end{aligned}$$

for a.e. $\omega \in \Omega$ and all $t \in [0,T]$. Taking into account the Burkholder-Davis-Gundy inequality (2.13), property (2.6) of A and the bounded growth (2.9) of g, we see

$$\begin{aligned}&E \sup_{t\in[0,T]} 2\operatorname{Re}\sum_{j=1}^{n}\int_0^t \big(g_n(s,Y(s))e_j, AY(s)\big)\, d\beta_j(s) \\ &\le 6E\left[\int_0^T \sum_{j=1}^{n} \left\|\frac{\partial}{\partial x}\left[g_n(s,Y(s))e_j\right]\right\|^2 \left\|\frac{\partial}{\partial x} Y(s)\right\|^2 ds\right]^{\frac{1}{2}} \\ &\le E\left[\left(\sup_{t\in[0,T]}\left\|\frac{\partial}{\partial x} Y(t)\right\|^2\right)^{\frac{1}{2}} \left(36\int_0^T \|g(s,Y(s))\|^2_{L_2(K,V)}\, ds\right)^{\frac{1}{2}}\right] \\ &\le \frac{1}{2} E \sup_{t\in[0,T]} \left\|\frac{\partial}{\partial x} Y(t)\right\|^2 + 18k_g E\int_0^T \left(1 + \|Y(s)\|_V^2\right) ds\end{aligned}$$

and

$$E \sup_{t\in[0,T]} \int_0^t \|g(s,Y(s))\|^2_{L_2(K,V)}\, ds \le k_g E\int_0^T \left(1 + \|Y(s)\|_V^2\right) ds.$$

Based on

$$\|Y(s)\|_V^2 = \|Y(s)\|^2 + \left\|\frac{\partial}{\partial x}Y(s)\right\|^2,$$

the inequality $\left\|\frac{d}{dx}\varphi_n\right\|^2 \leq \left\|\frac{d}{dx}\varphi\right\|^2$ (given by (2.3)), Theorem 2.2.2 and the relation (2.23), we conclude

$$\begin{aligned} E\sup_{t\in[0,T]}\left\|\frac{\partial}{\partial x}Y(t)\right\|^2 &\leq 2\left\|\frac{d}{dx}\varphi_n\right\|^2 + 38k_g E\int_0^T\left(1+\|Y(s)\|_V^2\right)ds \\ &\leq 2\left\|\frac{d}{dx}\varphi\right\|^2 + 38k_gT + 38k_gE\int_0^T\|Y(s)\|^2\,ds + 38k_gE\int_0^T\left\|\frac{\partial}{\partial x}Y(s)\right\|^2 ds \\ &\leq 2\left\|\frac{d}{dx}\varphi\right\|^2 + C(c_g,k_g,T)\left[1+\|\varphi\|^2\right] + 38k_g\int_0^T E\sup_{s\in[0,t]}\left\|\frac{\partial}{\partial x}Y(s)\right\|^2 dt. \end{aligned}$$

Hence, Gronwall's lemma entails

$$E\sup_{t\in[0,T]}\left\|\frac{\partial}{\partial x}Y(t)\right\|^2 \leq C(c_g,k_g,T)\left[1+\|\varphi\|_V^2\right].$$

Together with Theorem 2.2.2 and $Y(t) := X_n^M(t)$, we obtain the result of Theorem 2.2.3. □

Having shown the uniform boundedness of the solution X_n^M of (2.25) in $L^{2p}(\Omega;C([0,T];H))$ and $L^{2p}(\Omega;C([0,T];V))$ for $p=1$, we are also able to establish uniform boundedness results for $p>1$. These a priori estimates are necessary to prove the existence of the variational solution of the stochastic nonlinear Schrödinger problem (2.17). To ensure the existence of the integrals in the following theorems, we use a localizing argument.

Remark 2.2.4. *Let $(u(t))_{t\in[0,T]}$ be an H-valued process with*

$$\sup_{t\in[0,T]}\|u(t)\|^2 < \infty, \qquad \text{for a.e. } \omega\in\Omega.$$

Then we introduce for $R\in\mathbb{N}$ the stopping time

$$\tau_R^u := \begin{cases} T & : \sup_{t\in[0,T]}\|u(t)\|^2 < R^2, \\ \inf\left\{t\in[0,T] : \|u(t)\|^2 \geq R^2\right\} & : \sup_{t\in[0,T]}\|u(t)\|^2 \geq R^2. \end{cases} \tag{2.30}$$

Notice that $(\tau_R^u)_R$ is an increasing sequence with

$$\lim_{R\to\infty}\tau_R^u = T, \qquad \text{for a.e. } \omega\in\Omega. \tag{2.31}$$

Theorem 2.2.5. *Let $M,n\in\mathbb{N}$ be arbitrarily fixed and $p>1$. Then there exists a positive constant C depending on p,c_g,k_g and T such that*

$$E\sup_{t\in[0,T]}\left\|X_n^M(t)\right\|^{2p} \leq C(p,c_g,k_g,T)\left[1+\|\varphi\|^{2p}\right].$$

Proof. The beginning is identical to the proof of Theorem 2.2.2. We denote $Y(t) := X_n^M(t)$ and start with equation (2.28) given by

$$\begin{aligned} \|Y(t)\|^2 = \|\varphi_n\|^2 &+ 2\operatorname{Im}\int_0^t\left\langle AY(s),Y(s)\right\rangle ds - 2\lambda\operatorname{Re}\int_0^t\psi^M(\|Y(s)\|)\left(f_n(Y(s)),Y(s)\right)ds \\ &+ 2\operatorname{Re}\sum_{j=1}^n\int_0^t\left(g_n(s,Y(s))e_j,Y(s)\right)d\beta_j(s) + \int_0^t\|g_n(s,Y(s))\|_{L_2(K_n,H_n)}^2\,ds \end{aligned}$$

for a.e. $\omega \in \Omega$ and all $t \in [0,T]$, where $\operatorname{Im}\langle AY(s), Y(s)\rangle = 0$ because of the second property in (2.5). Then we use the Itô formula for $F(x) = x^p$ with $p > 1$ and obtain

$$\begin{aligned}
\|Y(t)\|^{2p} = \|\varphi_n\|^{2p} &- 2\lambda p \operatorname{Re} \int_0^t \psi^M(\|Y(s)\|)\big(f_n(Y(s)), Y(s)\big)\|Y(s)\|^{2(p-1)}\, ds \\
&+ 2p \operatorname{Re} \sum_{j=1}^n \int_0^t \big(g_n(s, Y(s))e_j, Y(s)\big)\, \|Y(s)\|^{2(p-1)}\, d\beta_j(s) \\
&+ p(p-1) \operatorname{Re} \int_0^t \sum_{j=1}^n \big(g_n(s, Y(s))e_j, Y(s)\big)^2\, \|Y(s)\|^{2(p-2)}\, ds \\
&+ p \int_0^t \|g_n(s, Y(s))\|^2_{L_2(K_n, H_n)}\, \|Y(s)\|^{2(p-1)}\, ds
\end{aligned} \tag{2.32}$$

for a.e. $\omega \in \Omega$ and all $t \in [0,T]$. Observe that the term with the nonlinearity f_n obeys

$$\begin{aligned}
&- 2\lambda p \operatorname{Re} \int_0^t \psi^M(\|Y(s)\|)\big(f_n(Y(s)), Y(s)\big)\|Y(s)\|^{2(p-1)}\, ds \\
= &- 2\lambda p \operatorname{Re} \int_0^t \psi^M(\|Y(s)\|)\big(f(Y(s)), Y(s)\big)\|Y(s)\|^{2(p-1)}\, ds \leq 0
\end{aligned}$$

because of the first property in (2.3) and relation (2.21). The last two terms in (2.32) reduce to

$$\begin{aligned}
&p(p-1) \operatorname{Re} \int_0^t \sum_{j=1}^n \big(g_n(s, Y(s))e_j, Y(s)\big)^2\, \|Y(s)\|^{2(p-2)}\, ds \\
&+ p \int_0^t \|g_n(s, Y(s))\|^2_{L_2(K_n, H_n)}\, \|Y(s)\|^{2(p-1)}\, ds \\
\leq &(p^2 - p) \int_0^t \|g_n(s, Y(s))\|^2_{L_2(K_n, H_n)}\, \|Y(s)\|^{2(p-1)}\, ds \\
&+ p \int_0^t \|g_n(s, Y(s))\|^2_{L_2(K_n, H_n)}\, \|Y(s)\|^{2(p-1)}\, ds \\
\leq &p^2 \int_0^t \|g_n(s, Y(s))\|^2_{L_2(K_n, H_n)}\, \|Y(s)\|^{2(p-1)}\, ds.
\end{aligned}$$

Consequently, it holds that

$$\begin{aligned}
\|Y(t)\|^{2p} \leq \|\varphi_n\|^{2p} &+ 2p \operatorname{Re} \sum_{j=1}^n \int_0^t \big(g_n(s, Y(s))e_j, Y(s)\big)\, \|Y(s)\|^{2(p-1)}\, d\beta_j(s) \\
&+ p^2 \int_0^t \|g_n(s, Y(s))\|^2_{L_2(K_n, H_n)}\, \|Y(s)\|^{2(p-1)}\, ds
\end{aligned} \tag{2.33}$$

for a.e. $\omega \in \Omega$ and all $t \in [0,T]$. In the following, we use the notation $a \wedge b := \min\{a, b\}$ for all $a, b \in \mathbb{R}$, the stopping time $\tau := \tau_R^Y$ for $R \in \mathbb{N}$ (compare (2.30) in Remark 2.2.4) and the estimate

$$\begin{aligned}
\gamma E\left(\int_0^{t\wedge\tau} \|Y(s)\|^{4p}\, ds\right)^{\frac{1}{2}} &\leq E\left[\left(\sup_{s\in[0,t\wedge\tau]} \|Y(s)\|^{2p}\right)^{\frac{1}{2}} \left(\gamma^2 \int_0^{t\wedge\tau} \|Y(s)\|^{2p}\, ds\right)^{\frac{1}{2}}\right] \\
&\leq \frac{1}{2} E \sup_{s\in[0,t\wedge\tau]} \|Y(s)\|^{2p} + \frac{\gamma^2}{2} E \int_0^{t\wedge\tau} \|Y(s)\|^{2p}\, ds
\end{aligned} \tag{2.34}$$

for $\gamma \geq 0$. The Burkholder-Davis-Gundy inequality (2.13), Young's inequality, estimate (2.10), Lemma D.1 and relation (2.34) lead to

$$
\begin{aligned}
&E \sup_{s\in[0,t\wedge\tau]} 2p\,\mathrm{Re}\sum_{j=1}^{n}\int_0^s \big(g_n(r,Y(r))e_j, Y(r)\big)\,\|Y(r)\|^{2(p-1)}\,d\beta_j(r)\\
&\le 6pE\left[\int_0^{t\wedge\tau} \|g(s,Y(s))\|^2_{L_2(K,H)}\,\|Y(s)\|^{4p-2}\,ds\right]^{\frac{1}{2}}\\
&\le 6pE\left[\frac{1}{2p}\int_0^{t\wedge\tau} \|g(s,Y(s))\|^{4p}_{L_2(K,H)}\,ds + \frac{2p-1}{2p}\int_0^{t\wedge\tau}\|Y(s)\|^{4p}\,ds\right]^{\frac{1}{2}}\\
&\le 6pE\left[\frac{2^{4p}}{2p}\int_0^{t\wedge\tau} \left(c_g^{2p}\|Y(s)\|^{4p} + k_g^{2p}\right)ds + \frac{2p-1}{2p}\int_0^{t\wedge\tau}\|Y(s)\|^{4p}\,ds\right]^{\frac{1}{2}}\\
&\le 2^{2p}6k_g^p\sqrt{p(t\wedge\tau)} + 6\sqrt{p}\left[2^{4p}c_g^{2p}+2p-1\right]^{\frac{1}{2}} E\left(\int_0^{t\wedge\tau}\|Y(s)\|^{4p}\,ds\right)^{\frac{1}{2}}\\
&\le 2^{2p}6k_g^p\sqrt{pT} + \frac{1}{2}E\sup_{s\in[0,t\wedge\tau]}\|Y(s)\|^{2p} + 18p\left[2^{4p}c_g^{2p}+2p-1\right]E\int_0^{t\wedge\tau}\|Y(s)\|^{2p}\,ds\\
&=: C(p,k_g,T) + \frac{1}{2}E\sup_{s\in[0,t\wedge\tau]}\|Y(s)\|^{2p} + C(p,c_g)E\int_0^{t\wedge\tau}\|Y(s)\|^{2p}\,ds.
\end{aligned}
$$

Moreover, Young's inequality with $p>1$, estimate (2.10) and Lemma D.1 yield

$$
\begin{aligned}
&E\sup_{s\in[0,t\wedge\tau]} p^2\int_0^s \|g_n(r,Y(r))\|^2_{L_2(K_n,H_n)}\,\|Y(r)\|^{2(p-1)}\,dr\\
&\le pE\int_0^{t\wedge\tau}\|g(s,Y(s))\|^{2p}_{L_2(K,H)}\,ds + p(p-1)E\int_0^{t\wedge\tau}\|Y(s)\|^{2p}\,ds\\
&\le 2^{2p}pE\int_0^{t\wedge\tau}\left(c_g^p\|Y(s)\|^{2p}+k_g^p\right)ds + p(p-1)E\int_0^{t\wedge\tau}\|Y(s)\|^{2p}\,ds\\
&\le 2^{2p}pk_g^pT + p(2^{2p}c_g^p+p-1)\,E\int_0^{t\wedge\tau}\|Y(s))\|^{2p}\,ds\\
&=: C(p,k_g,T) + C(p,c_g)\,E\int_0^{t\wedge\tau}\|Y(s)\|^{2p}\,ds.
\end{aligned}
$$

Based on (2.33), $\|\varphi_n\|^{2p}\le\|\varphi\|^{2p}$ (due to the second property in (2.3)) and

$$
E\int_0^{t\wedge\tau}\|u(s)\|^{2p}\,ds = \int_0^{t\wedge\tau}E\|u(s)\|^{2p}\,ds \le \int_0^t E\sup_{s\in[0,r\wedge\tau]}\|u(s)\|^{2p}\,dr \tag{2.35}
$$

for all $u\in L^{2p}(\Omega;C([0,T];H))$, we obtain

$$
\begin{aligned}
E\sup_{s\in[0,t\wedge\tau]}\|Y(s)\|^{2p} &\le 2\|\varphi_n\|^{2p} + C(p,k_g,T) + C(p,c_g)E\int_0^{t\wedge\tau}\|Y(s)\|^{2p}\,ds\\
&\le 2\|\varphi\|^{2p} + C(p,k_g,T) + C(p,c_g)\int_0^t E\sup_{s\in[0,r\wedge\tau]}\|Y(s)\|^{2p}\,dr.
\end{aligned}
$$

Applying Gronwall's lemma and replacing t by T, we receive

$$
E\sup_{s\in[0,T\wedge\tau]}\|Y(s)\|^{2p} \le C(p,c_g,k_g,T)\left[1+\|\varphi\|^{2p}\right].
$$

Finally, letting $R\to\infty$, using (2.31) and the notation $Y(t)=X_n^M(t)$, we get

$$
E\sup_{t\in[0,T]}\left\|X_n^M(t)\right\|^{2p} \le C(p,c_g,k_g,T)\left[1+\|\varphi\|^{2p}\right]. \qquad \square
$$

Theorem 2.2.6. *Let $M, n \in \mathbb{N}$ be arbitrarily fixed and $p > 1$. Then there exists a positive constant C depending on p, c_g, k_g and T such that*

$$E \sup_{t\in[0,T]} \left\| X_n^M(t) \right\|_V^{2p} \leq C(p, c_g, k_g, T) \left[1 + \|\varphi\|_V^{2p} \right].$$

Proof. Remembering the notation $Y(t) := X_n^M(t)$, we start our considerations with equation (2.29) from the proof of Theorem 2.2.3 given by

$$\begin{aligned}\left\| \frac{\partial}{\partial x} Y(t) \right\|^2 &= \left\| \frac{d}{dx} \varphi_n \right\|^2 - 2\lambda \operatorname{Re} \int_0^t \psi^M(\|Y(s)\|) \big(f(Y(s)), AY(s) \big)\, ds \\ &\quad + 2 \operatorname{Re} \sum_{j=1}^n \int_0^t \big(g_n(s, Y(s)) e_j, AY(s) \big)\, d\beta_j(s) + \int_0^t \sum_{j=1}^n \left\| \frac{\partial}{\partial x} \left[g_n(s, Y(s)) e_j \right] \right\|^2 ds\end{aligned}$$

for a.e. $\omega \in \Omega$ and all $t \in [0, T]$. The Itô formula for $F(x) := x^p$ with $p > 1$ entails

$$\begin{aligned}\left\| \frac{\partial}{\partial x} Y(t) \right\|^{2p} &= \left\| \frac{d}{dx} \varphi_n \right\|^{2p} - 2\lambda p \operatorname{Re} \int_0^t \psi^M(\|Y(s)\|) \big(f(Y(s)), AY(s) \big) \left\| \frac{\partial}{\partial x} Y(s) \right\|^{2(p-1)} ds \\ &\quad + 2p \operatorname{Re} \sum_{j=1}^n \int_0^t \big(g_n(s, Y(s)) e_j, AY(s) \big) \left\| \frac{\partial}{\partial x} Y(s) \right\|^{2(p-1)} d\beta_j(s) \\ &\quad + p(p-1) \operatorname{Re} \int_0^t \sum_{j=1}^n \left| \big(g_n(s, Y(s)) e_j, AY(s) \big) \right|^2 \left\| \frac{\partial}{\partial x} Y(s) \right\|^{2(p-2)} ds \\ &\quad + p \int_0^t \sum_{j=1}^n \left\| \frac{\partial}{\partial x} \left[g_n(s, Y(s)) e_j \right] \right\|^2 \left\| \frac{\partial}{\partial x} Y(s) \right\|^{2(p-1)} ds\end{aligned} \tag{2.36}$$

for a.e. $\omega \in \Omega$ and all $t \in [0, T]$. Due to Lemma D.6, the second term on the right-hand side is less than or equal to zero such that we only have to investigate the last three terms induced by noise. For the Itô integral we apply the Burkholder-Davis-Gundy inequality (2.13), property (2.6) of the operator A, the bounded growth (2.9) of g, Young's inequality and Lemma D.1 to receive for the stopping time $\tau := \tau_R^Y$ for $R \in \mathbb{N}$ (see (2.30) in Remark 2.2.4) that

$$\begin{aligned}& E \sup_{s\in[0,t\wedge\tau]} 2p \operatorname{Re} \sum_{j=1}^n \int_0^s \big(g_n(r, Y(r)) e_j, AY(r) \big) \left\| \frac{\partial}{\partial x} Y(r) \right\|^{2(p-1)} d\beta_j(r) \\ &\leq 6pE \left[\int_0^{t\wedge\tau} \sum_{j=1}^n \left\| \frac{\partial}{\partial x} \left[g_n(s, Y(s)) e_j \right] \right\|^2 \left\| \frac{\partial}{\partial x} Y(s) \right\|^{4p-2} ds \right]^{\frac{1}{2}} \\ &\leq 6pE \left[\int_0^{t\wedge\tau} \|g(s, Y(s))\|_{L_2(K,V)}^2 \left\| \frac{\partial}{\partial x} Y(s) \right\|^{4p-2} ds \right]^{\frac{1}{2}} \\ &\leq 6p\sqrt{k_g}\, E \left[\int_0^{t\wedge\tau} \left(1 + \|Y(s)\|^2 + \left\| \frac{\partial}{\partial x} Y(s) \right\|^2 \right) \left\| \frac{\partial}{\partial x} Y(s) \right\|^{4p-2} ds \right]^{\frac{1}{2}} \\ &\leq 6p\sqrt{k_g}\, E \left[\frac{1}{2p}(t \wedge \tau) + \frac{1}{2p} \int_0^{t\wedge\tau} \|Y(s)\|^{4p} ds + \frac{6p-2}{2p} \int_0^{t\wedge\tau} \left\| \frac{\partial}{\partial x} Y(s) \right\|^{4p} ds \right]^{\frac{1}{2}} \\ &\leq 6\sqrt{2pk_g} \left[\sqrt{T} + E \left(\int_0^T \|Y(s)\|^{4p} ds \right)^{\frac{1}{2}} + \sqrt{3p-1}\, E \left(\int_0^{t\wedge\tau} \left\| \frac{\partial}{\partial x} Y(s) \right\|^{4p} ds \right)^{\frac{1}{2}} \right].\end{aligned}$$

Therefore, relation (2.34) implies

$$
\begin{aligned}
&E \sup_{s\in[0,t\wedge\tau]} 2p \operatorname{Re}\sum_{j=1}^{n}\int_0^s \big(g_n(r,Y(r))e_j, AY(r)\big)\left\|\frac{\partial}{\partial x}Y(r)\right\|^{2(p-1)} d\beta_j(r)\\
&\le 6\sqrt{2pk_gT} + \frac{1}{2}E\sup_{s\in[0,T]}\|Y(s)\|^{2p} + 36pk_gE\int_0^T \|Y(s)\|^{2p}ds\\
&\quad + \frac{1}{2}E\sup_{s\in[0,t\wedge\tau]}\left\|\frac{\partial}{\partial x}Y(s)\right\|^{2p} + 36pk_g(3p-1)E\int_0^{t\wedge\tau}\left\|\frac{\partial}{\partial x}Y(s)\right\|^{2p}ds\\
&=: C(p,k_g,T) + \frac{1}{2}E\sup_{s\in[0,T]}\|Y(s)\|^{2p} + C(p,k_g)E\int_0^T \|Y(s)\|^{2p}\,ds\\
&\quad + \frac{1}{2}E\sup_{s\in[0,t\wedge\tau]}\left\|\frac{\partial}{\partial x}Y(s)\right\|^{2p} + C(p,k_g)E\int_0^{t\wedge\tau}\left\|\frac{\partial}{\partial x}Y(s)\right\|^{2p}ds.
\end{aligned}
$$

The same calculations like in the case of the stochastic integral indicate that the last two terms in (2.36) suffice

$$
\begin{aligned}
&E\sup_{s\in[0,t\wedge\tau]} p(p-1)\operatorname{Re}\int_0^s\sum_{j=1}^n\big|\big(g_n(r,Y(r))e_j, AY(r)\big)\big|^2\left\|\frac{\partial}{\partial x}Y(r)\right\|^{2(p-2)}dr\\
&+E\sup_{s\in[0,t\wedge\tau]} p\int_0^s\sum_{j=1}^n\left\|\frac{\partial}{\partial x}\left[g_n(r,Y(r))e_j\right]\right\|^2\left\|\frac{\partial}{\partial x}Y(r)\right\|^{2(p-1)}dr\\
&\le p^2E\int_0^{t\wedge\tau}\sum_{j=1}^n\left\|\frac{\partial}{\partial x}\left[g_n(s,Y(s))e_j\right]\right\|^2\left\|\frac{\partial}{\partial x}Y(s)\right\|^{2(p-1)}ds\\
&\le p^2E\int_0^{t\wedge\tau}\|g(s,Y(s))\|^2_{L_2(K,V)}\left\|\frac{\partial}{\partial x}Y(s)\right\|^{2(p-1)}ds\\
&\le p^2k_gE\int_0^{t\wedge\tau}\left(1+\|Y(s)\|^2+\left\|\frac{\partial}{\partial x}Y(s)\right\|^2\right)\left\|\frac{\partial}{\partial x}Y(s)\right\|^{2(p-1)}ds\\
&\le pk_gT + pk_gE\int_0^T\|Y(s)\|^{2p}\,ds + p(3p-2)k_gE\int_0^{t\wedge\tau}\left\|\frac{\partial}{\partial x}Y(s)\right\|^{2p}ds\\
&=: C(p,k_g,T) + C(p,k_g)E\int_0^T\|Y(s)\|^{2p}\,ds + C(p,k_g)E\int_0^{t\wedge\tau}\left\|\frac{\partial}{\partial x}Y(s)\right\|^{2p}ds.
\end{aligned}
$$

Combining these estimates, using the inequality $\left\|\frac{\partial}{\partial x}\varphi_n\right\|^{2p}\le\left\|\frac{\partial}{\partial x}\varphi\right\|^{2p}$ (compare the second property in (2.3)), relation (2.35) and the result of Theorem 2.2.5, we obtain from equation (2.36)

$$
\begin{aligned}
E\sup_{s\in[0,t\wedge\tau]}\left\|\frac{\partial}{\partial x}Y(s)\right\|^{2p} &\le 2\left\|\frac{\partial}{\partial x}\varphi_n\right\|^{2p} + C(p,k_g,T) + E\sup_{s\in[0,T]}\|Y(s)\|^{2p}\\
&\quad + C(p,k_g)E\int_0^T\|Y(s)\|^{2p}\,ds + C(p,k_g)E\int_0^{t\wedge\tau}\left\|\frac{\partial}{\partial x}Y(s)\right\|^{2p}ds\\
&\le 2\left\|\frac{\partial}{\partial x}\varphi\right\|^{2p} + C(p,c_g,k_g,T)\left[1+\|\varphi\|^{2p}\right]\\
&\quad + C(p,k_g)\int_0^t E\sup_{s\in[0,r\wedge\tau]}\left\|\frac{\partial}{\partial x}Y(s)\right\|^{2p}dr.
\end{aligned}
$$

Applying Gronwall's lemma and replacing t by T, it results that

$$
E\sup_{s\in[0,T\wedge\tau]}\left\|\frac{\partial}{\partial x}Y(s)\right\|^{2p}\le C(p,c_g,k_g,T)\left[1+\|\varphi\|_V^{2p}\right].
$$

Letting $R \to \infty$ and taking into account (2.31) and the notation $Y(t) = X_n^M(t)$, we finally get

$$E \sup_{t\in[0,T]} \left\| \frac{\partial}{\partial x} X_n^M(t) \right\|^{2p} \le C(p, c_g, k_g, T) \left[1 + \|\varphi\|_V^{2p}\right].$$

Hence, together with Theorem 2.2.5 it follows that

$$E \sup_{t\in[0,T]} \left\|X_n^M(t)\right\|_V^{2p} \le C(p, c_g, k_g, T) \left[1 + \|\varphi\|_V^{2p}\right].$$ □

Based on Theorems 2.2.5 and 2.2.6, we can conclude similar results of Theorems 2.2.2 and 2.2.3 as well. A short proof indicates how to proceed in this case.

Corollary 2.2.7. *From the uniform a priori estimates of the solution X_n^M of problem* (2.25) *in $L^{2p}(\Omega; C([0,T]; H))$ and $L^{2p}(\Omega; C([0,T]; V))$ for $p > 1$ we can deduce uniform estimates for $p = 1$, which are given by*

$$E \sup_{t\in[0,T]} \left\|X_n^M(t)\right\|^2 \le C(p, c_g, k_g, T) \left[1 + \|\varphi\|^{2p}\right],$$
$$E \sup_{t\in[0,T]} \left\|X_n^M(t)\right\|_V^2 \le C(p, c_g, k_g, T) \left[1 + \|\varphi\|_V^{2p}\right].$$

Proof. Using Young's inequality $ab \le \frac{1}{p}a^p + \frac{1}{q}b^q$ with $a, b \ge 0$, $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ for $a := \|X_n^M(t)\|^2$ and $b := 1$, we get

$$\left\|X_n^M(t)\right\|^2 \le \frac{1}{p} \left\|X_n^M(t)\right\|^{2p} + \frac{p-1}{p}.$$

Hence, the application of the supremum over all $t \in [0,T]$ and the mean value yield

$$E \sup_{t\in[0,T]} \left\|X_n^M(t)\right\|^2 \le \frac{1}{p} E \sup_{t\in[0,T]} \left\|X_n^M(t)\right\|^{2p} + \frac{p-1}{p} \le C(p, c_g, k_g, T) \left[1 + \|\varphi\|^{2p}\right].$$

By choosing $a := \left\|X_n^M(t)\right\|_V^2$, the analogue result for the norm in V is true. □

All the preceding estimates can be summarized in the following way.

Corollary 2.2.8. *For each $M, n \in \mathbb{N}$ arbitrarily fixed and $p \ge 1$, the solution X_n^M of the stochastic nonlinear Schrödinger problem* (2.25) *is uniformly bounded in $L^{2p}(\Omega; C([0,T]; H))$ and $L^{2p}(\Omega; C([0,T]; V))$ and, therefore, also in $L^{2p}(\Omega \times [0,T]; H)$ and $L^{2p}(\Omega \times [0,T]; V)$.*

2.2.2 Existence of the Variational Solution

Based on the unique existence and the uniform a priori estimates of the sequence of variational solutions $\left((X_n^M)_{n\in\mathbb{N}}\right)_{M\in\mathbb{N}}$ of the finite-dimensional stochastic nonlinear Schrödinger problem (2.25) in the last subsection, we are now able to show the unique existence of the variational solution of the finite-dimensional problem (2.24) and of the infinite-dimensional problem (2.17) thereafter. The proofs rely on the approach in [42, Section 3].

Theorem 2.2.9. *For each fixed $n \in \mathbb{N}$ and $p \ge 1$ there exists a unique variational solution $X_n \in L^{2p}(\Omega; C([0,T]; H)) \cap L^{2p}(\Omega \times [0,T]; V)$ of the stochastic nonlinear Schrödinger problem* (2.24)*, which satisfies the estimates*

$$E \sup_{t\in[0,T]} \|X_n(t)\|^{2p} \le C(p, c_g, k_g, T) \left[1 + \|\varphi\|^{2p}\right],$$
$$E \int_0^T \|X_n(t)\|_V^{2p} \, dt \le C(p, c_g, k_g, T) \left[1 + \|\varphi\|_V^{2p}\right].$$

Proof. Let $n \in \mathbb{N}$ be arbitrarily fixed. Thanks to the first and second property in (2.3), the uniqueness of the variational solution follows similarly to the proof of Theorem 2.2.1. Therefore, we first consider the stopping time $\tau_M := \tau_M^u$ for $u := X_n^M$, which is equal to the stopping time in (2.30) for $R = M$. From the definition of τ_M, Markov's inequality and Theorem 2.2.2 we obtain

$$\begin{aligned} P(\tau_M < T) &\le P\left(\sup_{t\in[0,T]} \left\| X_n^M(t) \right\|^2 \ge M^2 \right) \\ &\le \frac{1}{M^2} E \sup_{t\in[0,T]} \left\| X_n^M(t) \right\|^2 \le \frac{C(c_g,k_g,T)}{M^2} \left[1 + \|\varphi\|^2 \right]. \end{aligned} \tag{2.37}$$

Thus, the increasing sequence of stopping times $(\tau_M)_M$ converges P-a.s. to T. Let Ω^M be the set of all $\omega \in \Omega$ such that $X_n^M(\omega,\cdot)$ satisfies (2.25) for all $t \in [0,T]$ and all $k \in \{1,2,\dots,n\}$, $X_n^M(\omega,\cdot)$ has continuous trajectories in H and takes values in $L^2([0,T];V)$. We introduce $\Omega' := \bigcap_{M=1}^\infty \Omega^M$ with $P(\Omega') = 1$. Furthermore, we define

$$S := \bigcup_{M=1}^\infty \bigcup_{K=1}^M \left\{ \omega \in \Omega' : \tau_K = T \text{ and } \exists\, t \in [0,T] : X_n^K(\omega,t) \neq X_n^M(\omega,t) \right\}.$$

It holds that $P(S) = 0$ because otherwise there exist two natural numbers K_0, M_0 with $K_0 < M_0$ such that the set

$$S_{M_0,K_0} := \left\{ \omega \in \Omega' : \tau_{K_0} = T \text{ and } \exists\, t \in [0,T] : X_n^{K_0}(\omega,t) \neq X_n^{M_0}(\omega,t) \right\}$$

has the probability $P(S_{M_0,K_0}) > 0$. Denoting for all $t \in [0,T]$

$$X^*(\omega,t) := \begin{cases} X_n^{K_0}(\omega,t) & : \omega \in S_{M_0,K_0}, \\ X_n^{M_0}(\omega,t) & : \omega \in \Omega' \setminus S_{M_0,K_0}, \end{cases}$$

we see that for all $\omega \in S_{M_0,K_0}$ there exists a $t \in [0,T]$ such that $X^*(\omega,t) \neq X_n^{M_0}(\omega,t)$. This contradicts the almost sure uniqueness of the variational solution of (2.25) for $M = M_0$ and it follows that $P(S) = 0$. Letting

$$\Omega'' := \Omega' \cap \left(\bigcup_{M=1}^\infty \{\tau_M = T\} \setminus S \right),$$

using (2.37) and the definition of S, we get

$$P(\Omega'') = \lim_{M\to\infty} P\left(\{\tau_M = T\} \setminus S \right) = 1 - \lim_{M\to\infty} P(\tau_M < T) = 1.$$

Choosing now an $\omega \in \Omega''$, there exists an $M_0 \in \mathbb{N}$ such that $\tau_M = T$ for all $M \ge M_0$. Therefore, $\psi^M\left(\left\| X_n^M(s) \right\| \right) = 1$ for all $s \in [0,T]$ and all $M \ge M_0$, and consequently

$$\begin{aligned} \left(X_n^M(t), h_k \right) &= \left(\varphi_n, h_k \right) - i \int_0^t \left\langle A X_n^M(s), h_k \right\rangle ds - \lambda \int_0^t \left(f_n(X_n^M(s)), h_k \right) ds \\ &\quad + \left(\int_0^t g_n(s, X_n^M(s))\, dW_n(s), h_k \right) \end{aligned}$$

for all $t \in [0,T]$, all $M \ge M_0$ and all $k \in \{1,2,\dots,n\}$. For this fixed $\omega \in \Omega''$ we define

$$X_n(\omega,\cdot) := X_n^M(\omega,\cdot), \qquad \text{for all } t \in [0,T] \text{ and all } M \ge M_0. \tag{2.38}$$

Hence, we have

$$\begin{aligned} \left(X_n(t), h_k \right) &= \left(\varphi_n, h_k \right) - i \int_0^t \left\langle A X_n(s), h_k \right\rangle ds - \lambda \int_0^t \left(f_n(X_n(s)), h_k \right) ds \\ &\quad + \left(\int_0^t g_n(s, X_n(s))\, dW_n(s), h_k \right) \end{aligned}$$

for all $\omega \in \Omega''$, all $t \in [0,T]$ and all $k \in \{1,2,\ldots,n\}$. This equals equation (2.24), which does not only hold for the eigenfunctions $(h_k)_{k\in\{1,2,\ldots,n\}}$ but also for all $v \in V$ (since for all $u \in H_n$ it follows that $\big(u, \pi_n v\big) = \big(u,v\big)$ by the first property in (2.3)). Due to the properties of $(X_n^M(t))_{t\in[0,T]}$, the process $(X_n(t))_{t\in[0,T]}$ is H-valued, $\mathcal{F}\times\mathcal{B}([0,T])$-measurable, adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$, has almost surely continuous trajectories in H and takes values in $L^2([0,T];V)$. Because of (2.38), it results for all $p \geq 1$ that

$$\lim_{M\to\infty} \sup_{t\in[0,T]} \big\|X_n^M(t) - X_n(t)\big\|^{2p} = 0, \qquad \text{for a.e. } \omega \in \Omega,$$

$$\lim_{M\to\infty} \int_0^T \big\|X_n^M(t) - X_n(t)\big\|_V^{2p}\, dt = 0, \qquad \text{for a.e. } \omega \in \Omega.$$

Finally, the application of Fatou's lemma, Theorems 2.2.2 and 2.2.3 for $p = 1$ and Theorems 2.2.5 and 2.2.6 for $p > 1$ yield

$$\begin{aligned} E \sup_{t\in[0,T]} \|X_n(t)\|^{2p} &= E \lim_{M\to\infty} \sup_{t\in[0,T]} \big\|X_n^M(t)\big\|^{2p} \leq \liminf_{M\to\infty} E \sup_{t\in[0,T]} \big\|X_n^M(t)\big\|^{2p} \\ &\leq C(p,c_g,k_g,T)\left[1 + \|\varphi\|^{2p}\right], \\ E\int_0^T \|X_n(t)\|_V^{2p}\, dt &= E \lim_{M\to\infty} \int_0^T \big\|X_n^M(t)\big\|_V^{2p}\, dt \leq \liminf_{M\to\infty} E \int_0^T \big\|X_n^M(t)\big\|_V^{2p}\, dt \\ &\leq C(p,c_g,k_g,T)\left[1 + \|\varphi\|_V^{2p}\right]. \end{aligned}$$

Thus, $X_n \in L^{2p}(\Omega; C([0,T];H)) \cap L^{2p}(\Omega\times[0,T];V)$ is the unique variational solution of the finite-dimensional problem (2.24) for all $p \geq 1$. □

Now, we state our main result concerning the unique existence of the variational solution of the infinite-dimensional stochastic nonlinear Schrödinger problem (2.17).

Theorem 2.2.10. *For all $p \geq 1$ the stochastic nonlinear Schrödinger problem* (2.17) *possesses a unique variational solution $X \in L^{2p}(\Omega; C([0,T];H)) \cap L^{2p}(\Omega\times[0,T];V)$, which satisfies*

$$\begin{aligned} E \sup_{t\in[0,T]} \|X(t)\|^{2p} &\leq C(p,c_g,k_g,T)\left[1 + \|\varphi\|^{2p}\right], \\ E\int_0^T \|X(t)\|_V^{2p}\, dt &\leq C(p,c_g,k_g,T)\left[1 + \|\varphi\|_V^{2p}\right]. \end{aligned}$$

Moreover, the sequence of Galerkin approximations $(X_n)_{n\in\mathbb{N}}$ converges to the variational solution X of problem (2.17) *strongly in $L^2(\Omega; C([0,T];H))$ and weakly in $L^{2p}(\Omega\times[0,T];V)$.*

Proof. It suffices to focus on the existence of the variational solution because the uniqueness can be found in Theorem 2.2.1. We know from Theorem 2.2.9 that there exists a unique variational solution $X_n \in L^{2p}(\Omega; C([0,T];H)) \cap L^{2p}(\Omega\times[0,T];V)$ of problem (2.24) and its corresponding uniform a priori estimates. With respect to the definition of $f_n(u)$ and $g_n(\,\cdot\,,u)w$ for all $u \in H_n$ and all $w \in K_n$ and because of the first property in (2.3), we write the stochastic nonlinear Schrödinger problem (2.24) in the equivalent form

$$\begin{aligned} \big(X_n(t), h_k\big) &= \big(\varphi_n, h_k\big) - i\int_0^t \big\langle AX_n(s), h_k\big\rangle\, ds - \lambda \int_0^t \big(f(X_n(s)), h_k\big)\, ds \\ &\quad + \left(\int_0^t g(s, X_n(s))\, dW_n(s), h_k\right) \end{aligned} \tag{2.39}$$

for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $k \in \{1,2,\ldots,n\}$. Due to (2.18) and Theorem 2.2.9 (for $p = 2\sigma+1$), the nonlinear drift term obeys

$$E\int_0^T \|f(X_n(t))\|^2\, dt \leq 2^{2\sigma+1}\, E\int_0^T \|X_n(t)\|_V^{2(2\sigma+1)}\, dt \leq C(\sigma,c_g,k_g,T)\left[1 + \|\varphi\|_V^{2(2\sigma+1)}\right], \tag{2.40}$$

which implies the uniform boundedness of $(f(X_n))_n$ in $L^2(\Omega \times [0,T]; H)$. Hence, based on its structure, we do not need any condition of bounded growth for the nonlinear drift term. The property (2.10) and Theorem 2.2.9 (for $\sigma = 1$) lead to

$$E \int_0^T \|g(t, X_n(t))\|^2_{L_2(K,H)} \, dt \le 2k_g T + 2c_g E \int_0^T \|X_n(t)\|^2 \, dt \le C(c_g, k_g, T) \left[1 + \|\varphi\|^2\right],$$

which entails that $(g(\,\cdot\,, X_n))_n$ is uniformly bounded in $L^2(\Omega \times [0,T]; L_2(K,H))$.

First, we fix $p \ge 4\sigma$. By the above uniform boundedness properties and Lemma F.1, it follows that there exist a subsequence of $(X_n)_n$, which we denote for simplicity by $(X_n)_n$ as well, and functions $\tilde{Z} \in L^{2p}(\Omega \times [0,T]; V)$, $f^* \in L^2(\Omega \times [0,T]; H)$ and $g^* \in L^2(\Omega \times [0,T]; L_2(K,H))$ such that we receive for $n \to \infty$ that

$$X_n \rightharpoonup \tilde{Z} \qquad \text{in } L^2(\Omega \times [0,T]; H), L^2(\Omega \times [0,T]; V) \text{ and } L^{2p}(\Omega \times [0,T]; V), \tag{2.41}$$

$$f(X_n) \rightharpoonup f^* \qquad \text{in } L^2(\Omega \times [0,T]; H), \tag{2.42}$$

$$g(\,\cdot\,, X_n) \rightharpoonup g^* \qquad \text{in } L^2(\Omega \times [0,T]; L_2(K,H)). \tag{2.43}$$

Taking $n \to \infty$ in (2.39) and using these weak convergence results, we get for a.e. $(\omega, t) \in \Omega \times [0,T]$ and all $k \in \mathbb{N}$ that

$$\big(\tilde{Z}(t), h_k\big) = (\varphi, h_k) - i \int_0^t \big\langle A\tilde{Z}(s), h_k \big\rangle \, ds - \lambda \int_0^t \big(f^*(s), h_k\big) \, ds + \left(\int_0^t g^*(s) \, dW(s), h_k \right). \tag{2.44}$$

There exists an $\mathcal{F}_t$-measurable process for all $t \in [0,T]$, which has in H almost surely continuous trajectories, is equal to $\tilde{Z}(t)$ for a.e. $(\omega, t) \in \Omega \times [0,T]$ and is equal to the right-hand side of (2.44) for a.e. $\omega \in \Omega$ and all $t \in [0,T]$ (see [87, p. 73, Theorem 2]). We denote this process by $(Z(t))_{t \in [0,T]}$ and get for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $k \in \mathbb{N}$ that

$$\big(Z(t), h_k\big) = (\varphi, h_k) - i \int_0^t \big\langle AZ(s), h_k \big\rangle \, ds - \lambda \int_0^t \big(f^*(s), h_k\big) \, ds + \left(\int_0^t g^*(s) \, dW(s), h_k \right). \tag{2.45}$$

Next, we denote by $Z_n := \pi_n Z$ and $g_n^*(\,\cdot\,)w := \pi_n\{g^*(\,\cdot\,)w\}$ for all $w \in K_n$ the finite-dimensional approximations of Z and $g^*(\,\cdot\,)w$, respectively. Regarding (2.24) and (2.45) for $Z_n(t)$, using the stochastic energy equality (C.2) and summing up over all $k = 1, 2, \ldots, n$, it follows that

$$\begin{aligned}
\|X_n(t) - Z_n(t)\|^2 = {}& 2 \operatorname{Im} \int_0^t \big\langle A\left[X_n(s) - Z_n(s)\right], X_n(s) - Z_n(s) \big\rangle \, ds \\
& - 2\lambda \operatorname{Re} \int_0^t \big(f_n(X_n(s)) - f_n^*(s), X_n(s) - Z_n(s)\big) \, ds \\
& + 2 \operatorname{Re} \sum_{j=1}^n \int_0^t \big([g_n(s, X_n(s)) - g_n^*(s)]e_j, X_n(s) - Z_n(s)\big) \, d\beta_j(s) \\
& - 2 \operatorname{Re} \sum_{j=n+1}^\infty \int_0^t \big(g_n^*(s)e_j, X_n(s) - Z_n(s)\big) \, d\beta_j(s) \\
& + \int_0^t \|g_n(s, X_n(s)) - g_n^*(s)\|^2_{L_2(K_n, H_n)} \, ds \\
& + \int_0^t \sum_{j=n+1}^\infty \sum_{k=1}^n \big|\big(g_n^*(s)e_j, h_k\big)\big|^2 \, ds
\end{aligned}$$

for a.e. $\omega \in \Omega$ and all $t \in [0,T]$. Regarding each time the first property in (2.3) and (2.5) of the orthogonal projection π_n and the operator A, choosing $\xi(t) := \exp\{-2(1 + c_g)t\}$ for all

$t \in [0,T]$, applying the Itô formula to $F(t,\eta(t)) := \xi(t)\eta(t)$ with $\eta(t) := \|X_n(t) - Z_n(t)\|^2$ and taking expectation, we obtain

$$\begin{aligned} E\,\xi(T)\,\|X_n(T) - Z_n(T)\|^2 = &- 2(1+c_g)E\int_0^T \xi(t)\,\|X_n(t) - Z_n(t)\|^2\,dt \\ &- 2\lambda E\,\mathrm{Re}\int_0^T \xi(t)\,\big(f(X_n(t)) - f^*(t), X_n(t) - Z_n(t)\big)\,dt \\ &+ E\int_0^T \xi(t)\,\|g_n(t,X_n(t)) - g_n^*(t)\|^2_{L_2(K_n,H_n)}\,dt \\ &+ E\int_0^T \xi(t)\sum_{j=n+1}^{\infty}\sum_{k=1}^{n}\big|\big(g^*(t)e_j, h_k\big)\big|^2\,dt. \end{aligned} \tag{2.46}$$

For the sake of brevity, we omit to write the dependence on $t \in [0,T]$ in the following two auxiliary results. Property (2.20) entails

$$\begin{aligned} &- 2\lambda\,\mathrm{Re}\big(f(X_n) - f^*, X_n - Z_n\big) \\ =&- 2\lambda\,\mathrm{Re}\big(f(X_n) - f(Z_n), X_n - Z_n\big) - 2\lambda\,\mathrm{Re}\big(f(Z_n) - f(Z), X_n - Z_n\big) \\ &- 2\lambda\,\mathrm{Re}\big(f(Z) - f^*, X_n - Z_n\big) \\ \le&- 2\lambda\,\mathrm{Re}\big(f(Z_n) - f(Z), X_n - Z_n\big) - 2\lambda\,\mathrm{Re}\big(f(Z) - f^*, X_n - Z_n\big) \\ \le&\,\lambda^2\|f(Z_n) - f(Z)\|^2 + \|X_n - Z_n\|^2 - 2\lambda\,\mathrm{Re}\big(f(Z) - f^*, X_n - Z_n\big), \end{aligned}$$

and, regarding the second property in (2.3) and the Lipschitz continuity (2.8) of g, it results that

$$\begin{aligned} &\|g_n(\,\cdot\,,X_n) - g_n^*\|^2_{L_2(K_n,H_n)} \le \|g(\,\cdot\,,X_n) - g^*\|^2_{L_2(K,H)} \\ =&\big(g(\,\cdot\,,X_n) - g(\,\cdot\,,Z), g(\,\cdot\,,X_n) - g(\,\cdot\,,Z)\big)_{L_2(K,H)} \\ &+ \big(g(\,\cdot\,,Z) - g^*, g(\,\cdot\,,X_n) - g(\,\cdot\,,Z)\big)_{L_2(K,H)} \\ &+ \big(g(\,\cdot\,,X_n) - g^*, g(\,\cdot\,,Z) - g^*\big)_{L_2(K,H)} \\ =&\,\|g(\,\cdot\,,X_n) - g(\,\cdot\,,Z)\|^2_{L_2(K,H)} + \big(g(\,\cdot\,,X_n) - g^*, g(\,\cdot\,,Z) - g^*\big)_{L_2(K,H)} \\ &+ \big(g(\,\cdot\,,Z) - g^*, g(\,\cdot\,,X_n) - g^*\big)_{L_2(K,H)} - \|g(\,\cdot\,,Z) - g^*\|^2_{L_2(K,H)} \\ \le&\,2c_g\|X_n - Z_n\|^2 + 2c_g\|Z_n - Z\|^2 + \big(g(\,\cdot\,,X_n) - g^*, g(\,\cdot\,,Z) - g^*\big)_{L_2(K,H)} \\ &+ \big(g(\,\cdot\,,Z) - g^*, g(\,\cdot\,,X_n) - g^*\big)_{L_2(K,H)} - \|g(\,\cdot\,,Z) - g^*\|^2_{L_2(K,H)}. \end{aligned}$$

Hence, it remains from (2.46) that

$$\begin{aligned} &E\,\xi(T)\,\|X_n(T) - Z_n(T)\|^2 \\ \le&- E\int_0^T \xi(t)\,\|X_n(t) - Z_n(t)\|^2\,dt + \lambda^2 E\int_0^T \xi(t)\,\|f(Z_n(t)) - f(Z(t))\|^2\,dt \\ &- 2\lambda E\,\mathrm{Re}\int_0^T \xi(t)\,\big(f(Z(t)) - f^*(t), X_n(t) - Z_n(t)\big)\,dt \\ &+ 2c_g E\int_0^T \xi(t)\,\|Z_n(t) - Z(t)\|^2\,dt \\ &+ E\int_0^T \xi(t)\,\big(g(t,X_n(t)) - g^*(t), g(t,Z(t)) - g^*(t)\big)_{L_2(K,H)}\,dt \\ &+ E\int_0^T \xi(t)\,\big(g(t,Z(t)) - g^*(t), g(t,X_n(t)) - g^*(t)\big)_{L_2(K,H)}\,dt \\ &- E\int_0^T \xi(t)\,\|g(t,Z(t)) - g^*(t)\|^2_{L_2(K,H)}\,dt + E\int_0^T \xi(t)\sum_{j=n+1}^{\infty}\sum_{k=1}^{n}\big|\big(g^*(t)e_j, h_k\big)\big|^2\,dt. \end{aligned} \tag{2.47}$$

Due to the monotone convergence theorem and since $g^* \in L^2(\Omega \times [0,T]; L_2(K,H))$, we have for $n \to \infty$ that

$$E\int_0^T \xi(t) \sum_{j=n+1}^{\infty} \sum_{k=1}^{n} \left|\left(g^*(t)e_j, h_k\right)\right|^2 dt \le E\int_0^T \xi(t) \sum_{j=n+1}^{\infty} \|g^*(t)e_j\|^2\, dt \to 0.$$

We use that $Z_n = \pi_n Z$ and $2p \ge 8\sigma \ge 4$ to state

$$Z_n, Z \in L^{2p}(\Omega \times [0,T]; V) \hookrightarrow L^4(\Omega \times [0,T]; V) \hookrightarrow L^4(\Omega \times [0,T]; H)$$

and

$$\begin{aligned}\|Z_n(t) - Z(t)\|^4 &= \left(\|Z_n(t) - Z(t)\|^2\right)^2 \le \left(2\,\|Z_n(t)\|^2 + 2\,\|Z(t)\|^2\right)^2 \\ &\le \left(4\,\|Z(t)\|^2\right)^2 = 16\,\|Z(t)\|^4.\end{aligned}$$

Observe that $16\|Z(t)\|^4$ is an integrable majorant for $\|Z_n(t) - Z(t)\|^4$ over $\Omega \times [0,T]$ and that (by the third property in (2.3)) $\|Z_n(t) - Z(t)\| \to 0$ for a.e. $\omega \in \Omega$ and all $t \in [0,T]$ as $n \to \infty$ such that Lebesgue's dominated convergence theorem entails

$$\lim_{n\to\infty} E\int_0^T \|Z_n(t) - Z(t)\|^4\, dt = E\int_0^T \lim_{n\to\infty} \|Z_n(t) - Z(t)\|^4\, dt = 0. \tag{2.48}$$

Based on relation (2.19), the fact that $\xi(t) \le 1$ for all $t \in [0,T]$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &E\int_0^T \xi(t)\, \|f(Z_n(t)) - f(Z(t))\|^2\, dt \\ &\le 2^{2\sigma+3}\sigma^2 E\int_0^T \left(\|Z_n(t)\|_V^{4\sigma} + \|Z(t)\|_V^{4\sigma}\right) \|Z_n(t) - Z(t)\|^2\, dt \\ &\le 2^{\frac{4\sigma+7}{2}}\sigma^2 \left(E\int_0^T \left(\|Z_n(t)\|_V^{8\sigma} + \|Z(t)\|_V^{8\sigma}\right) dt\right)^{\frac{1}{2}} \left(E\int_0^T \|Z_n(t) - Z(t)\|^4\, dt\right)^{\frac{1}{2}}. \end{aligned} \tag{2.49}$$

This yields

$$f(Z_n) \to f(Z) \qquad \text{in } L^2(\Omega \times [0,T]; H) \text{ as } n \to \infty \tag{2.50}$$

due to $Z_n, Z \in L^{2p}(\Omega \times [0,T]; V) \hookrightarrow L^{8\sigma}(\Omega \times [0,T]; V)$ and (2.48). Moreover, it holds that $Z_n - Z \to 0$ in $L^2(\Omega \times [0,T]; H)$ (since $Z_n = \pi_n Z$), $X_n - Z_n = (X_n - Z) + (Z - Z_n) \rightharpoonup 0$ in $L^2(\Omega \times [0,T]; H)$ (by the last result and (2.41)) and $g(\,\cdot\,, X_n) \rightharpoonup g^*$ in $L^2(\Omega \times [0,T]; L_2(K,H))$ (see (2.43)). Thus, writing the non-positive terms in (2.47) on the left-hand side, it follows for $n \to \infty$ that

$$E\,\xi(T)\,\|X_n(T) - Z_n(T)\|^2 \to 0, \qquad E\int_0^T \xi(t)\,\|X_n(t) - Z_n(t)\|^2\, dt \to 0,$$

and, therefore,

$$E\int_0^T \|X_n(t) - Z_n(t)\|^2\, dt \to 0. \tag{2.51}$$

Furthermore, we obtain

$$E\int_0^T \xi(t)\,\|g(t, Z(t)) - g^*(t)\|_{L_2(K,H)}^2\, dt = 0,$$

which implies

$$g(t, Z(t)) = g^*(t), \qquad \text{for a.e. } (\omega, t) \in \Omega \times [0,T].$$

Now, consider $\eta \in L^2(\Omega \times [0,T]; H)$ to be a simple function. Hence, it is uniformly bounded with respect to the variables ω and t. Relation (2.19) and the Cauchy-Schwarz inequality lead to

$$\begin{aligned}
&\left| E \int_0^T \big(f(X_n(t)) - f(Z_n(t)), \eta(t)\big)\, dt \right| \\
&\leq E \int_0^T \|\eta(t)\| \, \|f(X_n(t)) - f(Z_n(t))\| \, dt \\
&\leq 2^{\sigma+2}\sigma \, E \int_0^T \|\eta(t)\| \left(\|X_n(t)\|_V^{2\sigma} + \|Z_n(t)\|_V^{2\sigma}\right) \|X_n(t) - Z_n(t)\| \, dt \\
&\leq 2^{\sigma+2}\sigma \left(E \int_0^T \|\eta(t)\|^2 \left(\|X_n(t)\|_V^{2\sigma} + \|Z_n(t)\|_V^{2\sigma}\right)^2 dt \right)^{\frac{1}{2}} \left(E \int_0^T \|X_n(t) - Z_n(t)\|^2 \, dt \right)^{\frac{1}{2}} \\
&\leq 2^{\sigma+\frac{5}{2}}\sigma \left(E \int_0^T \|\eta(t)\|^2 \left(\|X_n(t)\|_V^{4\sigma} + \|Z_n(t)\|_V^{4\sigma}\right) dt \right)^{\frac{1}{2}} \left(E \int_0^T \|X_n(t) - Z_n(t)\|^2 \, dt \right)^{\frac{1}{2}}.
\end{aligned}$$

As a result of (2.51) and the fact that $(X_n)_n$ and $(Z_n)_n$ are bounded sequences in $L^{4\sigma}(\Omega \times [0,T]; V)$ (due to Theorem 2.2.9 and the embedding $L^{2p}(\Omega \times [0,T]; V) \hookrightarrow L^{4\sigma}(\Omega \times [0,T]; V)$), we conclude

$$E \int_0^T \big(f(X_n(t)) - f(Z_n(t)), \eta(t)\big)\, dt \to 0 \qquad \text{as } n \to \infty$$

for each simple function $\eta \in L^2(\Omega \times [0,T]; H)$. Based on the weak convergences $f(X_n) \rightharpoonup f^*$ (by (2.42)) and $f(Z_n) - f(Z) \to 0$ (by (2.50)) in $L^2(\Omega \times [0,T]; H)$, it follows for each simple function $\eta \in L^2(\Omega \times [0,T]; H)$ that

$$\begin{aligned}
0 &= \lim_{n\to\infty} E \int_0^T \big(f(X_n(t)) - f(Z_n(t)), \eta(t)\big)\, dt \\
&= \lim_{n\to\infty} E \int_0^T \big(f(X_n(t)) - f^*(t), \eta(t)\big)\, dt + E \int_0^T \big(f^*(t) - f(Z(t)), \eta(t)\big)\, dt \\
&\quad + \lim_{n\to\infty} E \int_0^T \big(f(Z(t)) - f(Z_n(t)), \eta(t)\big)\, dt \\
&= E \int_0^T \big(f^*(t) - f(Z(t)), \eta(t)\big)\, dt.
\end{aligned}$$

However, the set of simple functions is dense in the space $L^2(\Omega \times [0,T]; H)$, so we deduce

$$E \int_0^T \big(f^*(t) - f(Z(t)), \eta(t)\big)\, dt = 0, \qquad \text{for all } \eta \in L^2(\Omega \times [0,T]; H).$$

Thus,

$$f(Z(t)) = f^*(t), \qquad \text{for a.e. } (\omega, t) \in \Omega \times [0,T].$$

Then equation (2.45) coincides for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $k \in \mathbb{N}$ with

$$\big(Z(t), h_k\big) = \big(\varphi, h_k\big) - i \int_0^t \langle AZ(s), h_k \rangle \, ds - \lambda \int_0^t \big(f(Z(s)), h_k\big)\, ds + \left(\int_0^t g(s, Z(s))\, dW(s), h_k \right).$$

Since $\operatorname{span}\{h_1, h_2, \ldots, h_n, \ldots\}$ is dense in V, the above equation also holds for all $v \in V$. Hence, $X := Z$ is the variational solution of the stochastic nonlinear Schrödinger problem (2.17), and $X \in L^2(\Omega; C([0,T]; H)) \cap L^{2p}(\Omega \times [0,T]; V)$ for fixed $p \geq 4\sigma$.

For $p \in [1, 4\sigma)$ we use the continuous embedding result $L^{8\sigma}(\Omega \times [0,T]; V) \hookrightarrow L^{2p}(\Omega \times [0,T]; V)$. Consequently, weak convergence in $L^{8\sigma}(\Omega \times [0,T]; V)$ implies weak convergence in $L^{2p}(\Omega \times [0,T]; V)$

(see [104, p. 265, Proposition 21.35 (c)]). By now, we only know that a subsequence of $(X_n)_n$ converges to X strongly in $L^2(\Omega \times [0,T]; H)$ (since $X_n - X = (X_n - Z_n) + (Z_n - X) \to 0$ in $L^2(\Omega \times [0,T]; H)$ by $Z_n = \pi_n Z = \pi_n X$ and (2.51)) and weakly in $L^{2p}(\Omega \times [0,T]; V)$. In fact, the whole sequence has these properties. Observe that every subsequence of $(X_n)_n$ has a subsequence which converges strongly to the same limit X (the unique solution of the Schrödinger problem) in $L^2(\Omega \times [0,T]; H)$. Hence, the whole sequence $(X_n)_n$ converges strongly to X in $L^2(\Omega \times [0,T]; H)$ (compare [103, p. 480, Proposition 10.13 (1)]). Furthermore, every subsequence of $(X_n)_n$ has, in turn, a subsequence which converges weakly to the same limit X in $L^2(\Omega \times [0,T]; V)$. Hence, the whole sequence $(X_n)_n$ converges weakly to X in $L^2(\Omega \times [0,T]; V)$ (see [103, p. 480, Proposition 10.13 (2)]). Using the weak convergence of $(X_n)_n$ to X in $L^{2p}(\Omega \times [0,T]; V)$ for all $p \geq 1$ and the result of Theorem 2.2.9, we get

$$E \int_0^T \|X(t)\|_V^{2p} \, dt \leq \liminf_{n \to \infty} E \int_0^T \|X_n(t)\|_V^{2p} \, dt \leq C(p, c_g, k_g, T) \left[1 + \|\varphi\|_V^{2p}\right]. \tag{2.52}$$

Similarly to Theorem 2.2.2 and Theorem 2.2.5, the estimate

$$E \sup_{t \in [0,T]} \|X(t)\|^{2p} \leq C(p, c_g, k_g, T) \left[1 + \|\varphi\|^{2p}\right]$$

for all $p \geq 1$ can be shown. Therefore, it holds that $X \in L^{2p}(\Omega; C([0,T]; H)) \cap L^{2p}(\Omega \times [0,T]; V)$.

To verify the strong convergence of $(X_n)_n$ to X in $L^2(\Omega; C([0,T]; H))$, we take equations (2.17) and (2.24), apply the stochastic energy equality (C.2) to their difference and obtain

$$\begin{aligned}
\|X(t) - X_n(t)\|^2 = {} & \|\varphi - \varphi_n\|^2 + 2 \operatorname{Im} \int_0^t \big\langle A\left[X(s) - X_n(s)\right], X(s) - X_n(s) \big\rangle \, ds \\
& - 2\lambda \operatorname{Re} \int_0^t \big(f(X(s)) - f_n(X_n(s)), X(s) - X_n(s)\big) \, ds \\
& + 2 \operatorname{Re} \sum_{j=1}^{n} \int_0^t \big(\left[g(s, X(s)) - g_n(s, X_n(s))\right] e_j, X(s) - X_n(s)\big) \, d\beta_j(s) \\
& + 2 \operatorname{Re} \sum_{j=n+1}^{\infty} \int_0^t \big(g(s, X(s)) e_j, X(s) - X_n(s)\big) \, d\beta_j(s) \\
& + \int_0^t \sum_{j=1}^{n} \|\left[g(s, X(s)) - g_n(s, X_n(s))\right] e_j\|^2 \, ds \\
& + \int_0^t \sum_{j=n+1}^{\infty} \|g(s, X(s)) e_j\|^2 \, ds
\end{aligned} \tag{2.53}$$

for a.e. $\omega \in \Omega$ and all $t \in [0,T]$. Based on the series representation, the first term on the right-hand side converges to zero as $n \to \infty$ and the second one vanishes because of

$$X(s) - X_n(s) = \sum_{k=1}^{n} \big(X(s) - X_n(s), h_k\big) h_k + \sum_{k=n+1}^{\infty} \big(X(s), h_k\big) h_k,$$

the eigenvalue equation $Ah_k = \mu_k h_k$ for all $k \in \mathbb{N}$ and the orthogonality of the eigenfunctions $(h_k)_{k \in \mathbb{N}}$. Due to the third property in (2.3) and relation (2.51) with $Z_n(s) = \pi_n Z(s) = \pi_n X(s)$, it results for $n \to \infty$ that

$$\begin{aligned}
& E \int_0^T \|X(s) - X_n(s)\|^2 \, ds \\
& \leq 2E \int_0^T \|X(s) - \pi_n X(s)\|^2 \, ds + 2E \int_0^T \|\pi_n X(s) - X_n(s)\|^2 \, ds \to 0.
\end{aligned} \tag{2.54}$$

The Cauchy-Schwarz inequality yields

$$\begin{aligned}
&E \sup_{t\in[0,T]} -2\lambda \operatorname{Re} \int_0^t \big(f(X(s)) - f_n(X_n(s)), X(s) - X_n(s)\big)\, ds \\
&\le 2\lambda E \int_0^T \|f(X(s)) - f_n(X_n(s))\| \, \|X(s) - X_n(s)\| \, ds \\
&\le 2\lambda \left(E \int_0^T \|f(X(s)) - f_n(X_n(s))\|^2 \, ds\right)^{\frac{1}{2}} \left(E \int_0^T \|X(s) - X_n(s)\|^2 \, ds\right)^{\frac{1}{2}},
\end{aligned}$$

which converges to zero as $n \to \infty$ because the first expression in parentheses is bounded (compare (2.3), (2.40) and (2.18), (2.52)), while the second one goes to zero (by (2.54)). Using the Burkholder-Davis-Gundy inequality (2.13), we get

$$\begin{aligned}
&E \sup_{t\in[0,T]} 2 \operatorname{Re} \sum_{j=1}^{n} \int_0^t \big([g(s, X(s)) - g_n(s, X_n(s))]\, e_j, X(s) - X_n(s)\big)\, d\beta_j(s) \\
&\le 6E \left[\int_0^T \sum_{j=1}^{n} \|[g(s, X(s)) - g_n(s, X_n(s))]\, e_j\|^2 \, \|X(s) - X_n(s)\|^2 \, ds\right]^{\frac{1}{2}} \\
&\le E \left[\left(\frac{1}{2} \sup_{t\in[0,T]} \|X(t) - X_n(t)\|^2\right)^{\frac{1}{2}} \left(72 \int_0^T \sum_{j=1}^{n} \|[g(s, X(s)) - g_n(s, X_n(s))]\, e_j\|^2 \, ds\right)^{\frac{1}{2}}\right] \\
&\le \frac{1}{4} E \sup_{t\in[0,T]} \|X(t) - X_n(t)\|^2 + 36E \int_0^T \sum_{j=1}^{n} \|[g(s, X(s)) - g_n(s, X_n(s))]\, e_j\|^2 \, ds.
\end{aligned}$$

Relations (2.2) and (2.3), the definition of $g_n(\,\cdot\,, u)w$ for all $u \in H_n$ and all $w \in K_n$, Cauchy's Double Series Theorem (see [30, p. 22]) and the Lipschitz continuity (2.8) of g entail

$$\begin{aligned}
&E \int_0^T \sum_{j=1}^{n} \|[g(s, X(s)) - g_n(s, X_n(s))]\, e_j\|^2 \, ds \\
&\le 2E \int_0^T \sum_{j=1}^{n} \Big(\|g(s, X(s))e_j - \pi_n\{g(s, X(s))e_j\}\|^2 + \|\pi_n\{g(s, X(s))e_j\} - g_n(s, X_n(s))e_j\|^2\Big)\, ds \\
&\le 2E \int_0^T \sum_{j=1}^{\infty} \left\| \sum_{k=n+1}^{\infty} \big(g(s, X(s))e_j, h_k\big)\, h_k \right\|^2 ds \\
&\quad + 2E \int_0^T \sum_{j=1}^{\infty} \left\| \sum_{k=1}^{n} \big([g(s, X(s)) - g(s, X_n(s))]e_j, h_k\big)\, h_k \right\|^2 ds \\
&\le 2E \int_0^T \sum_{j=1}^{\infty} \sum_{k=n+1}^{\infty} \big|\big(g(s, X(s))e_j, h_k\big)\big|^2 \, ds \\
&\quad + 2E \int_0^T \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \big|\big([g(s, X(s)) - g(s, X_n(s))]e_j, h_k\big)\big|^2 \, ds \\
&= 2E \int_0^T \sum_{k=n+1}^{\infty} \sum_{j=1}^{\infty} \big|\big(g(s, X(s))e_j, h_k\big)\big|^2 \, ds + 2E \int_0^T \|g(s, X(s)) - g(s, X_n(s))\|^2_{L_2(K,H)} \, ds \\
&\le 2E \int_0^T \sum_{k=n+1}^{\infty} \sum_{j=1}^{\infty} \big|\big(g(s, X(s))e_j, h_k\big)\big|^2 \, ds + 2c_g E \int_0^T \|X(s) - X_n(s)\|^2 \, ds.
\end{aligned}$$

This expression also converges to zero as $n \to \infty$ because of relation (2.54), the fact that $g \in L^2(\Omega \times [0,T]; L_2(K,H))$ and the rest of a convergent series goes to zero. Furthermore, the Burkholder-Davis-Gundy inequality (2.13) leads to

$$
\begin{aligned}
&E \sup_{t\in[0,T]} 2\,\mathrm{Re} \sum_{j=n+1}^{\infty} \int_0^t \big(g(s,X(s))e_j, X(s)-X_n(s)\big)\, d\beta_j(s) \\
&\leq 6E \left[\int_0^T \sum_{j=n+1}^{\infty} \|g(s,X(s))e_j\|^2 \, \|X(s)-X_n(s)\|^2 \, ds \right]^{\frac{1}{2}} \\
&\leq E \left[\left(\frac{1}{2} \sup_{t\in[0,T]} \|X(t)-X_n(t)\|^2 \right)^{\frac{1}{2}} \left(72 \int_0^T \sum_{j=n+1}^{\infty} \|g(s,X(s))e_j\|^2 \, ds \right)^{\frac{1}{2}} \right] \\
&\leq \frac{1}{4} E \sup_{t\in[0,T]} \|X(t)-X_n(t)\|^2 + 36E \int_0^T \sum_{j=n+1}^{\infty} \|g(s,X(s))e_j\|^2 \, ds,
\end{aligned}
$$

where the same argument of a convergent series is valid such that

$$
E \int_0^T \sum_{j=n+1}^{\infty} \|g(s,X(s))e_j\|^2 \, ds \to 0 \qquad \text{as } n \to \infty.
$$

Finally, equation (2.53) given by the stochastic energy equality results in

$$
E \sup_{t\in[0,T]} \|X(t)-X_n(t)\|^2 \to 0 \qquad \text{as } n \to \infty.
$$

Thus, the sequence of Galerkin approximations $(X_n)_n$ converges to the variational solution X of problem (2.17) strongly in $L^2(\Omega; C([0,T]; H))$ and weakly in $L^{2p}(\Omega \times [0,T]; V)$. □

2.2.3 Generalizations

Instead of homogeneous Neumann boundary conditions, we can also think of homogeneous Dirichlet boundary conditions (for the deterministic two-dimensional case see [11, 96]) or periodic boundary conditions (the deterministic one-dimensional case can be found in [36, 94]). The only thing that changes is the explicit form of the eigenvalues and eigenfunctions of the operator A. However, all their properties are retained which means that there still exists an increasing sequence of real-valued eigenvalues and the corresponding eigenfunctions are orthonormal in H and orthogonal in a space V that is adjusted to the boundary conditions. Then all results of this section for the stochastic nonlinear Schrödinger problem remain the same.

Furthermore, during this work, we use Lipschitz continuity and bounded growth conditions of the diffusion function g. These assumptions can be weakened to local Lipschitz continuity in $L_2(K,H)$ and bounded growth in $L_2(K,H)$ and $L_2(K,V)$.

Corollary 2.2.11. *The results of Section 2.2 and, therefore, especially of Theorem 2.2.10 also hold if we replace conditions* (2.8) *and* (2.9) *by the following assumptions:*

- *for each $L \in \mathbb{N}$ there exists a constant $c_{g,L} > 0$ such that for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $u, v \in H$ with $\|u\| \leq L$ and $\|v\| \leq L$*

$$
\|g(t,u) - g(t,v)\|^2_{L_2(K,H)} \leq c_{g,L} \|u-v\|^2,
$$

- *there exists a constant $k_g > 0$ such that for a.e. $\omega \in \Omega$, all $t \in [0,T]$, all $u \in H$ and all $v \in V$*

$$
\|g(t,u)\|^2_{L_2(K,H)} \leq k_g \left(1 + \|u\|^2\right) \qquad \text{and} \qquad \|g(t,v)\|^2_{L_2(K,V)} \leq k_g \left(1 + \|v\|_V^2\right).
$$

Proof. At first, we use the Galerkin method and, secondly, we apply the truncation function to the drift term f as well as to the diffusion term g. Then we obtain again globally Lipschitz continuous functions f_n^M and g_n^M for fixed $M, n \in \mathbb{N}$ and can proceed as in the present issue. We immediately get the unique existence of the variational solution of the truncated finite-dimensional problem and can extend this result to the finite-dimensional and, thereafter, to the infinite-dimensional problem as described in the current section. Hence, the same results as in the case of a globally Lipschitz continuous function g hold. $\square$

The discussion of more general types of nonlinearities f is another possible extension of our stochastic Schrödinger problem.

Corollary 2.2.12. *All assertions of the current section remain true if we exchange the power-term $f(v) = |v|^{2\sigma}v$ for all $v \in V$ and $\sigma \geq 1$ by a nonlinear function $f : V \to H$ defined by $f(v) := F(|v|^2)v$, where $F : [0, \infty) \to [0, \infty)$ is once continuously differentiable with $F'(x) \geq 0$ for each $x \geq 0$, and there exist $C > 0$ and $\sigma > 1$ such that*

$$|F(x_1) - F(x_2)| \leq C \left(1 + |x_1|^{\sigma-1} + |x_2|^{\sigma-1}\right) |x_1 - x_2|, \qquad \text{for all } x_1, x_2 \geq 0. \tag{2.55}$$

The case $\sigma = 1$ may also be included by assuming that F is globally Lipschitz continuous.

Proof. Assumption (2.55) substitutes the inequality from Lemma D.4 (b). With the help of Lemma D.2 and Young's inequality, one can verify the analogues of (2.18) and (2.19)

$$\|f(v)\| \leq C(\sigma) \left(1 + \|v\|_V^{2\sigma+1}\right), \qquad \text{for all } v \in V,$$
$$\|f(u) - f(v)\| \leq C(\sigma) \left(1 + \|u\|_V^{2\sigma} + \|v\|_V^{2\sigma}\right) \|u - v\|, \qquad \text{for all } u, v \in V.$$

These inequalities permit to derive similar estimates as (2.40) and (2.49) needed in Theorem 2.2.10. The result from Lemma D.5 is replaced by

$$\operatorname{Re}\left\{\left(F(|z_1|^2)z_1 - F(|z_2|^2)z_2\right)\left(\overline{z_1} - \overline{z_2}\right)\right\} \geq 0, \qquad \text{for all } z_1, z_2 \in \mathbb{C},$$

which is proved analogously to Lemma D.5 while using the fact that F is an increasing and positive function. Moreover, the inequality from Lemma D.6 is exchanged by

$$\operatorname{Re}\left\{\left(F(|v|^2)v, Av\right)\right\} \geq 0, \qquad \text{for each } v \in V \text{ such that } Av \in H,$$

which is shown similarly to Lemma D.6 since F and F' are positive functions. Further details of the derivation of these inequalities can be found in Appendix E. $\square$

The case $F(x) = x^\sigma$ with $\sigma \geq 1$ corresponds to $f(v) = |v|^{2\sigma}v$ which obeys (2.55) since $|x_1^\sigma - x_2^\sigma| \leq \sigma|x_1|^{\sigma-1}|x_1 - x_2|$, which can be seen by applying inequality (D.1) for $x = \frac{x_1}{x_2} \geq 1$ with $x_2 \neq 0$ and $s = \sigma$. Such nonlinearities appear, for example, in the deterministic articles [55, 79, 80]. We can also take a polynomial of the form $F(x) = \lambda_0 + \lambda_1 x + \lambda_2 x^2$ with $\lambda_k \geq 0$ for $k = 0, 1, 2$, which represents a cubic-quintic nonlinearity mentioned in [7, 17]. In that case our method also works and yields the same results. Without loss of generality, this idea can be transferred to polynomials of finite degree with positive coefficients and also to linear combinations of power-type nonlinearities of the form

$$F(x) = \sum_{k=1}^{m} \lambda_k x^{\sigma_k}, \qquad \text{for fixed } m \in \mathbb{N},$$

with $\lambda_k > 0$ and $\sigma_k \geq 1$ for $k = 1, 2, \ldots, m$ like in [78, 93, for $m = 2$].

Due to our approach, we have considered the stochastic nonlinear Schrödinger problem over a bounded one-dimensional domain. It would also be interesting to investigate this problem over an unbounded domain $G \subseteq \mathbb{R}$. However, notice that there is a lack of compactness of the embedding $H^1(G) \hookrightarrow L^2(G)$ such that the main idea of Lemma D.2 does not work any longer and we do not have a countable spectrum of eigenvalues and eigenfunctions in this case.

Remark 2.2.13. *Taking* $\mathbb{R}$ *instead of the interval* $(0,1)$ *with* $H := L^2(\mathbb{R})$ *and* $V := H^1(\mathbb{R})$, *we can further ensure the same results of the variational solution as in Section 2.2 if we*

- *use the continuous embedding* $V \hookrightarrow C(\mathbb{R})$ *(see [105, p. 1027, (45d)]) such that we have an analogue of Lemma D.2 in form of*

$$\sup_{x\in\mathbb{R}} |v(x)|^2 \le c\|v\|_V^2, \qquad \text{for all } v \in V,$$

- *regard the continuity of the embedding* $V \hookrightarrow H$ *(see [105, p. 1027, (45e)]),*
- *replace the operator* A *in our Schrödinger equation by* $\mathcal{A} := A + P$, *where* $P \in L^1_{\text{loc}}(\mathbb{R})$ *is bounded from below and satisfies* $P(x) \to \infty$ *as* $|x| \to \infty$ *such that* $\mathcal{A}$ *has a purely discrete spectrum of eigenvalues and a complete set of eigenfunctions (see [84, p. 249, Theorem XIII.67]).*

2.3 A Pathwise Approach for Linear Multiplicative Noise

Based on [60], we regard the nonlinear Schrödinger equation perturbed by linear multiplicative Gaussian noise

$$dX(t) = -\,iAX(t)\,dt + i\lambda f(X(t))\,dt + i\sum_{j=1}^{\infty} b_j(t)X(t)\,d\beta_j(t) \tag{2.56}$$

for a.e. $\omega \in \Omega$ and all $t \in [0,T]$ with initial condition $X(0) = \varphi \in V$ and the notations from Section 2.1. Let $\lambda \in \mathbb{R}_+$, the nonlinear drift function $f : V \to H$ be again of the form $f(v) := |v|^{2\sigma}v$ for all $v \in V$ with $\sigma \in (0,2)$, $(\beta_j)_{j\in\mathbb{N}}$ be independent real-valued Wiener processes and $\mathfrak{B} := [\beta_1, \beta_2, \ldots]'$ such that $(\mathfrak{B}(t))_{t\in[0,T]}$ generates an increasing family of σ-algebras $(\mathcal{F}_t)_{t\in[0,T]}$. Furthermore, for all $j \ge 1$ let $b_j : \Omega \times [0,T] \to \mathbb{R}$ be $\mathcal{F}_t$-adapted processes satisfying

$$E\left(\exp\left\{\frac{2+\sigma}{2-\sigma}\sum_{j=1}^{\infty}\int_0^T b_j^2(s)\,ds\right\}\right) < \infty, \tag{2.57}$$

which especially implies that

$$\sum_{j=1}^{\infty}\int_0^T b_j^2(s)\,ds < \infty, \qquad \text{for a.e. } \omega \in \Omega. \tag{2.58}$$

Thus, the Schrödinger equation (2.56) is a special case of the Schrödinger equation in (2.11) with a power-type nonlinearity $f(t, X(t)) := |X(t)|^{2\sigma}X(t)$ and the noise term

$$\int_0^T g(t, X(t))\,dW(t) := \sum_{j=1}^{\infty}\int_0^T b_j(t)X(t)\,d\beta_j(t),$$

which is composed of a countably infinite set of linear multiplicative Wiener noises. The case of a finite set is included by assuming that $b_j(t) \equiv 0$ for all $j > m$ with $m \in \mathbb{N}$. Analogously, we define a variational solution of the stochastic nonlinear Schrödinger equation (2.56) with initial condition $\varphi \in V$ as a process $X \in L^2(\Omega; C([0,T]; H)) \cap L^2(\Omega \times [0,T]; V)$ which fulfills

$$\begin{aligned}\big(X(t), v\big) = \big(\varphi, v\big) - i\int_0^t \langle AX(s), v\rangle\,ds + i\lambda\int_0^t \big(f(X(s)), v\big)\,ds \\ + i\sum_{j=1}^{\infty}\int_0^t b_j(s)\big(X(s), v\big)\,d\beta_j(s)\end{aligned} \tag{2.59}$$

for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $v \in V$.

Remark 2.3.1. *Let* (2.57) *and by association* (2.58) *be satisfied. Then the H-valued integral*

$$I(t) := \sum_{j=1}^{\infty} \int_0^t b_j(s)\, d\beta_j(s), \qquad \text{for all } t \in [0,T],$$

defines a continuous local martingale with respect to the sequence of stopping times $(\mathcal{T}_M)_{M\in\mathbb{N}}$ given by

$$\mathcal{T}_M := \begin{cases} T & : \sum_{j=1}^{\infty} \int_0^T b_j^2(s)\, ds < M, \\ \inf\left\{ t \in [0,T] : \sum_{j=1}^{\infty} \int_0^t b_j^2(s)\, ds \geq M \right\} & : \sum_{j=1}^{\infty} \int_0^T b_j^2(s)\, ds \geq M \end{cases}$$

for all $M \in \mathbb{N}$ (compare [12, Lemma 2.1]). Notice that $(\mathcal{T}_M)_{M\in\mathbb{N}}$ is an increasing sequence which converges P-a.s. to T and (due to (2.58)*) it holds that*

$$P\left(\bigcup_{M=1}^{\infty} \{\mathcal{T}_M = T\} \right) = 1.$$

If, additionally, $X \in L^2(\Omega; C([0,T];H))$, one can also show that

$$\tilde{I}(t) := \sum_{j=1}^{\infty} \int_0^t b_j(s) X(s)\, d\beta_j(s), \qquad \text{for all } t \in [0,T],$$

is a local martingale with the same sequence of stopping times $(\mathcal{T}_M)_{M\in\mathbb{N}}$ (compare Appendix G).

We investigate the existence and uniqueness of the variational solution of (2.59) by transforming this problem into a pathwise one, derive the properties of the solution of the pathwise problem and then transfer the results to the stochastic problem (2.59). Therefore, we first regard the process

$$Y(t) := \exp\left\{ -\frac{1}{2} \sum_{j=1}^{\infty} \int_0^t b_j^2(s)\, ds - i \sum_{j=1}^{\infty} \int_0^t b_j(s)\, d\beta_j(s) \right\} \tag{2.60}$$

for a.e. $\omega \in \Omega$ and all $t \in [0,T]$, whose absolute value (needed later on) is given by

$$|Y(t)| = \exp\left\{ -\frac{1}{2} \sum_{j=1}^{\infty} \int_0^t b_j^2(s)\, ds \right\}, \qquad \text{for a.e. } \omega \in \Omega \text{ and all } t \in [0,T].$$

Observe that it is possible to write $Y(t) := \exp\{-Z(t)\}$ with

$$Z(t) := \frac{1}{2} \sum_{j=1}^{\infty} \int_0^t b_j^2(s)\, ds + i \sum_{j=1}^{\infty} \int_0^t b_j(s)\, d\beta_j(s).$$

Applying the Itô formula for $F(x) := e^{-x}$ yields the differential

$$dY(t) = -\sum_{j=1}^{\infty} b_j^2(t) Y(t)\, dt - i \sum_{j=1}^{\infty} b_j(t) Y(t)\, d\beta_j(t), \qquad Y(0) = 1.$$

Thus, the process $(Y(t))_{t\in[0,T]}$ is the solution of the stochastic linear differential equation

$$Y(t) = 1 - \sum_{j=1}^{\infty} \int_0^t b_j^2(s) Y(s)\, ds - i \sum_{j=1}^{\infty} \int_0^t b_j(s) Y(s)\, d\beta_j(s) \tag{2.61}$$

for a.e. $\omega \in \Omega$ and all $t \in [0,T]$, where the stochastic integrals in (2.60) and (2.61) are real-valued local martingales with a sequence of stopping times $(\mathcal{T}_M)_{M\in\mathbb{N}}$ as stated in Remark 2.3.1. Moreover, we introduce for a.e. $\omega \in \Omega$ and all $t \in [0,T]$

$$B(t) := \frac{1}{|Y(t)|^{2\sigma}} = \exp\left\{\sigma \sum_{j=1}^{\infty} \int_0^t b_j^2(s)\, ds\right\}, \tag{2.62}$$

where $(B(t))_{t\in[0,T]}$ is $\mathcal{F}_t$-adapted and $0 < B(t) \le B(T) < \infty$ for a.e. $\omega \in \Omega$ and all $t \in [0,T]$. Now, we consider the pathwise nonlinear Schrödinger problem

$$\big(Z(t), v\big) = \big(\varphi, v\big) - i \int_0^t \big\langle AZ(s), v\big\rangle\, ds + i\lambda \int_0^t B(s)\big(f(Z(s)), v\big)\, ds \tag{2.63}$$

for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $v \in V$. Initially, the focus is on the existence and uniqueness of the variational solution Z of the pathwise Schrödinger problem (2.63), which can be extended to the variational solution X of the stochastic Schrödinger problem (2.59) thereafter. As we verify in Subsection 2.3.2, the solution of the stochastic problem (2.59) is given by

$$X(t,\,\cdot\,) := \lim_{M\to\infty} \frac{Z(t\wedge \mathcal{T}_M,\,\cdot\,)}{Y(t\wedge\mathcal{T}_M)}, \qquad \text{for a.e. } \omega \in \Omega \text{ and all } t \in [0,T].$$

2.3.1 Investigation of the Pathwise Problem

First, we investigate the uniqueness of the variational solution of the pathwise nonlinear Schrödinger problem (2.63) (for a.e. $\omega \in \Omega$ arbitrarily fixed).

Theorem 2.3.2. *If $Z \in C([0,T];H) \cap L^2([0,T];V)$ is a variational solution of the pathwise Schrödinger problem* (2.63)*, then it is unique.*

Proof. Assume that there are two variational solutions Z, $\hat{Z} \in C([0,T];H)\cap L^2([0,T];V)$ of problem (2.63). By denoting $\delta Z := Z - \hat{Z}$, we get

$$\big(\delta Z(t), v\big) = -\,i \int_0^t \big\langle A\delta Z(s), v\big\rangle\, ds + i\lambda \int_0^t B(s)\big(f(Z(s)) - f(\hat{Z}(s)), v\big)\, ds$$

for all $t \in [0,T]$ and all $v \in V$. Applying the energy equality, we obtain

$$\|\delta Z(t)\|^2 = 2\,\mathrm{Im} \int_0^t \big\langle A\delta Z(s), \delta Z(s)\big\rangle\, ds - 2\lambda\, \mathrm{Im} \int_0^t B(s)\big(f(Z(s)) - f(\hat{Z}(s)), \delta Z(s)\big)\, ds$$

for all $t \in [0,T]$, where the first addend on the right-hand side vanishes immediately because $\mathrm{Im}\,\big\langle Av, v\big\rangle = 0$ for all $v \in V$. Due to $0 < B(s)$ for all $s \in [0,t]$ and Lemma D.4 (a), we conclude

$$\begin{aligned}
\|\delta Z(t)\|^2 &= -\,2\lambda\, \mathrm{Im} \int_0^t B(s)\big(f(Z(s)) - f(\hat{Z}(s)), \delta Z(s)\big)\, ds \\
&\le 2\lambda \int_0^t B(s) \left\| f(Z(s)) - f(\hat{Z}(s)) \right\| \|\delta Z(s)\|\, ds \\
&\le 5\lambda \int_0^t B(s) \left[\int_0^1 \Big(|Z(s,x)|^{2\sigma} + |\hat{Z}(s,x)|^{2\sigma}\Big)^2 |\delta Z(s,x)|^2\, dx\right]^{\frac{1}{2}} \|\delta Z(s)\|\, ds \\
&\le 5\lambda \int_0^t B(s) \left(\sup_{x\in[0,1]} |Z(s,x)|^{2\sigma} + \sup_{x\in[0,1]} |\hat{Z}(s,x)|^{2\sigma}\right) \|\delta Z(s)\|^2\, ds
\end{aligned}$$

for all $t \in [0,T]$. Lemma D.3 further yields

$$\int_0^T \sup_{x\in[0,1]} |Z(s,x)|^{2\sigma}\, ds < \infty \qquad \text{and} \qquad \int_0^T \sup_{x\in[0,1]} |\hat{Z}(s,x)|^{2\sigma}\, ds < \infty.$$

Hence, we deduce by Gronwall's lemma for integrable functions (see [53, p. 479, Lemma A.1]) that $\|\delta Z(t)\|^2 = 0$ for all $t \in [0,T]$ and, consequently, $Z(t) = \hat{Z}(t)$ for all $t \in [0,T]$. □

To show the existence of the variational solution of the pathwise nonlinear Schrödinger problem (2.63), we adapt the Galerkin method for deterministic nonlinear Schrödinger equations (compare [36] with $\sigma = 1$) to the case of problem (2.63) with $\sigma \in (0,2)$. For each $n \in \mathbb{N}$, we consider the finite-dimensional Galerkin equations corresponding to the pathwise Schrödinger problem (2.63)

$$\big(Z_n(t), h_k\big) = \big(\varphi_n, h_k\big) - i \int_0^t \big\langle A Z_n(s), h_k \big\rangle \, ds + i\lambda \int_0^t B(s)\big(f_n(Z_n(s)), h_k\big) \, ds \tag{2.64}$$

for all $t \in [0,T]$ and all $k \in \{1,2,\ldots,n\}$, where $\varphi_n := \pi_n \varphi$ and $f_n(u) := \pi_n f(u)$ for all $u \in H_n$ are the orthogonal projections of $\varphi \in V$ and $f(u) \in H$ on the finite-dimensional space H_n (see (2.2)). The subsequent theorems state uniform a priori estimates of the Galerkin approximations Z_n in the spaces H and V.

Theorem 2.3.3. *Let $n \in \mathbb{N}$ be arbitrarily fixed. Then the Schrödinger problem* (2.64) *possesses a unique solution $Z_n \in C([0,T];H)$ and*

$$\|Z_n(t)\|^2 \leq \|\varphi\|^2, \qquad \text{for all } t \in [0,T].$$

Proof. We introduce the mapping $F_n : [0,T] \times H_n \to H_n$ defined by $F_n(t,u) := B(t) f_n(u)$ for all $t \in [0,T]$ and all $u \in H_n$. Regard that for each fixed $t \in [0,T]$ and each $u_1, u_2 \in H_n$, we have by the second property in (2.3), Lemma D.4 (a), Lemma D.2 and the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_V$ on H_n (see (2.4))

$$\begin{aligned}
\|F_n(t,u_1) - F_n(t,u_2)\|^2 &= B^2(t)\|f_n(u_1) - f_n(u_2)\|^2 \leq B^2(T)\|f(u_1) - f(u_2)\|^2 \\
&\leq \frac{25}{4} B^2(T) \int_0^1 \left(|u_1(x)|^{2\sigma} + |u_2(x)|^{2\sigma}\right)^2 |u_1(x) - u_2(x)|^2 \, dx \\
&\leq \frac{25}{2} B^2(T) \sup_{x \in [0,1]} \left(|u_1(x)|^{4\sigma} + |u_2(x)|^{4\sigma}\right) \|u_1 - u_2\|^2 \\
&\leq 25 \cdot 2^{2\sigma-1} B^2(T) \left(\|u_1\|_V^{4\sigma} + \|u_2\|_V^{4\sigma}\right) \|u_1 - u_2\|^2 \\
&\leq 25 \cdot 2^{2\sigma-1} B^2(T) \left(1+\mu_n\right)^{2\sigma} \left(\|u_1\|^{4\sigma} + \|u_2\|^{4\sigma}\right) \|u_1 - u_2\|^2.
\end{aligned}$$

Hence, the mapping $F_n(t,\cdot) : H_n \to H_n$ is locally Lipschitz continuous for each fixed $t \in [0,T]$. Combining this result with definition (2.62) of $B(t)$, it follows that $F_n : [0,T] \times H_n \to H_n$ is continuous in time and space. Therefore, the existence of a local (strong) solution $Z_n \in C([0,\delta];H_n)$ with $\delta \in (0,T]$ of

$$Z_n(t) = \varphi_n - i \int_0^t A Z_n(s) \, ds + i\lambda \int_0^t F_n(s, Z_n(s)) \, ds, \qquad \text{for all } t \in [0,\delta], \tag{2.65}$$

in H_n is ensured by Peano's theorem for mappings which are locally Lipschitz continuous on H_n (see [28, p. 214–216, especially Theorems 2.3.1 and 2.3.4]). With the help of the second property in (2.3), the uniqueness of the solution of (2.65) can be established analogously to the proof of Theorem 2.3.2. Obviously, Z_n fulfills (2.64) and we can apply the energy equality to (2.64). Summing over all $k = 1,2,\ldots,n$, we receive

$$\|Z_n(t)\|^2 = \|\varphi_n\|^2 + 2\,\mathrm{Im} \int_0^t \big\langle A Z_n(s), Z_n(s) \big\rangle \, ds - 2\lambda\,\mathrm{Im} \int_0^t B(s)\big(f_n(Z_n(s)), Z_n(s)\big) \, ds$$

for all $t \in [0,\delta]$. Since $\big\langle A Z_n(s), Z_n(s) \big\rangle \geq 0$ (by the second property in (2.5)) and

$$B(s)\big(f_n(Z_n(s)), Z_n(s)\big) = B(s)\big(f(Z_n(s)), Z_n(s)\big) \geq 0,$$

(due to the first property in (2.3)), both imaginary parts vanish and it only remains

$$\|Z_n(t)\|^2 = \|\varphi_n\|^2, \qquad \text{for all } t \in [0,\delta].$$

Because of $\|\varphi_n\|^2 \le \|\varphi\|^2$ (see the second property in (2.3)), it further holds that

$$\|Z_n(t)\|^2 \le \|\varphi\|^2, \qquad \text{for all } t \in [0,\delta].$$

In order to get a global solution, we successively consider this problem over the interval $[k\delta,(k+1)\delta]$ for all $k = 1,2,\dots,m$ until we reach the final time $T \in (m\delta,(m+1)\delta]$ after finitely many steps $m \in \mathbb{N}$. For each fixed $k \in \{1,2,\dots,m\}$ the same procedure with the initial condition $Z_n(k\delta)$ yields a unique local (strong) solution $Z_n \in C([k\delta,(k+1)\delta];H_n)$ and the estimate $\|Z_n(t)\|^2 \le \|\varphi\|^2$ for all $t \in [k\delta,(k+1)\delta]$. Hence, the composition of these solutions enables us to extend the local solution Z_n on $[0,\delta]$ to a global solution on $[0,T]$. □

Corollary 2.3.4. *The uniform a priori estimate of the sequence of Galerkin approximations $(Z_n)_n$ in H can be raised to the higher power $2p$ for each $p \ge 1$. Thus, we additionally get the boundedness of $(Z_n)_n$ in $L^{2p}([0,T];H)$ for all $p \ge 1$.*

Theorem 2.3.5. *Let $n \in \mathbb{N}$ be arbitrarily fixed. Then there exists a positive constant C depending on σ such that the solution Z_n of (2.64) satisfies the estimate*

$$\|Z_n(t)\|_V^2 \le C(\sigma)\left(\|\varphi\|_V^2 + \lambda B(T)\,\|\varphi\|^{2(1+\sigma)} + (\lambda B(T))^{\frac{2}{2-\sigma}}\,\|\varphi\|^{\frac{2(2+\sigma)}{2-\sigma}}\right), \qquad \text{for all } t \in [0,T].$$

Proof. To prove the boundedness of Z_n in V as well, remember the equivalent representation of (2.64) as the integral equation

$$Z_n(t) = \varphi_n - i\int_0^t AZ_n(s)\,ds + i\lambda\int_0^t B(s)f_n(Z_n(s))\,ds$$

in V^* for all $t \in [0,T]$. Using the absence of noise, we write

$$\frac{\partial}{\partial t}Z_n(t) = -iAZ_n(t) + i\lambda B(t)f_n(Z_n(t)), \qquad Z_n(0) = \varphi_n.$$

Unless noted otherwise, each relation in this proof is valid for a.e. $t \in [0,T]$. Multiplication with $\frac{\partial}{\partial t}\overline{Z_n}(t)$ and the properties of the Gelfand triple (V,H,V^*) yield

$$\left\|\frac{\partial}{\partial t}Z_n(t)\right\|^2 = -i\left\langle AZ_n(t),\frac{\partial}{\partial t}Z_n(t)\right\rangle + i\lambda B(t)\Big(f_n(Z_n(t)),\frac{\partial}{\partial t}Z_n(t)\Big).$$

The definition (2.1) of A and the first property in (2.3) entail

$$\left\|\frac{\partial}{\partial t}Z_n(t)\right\|^2 = -i\left(\frac{\partial}{\partial x}Z_n(t),\frac{\partial}{\partial x}\frac{\partial}{\partial t}Z_n(t)\right) + i\lambda B(t)\Big(f(Z_n(t)),\frac{\partial}{\partial t}Z_n(t)\Big).$$

Regarding that $\lambda, B(t) \in \mathbb{R}_+$ and taking the imaginary part, the left-hand side vanishes and

$$\begin{aligned}0 &= -\operatorname{Im}\left\{i\left(\frac{\partial}{\partial x}Z_n(t),\frac{\partial}{\partial x}\frac{\partial}{\partial t}Z_n(t)\right)\right\} + \lambda B(t)\operatorname{Im}\left\{i\left(f(Z_n(t)),\frac{\partial}{\partial t}Z_n(t)\right)\right\}\\ &= -\operatorname{Re}\left(\frac{\partial}{\partial x}Z_n(t),\frac{\partial}{\partial x}\frac{\partial}{\partial t}Z_n(t)\right) + \lambda B(t)\operatorname{Re}\left(f(Z_n(t)),\frac{\partial}{\partial t}Z_n(t)\right).\end{aligned} \tag{2.66}$$

Due to $\operatorname{Re}\{a\bar{b}\} = \frac{1}{2}(\bar{a}b + a\bar{b})$ for $a,b \in \mathbb{C}$, it holds for $b := a_t = \frac{d}{dt}a$ that

$$\operatorname{Re}\{a\bar{a}_t\} = \frac{1}{2}(\bar{a}a_t + a\bar{a}_t) = \frac{1}{2}\frac{d}{dt}(a\bar{a}) = \frac{1}{2}\frac{d}{dt}|a|^2. \tag{2.67}$$

Using this result for $a := \frac{\partial}{\partial x}Z_n(t,x)$ and the fact that the representation of the Galerkin approximations is separated in time and space such that the order of differentiation and integration can be simply changed, the first term on the right-hand side suffices

$$\begin{aligned}\mathrm{Re}\left(\frac{\partial}{\partial x}Z_n(t),\frac{\partial}{\partial x}\frac{\partial}{\partial t}Z_n(t)\right)&=\mathrm{Re}\left(\frac{\partial}{\partial x}Z_n(t),\frac{\partial}{\partial t}\frac{\partial}{\partial x}Z_n(t)\right)\\&=\int_0^1\mathrm{Re}\left\{\frac{\partial}{\partial x}Z_n(t,x)\,\frac{\partial}{\partial t}\frac{\partial}{\partial x}\overline{Z_n}(t,x)\right\}dx\\&=\frac{1}{2}\int_0^1\frac{\partial}{\partial t}\left|\frac{\partial}{\partial x}Z_n(t,x)\right|^2dx=\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial}{\partial x}Z_n(t)\right\|^2.\end{aligned}$$

Moreover, applying (2.67) to $a:=Z_n(t,x)$, the second term results in

$$\begin{aligned}&\mathrm{Re}\left(f(Z_n(t)),\frac{\partial}{\partial t}Z_n(t)\right)=\int_0^1|Z_n(t,x)|^{2\sigma}\,\mathrm{Re}\left\{Z_n(t,x)\,\frac{\partial}{\partial t}\overline{Z_n}(t,x)\right\}dx\\&=\frac{1}{2}\int_0^1|Z_n(t,x)|^{2\sigma}\left(\overline{Z_n}(t,x)\,\frac{\partial}{\partial t}Z_n(t,x)+Z_n(t,x)\,\frac{\partial}{\partial t}\overline{Z_n}(t,x)\right)dx\\&=\frac{1}{2(1+\sigma)}\int_0^1\left(|Z_n(t,x)|^2\,\frac{\partial}{\partial t}\left[Z_n(t,x)^\sigma\,\overline{Z_n}(t,x)^\sigma\right]+|Z_n(t,x)|^{2\sigma}\,\frac{\partial}{\partial t}\left[Z_n(t,x)\,\overline{Z_n}(t,x)\right]\right)dx\\&=\frac{1}{2(1+\sigma)}\int_0^1\left(|Z_n(t,x)|^2\,\frac{\partial}{\partial t}|Z_n(t,x)|^{2\sigma}+|Z_n(t,x)|^{2\sigma}\,\frac{\partial}{\partial t}|Z_n(t,x)|^2\right)dx\\&=\frac{1}{2(1+\sigma)}\int_0^1\frac{\partial}{\partial t}\left[|Z_n(t,x)|^{2\sigma}|Z_n(t,x)|^2\right]dx=\frac{1}{2(1+\sigma)}\frac{d}{dt}\left(\int_0^1|Z_n(t,x)|^{2+2\sigma}dx\right).\end{aligned}$$

Thus, we conclude from equation (2.66) that

$$\frac{d}{dt}\left\|\frac{\partial}{\partial x}Z_n(t)\right\|^2=\frac{\lambda}{1+\sigma}B(t)\,\frac{d}{dt}\left(\int_0^1|Z_n(t,x)|^{2+2\sigma}\,dx\right).$$

Integration with respect to the time variable yields

$$\left\|\frac{\partial}{\partial x}Z_n(t)\right\|^2-\left\|\frac{\partial}{\partial x}Z_n(0)\right\|^2=\frac{\lambda}{1+\sigma}\int_0^t B(s)\,\frac{d}{ds}\left(\int_0^1|Z_n(s,x)|^{2+2\sigma}\,dx\right)ds,$$

where from now on each relation is valid for all $t\in[0,T]$ in this proof. Due to integration by parts on the right-hand side, we receive

$$\begin{aligned}\left\|\frac{\partial}{\partial x}Z_n(t)\right\|^2-\left\|\frac{\partial}{\partial x}Z_n(0)\right\|^2&=\frac{\lambda}{1+\sigma}B(t)\left(\int_0^1|Z_n(t,x)|^{2+2\sigma}\,dx\right)\\&\quad-\frac{\lambda}{1+\sigma}\left(\int_0^1|Z_n(0,x)|^{2+2\sigma}\,dx\right)\\&\quad-\frac{\lambda}{1+\sigma}\int_0^t\left[\frac{d}{ds}B(s)\right]\left(\int_0^1|Z_n(s,x)|^{2+2\sigma}\,dx\right)ds\\&\le\frac{\lambda}{1+\sigma}B(t)\left(\int_0^1|Z_n(t,x)|^{2+2\sigma}\,dx\right)\end{aligned}\tag{2.68}$$

since (by assumption (2.57) and notation (2.62)) we have

$$0\le\frac{d}{ds}B(s)=\exp\left\{\sigma\sum_{j=1}^\infty\int_0^s b_j^2(r)\,dr\right\}\cdot\sigma\sum_{j=1}^\infty b_j^2(s)<\infty,\qquad\text{for a.e. } s\in[0,T].$$

Because of the initial condition $Z_n(0)=\varphi_n$, it follows that

$$\left\|\frac{\partial}{\partial x}Z_n(t)\right\|^2\le\left\|\frac{d}{dx}\varphi_n\right\|^2+\frac{\lambda}{1+\sigma}B(T)\left(\int_0^1|Z_n(t,x)|^{2+2\sigma}\,dx\right).$$

Based on Lemma D.2 and Lemma D.1, we see

$$\begin{aligned}\int_0^1 |Z_n(t,x)|^{2+2\sigma}\,dx &\le \left(\int_0^1 |Z_n(t,x)|^2\,dx\right)\sup_{x\in[0,1]}|Z_n(t,x)|^{2\sigma}\\ &\le \|Z_n(t)\|^2\,\|Z_n(t)\|^\sigma\left(\|Z_n(t)\| + 2\left\|\frac{\partial}{\partial x}Z_n(t)\right\|\right)^\sigma \\ &\le 2^\sigma\|Z_n(t)\|^{2(1+\sigma)} + 2^{2\sigma}\|Z_n(t)\|^{2+\sigma}\left\|\frac{\partial}{\partial x}Z_n(t)\right\|^\sigma.\end{aligned} \tag{2.69}$$

This implies that

$$\begin{aligned}\left\|\frac{\partial}{\partial x}Z_n(t)\right\|^2 &\le \left\|\frac{d}{dx}\varphi_n\right\|^2 + \frac{2^\sigma}{1+\sigma}\lambda B(T)\|Z_n(t)\|^{2(1+\sigma)} + \frac{2^{2\sigma}}{1+\sigma}\lambda B(T)\|Z_n(t)\|^{2+\sigma}\left\|\frac{\partial}{\partial x}Z_n(t)\right\|^\sigma\\ &\le \left\|\frac{d}{dx}\varphi_n\right\|^2 + \frac{2^\sigma}{1+\sigma}\lambda B(T)\|Z_n(t)\|^{2(1+\sigma)}\\ &\quad + \frac{2-\sigma}{2}\left(\frac{2^{2\sigma}}{1+\sigma}\right)^{\frac{2}{2-\sigma}}(\lambda B(T))^{\frac{2}{2-\sigma}}\|Z_n(t)\|^{\frac{2(2+\sigma)}{2-\sigma}} + \frac{\sigma}{2}\left\|\frac{\partial}{\partial x}Z_n(t)\right\|^2\end{aligned}$$

due to Young's inequality. Since $\|Z_n(t)\|^2 \le \|\varphi\|^2$ for all $t\in[0,T]$ (compare Theorem 2.3.3) and the exponents $2(1+\sigma)$ and $\frac{2(2+\sigma)}{2-\sigma}$ with $\sigma\in(0,2)$ are positive, it also holds that

$$\|Z_n(t)\|^{2(1+\sigma)} \le \|\varphi\|^{2(1+\sigma)}, \qquad \|Z_n(t)\|^{\frac{2(2+\sigma)}{2-\sigma}} \le \|\varphi\|^{\frac{2(2+\sigma)}{2-\sigma}}, \qquad \text{for all } t\in[0,T],$$

(see Corollary 2.3.4) and, therefore,

$$\begin{aligned}\left(1-\frac{\sigma}{2}\right)\left\|\frac{\partial}{\partial x}Z_n(t)\right\|^2 \le{}& \left\|\frac{d}{dx}\varphi_n\right\|^2 + \frac{2^\sigma}{1+\sigma}\lambda B(T)\,\|\varphi\|^{2(1+\sigma)}\\ &+ \frac{2-\sigma}{2}\left(\frac{2^{2\sigma}}{1+\sigma}\right)^{\frac{2}{2-\sigma}}(\lambda B(T))^{\frac{2}{2-\sigma}}\,\|\varphi\|^{\frac{2(2+\sigma)}{2-\sigma}}.\end{aligned}$$

By using the relation $\left\|\frac{d}{dx}\varphi_n\right\|^2 \le \left\|\frac{d}{dx}\varphi\right\|^2$ (compare the second property in (2.3)), we infer for all $t\in[0,T]$ that

$$\begin{aligned}\left\|\frac{\partial}{\partial x}Z_n(t)\right\|^2 &\le \frac{2}{2-\sigma}\left(\left\|\frac{d}{dx}\varphi\right\|^2 + \frac{2^\sigma}{1+\sigma}\lambda B(T)\,\|\varphi\|^{2(1+\sigma)}\right) + \left(\frac{2^{2\sigma}}{1+\sigma}\right)^{\frac{2}{2-\sigma}}(\lambda B(T))^{\frac{2}{2-\sigma}}\,\|\varphi\|^{\frac{2(2+\sigma)}{2-\sigma}}\\ &\le C(\sigma)\left(\left\|\frac{d}{dx}\varphi\right\|^2 + \lambda B(T)\,\|\varphi\|^{2(1+\sigma)} + (\lambda B(T))^{\frac{2}{2-\sigma}}\,\|\varphi\|^{\frac{2(2+\sigma)}{2-\sigma}}\right).\end{aligned}$$

Since $\|Z_n(t)\|_V^2 = \|Z_n(t)\|^2 + \left\|\frac{\partial}{\partial x}Z_n(t)\right\|^2$, it follows together with Theorem 2.3.3 that

$$\|Z_n(t)\|_V^2 \le C(\sigma)\left(\|\varphi\|_V^2 + \lambda B(T)\,\|\varphi\|^{2(1+\sigma)} + (\lambda B(T))^{\frac{2}{2-\sigma}}\,\|\varphi\|^{\frac{2(2+\sigma)}{2-\sigma}}\right), \qquad \text{for all } t\in[0,T]. \ \square$$

Corollary 2.3.6. *The uniform a priori estimate in V can also be raised to the power $2p$ for each $p\ge 1$. Hence, we get the boundedness of $(Z_n)_n$ in $L^{2p}([0,T];V)$ as well.*

Next, we investigate some special properties of $(Z_n)_{n\in\mathbb{N}}$ to conclude the existence of the variational solution of the Schrödinger problem (2.63).

Corollary 2.3.7. *The sequence of Galerkin approximations $(Z_n)_n$ of the pathwise Schrödinger problem (2.64) is bounded in $C([0,T];H)$, $L^2([0,T];H)$ and $L^2([0,T];V)$. Moreover, $(Z_n)_n$ is relatively compact in $L^2([0,T];H)$.*

Proof. Theorems 2.3.3 and 2.3.5 imply the boundedness of $(Z_n)_n$ in $C([0,T];H)$, $L^2([0,T];H)$ and $L^2([0,T];V)$. Referring to the Galerkin equations (2.64), applying the Cauchy-Schwarz inequality for square integrable functions and the triangle inequality, we get for all $s,t \in [0,T]$ with $s < t$

$$\begin{aligned}\|Z_n(t) - Z_n(s)\|_{V^*}^2 &\leq \left\| -i \int_s^t \big(AZ_n(r) - \lambda B(r) f_n(Z_n(r))\big)\, dr \right\|_{V^*}^2 \\ &\leq (t-s) \int_s^t \|AZ_n(r) - \lambda B(r) f_n(Z_n(r))\|_{V^*}^2\, dr \\ &\leq 2(t-s) \int_s^t \left(\|AZ_n(r)\|_{V^*}^2 + \lambda^2 B^2(r) \left\|f_n(Z_n(r))\right\|_{V^*}^2 \right) dr.\end{aligned}$$

The continuity of the operator A (compare Section 2.1) and of the embedding $H \hookrightarrow V^*$ with embedding constant $\tilde{C} := C_{H,V^*}$ and the second property in (2.3) lead to

$$\begin{aligned}\|Z_n(t) - Z_n(s)\|_{V^*}^2 &\leq 2(t-s) \int_s^t \left(\|Z_n(r)\|_V^2 + \lambda^2 B^2(r) \tilde{C}^2 \left\|f_n(Z_n(r))\right\|^2 \right) dr \\ &\leq (t-s)\, C \int_s^t \left(\|Z_n(r)\|_V^2 + \lambda^2 B^2(r) \left\|f(Z_n(r))\right\|^2 \right) dr.\end{aligned}$$

Lemma D.2 entails for all $r \in [0,T]$ the estimate

$$\|f(Z_n(r))\|^2 = \left\| |Z_n(r)|^{2\sigma} Z_n(r) \right\|^2 \leq \sup_{x \in [0,1]} |Z_n(r,x)|^{2(2\sigma+1)} \leq 2^{2\sigma+1} \|Z_n(r)\|_V^{2(2\sigma+1)}$$

such that

$$\|Z_n(t) - Z_n(s)\|_{V^*}^2 \leq (t-s)\, C \int_s^t \left(\|Z_n(r)\|_V^2 + 2^{2\sigma+1} \lambda^2 B^2(T) \|Z_n(r)\|_V^{2(2\sigma+1)} \right) dr.$$

Hence, the boundedness results in Theorem 2.3.5 and Corollary 2.3.6 yield the equicontinuity of $(Z_n)_n$ in $C([0,T];V^*)$. Since the embedding $V \hookrightarrow H$ is compact, we finally obtain by a theorem due to Dubinskij that $(Z_n)_n$ is relatively compact in $L^2([0,T];H)$ (see [99, p. 132, Theorem 4.1]). □

Now, we proceed with the existence of the variational solution Z of problem (2.63).

Theorem 2.3.8. *Let $(Z_n)_n$ be the sequence of variational solutions of the finite-dimensional Galerkin equations* (2.64)*. Then it holds that*

- **(a)** *$(Z_n)_n$ converges strongly in $L^2([0,T];H)$ and weakly in $L^2([0,T];V)$ to the variational solution Z of the pathwise nonlinear Schrödinger problem* (2.63)*,*
- **(b)** *$Z \in L^\infty([0,T];V)$ and especially*

$$\operatorname*{ess\,sup}_{t \in [0,T]} \|Z(t)\|_V^2 \leq C(\sigma) \left(\|\varphi\|_V^2 + \lambda B(T) \|\varphi\|^{2(1+\sigma)} + (\lambda B(T))^{\frac{2}{2-\sigma}} \|\varphi\|^{\frac{2(2+\sigma)}{2-\sigma}} \right),$$

- **(c)** *$(Z_n)_n$ also converges to Z in $C([0,T];H)$.*

Proof. **(a)** We derive some convergence results in $L^2([0,T];H)$ and $L^2([0,T];V)$ and then we take $n \to \infty$ in the Galerkin equations

$$\big(Z_n(t), h_k\big) = \big(\varphi_n, h_k\big) - i \int_0^t \langle AZ_n(s), h_k \rangle\, ds + i\lambda \int_0^t B(s) \big(f_n(Z_n(s)), h_k\big)\, ds$$

for all $t \in [0,T]$ and all $k \in \{1,2,\ldots,n\}$. First, observe that

$$\varphi_n \to \varphi \qquad \text{in } H \text{ as } n \to \infty.$$

By Corollary 2.3.7 there exist a subsequence $(Z_{n'})_{n'}$ of $(Z_n)_n$ and a function $\tilde{Z} \in L^2([0,T];V)$ such that

$$Z_{n'} \rightharpoonup \tilde{Z} \qquad \text{in } L^2([0,T];H) \text{ as } n' \to \infty, \tag{2.70}$$

$$Z_{n'} \rightharpoonup \tilde{Z} \qquad \text{in } L^2([0,T];V) \text{ as } n' \to \infty \tag{2.71}$$

and

$$\int_0^T \|Z_{n'}(t) - \tilde{Z}(t)\|^2 \, dt \to 0 \qquad \text{as } n' \to \infty. \tag{2.72}$$

The strong convergence implies the almost sure convergence of a further subsequence $(Z_{n''})_{n''}$ of $(Z_{n'})_{n'}$ with

$$Z_{n''}(t,x) \to \tilde{Z}(t,x), \qquad \text{for a.e. } (t,x) \in [0,T] \times [0,1] \text{ as } n'' \to \infty.$$

Therefore, we obtain

$$B(t)f(Z_{n''}(t,x)) \to B(t)f(\tilde{Z}(t,x)), \qquad \text{for a.e. } (t,x) \in [0,T] \times [0,1] \text{ as } n'' \to \infty.$$

Lemma D.2 and the estimate given in Theorem 2.3.5 yield the uniform integrability condition

$$\begin{aligned}
\int_0^T \|B(t)f(Z_{n''}(t))\|^2 \, dt &= \int_0^T \int_0^1 \left|B(t)|Z_{n''}(t,x)|^{2\sigma} Z_{n''}(t,x)\right|^2 dx \, dt \\
&\leq B^2(T) \int_0^T \left(\sup_{x \in [0,1]} |Z_{n''}(t,x)|^{2(2\sigma+1)} \right) dt \\
&\leq 2^{2\sigma+1} B^2(T) \int_0^T \|Z_{n''}(t)\|_V^{2(2\sigma+1)} dt \leq C(\sigma, \lambda, \varphi, B(T))
\end{aligned}$$

for all $n'' \in \mathbb{N}$. Then by [63, p. 72, Lemma 2.3] we receive

$$Bf(Z_{n''}) \rightharpoonup Bf(\tilde{Z}) \qquad \text{in } L^2([0,T];H) \text{ as } n'' \to \infty. \tag{2.73}$$

Abbreviating $Y_{n''}(s) := B(s)f(Z_{n''}(s))$ and $\tilde{Y}(s) := B(s)f(\tilde{Z}(s))$ for all $s \in [0,t]$ and choosing $t \in [0,T]$, we deduce with the help of the Cauchy-Schwarz inequality and $Y_{n''} \in L^2([0,T];H)$ that

$$\int_0^T \left\| \int_0^t Y_{n''}(s) \, ds \right\|^2 dt \leq \int_0^T t \int_0^t \|Y_{n''}(s)\|^2 \, ds \, dt \leq T^2 \int_0^T \|Y_{n''}(t)\|^2 \, dt < \infty.$$

This result also holds for $\tilde{Y}$ such that we infer

$$\int_0^t Y_{n''}(s) \, ds \in L^2([0,T];H) \qquad \text{and} \qquad \int_0^t \tilde{Y}(s) \, ds \in L^2([0,T];H).$$

Because of (2.73), we know that

$$\int_0^T \big(Y_{n''}(s) - \tilde{Y}(s), \xi(s)\big) \, ds \to 0, \qquad \text{for all } \xi \in L^2([0,T];H) \text{ as } n'' \to \infty.$$

Choosing $\xi(s) = \mathbb{1}_{[0,t]}(s)v$ with $v \in V$ and $t \in [0,T]$, we have

$$\int_0^T \big(Y_{n''}(s) - \tilde{Y}(s), \mathbb{1}_{[0,t]}(s)v\big) \, ds = \int_0^t \big(Y_{n''}(s) - \tilde{Y}(s), v\big) \, ds \to 0 \qquad \text{as } n'' \to \infty,$$

which means that

$$\left(\int_0^t Y_{n''}(s) \, ds, v \right) = \int_0^t \big(Y_{n''}(s), v\big) \, ds \to \int_0^t \big(\tilde{Y}(s), v\big) \, ds = \left(\int_0^t \tilde{Y}(s) \, ds, v \right)$$

for all $t \in [0,T]$ and all $v \in V$ as $n'' \to \infty$. Furthermore, since each weakly convergent sequence is bounded, there exists a majorant

$$\left|\left(\int_0^t Y_{n''}(s)\,ds, v\right)\right| \le \left\|\int_0^t Y_{n''}(s)\,ds\right\| \|v\| \le T^{\frac{1}{2}} \left(\int_0^T \|Y_{n''}(s)\|^2\,ds\right)^{\frac{1}{2}} \|v\| \le C,$$

which is independent of $t \in [0,T]$ and, therefore, integrable. Thus, Lebesgue's dominated convergence theorem implies for all $v \in V$ that

$$\lim_{n''\to\infty} \int_0^T \left(\int_0^t Y_{n''}(s)\,ds, v\right) dt = \int_0^T \lim_{n''\to\infty} \left(\int_0^t Y_{n''}(s)\,ds, v\right) dt = \int_0^T \left(\int_0^t \tilde{Y}(s)\,ds, v\right) dt.$$

This especially entails the convergence

$$\int_0^T \int_0^t B(s)\big(f(Z_{n''}(s)), v\big)\,ds\,dt \to \int_0^T \int_0^t B(s)\big(f(\tilde{Z}(s)), v\big)\,ds\,dt$$

for all $v \in V$ as $n'' \to \infty$. Let $k \in \mathbb{N}$ be fixed and consider $v = h_k$ in the above convergence (for n'' sufficiently large such that $n'' \ge k$). Then using the first property of (2.3) with $n := n''$, it results for $n'' \to \infty$ that

$$\begin{aligned}\int_0^T \int_0^t B(s)\big(f_{n''}(Z_{n''}(s)), h_k\big)\,ds\,dt &= \int_0^T \int_0^t B(s)\big(f(Z_{n''}(s)), h_k\big)\,ds\,dt \\ &\to \int_0^T \int_0^t B(s)\big(f(\tilde{Z}(s)), h_k\big)\,ds\,dt.\end{aligned}$$

From (2.70) we get

$$\int_0^T \big(Z_{n''}(t), h_k\big)\,dt \to \int_0^T \big(\tilde{Z}(t), h_k\big)\,dt \qquad \text{as } n'' \to \infty,$$

and (2.1) and (2.71) lead to

$$\begin{aligned}\int_0^T \int_0^t \big\langle AZ_{n''}(s), h_k\big\rangle\,ds\,dt &= \int_0^T \int_0^t \left(\frac{\partial}{\partial x} Z_{n''}(s), \frac{d}{dx} h_k\right) ds\,dt \\ &\to \int_0^T \int_0^t \left(\frac{\partial}{\partial x} \tilde{Z}(s), \frac{d}{dx} h_k\right) ds\,dt = \int_0^T \int_0^t \big\langle A\tilde{Z}(s), h_k\big\rangle\,ds\,dt\end{aligned}$$

as $n'' \to \infty$. Then we have by (2.64) that

$$\begin{aligned}\int_0^T \big(Z_{n''}(t), h_k\big)\,dt &= \int_0^T \big(\varphi_{n''}, h_k\big)\,dt - i \int_0^T \int_0^t \big\langle AZ_{n''}(s), h_k\big\rangle\,ds\,dt \\ &\quad + i\lambda \int_0^T \int_0^t B(s)\big(f_{n''}(Z_{n''}(s)), h_k\big)\,ds\,dt\end{aligned}$$

for all $k \in \{1, 2, \ldots, n''\}$. Taking $n'' \to \infty$ and applying the above convergence results, we obtain

$$\begin{aligned}\int_0^T \big(\tilde{Z}(t), h_k\big)\,dt &= \int_0^T \big(\varphi, h_k\big)\,dt - i \int_0^T \int_0^t \big\langle A\tilde{Z}(s), h_k\big\rangle\,ds\,dt \\ &\quad + i\lambda \int_0^T \int_0^t B(s)\big(f(\tilde{Z}(s)), h_k\big)\,ds\,dt\end{aligned}$$

for all $k \in \mathbb{N}$. We deduce for a.e. $t \in [0,T]$ and all $k \in \mathbb{N}$ that

$$\big(\tilde{Z}(t), h_k\big) = \big(\varphi, h_k\big) - i \int_0^t \big\langle A\tilde{Z}(s), h_k\big\rangle\,ds + i\lambda \int_0^t B(s)\big(f(\tilde{Z}(s)), h_k\big)\,ds.$$

Since $\operatorname{span}\{h_1, h_2, \ldots, h_n, \ldots\}$ is dense in V, the last equation holds for all $v \in V$ as well. Moreover, there exists a process $Z \in C([0,T];H) \cap L^2([0,T];V)$ such that $Z(t,x) = \tilde{Z}(t,x)$ for a.e. $(t,x) \in [0,T] \times [0,1]$ and

$$\big(Z(t), v\big) = \big(\varphi, v\big) - i \int_0^t \big\langle AZ(s), v\big\rangle \, ds + i\lambda \int_0^t B(s)\big(f(Z(s)), v\big) \, ds$$

for all $t \in [0,T]$ and all $v \in V$ (compare [87, p. 73, Theorem 2]). Thus, Z is the variational solution of the pathwise Schrödinger problem (2.63) that is especially unique by Theorem 2.3.2. Observe that every subsequence of $(Z_n)_n$ has a subsequence which converges strongly to the same limit Z in $L^2([0,T];H)$. Hence, the whole sequence $(Z_n)_n$ converges strongly to Z (compare [103, p. 480, Proposition 10.13 (1)]). Furthermore, every subsequence of $(Z_n)_n$ has a subsequence which converges weakly to the same limit Z in $L^2([0,T];V)$. Hence, the whole sequence $(Z_n)_n$ converges weakly to Z (see [103, p. 480, Proposition 10.13 (2)]).

(b) Now, we establish that $Z \in L^\infty([0,T];V)$. On the one hand, we apply Lemma F.2 for $(Z_n)_n$, which is bounded in $L^\infty([0,T];V)$ due to the estimate given in Theorem 2.3.5. And on the other hand, we take into consideration that $(Z_n)_n$ converges weakly to Z (the unique variational solution of the pathwise Schrödinger equation) in $L^2([0,T];V)$ (compare part (a) of this theorem). Thus, there exist a subsequence $(Z_{\bar{n}})_{\bar{n}}$ of $(Z_n)_n$ and a function $\hat{Z} \in L^\infty([0,T];V)$ such that for all $v \in L^2([0,T];V) \hookrightarrow L^1([0,T];V)$

$$\int_0^T \big(Z_{\bar{n}}(t), v\big)_V \, dt \to \int_0^T \big(\hat{Z}(t), v\big)_V \, dt \qquad \text{as } \bar{n} \to \infty$$

and

$$\int_0^T \big(Z_{\bar{n}}(t), v\big)_V \, dt \to \int_0^T \big(Z(t), v\big)_V \, dt \qquad \text{as } \bar{n} \to \infty.$$

Accordingly,

$$\int_0^T \|\hat{Z}(t) - Z(t)\|_V^2 \, dt = 0,$$

which means that Z can be a.s. identified with a function belonging to $L^\infty([0,T];V)$. By Lemma F.2 and Theorem 2.3.5, we also have

$$\begin{aligned}\operatorname*{ess\,sup}_{t\in[0,T]} \|Z(t)\|_V^2 &\leq \liminf_{\bar{n}\to\infty} \operatorname*{ess\,sup}_{t\in[0,T]} \|Z_{\bar{n}}(t)\|_V^2 \\ &\leq C(\sigma)\left(\|\varphi\|_V^2 + \lambda B(T)\,\|\varphi\|^{2(1+\sigma)} + (\lambda B(T))^{\frac{2}{2-\sigma}}\,\|\varphi\|^{\frac{2(2+\sigma)}{2-\sigma}}\right).\end{aligned}$$

(c) Next, we show that the sequence of Galerkin approximations $(Z_n)_n$ converges to the variational solution Z of the pathwise problem (2.63) in $C([0,T];H)$. In part (a), we proved the strong convergence

$$Z_n \to Z \qquad \text{in } L^2([0,T];H) \text{ as } n \to \infty. \tag{2.74}$$

Notice that (2.64) holds for all $k \in \{1, 2, \ldots, n\}$ but it is easy to deduce that it also holds for all $v \in V$ since $\big(u, \pi_n v\big) = \big(u, v\big)$ for all $u \in H_n$ (by the first property in (2.3)). Then we use (2.63) and (2.64), apply the energy equality and get for all $t \in [0,T]$

$$\begin{aligned}\|Z(t) - Z_n(t)\|^2 &= \|\varphi - \varphi_n\|^2 + 2\operatorname{Im}\int_0^t \big\langle A(Z(s) - Z_n(s)), Z(s) - Z_n(s)\big\rangle \, ds \\ &\quad - 2\lambda \operatorname{Im}\int_0^t B(s)\big(f(Z(s)) - f_n(Z_n(s)), Z(s) - Z_n(s)\big) \, ds.\end{aligned}$$

The second term on the right-hand side vanishes because of

$$Z(s) - Z_n(s) = \sum_{k=1}^{n} \big(Z(s) - Z_n(s), h_k\big)\, h_k + \sum_{k=n+1}^{\infty} \big(Z(s), h_k\big)\, h_k,$$

the eigenvalue equation $Ah_k = \mu_k h_k$ for all $k \in \mathbb{N}$ and the orthogonality of the eigenfunctions $(h_k)_{k\in\mathbb{N}}$. With the notation $\Pi_n(s) := f(Z(s)) - f_n(Z_n(s))$, we obtain

$$\|Z(t) - Z_n(t)\|^2 = \|\varphi - \varphi_n\|^2 - 2\lambda \operatorname{Im} \int_0^t B(s)\big(\Pi_n(s), Z(s) - Z_n(s)\big)\, ds,$$

and the Cauchy-Schwarz inequality yields

$$\begin{aligned} &\sup_{t\in[0,T]} \|Z(t) - Z_n(t)\|^2 \\ &\le \|\varphi - \varphi_n\|^2 + 2\lambda B(T) \left(\int_0^T \|\Pi_n(s)\|^2\, ds \right)^{\frac{1}{2}} \left(\int_0^T \|Z(s) - Z_n(s)\|^2\, ds \right)^{\frac{1}{2}}. \end{aligned} \tag{2.75}$$

By the second property in (2.3) and Lemma D.2, it follows that

$$\begin{aligned} \int_0^T \|\Pi_n(s)\|^2\, ds &= \int_0^T \|f(Z(s)) - f_n(Z_n(s))\|^2\, ds \\ &\le 2 \int_0^T \left(\|f(Z(s))\|^2 + \|f_n(Z_n(s))\|^2 \right) ds \\ &\le 2 \int_0^T \left(\|f(Z(s))\|^2 + \|f(Z_n(s))\|^2 \right) ds \\ &= 2 \int_0^T \left(\int_0^1 |Z(s,x)|^{2(2\sigma+1)}\, dx + \int_0^1 |Z_n(s,x)|^{2(2\sigma+1)}\, dx \right) ds \\ &\le 2 \int_0^T \left(\sup_{x\in[0,1]} |Z(s,x)|^{2(2\sigma+1)} + \sup_{x\in[0,1]} |Z_n(s,x)|^{2(2\sigma+1)} \right) ds \\ &\le 2^{2(\sigma+1)} \int_0^T \left(\|Z(s)\|_V^{2(2\sigma+1)} + \|Z_n(s)\|_V^{2(2\sigma+1)} \right) ds \\ &\le 2^{2(\sigma+1)}\, T \operatorname*{ess\,sup}_{t\in[0,T]} \left(\|Z(t)\|_V^{2(2\sigma+1)} + \|Z_n(t)\|_V^{2(2\sigma+1)} \right), \end{aligned}$$

which implies the boundedness of the sequence

$$\left(\int_0^T \|\Pi_n(s)\|^2\, ds \right)_n = \left(\int_0^T \|f(Z(s)) - f_n(Z_n(s))\|^2\, ds \right)_n$$

by a constant independent of $n \in \mathbb{N}$ (due to Theorem 2.3.5 and statement (b) of this theorem). Since $\|\varphi - \varphi_n\| \to 0$ as $n \to \infty$ (compare the third property in (2.3)), it finally results from (2.74) and (2.75) that

$$\sup_{t\in[0,T]} \|Z(t) - Z_n(t)\|^2 \to 0 \qquad \text{as } n \to \infty.$$ □

2.3.2 Results of the Stochastic Schrödinger Problem

Finally, we indicate an approximation result which implies the existence and uniqueness of the variational solution of the stochastic nonlinear Schrödinger problem (2.59) with linear multiplicative Gaussian noise.

Theorem 2.3.9. *Let $(\mathcal{T}_M)_{M\in\mathbb{N}}$ be the sequence of stopping times in Remark 2.3.1. Then it holds that*

$$X(t,\cdot) = \lim_{M\to\infty} \frac{Z(t \wedge \mathcal{T}_M, \cdot)}{Y(t \wedge \mathcal{T}_M)}, \qquad \text{for a.e. } \omega \in \Omega \text{ and all } t \in [0,T].$$

Thus, if (2.57) is satisfied, $X \in L^2(\Omega; C([0,T];H)) \cap L^2(\Omega \times [0,T]; V)$ is the unique variational solution of the stochastic nonlinear Schrödinger problem (2.59) and $X \in L^2(\Omega; L^\infty([0,T];V))$.

Proof. Using stochastic analysis (especially the Itô calculus) for (2.59), we compute

$$\begin{aligned}\big(X(t)Y(t),v\big) &= \big(\varphi,v\big) - \sum_{j=1}^{\infty}\int_0^t b_j^2(s)Y(s)\big(X(s),v\big)\,ds - i\sum_{j=1}^{\infty}\int_0^t b_j(s)Y(s)\big(X(s),v\big)\,d\beta_j(s)\\ &\quad - i\int_0^t Y(s)\big\langle AX(s),v\big\rangle\,ds + i\lambda\int_0^t Y(s)\big(f(X(s)),v\big)\,ds\\ &\quad + i\sum_{j=1}^{\infty}\int_0^t b_j(s)Y(s)\big(X(s),v\big)\,d\beta_j(s) + \sum_{j=1}^{\infty}\int_0^t b_j^2(s)Y(s)\big(X(s),v\big)\,ds\\ &= \big(\varphi,v\big) - i\int_0^t \big\langle AX(s)\,Y(s),v\big\rangle\,ds + i\lambda\int_0^t \big(f(X(s))\,Y(s),v\big)\,ds\end{aligned}$$

for a.e. $\omega \in \Omega$, all $t \in [0,\mathcal{T}_M]$ with fixed $M \in \mathbb{N}$ and all $v \in V$. The appearance of a quadratic covariation term is justified by [77, pp. 48 f.]. The notation $Z(t,\,\cdot\,) := X(t,\,\cdot\,)Y(t)$ leads to the pathwise nonlinear Schrödinger problem

$$\big(Z(t),v\big) = \big(\varphi,v\big) - i\int_0^t \big\langle AZ(s),v\big\rangle\,ds + i\lambda\int_0^t B(s)\big(f(Z(s)),v\big)\,ds$$

for a.e. $\omega \in \Omega$, all $t \in [0,\mathcal{T}_M]$ with fixed $M \in \mathbb{N}$ and all $v \in V$, which we have investigated in Subsection 2.3.1. Thus, the statements of this theorem result from similar statements of the pathwise Schrödinger problem (2.63). The uniqueness of the solution bases on Theorem 2.3.2 and the existence relies on Theorem 2.3.8 and the transformation formula

$$X(t,\,\cdot\,) = \frac{Z(t,\,\cdot\,)}{Y(t)} = Z(t,\,\cdot\,)\exp\left\{\frac{1}{2}\sum_{j=1}^{\infty}\int_0^t b_j^2(s)\,ds + i\sum_{j=1}^{\infty}\int_0^t b_j(s)\,d\beta_j(s)\right\}$$

for a.e. $\omega \in \Omega$ and all $t \in [0,\mathcal{T}_M]$ with fixed $M \in \mathbb{N}$. In addition, for a.e. $\omega \in \Omega$ there exists an $M_\omega \in \mathbb{N}$ such that for all $M \geq M_\omega$ it holds that $\mathcal{T}_M = T$ (see Remark 2.3.1) and we can define the solution of (2.59) by

$$X(t,\,\cdot\,) = \lim_{M\to\infty}\frac{Z(t\wedge\mathcal{T}_M,\,\cdot\,)}{Y(t\wedge\mathcal{T}_M)}, \qquad \text{for a.e. } \omega \in \Omega \text{ and all } t \in [0,T].$$

Hence, it holds that

$$E\sup_{t\in[0,T]}\|X(t)\|^2 = E\lim_{M\to\infty}\sup_{t\in[0,\mathcal{T}_M]}\|X(t)\|^2 \leq E\left(B^{\frac{1}{\sigma}}(T)\sup_{t\in[0,T]}\|Z(t)\|^2\right) = \|\varphi\|^2 E\left(B^{\frac{1}{\sigma}}(T)\right)$$

since $\|Z(t)\|^2 = \|\varphi\|^2$ for all $t \in [0,T]$ (which follows from the energy equality applied to the variational solution Z of the pathwise problem (2.63)). In order to prove that $X \in L^2(\Omega\times[0,T];V)$, we use that $(Z_n)_n$ converges weakly to Z in $L^2([0,T];V)$ (compare Theorem 2.3.8 (a)) and satisfies the estimate of Theorem 2.3.5 such that

$$\begin{aligned}&E\int_0^T \|X(t)\|_V^2\,dt = E\lim_{M\to\infty}\int_0^{\mathcal{T}_M}\|X(t)\|_V^2\,dt\\ &\leq E\left(B^{\frac{1}{\sigma}}(T)\int_0^T\|Z(t)\|_V^2\,dt\right) \leq E\left(B^{\frac{1}{\sigma}}(T)\liminf_{n\to\infty}\int_0^T\|Z_n(t)\|_V^2\,dt\right)\\ &\leq C(\sigma)T\left[\|\varphi\|_V^2\,E\left(B^{\frac{1}{\sigma}}(T)\right) + \lambda\,\|\varphi\|^{2(1+\sigma)}\,E\left(B^{\frac{1}{\sigma}(1+\sigma)}(T)\right) + \lambda^{\frac{2}{2-\sigma}}\,\|\varphi\|^{\frac{2(2+\sigma)}{2-\sigma}}\,E\left(B^{\frac{2+\sigma}{\sigma(2-\sigma)}}(T)\right)\right].\end{aligned}$$

From these results we get $X \in L^2(\Omega;C([0,T];H)) \cap L^2(\Omega\times[0,T];V)$. Moreover, we obtain by

Theorem 2.3.8 (b) that

$$E \operatorname*{ess\,sup}_{t\in[0,T]} \|X(t)\|_V^2 = E \lim_{M\to\infty} \operatorname*{ess\,sup}_{t\in[0,\mathcal{T}_M]} \|X(t)\|_V^2 \le E\left(B^{\frac{1}{\sigma}}(T) \operatorname*{ess\,sup}_{t\in[0,T]} \|Z(t)\|_V^2 \right)$$
$$\le C(\sigma) \left[\|\varphi\|_V^2 \, E\left(B^{\frac{1}{\sigma}}(T)\right) + \lambda \, \|\varphi\|^{2(1+\sigma)} \, E\left(B^{\frac{1}{\sigma}(1+\sigma)}(T)\right) + \lambda^{\frac{2}{2-\sigma}} \, \|\varphi\|^{\frac{2(2+\sigma)}{2-\sigma}} \, E\left(B^{\frac{2+\sigma}{\sigma(2-\sigma)}}(T)\right)\right],$$

and, therefore, $X \in L^2(\Omega; L^\infty([0,T]; V))$ as well. □

2.3.3 Extensions

Analogously to Subsection 2.2.3, all results of this section are also valid in the case of homogeneous Dirichlet and periodic boundary conditions. Furthermore, the stochastic nonlinear Schrödinger problem is necessarily considered over a finite time horizon and a bounded one-dimensional domain throughout this section. This is important since we use the compact embedding $V \hookrightarrow H$ at the end of the proof of Corollary 2.3.7 to show the relative compactness of the sequence of Galerkin approximations $(Z_n)_n$ in $L^2([0,T]; H)$. However, due to this relative compactness result, we can not simply generalize $x \in (0,1)$ to the unbounded space domain $x \in \mathbb{R}$ like in the last paragraph of Subsection 2.2.3.

If $\lambda = 0$, we are in the case of the linear Schrödinger problem that simplifies our considerations. Moreover, the case $\lambda < 0$ in (2.59) can be investigated with the methods of the present issue.

Corollary 2.3.10. *Regarding $\lambda < 0$ in the stochastic nonlinear Schrödinger problem* (2.59)*, we obtain analogue results as in Section 2.3. However, one has to be careful in deriving an analogue estimate as in Theorem 2.3.5 which requires, besides* (2.57)*, another similar assumption on the coefficients in front of the Wiener process in form of*

$$\int_0^T \left(\sum_{j=1}^\infty b_j^2(s) \right)^{\frac{2}{2-\sigma}} ds < \infty, \qquad \text{for a.e. } \omega \in \Omega. \tag{2.76}$$

Analogue statements of Theorem 2.3.9 apply as well under the additional assumptions

$$\begin{aligned} &E\left(\exp\left\{ (1+\sigma) \sum_{j=1}^\infty \int_0^T b_j^2(s)\,ds \right\} \int_0^T \sum_{j=1}^\infty b_j^2(s)\,ds \right) < \infty, \\ &E\left(\exp\left\{ \frac{2+\sigma}{2-\sigma} \sum_{j=1}^\infty \int_0^T b_j^2(s)\,ds \right\} \int_0^T \left(\sum_{j=1}^\infty b_j^2(s) \right)^{\frac{2}{2-\sigma}} ds \right) < \infty. \end{aligned} \tag{2.77}$$

Proof. Investigating the case $\lambda < 0$ in the stochastic nonlinear Schrödinger problem (2.59), we usually have to regard the absolute value of λ while estimating from above, for example, in the proof of Theorem 2.3.2, Corollary 2.3.7 and part (c) of Theorem 2.3.8. However, what makes the difference for $\lambda < 0$ is the proof of Theorem 2.3.5. We need to have a closer look at relation (2.68) since for $\lambda < 0$ the two terms that vanish in (2.68) are now the relevant terms. Using inequality (2.69), the result

$$0 \le \frac{d}{ds} B(s) = B(s)\, \sigma \sum_{j=1}^\infty b_j^2(s) < \infty, \qquad \text{for a.e. } s \in [0,T],$$

and Young's inequality, we get coefficients containing the expressions

$$\rho_1(t) := \int_0^t \sum_{j=1}^\infty b_j^2(s)\,ds \qquad \text{and} \qquad \rho_2(t) := \int_0^t \left(\sum_{j=1}^\infty b_j^2(s) \right)^{\frac{2}{2-\sigma}} ds.$$

Applying Gronwall's lemma, we estimate them by the majorants $\rho_1(T)$ and $\rho_2(T)$ and that is why we have to assume the finiteness of these majorants for a.e. $\omega \in \Omega$. The first integral $\rho_1(T)$ is finite by (2.58), which follows from (2.57), and the finiteness of the second integral $\rho_2(T)$ is the new assumption (2.76). Thus, the solutions Z_n of the Galerkin equations satisfy for all $t \in [0,T]$ and each $n \in \mathbb{N}$ (arbitrarily fixed) the estimate

$$\|Z_n(t)\|_V^2 \le C(\sigma,T)\Big(\|\varphi\|_V^2 + |\lambda| \Big[1+\sigma B(T)\rho_1(T)\Big] \|\varphi\|^{2(1+\sigma)} \\ + |\lambda|^{\frac{2}{2-\sigma}} \Big[1+(\sigma B(T))^{\frac{2}{2-\sigma}} \rho_2(T)\Big] \|\varphi\|^{\frac{2(2+\sigma)}{2-\sigma}} \Big).$$

To further ensure analogue results of Theorem 2.3.9 for $\lambda < 0$, we use the last inequality and have to guarantee the existence of

$$E\left(B^{\frac{1}{\sigma}}(T)\right), \qquad E\left(B^{\frac{1}{\sigma}(1+\sigma)}(T)\rho_1(T)\right) \qquad \text{and} \qquad E\left(B^{\frac{2+\sigma}{\sigma(2-\sigma)}}(T)\rho_2(T)\right).$$

Plugging in $B(T), \rho_1(T)$ and $\rho_2(T)$, one observes that condition (2.57) implies the existence of the first mean value and the two conditions in (2.77) entail the existence of the last two mean values. □

Knowing now that we can choose $\lambda \in \mathbb{R}$ without any restrictions, it arises the question if the problem is also well-defined for $\lambda \in i\mathbb{R}$ or $\lambda \in \mathbb{C}$.

Corollary 2.3.11. *We can also take $\lambda := i\tilde{\lambda}$ and $\tilde{\lambda} \in \mathbb{R}_+$ in the stochastic nonlinear Schrödinger problem* (2.59) *to obtain the statements of Theorem 2.3.9 under the restriction that $\sigma \in [1,2)$.*

Proof. Using the ideas from Section 2.2, we obtain for $\sigma \in [1,2)$ (because of Lemma D.6) that there exists a unique variational solution $X \in L^2(\Omega; C([0,T];H)) \cap L^2(\Omega \times [0,T]; V)$ with

$$E \sup_{t\in[0,T]} \|X(t)\|^2 \le \|\varphi\|^2 E\left(B^{\frac{1}{\sigma}}(T)\right), \qquad E\int_0^T \|X(t)\|_V^2\, dt \le T\, \|\varphi\|_V^2 E\left(B^{\frac{1}{\sigma}}(T)\right).$$

Moreover, $X \in L^2(\Omega; L^\infty([0,T];V))$ since

$$E \operatorname*{ess\,sup}_{t\in[0,T]} \|X(t)\|_V^2 \le \|\varphi\|_V^2 E\left(B^{\frac{1}{\sigma}}(T)\right).$$

Due to this approach, it is not possible to choose $\tilde{\lambda} < 0$ here. □

Inspired by [78, 93], we can treat combined power-type nonlinearities as well.

Corollary 2.3.12. *By considering combined power-type nonlinearities*

$$f(v) := \sum_{k=1}^m \lambda_k |v|^{2\sigma_k} v, \qquad \textit{for all } v \in V \textit{ and fixed } m \in \mathbb{N},$$

with $\sigma_k \in (0,2)$ and $\lambda_k \in \mathbb{R}\setminus\{0\}$ or $\lambda_k = i\tilde{\lambda}_k$ with $\tilde{\lambda}_k \in \mathbb{R}_+$ for $k = 1,2,\ldots,m$, we obtain similar results as in the current section.

Proof. For each $t \in [0,T]$ and all $k = 1,2,\ldots,m$ we have to introduce

$$B_k(t) := \frac{1}{|Y(t)|^{2\sigma_k}} = \exp\left\{\sigma_k \sum_{j=1}^\infty \int_0^t b_j^2(s)\, ds\right\}$$

and replace the nonlinear term in the current section by m nonlinear terms. Thus, the pathwise problem has the form

$$\big(Z(t),v\big) = \big(\varphi,v\big) - i\int_0^t \big\langle AZ(s),v\big\rangle\, ds + i\sum_{k=1}^m \lambda_k \int_0^t B_k(s)\big(|Z(s)|^{2\sigma_k} Z(s), v\big)\, ds$$

for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $v \in V$. One determines that under the assumption

$$E\left(\exp\left\{\frac{2+\sigma_k}{2-\sigma_k}\sum_{j=1}^{\infty}\int_0^T b_j^2(s)\,ds\right\}\right) < \infty, \qquad \text{for all } k = 1,2,\dots,m, \tag{2.78}$$

the approach from this section also works and yields similar results. Notice that also a combination of positive and negative $(\lambda_k)_{k\in\{1,2,\dots,m\}}$ can be considered. For the sake of simplicity, one can arrange them for $r < m$ in the way that $(\lambda_k)_{k\in\{1,2,\dots,r\}}$ are positive and $(\lambda_k)_{k\in\{r+1,r+2,\dots,m\}}$ are negative. Then one needs condition (2.78) for all $k = 1,2,\dots,m$ and one has to generalize the assumptions (2.76) and (2.77) for all $k = r+1, r+2, \dots, m$ analogously. However, in the case of combined power-type nonlinearities we cannot mix $\lambda_k \in \mathbb{R}\setminus\{0\}$ and $\lambda_k := i\tilde{\lambda}_k$ with $\tilde{\lambda}_k \in \mathbb{R}_+$. We have to assume that for all $\lambda_k := \operatorname{Re}\lambda_k + i\operatorname{Im}\lambda_k$ it holds that $\operatorname{Re}\lambda_k \cdot \operatorname{Im}\lambda_k = 0$, which implies that it is not allowed to choose complex-valued numbers. $\square$

2.4 Case of a Lipschitz Continuous Drift Term

After the investigation of the power-type nonlinearity $f : V \to H$ given by $f(v) = |v|^{2\sigma}v$ with $\sigma \geq 1$ in Section 2.2 and $\sigma \in (0,2)$ in Section 2.3, we are interested in nonlinear terms f that are Lipschitz continuous and of bounded growth. We still regard the stochastic nonlinear Schrödinger problem

$$\big(X(t),v\big) = \big(\varphi,v\big) - i\int_0^t \langle AX(s),v\rangle\,ds + i\lambda\int_0^t \big(f(s,X(s)),v\big)\,ds + i\bigg(\int_0^t g(s,X(s))\,dW(s),v\bigg)$$

for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $v \in V$ with the notations from Section 2.1. For the sake of brevity, we denote $f(t,z) := f(t,z,\overline{z}) := f(\omega,t,z,\overline{z})$ for all $\omega \in \Omega$, all $t \in [0,T]$ and all $z \in \mathbb{C}$. We assume that $f : \Omega \times [0,T] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ satisfies $|f(\,\cdot\,,0)| \leq C$, is measurable (which means that for all $s \in [0,t]$ it holds that $\{(\omega,s,z) : f(\omega,s,z) \in A\} \in \mathcal{F}_t \times \mathcal{B}([0,t]\times\mathbb{C})$ for all $A \in \mathcal{B}(\mathbb{C})$ and all $t \in [0,T]$), differentiable in the sense of Wirtinger and fulfills

$$\left|\frac{\partial}{\partial z}f(t,z)\right| \leq C, \qquad \text{and} \qquad \left|\frac{\partial}{\partial \overline{z}}f(t,z)\right| \leq C, \qquad \text{for all } t \in [0,T] \text{ and all } z \in \mathbb{C}. \tag{2.79}$$

Now, we consider $f(\,\cdot\,,z)$ for $z = X(t)$ and write $f(\,\cdot\,,X(t))$. Hence, $f(t,X(t))$ is in $L^2(\Omega\times[0,T];H)$ if $X(t)$ satisfies these assumptions. Initially, we indicate that f is indeed Lipschitz continuous and of bounded growth. For the sake of simplicity, the dependence on the time variable $t \in [0,T]$ is neglected.

Lemma 2.4.1. *The condition $|f(\,\cdot\,,0)| \leq C$ and the boundedness of the absolute values of the first Wirtinger derivatives* (2.79) *imply that the nonlinear function f is Lipschitz continuous in H and of bounded growth in H and V, which means that there exist positive constants c_f, k_f such that*

$$\begin{aligned}
\|f(\,\cdot\,,u) - f(\,\cdot\,,v)\|^2 &\leq c_f\|u-v\|^2, && \text{for all } u,v \in H,\\
\|f(\,\cdot\,,u)\|^2 &\leq k_f\left(1+\|u\|^2\right), && \text{for all } u \in H,\\
\|f(\,\cdot\,,v)\|_V^2 &\leq k_f\left(1+\|v\|_V^2\right), && \text{for all } v \in V.
\end{aligned}$$

Proof. In advance, remember that $f(\,\cdot\,,z) = f(\,\cdot\,,z,\overline{z})$ for all $z \in \mathbb{C}$. With the help of the differential quotients

$$\lim_{z_1\to z_2}\frac{f(\,\cdot\,,z_1,\overline{z}_1) - f(\,\cdot\,,z_2,\overline{z}_1)}{z_1 - z_2} = \frac{\partial}{\partial z}f(\,\cdot\,,z_2,\overline{z}_1)$$

and

$$\lim_{\overline{z}_1\to \overline{z}_2}\frac{f(\,\cdot\,,z_2,\overline{z}_1) - f(\,\cdot\,,z_2,\overline{z}_2)}{\overline{z}_1 - \overline{z}_2} = \frac{\partial}{\partial \overline{z}}f(\,\cdot\,,z_2,\overline{z}_2)$$

for all $z_1, z_2 \in \mathbb{C}$ and the boundedness of the first absolute Wirtinger derivatives by

$$\left|\frac{\partial}{\partial z} f(\,\cdot\,, z_2, \overline{z}_1)\right| \leq C \qquad \text{and} \qquad \left|\frac{\partial}{\partial \overline{z}} f(\,\cdot\,, z_2, \overline{z}_2)\right| \leq C,$$

we get the Lipschitz continuity (by regarding only the positive part of the square root)

$$\begin{aligned}|f(\,\cdot\,, z_1) - f(\,\cdot\,, z_2)| &= |f(\,\cdot\,, z_1, \overline{z}_1) - f(\,\cdot\,, z_2, \overline{z}_2)| \\ &\leq |f(\,\cdot\,, z_1, \overline{z}_1) - f(\,\cdot\,, z_2, \overline{z}_1)| + |f(\,\cdot\,, z_2, \overline{z}_1) - f(\,\cdot\,, z_2, \overline{z}_2)| \\ &\leq C\,|z_1 - z_2| + C\,|\overline{z}_1 - \overline{z}_2| \leq \sqrt{c_f}\,|z_1 - z_2|\,.\end{aligned}$$

Taking $z_2 = 0$, the bounded growth especially follows by $|f(\,\cdot\,, 0)| \leq C$ since

$$|f(\,\cdot\,, z_1)| \leq |f(\,\cdot\,, z_1) - f(\,\cdot\,, 0)| + |f(\,\cdot\,, 0)| \leq \sqrt{c_f}\,|z_1| + C \leq \sqrt{\frac{k_f}{2}}\,(1 + |z_1|)\,. \tag{2.80}$$

Accordingly, the Lipschitz continuity and the bounded growth of the nonlinear function f entail the Lipschitz continuity and the bounded growth in the square of its H-norm since for all $u, v \in H$ it holds that

$$\begin{aligned}\|f(\,\cdot\,, u) - f(\,\cdot\,, v)\|^2 &= \int_0^1 |f(\,\cdot\,, u(x)) - f(\,\cdot\,, v(x))|^2\,dx \leq c_f \int_0^1 |u(x) - v(x)|^2\,dx = c_f\,\|u - v\|^2\,, \\ \|f(\,\cdot\,, u)\|^2 &= \int_0^1 |f(\,\cdot\,, u(x))|^2\,dx \leq k_f \int_0^1 \left(1 + |u(x)|^2\right) dx = k_f\left(1 + \|u\|^2\right).\end{aligned}$$

Moreover, we obtain for all functions $z(x) \in \mathbb{C}$ which are once continuously differentiable with respect to x that

$$\begin{aligned}\left|\frac{d}{dx} f(\,\cdot\,, z(x))\right| &= \left|\frac{d}{dx} f(\,\cdot\,, z(x), \overline{z}(x))\right| \\ &= \left|\left(\frac{\partial}{\partial z} f(\,\cdot\,, z(x), \overline{z}(x))\right)\left(\frac{d}{dx} z(x)\right) + \left(\frac{\partial}{\partial \overline{z}} f(\,\cdot\,, z(x), \overline{z}(x))\right)\left(\frac{d}{dx}\overline{z}(x)\right)\right| \\ &\leq \left|\frac{\partial}{\partial z} f(\,\cdot\,, z(x), \overline{z}(x))\right| \left|\frac{d}{dx} z(x)\right| + \left|\frac{\partial}{\partial \overline{z}} f(\,\cdot\,, z(x), \overline{z}(x))\right| \left|\frac{d}{dx}\overline{z}(x)\right| \\ &\leq C\left|\frac{d}{dx} z(x)\right| + C\left|\frac{d}{dx}\overline{z}(x)\right| \leq \sqrt{c_f}\left|\frac{d}{dx} z(x)\right| \leq \sqrt{\frac{k_f}{2}}\left|\frac{d}{dx} z(x)\right|.\end{aligned}$$

Together with the bounded growth (2.80), we deduce

$$|f(\,\cdot\,, z(x))|^2 + \left|\frac{d}{dx} f(\,\cdot\,, z(x))\right|^2 \leq k_f\left(1 + |z(x)|^2 + \left|\frac{d}{dx} z(x)\right|^2\right).$$

Hence, the bounded growth of f in the square of its V-norm results for all $v \in V$ since

$$\begin{aligned}\|f(\,\cdot\,, v)\|_V^2 = \|f(\,\cdot\,, v)\|^2 + \left\|\frac{\partial}{\partial x} f(\,\cdot\,, v)\right\|^2 &= \int_0^1 \left(|f(\,\cdot\,, v(x))|^2 + \left|\frac{\partial}{\partial x} f(\,\cdot\,, v(x))\right|^2\right) dx \\ &\leq k_f \int_0^1 \left(1 + |v(x)|^2 + \left|\frac{\partial}{\partial x} v(x)\right|^2\right) dx = k_f\left(1 + \|v\|_V^2\right).\end{aligned}$$ □

To illustrate that the conditions on the nonlinear complex-valued function f are satisfiable, some examples shall be given here. Without loss of generality, we consider functions $f(z)$ which do not explicitly depend on the time variable $t \in [0, T]$. If all conditions are fulfilled, we multiply $f(z)$ by a function $h(t)$ being bounded and $\mathcal{F}_t$-measurable for all $t \in [0, T]$ such that all assumptions on f in Lemma 2.4.1 transfer to $f(z)h(t)$. The properties we have to check are that $|f(0)| \leq C$, measurability, differentiability in the sense of Wirtinger and boundedness of the absolute values of its first derivatives. In [83] the functions

$$f_1(z) = \frac{|z|^2}{1+|z|^2}z \qquad \text{and} \qquad f_2(z) = \left(1 - e^{-|z|^2}\right)z, \qquad \text{for all } z \in \mathbb{C},$$

are indicated as corrections to the cubic nonlinearity $|z|^2 z$ for large wave amplitudes. Indeed, they behave like that for $z \in \mathbb{C}$ with small absolute values $|z|$ and, because of its physical importance, we emphasize that this cubic nonlinearity, also called Kerr nonlinearity, is a special case of this thesis. Back on the examples, one immediately sees that $f_1(0) = f_2(0) \equiv 0$, that especially implies its boundedness from above, and that the functions are measurable since they are continuous and deterministic. From the continuity of the functions we also get their real differentiability and, since the considered functions f_1 and f_2 are not complex differentiable in each point of an open set (as shown in the following), the introduction of Wirtinger derivatives is a useful kind of representation. Now, let $z := x + iy$ and let u_j and v_j denote the real and imaginary part of f_j for $j = 1, 2$, respectively. We verify that the Cauchy-Riemann equations $(u_j)_x = (v_j)_y$ and $(u_j)_y = -(v_j)_x$ are not satisfied (compare Appendix H). Therefore, we write

$$f_1(z) = \frac{|z|^2}{1+|z|^2}z = \frac{x^2+y^2}{1+x^2+y^2}(x+iy)$$

and deduce

$$u_1 = \operatorname{Re} f_1(z) = \frac{x^3+xy^2}{1+x^2+y^2}, \qquad v_1 = \operatorname{Im} f_1(z) = \frac{x^2y+y^3}{1+x^2+y^2}.$$

Its partial derivatives with respect to x and y are given by

$$\begin{aligned} (u_1)_x &= \frac{3x^2+y^2+x^4+2x^2y^2+y^4}{(1+x^2+y^2)^2}, \qquad & (u_1)_y &= \frac{2xy}{(1+x^2+y^2)^2}, \\ (v_1)_y &= \frac{x^2+3y^2+x^4+2x^2y^2+y^4}{(1+x^2+y^2)^2}, \qquad & (v_1)_x &= \frac{2xy}{(1+x^2+y^2)^2}. \end{aligned}$$

The second function

$$f_2(z) = \left(1 - e^{-|z|^2}\right)z = \left(1 - e^{-x^2-y^2}\right)(x+iy)$$

possesses the real and imaginary part

$$u_2 = \operatorname{Re} f_2(z) = \left(1 - e^{-x^2-y^2}\right)x, \qquad v_2 = \operatorname{Im} f_2(z) = \left(1 - e^{-x^2-y^2}\right)y.$$

Calculating its derivatives with respect to x and y, it results that

$$\begin{aligned} (u_2)_x &= 1 + e^{-x^2-y^2}(2x^2-1), \qquad & (u_2)_y &= e^{-x^2-y^2}2xy, \\ (v_2)_y &= 1 + e^{-x^2-y^2}(2y^2-1), \qquad & (v_2)_x &= e^{-x^2-y^2}2xy. \end{aligned}$$

Hence, the two nonlinear functions $f_j(z)$ with $j = 1, 2$ fulfill the first Cauchy-Riemann equation $(u_j)_x = (v_j)_y$ for $x = y$ and the second one $(u_j)_y = -(v_j)_x$ either for $x = 0$ or for $y = 0$. Only if both equations are satisfied, the complex differentiability is guaranteed, which is just given in the point of origin $x = y = 0$. Since this is no open set, the functions $f_j(z)$ for $j = 1, 2$ are nowhere holomorphic (compare Appendix H). Nevertheless, the real differentiability in the sense of Wirtinger can be considered. Using z and its complex conjugated variable $\overline{z}$ as if they were independent of each other (compare Corollary H.7), one can calculate the absolute values of the first Wirtinger derivatives to see their boundedness. The first function obeys

$$f_1(z) = \frac{|z|^2}{1+|z|^2}z = \frac{z^2\overline{z}}{1+z\overline{z}}$$

and, therefore,

$$\left|\frac{\partial}{\partial z} f_1(z)\right| = \left|\frac{2|z|^2(1+|z|^2) - |z|^4}{(1+|z|^2)^2}\right| = \frac{2|z|^2 + |z|^4}{1 + 2|z|^2 + |z|^4} \leq 1,$$

$$\left|\frac{\partial}{\partial \overline{z}} f_1(z)\right| = \left|\frac{z^2(1+|z|^2) - z^2|z|^2}{(1+|z|^2)^2}\right| = \frac{|z|^2}{1 + 2|z|^2 + |z|^4} \leq \frac{2|z|^2}{1+2|z|^2} \leq 1.$$

Moreover, we get that

$$f_2(z) = \left(1 - e^{-|z|^2}\right)z = \left(1 - e^{-z\overline{z}}\right)z$$

can be estimated by

$$\left|\frac{\partial}{\partial z} f_2(z)\right| = \left|e^{-|z|^2}|z|^2 + 1 - e^{-|z|^2}\right| \leq \frac{|z|^2}{e^{|z|^2}} + 1 \leq \frac{|z|^2}{1+|z|^2} + 1 \leq 2,$$

$$\left|\frac{\partial}{\partial \overline{z}} f_2(z)\right| = \left|e^{-|z|^2} z^2\right| = \frac{|z|^2}{e^{|z|^2}} \leq \frac{|z|^2}{1+|z|^2} \leq 1.$$

2.4.1 Effect of Lipschitz Continuous Noise

Based on the notations in Section 2.1, we focus on the stochastic nonlinear Schrödinger equation

$$dX(t) = -\,iAX(t)\,dt + i\lambda f(t, X(t))\,dt + ig(t, X(t))\,dW(t), \qquad \text{for all } t \in [0,T],$$

with initial condition $X(0) = \varphi \in V$, $\lambda \in \mathbb{C}$ and a drift function $f : \Omega \times [0,T] \times H \to H$ as described at the beginning of this section. Our aim is to investigate the unique existence and some smoothness properties of the variational solution of

$$\begin{aligned} \big(X(t), v\big) = \big(\varphi, v\big) - i \int_0^t \langle AX(s), v\rangle\, ds + i\lambda \int_0^t \big(f(s, X(s)), v\big)\, ds \\ + i\left(\int_0^t g(s, X(s))\, dW(s), v\right) \end{aligned} \tag{2.81}$$

for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $v \in V$, which reminds of the results in [42]. Therefore, we show that the conditions of Theorem 1 and 2 in [42] are fulfilled. At first, the initial condition $\varphi \in V$ is $\mathcal{F}_0$-measurable. As already stated in Section 2.1, the one-dimensional negative Laplacian $A : V \to V^*$, formally defined by the symmetric bilinear form (2.1), is a linear continuous operator which possesses a discrete spectrum of eigenvalues and a complete orthonormal set of corresponding eigenfunctions in H. Referring to (2.8) and (2.9), the diffusion function g is assumed to be Lipschitz continuous in $L_2(K,H)$ and of bounded growth in $L_2(K,V)$. Analogue properties of the drift function f are inferred by Lemma 2.4.1.

Choosing now $\tilde{f} := \lambda f$ with $\lambda \in \mathbb{C}$ as the nonlinear diffusion function in (2.81), the statements of Lemma 2.4.1 transfer to $\tilde{f}$ and, therefore, the assumptions of Theorem 1 and 2 in [42] are satisfied. Hence, we conclude some useful results:

- The stochastic nonlinear Schrödinger problem (2.81) possesses a unique variational solution $X \in L^2(\Omega; C([0,T]; H)) \cap L^2(\Omega \times [0,T]; V)$ (see [42, Theorem 1]). To mention only some of the main arguments, this assertion is shown by the application of the Galerkin method, the stochastic energy equality (C.2), the Burkholder-Davis-Gundy inequality (2.13), Gronwall's lemma and some weak convergence results.
- It holds that $X \in L^{2p}(\Omega; C([0,T]; H))$ for all $p \geq 1$ (compare [42, Theorem 2]). There, the Itô formula, Hölder's inequality and the method of stopping times are additionally used.

Besides, these results are also true for locally Lipschitz continuous nonlinear drift and diffusion functions f and g that are of bounded growth as above. This is proved in [42, Theorem 5] with the help of a truncation function yielding globally Lipschitz continuous drift and diffusion terms.

Then the above existence and uniqueness statements are applied to the truncated equation and due to stopping times, Markov's inequality, an appropriate change of the universal set Ω and again some weak convergence results (see the approach in Subsection 2.2.2), the claim is verified.

Proceeding with the stochastic nonlinear Schrödinger problem (2.81) as in Section 2.2, we obtain similar boundedness results. By regarding that $\lambda \in \mathbb{C}$ (that is reflected in the norm square Itô formula generating the real and the imaginary part of the nonlinear term) and exploiting Lemma 2.4.1 (especially the Lipschitz continuity in H and the bounded growth in H and V), it follows that the solution X_n of the corresponding Galerkin equations of (2.81) fulfill

$$\begin{aligned} E \sup_{t\in[0,T]} \|X_n(t)\|^{2p} &\le C(p,|\lambda|,k_f,c_g,k_g,T)\left[1+\|\varphi\|^{2p}\right],\\ E \sup_{t\in[0,T]} \|X_n(t)\|_V^{2p} &\le C(p,|\lambda|,k_f,c_g,k_g,T)\left[1+\|\varphi\|_V^{2p}\right] \end{aligned}$$

for each $p \ge 1$ and each $n \in \mathbb{N}$ arbitrarily fixed. Based on Theorem 2.2.10, we extend these results to the solution X of the stochastic nonlinear Schrödinger problem (2.81) by verifying that for each $p \ge 1$ there exists a unique variational solution $X \in L^{2p}(\Omega; C([0,T];H)) \cap L^{2p}(\Omega \times [0,T]; V)$ with

$$\begin{aligned} E \sup_{t\in[0,T]} \|X(t)\|^{2p} &\le C(p,|\lambda|,k_f,c_g,k_g,T)\left[1+\|\varphi\|^{2p}\right],\\ E\int_0^T \|X(t)\|_V^{2p}\,dt &\le C(p,|\lambda|,k_f,c_g,k_g,T)\left[1+\|\varphi\|_V^{2p}\right]. \end{aligned}$$

2.4.2 Perturbation by Linear Multiplicative Noise

Since the approach of Section 2.3 is restricted to Schrödinger equations with linear multiplicative noise, it can also be applied to

$$\begin{aligned} \big(X(t),v\big) = \big(\varphi,v\big) - i\int_0^t \big\langle AX(s),v\big\rangle\,ds + i\lambda\int_0^t \big(f(s,X(s)),v\big)\,ds\\ + i\sum_{j=1}^{\infty}\int_0^t b_j(s)\big(X(s),v\big)\,d\beta_j(s) \end{aligned} \tag{2.82}$$

for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $v \in V$, where the power-type nonlinearity f in equation (2.59) is replaced by a nonlinear function $f(t,z) := f(t,z,\overline{z})$ for all $z \in \mathbb{C}$ which satisfies the conditions of the current section. Resuming Lemma 2.4.1, it follows from $|f(\,\cdot\,,0)| \le C$ and the differentiability of $f : \Omega \times [0,T] \times H \to H$ in the sense of Wirtinger with bounded absolute values of its first derivatives that f is Lipschitz continuous in H and of bounded growth in H and V. Using these properties, we work with absolute values and that is why we can take $\lambda \in \mathbb{C}$ here. Therefore, the real as well as the imaginary part of the nonlinear term appear while applying the energy equality.

Regarding again the process $(Y(t))_{t\in[0,T]}$ given by (2.60), the pathwise nonlinear Schrödinger problem for $Z(t,\,\cdot\,) = X(t,\,\cdot\,)Y(t)$ has the form

$$\big(Z(t),v\big) = \big(\varphi,v\big) - i\int_0^t \big\langle AZ(s),v\big\rangle\,ds + i\lambda\int_0^t \left(f\left(s,\frac{Z(s)}{Y(s)}\right)Y(s),v\right)ds \tag{2.83}$$

for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $v \in V$. Using the Lipschitz continuity, one shows the uniqueness of the variational solution $Z \in C([0,T];H) \cap L^2([0,T];V)$ analogously to Theorem 2.3.2. The corresponding Galerkin equations for each $n \in \mathbb{N}$ are stated by

$$\big(Z_n(t),h_k\big) = \big(\varphi_n,h_k\big) - i\int_0^t \big\langle AZ_n(s),h_k\big\rangle\,ds + i\lambda\int_0^t \left(f_n\left(s,\frac{Z_n(s)}{Y(s)}\right)Y(s),h_k\right)ds \tag{2.84}$$

for all $t \in [0,T]$ and all $k \in \{1,2,\dots,n\}$, where $\varphi_n := \pi_n\varphi$ and $f_n(\,\cdot\,,u) := \pi_n f(\,\cdot\,,u)$ for all $u \in H_n$ (compare (2.2)). Exploiting the properties (2.3) of the orthogonal projection π_n, the

Lipschitz continuity and the bounded growth of f, the theory of finite-dimensional stochastic differential equations (see [61, pp. 127–141, Theorem 4.3.5 and Exercise 4.5.5]) yields that there exists a unique solution $Z_n \in C([0,T];H_n)$ of problem (2.84). Due to the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_V$ on H_n (compare (2.4)), the Galerkin approximations Z_n are also continuous in V and we further get $Z_n \in L^2([0,T];V)$. That is why we can apply the energy equality. Similarly to the proofs of Theorems 2.3.3 and 2.3.5, we get for all $t \in [0,T]$ and each $n \in \mathbb{N}$ arbitrarily fixed

$$\|Z_n(t)\|^2 \le C(|\lambda|, k_f, T)\left[1+\|\varphi\|^2\right], \qquad \|Z_n(t)\|_V^2 \le C(|\lambda|, k_f, T)\left[1+\|\varphi\|_V^2\right] \tag{2.85}$$

(by using Lemma 2.4.1 and $0 \le |Y(t)|^2 \le 1$ for all $t \in [0,T]$). Furthermore, it is possible to verify the same results as in Corollaries 2.3.4, 2.3.6 and 2.3.7. The sequence of Galerkin approximations $(Z_n)_n$ of the pathwise nonlinear Schrödinger problem (2.84) is bounded in $C([0,T];H)$, $L^{2p}([0,T];H)$ and $L^{2p}([0,T];V)$ for all $p \ge 1$ and relatively compact in $L^2([0,T];H)$.

Now, we need to have a closer look at the results of Theorem 2.3.8, especially the convergence of the nonlinear drift term. Since in a reflexive space each bounded sequence possesses a weakly convergent subsequence, and f is of bounded growth, we conclude

$$\begin{aligned}\int_0^T\int_0^t \left(f_{n'}\left(s,\frac{Z_{n'}(s)}{Y(s)}\right)Y(s),h_k\right)ds\,dt &= \int_0^T\int_0^t \left(f\left(s,\frac{Z_{n'}(s)}{Y(s)}\right)Y(s),h_k\right)ds\,dt\\ &\to \int_0^T\int_0^t \left(f^*(s)Y(s),h_k\right)ds\,dt \qquad \text{as } n'\to\infty.\end{aligned}$$

We wish to get $f^*(s) = f\left(s,\frac{Z(s)}{Y(s)}\right)$ and that is why we consider

$$\begin{aligned}&\int_0^T\int_0^t \left(f\left(s,\frac{Z_{n'}(s)}{Y(s)}\right)Y(s),h_k\right)ds\,dt - \int_0^T\int_0^t \left(f\left(s,\frac{Z(s)}{Y(s)}\right)Y(s),h_k\right)ds\,dt\\
&\le \left|\int_0^T\int_0^t \left(\left[f\left(s,\frac{Z_{n'}(s)}{Y(s)}\right)-f\left(s,\frac{Z(s)}{Y(s)}\right)\right]Y(s),h_k\right)ds\,dt\right|\\
&\le \int_0^T\int_0^t \left\|f\left(s,\frac{Z_{n'}(s)}{Y(s)}\right)-f\left(s,\frac{Z(s)}{Y(s)}\right)\right\|\,|Y(s)|\,\|h_k\|\,ds\,dt\\
&\le \int_0^T\int_0^t \sqrt{c_f}\|Z_{n'}(s)-Z(s)\|\,ds\,dt \le \sqrt{c_f}\,T\left(\int_0^T 1\,dt\right)^{\frac12}\left(\int_0^T \|Z_{n'}(t)-Z(t)\|^2\,dt\right)^{\frac12}\\
&= \sqrt{c_f}\,T^{\frac32}\left(\int_0^T \|Z_{n'}(t)-Z(t)\|^2\,dt\right)^{\frac12} \to 0 \qquad \text{as } n'\to\infty\end{aligned}$$

(due to (2.72)). Hence, we know that

$$\int_0^T\int_0^t \left(f\left(s,\frac{Z_{n'}(s)}{Y(s)}\right)Y(s),h_k\right)ds\,dt \to \int_0^T\int_0^t \left(f\left(s,\frac{Z(s)}{Y(s)}\right)Y(s),h_k\right)ds\,dt \qquad \text{as } n'\to\infty.$$

Together with the results from the proof of Theorem 2.3.8 (a), we deduce that the sequence of variational solutions $(Z_n)_n$ of the finite-dimensional Galerkin equations (2.84) converges strongly in $L^2([0,T];H)$ and weakly in $L^2([0,T];V)$ to the variational solution Z of the pathwise problem (2.83). Statements (b) and (c) of Theorem 2.3.8 are deduced similarly as in Section 2.3 such that $Z \in L^\infty([0,T];V)$ with

$$\operatorname*{ess\,sup}_{t\in[0,T]}\|Z(t)\|_V^2 \le \liminf_{n'\to\infty}\operatorname*{ess\,sup}_{t\in[0,T]}\|Z_{n'}(t)\|_V^2 \le C(|\lambda|,k_f,T)\left[1+\|\varphi\|_V^2\right],$$

and the sequence $(Z_n)_n$ also converges to Z in $C([0,T];H)$ (because of the Lipschitz continuity).

Finally, we transfer the results from the pathwise problem (2.83) to the stochastic nonlinear Schrödinger problem (2.82). Following the ideas of the proof of Theorem 2.3.9, one obtains that there exists a unique variational solution $X \in L^2(\Omega;C([0,T];H)) \cap L^2(\Omega\times[0,T];V)$ of problem (2.82), if condition (2.57) is satisfied, and especially $X \in L^2(\Omega;L^\infty([0,T];V))$.

3 On a Problem of Optimal Control

3.1 Controlled Schrödinger Problem

In order to deal with optimal control problems, we have to introduce some further notations. Let $\mathcal{U}$ be the set of all admissible controls defined by

$$\begin{aligned}\mathcal{U} := \Big\{ & U : \Omega \times [0,T] \times [0,1] \to \mathbb{R}_+ : U(t) \in V \text{ is } \mathcal{F}_t\text{-adapted}, \\ & \exists\, \alpha_1 > 0 : U(t,x) \le \alpha_1 \ P\text{-a.s.}, \exists\, \alpha_2 \ge 0 : \left\| \frac{\partial}{\partial x} U(t) \right\| \le \alpha_2 \ P\text{-a.s.} \Big\}\end{aligned} \tag{3.1}$$

such that $U \in \mathcal{U}$ is a bounded stochastic process. Then it is possible to formulate the stochastic nonlinear Schrödinger problem corresponding to the control $U \in \mathcal{U}$ constituted by

$$\begin{aligned}\big(X^U(t), v\big) = (\varphi, v) - i \int_0^t \big\langle A X^U(s), v \big\rangle \, ds + i \int_0^t \big(U(s) X^U(s), v\big) \, ds \\ + i\lambda \int_0^t \big(f(s, X^U(s)), v\big) \, ds + i \Big(\int_0^t g(s, X^U(s)) \, dW(s), v \Big)\end{aligned} \tag{3.2}$$

for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $v \in V$. Since the complex-valued wave function now depends on an arbitrary admissible control $U \in \mathcal{U}$, it is particularly called X^U. For the sake of brevity, problem (3.2) will be referred to as controlled Schrödinger problem in the following. To further ensure the unique existence of the variational solution of problem (3.2) as in the previous chapter, we have to state some results and estimates of the additional control term

$$i \int_0^t \big(U(s) X^U(s), v\big) \, ds, \qquad \text{for a.e. } \omega \in \Omega, \text{ all } t \in [0,T] \text{ and all } v \in V. \tag{3.3}$$

At first, we emphasize the Lipschitz continuity in H and the bounded growth in H and V. Because of the boundedness of $U(s,x)$ from above by $\alpha_1 > 0$ P-a.s. for all $s \in [0,T]$ and all $x \in [0,1]$, it follows that

$$\begin{aligned}\|U(s)\,[h(s) - v(s)]\|^2 &\le \alpha_1^2 \, \|h(s) - v(s)\|^2, && \text{for all } h(s), v(s) \in H, \\ \|U(s) h(s)\|^2 &\le \alpha_1^2 \|h(s)\|^2 \le \alpha_1^2 \left(1 + \|h(s)\|^2\right), && \text{for all } h(s) \in H.\end{aligned}$$

Due to Lemma D.2 and the upper bounds $\alpha_1 > 0$ and $\alpha_2 \ge 0$ given in (3.1), we further get

$$\begin{aligned}\|U(s) v(s)\|_V^2 &= \|U(s) v(s)\|^2 + \left\| \left[\frac{\partial}{\partial x} U(s)\right] v(s) + U(s) \left[\frac{\partial}{\partial x} v(s)\right] \right\|^2 \\ &\le \|U(s) v(s)\|^2 + 2 \left\| \frac{\partial}{\partial x} U(s) \right\|^2 \left(\sup_{x \in [0,1]} |v(s,x)|^2 \right) + 2 \left\| U(s) \left[\frac{\partial}{\partial x} v(s)\right] \right\|^2 \\ &\le \left(2\alpha_1^2 + 4\alpha_2^2\right) \|v(s)\|_V^2 \le 2 \left(\alpha_1^2 + 2\alpha_2^2\right) \left(1 + \|v(s)\|_V^2\right), \qquad \text{for all } v(s) \in V.\end{aligned}$$

Now, we indicate the inequalities that are necessary to obtain the uniqueness of the variational solution and a priori estimates in H and V. Since $U \in \mathcal{U}$ is real-valued, we deduce for all $h(s) \in H$ that

$$2 \operatorname{Im} \int_0^t \big(U(s)h(s), h(s)\big)\, ds = 2 \int_0^t \int_0^1 U(s,x) \operatorname{Im} |h(s,x)|^2 \, dx \, ds = 0. \tag{3.4}$$

As will be seen in the following, we are limited to the pathwise approach of the controlled Schrödinger problem (see Section 2.3). Since one cannot obtain uniform a priori estimates in V without any restriction to the control $U \in \mathcal{U}$, we state a lemma including an important case differentiation. A further case will be treated in Subsection 3.2.5.

Lemma 3.1.1. *Let the control $U \in \mathcal{U}$ satisfy one of the following conditions*

$$\textbf{(i)}\ U(t,x) = U_1(t), \qquad \textbf{(ii)}\ U(t,x) = U_2(x).$$

Then for all $v \in L^2([0,T]; H') := \left\{ v \in L^2([0,T]; H) : \frac{\partial}{\partial t} v \in L^2([0,T]; H) \right\}$ with a representation that is separated in time and space and $\|v(t)\|^2 = C$ (independent of t) for all $t \in [0,T]$ it holds that

$$2 \operatorname{Re} \int_0^t \left(U(s)v(s), \frac{\partial}{\partial s} v(s) \right) ds \le \alpha_1 \|v(t)\|^2, \qquad \textit{for all } t \in [0,T].$$

Proof. Let $v \in L^2([0,T]; H')$ with a representation that is separated in time and space (in this thesis we consider Galerkin approximations) and $\|v(t)\|^2 = C$ (independent of t) for all $t \in [0,T]$.

(i) At first, let $U(s,x) = U_1(s)$ such that there is no dependence on the space variable. Because of relation (2.67), it follows for all $t \in [0,T]$ that

$$\begin{aligned} 2 \operatorname{Re} \int_0^t \left(U(s)v(s), \frac{\partial}{\partial s} v(s) \right) ds &= 2 \int_0^t U_1(s) \operatorname{Re} \left(v(s), \frac{\partial}{\partial s} v(s) \right) ds \\ = \int_0^t U_1(s) \left(\frac{d}{ds} \|v(s)\|^2 \right) ds &= \int_0^t U_1(s) \left(\frac{d}{ds} C \right) ds = 0 \le \alpha_1 \|v(t)\|^2 . \end{aligned}$$

(ii) The second case $U(s,x) = U_2(x)$ implies that there is no dependence on the time variable. Moreover, the control does not depend on $\omega \in \Omega$ due to the presumed $\mathcal{F}_t$-adaptedness. Thus, equation (2.67) and the boundedness $U_2(x) \le \alpha_1$ for all $x \in [0,1]$ yield for all $t \in [0,T]$

$$\begin{aligned} 2 \operatorname{Re} \int_0^t \left(U(s)v(s), \frac{\partial}{\partial s} v(s) \right) ds &= 2 \int_0^t \int_0^1 U_2(x) \operatorname{Re} \left\{ v(s,x) \frac{\partial}{\partial s} \overline{v}(s,x) \right\} dx \, ds \\ = \int_0^t \int_0^1 U_2(x) \left(\frac{\partial}{\partial s} |v(s,x)|^2 \right) dx \, ds &= \int_0^t \frac{d}{ds} \left(\int_0^1 U_2(x) |v(s,x)|^2 \, dx \right) ds \\ = \int_0^1 U_2(x) |v(t,x)|^2 \, dx - \int_0^1 U_2(x) |v(0,x)|^2 \, dx &\le \alpha_1 \|v(t)\|^2 . \end{aligned}$$

□

Based on the continuous embedding $H \hookrightarrow V^*$ with the embedding constant $\tilde{C} = C_{H,V^*}$, we show the equicontinuity in $C([0,T]; V^*)$ (for the pathwise controlled Schrödinger problem) by

$$\|U(r)v(r)\|_{V^*}^2 \le \tilde{C}^2 \|U(r)v(r)\|^2 \le \tilde{C}^2 \alpha_1^2 \|v(r)\|^2 \le \tilde{C}^2 \alpha_1^2 \|v(r)\|_V^2 , \qquad \text{for all } v(r) \in V.$$

The control term (3.3) also converges to the desired expression. If $v_n \rightharpoonup v$ in $L^2([0,T]; H)$ as $n \to \infty$, we get for all $k \in \{1, 2, \dots, n\}$ that

$$\int_0^t \big(U(s)v_n(s), h_k\big)\, ds = \int_0^t \big(v_n(s), U(s)h_k\big)\, ds \to \int_0^t \big(v(s), U(s)h_k\big)\, ds = \int_0^t \big(U(s)v(s), h_k\big)\, ds$$

for a.e. $t \in [0,T]$ as $n \to \infty$. Next, while proving the convergence in $C([0,T]; H)$, the term resulting from the control is treated in the same way like the term induced by the nonlinearity. Resumed, the preceding statements of the additional control term (3.3) in the Schrödinger problem (3.2) compared to (2.12) fit into the approaches of the existence and uniqueness results of Chapter 2.

Now, we concentrate on minimizing the objective functional

$$J(U) := \gamma E \left\|X^U(T) - y\right\|^2 + \beta E \int_0^T \|U(t) - \Upsilon(t)\|^2 \, dt \tag{3.5}$$

relative to the control $U \in \mathcal{U}$ (compare [3, 102]). Here, X^U is the variational solution of the controlled Schrödinger problem (3.2), the coefficients $\gamma, \beta \in \mathbb{R}_+$ and the functions $y \in V$ and $\Upsilon \in L^2(\Omega \times [0,T]; L^2([0,1];\mathbb{R})) \subset L^2(\Omega \times [0,T]; H)$ are given. Notice that the functional (3.5) does not only depend on the control U but also on the solution X^U of problem (3.2). Hence, minimizing the objective functional (3.5) is equivalent to finding a variational solution X^U of the controlled Schrödinger problem (3.2) which is a best approximation of a given function y with respect to the final time T, while U shall not vary too much from Υ.

Optimal control problems for solutions of partial differential equations do not always have a solution. Thus, we are first interested in the solvability of the control problem (3.5). We denote $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ and refer to an existence and uniqueness theorem that is signified in [41, 68, 90] and explicitly shown in [9].

Theorem 3.1.2. [1] *Let B be a uniformly convex Banach space and S a bounded closed subset of B. Furthermore, let $F : S \to \overline{\mathbb{R}}$ be a lower semi-continuous functional which is bounded from below and $p \geq 1$. Then there exists a dense subset $D \subset B$ such that for each $x \in D$ the functional $F(s) + \|s - x\|_B^p$ attains its minimum over S, which means that there exists an $s(x) \in S$ such that*

$$F(s(x)) + \|s(x) - x\|_B^p = \inf_{s \in S} \left\{ F(s) + \|s - x\|_B^p \right\}.$$

If $p > 1$, then $s(x)$ is unique. Besides, each minimizing sequence converges strongly and the function $x \mapsto s(x)$ is continuous on D.

Here, the Banach space $B := L^2(\Omega \times [0,T]; H) = L^2(\Omega \times [0,T] \times [0,1])$ is especially a Hilbert space and, therefore, uniformly convex. The set of all admissible controls $\mathcal{U}$ represents the subset $S \subset B$ which is bounded and closed by definition. Moreover, the functional $F := \gamma E \|X^U(T) - y\|^2$ is a mapping from $\mathcal{U}$ into $\overline{\mathbb{R}}$, bounded from below by $F \geq 0$ and $p = 2$. Hence, if we succeed in showing the lower semi-continuity of F (see Subsection 3.2.2), the above theorem states that there exists a dense subset $D \subset B$ such that for each $\Upsilon \in D$ the functional $J(U) = F(U) + \beta \|U - \Upsilon\|_B^2$ attains its unique minimum over $\mathcal{U}$. This means that there exists a unique element $U^* \in \mathcal{U}$ such that

$$J(U^*) = F(U^*) + \beta \|U^* - \Upsilon\|_B^2 = \inf_{U \in \mathcal{U}} \left\{ F(U) + \beta \|U - \Upsilon\|_B^2 \right\} = \inf_{U \in \mathcal{U}} J(U),$$

where U^* especially depends on Υ.

In the following, we consider the pathwise controlled Schrödinger problem, which results from the stochastic one perturbed by linear multiplicative noise, with power-type nonlinearity (compare Section 2.3). Thereafter, we resume the results for a nonlinear Lipschitz continuous drift term in the case of the pathwise controlled Schrödinger equation (compare Subsection 2.4.2).

3.2 Pathwise Problem with Power-Type Nonlinearity

The controlled Schrödinger problem with power-type nonlinearity $f(v) = |v|^{2\sigma} v$ for $\sigma \in (0,2)$ and linear multiplicative noise is given by

$$\begin{aligned} \big(X^U(t), v\big) &= \big(\varphi, v\big) - i \int_0^t \big\langle AX^U(s), v \big\rangle \, ds + i \int_0^t \big(U(s)X^U(s), v\big) \, ds \\ &\quad + i\lambda \int_0^t \big(f(X^U(s)), v\big) \, ds + i \sum_{j=1}^{\infty} \int_0^t b_j(s) \big(X^U(s), v\big) \, d\beta_j(s) \end{aligned} \tag{3.6}$$

[1] Bidaut [9], p. 23, Théorème 4.2

for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $v \in V$ (see Section 2.3). Without loss of generality, we first observe the case $\lambda \in \mathbb{R}_+$ that is generalized in Subsection 3.2.5. Additionally to the former assumptions, we require

$$\sum_{j=1}^{\infty} \int_0^T b_j^2(s)\,ds \leq C$$

independent of $\omega \in \Omega$, which represents a specialization of (2.58) and particularly implies that

$$B(T) = \exp\left\{ \sigma \sum_{j=1}^{\infty} \int_0^T b_j^2(s)\,ds \right\} \leq \exp\{\sigma C\} =: C_B. \tag{3.7}$$

Due to the transformation formula $Z^U(t,\,\cdot\,) = X^U(t,\,\cdot\,)Y(t)$ (compare Section 2.3), problem (3.6) is equivalent to the pathwise controlled Schrödinger problem

$$\begin{aligned} \big(Z^U(t), v\big) = \big(\varphi, v\big) - i \int_0^t \big\langle AZ^U(s), v\big\rangle\,ds + i \int_0^t \big(U(s)Z^U(s), v\big)\,ds \\ + i\lambda \int_0^t B(s)\big(f(Z^U(s)), v\big)\,ds \end{aligned} \tag{3.8}$$

for all $t \in [0,T]$ and all $v \in V$. Observe that all relations are valid for a.e. $\omega \in \Omega$ arbitrarily fixed while considering pathwise problems. As indicated in Sections 2.3 and 3.1, problem (3.8) possesses a unique variational solution $Z^U \in C([0,T];H) \cap L^2([0,T];V)$ if the admissible control satisfies $U(s,x) = U_1(s)$ or $U(s,x) = U_2(x)$.

Since we are not interested in the explicit form of constants below, we generalize the results of Section 2.3 and hereafter based on constants depending on various parameters. Therefore, we list the crucial arguments that are

- the power $\sigma \in (0,2)$ and the prefactor $\lambda \in \mathbb{R}_+$ of the nonlinearity $f : V \to H$ in (3.6), (3.8),
- the initial condition $\varphi \in V$ of the controlled Schrödinger problems (3.6), (3.8),
- the constant $\gamma \in \mathbb{R}_+$ and the function $y \in V$ given in the objective functional (3.5),
- the bounds $\alpha_1 > 0$, $\alpha_2 \geq 0$ characterizing the set of admissible controls $\mathcal{U}$ (compare (3.1)),
- the upper bound C_B of $B(T)$ given in (3.7) and the final time $T > 0$.

Applying the energy equality to (3.8), we remember the well-known relation

$$\big\|Z^U(t)\big\|^2 = \|\varphi\|^2, \qquad \text{for all } t \in [0,T]. \tag{3.9}$$

Due to the uniform a priori estimate in Theorem 2.3.5, the variational solutions Z_n^U for each $n \in \mathbb{N}$ of the corresponding Galerkin equations of the pathwise controlled Schrödinger problem (3.8) fulfill

$$\begin{aligned} \big\|Z_n^U(t)\big\|_V^2 \leq& C(\sigma)\left(\|\varphi\|_V^2 + \alpha_1 \|\varphi\|^2 + \lambda B(T)\|\varphi\|^{2(1+\sigma)} + (\lambda B(T))^{\frac{2}{2-\sigma}} \|\varphi\|^{\frac{2(2+\sigma)}{2-\sigma}} \right) \\ \leq& C(\sigma, \lambda, \varphi, \alpha_1, C_B), \qquad \text{for all } t \in [0,T]. \end{aligned}$$

Taking into account the convergence results of Theorem 2.3.8 (a) and (b), it follows that

$$\int_0^T \big\|Z^U(t)\big\|_V^2\,dt \leq \liminf_{n\to\infty} \int_0^T \big\|Z_n^U(t)\big\|_V^2\,dt \leq C(\sigma, \lambda, \varphi, \alpha_1, C_B, T) \tag{3.10}$$

and, analogously,

$$\operatorname*{ess\,sup}_{t\in[0,T]} \big\|Z^U(t)\big\|_V^2 \leq \liminf_{n\to\infty} \operatorname*{ess\,sup}_{t\in[0,T]} \big\|Z_n^U(t)\big\|_V^2 \leq C(\sigma, \lambda, \varphi, \alpha_1, C_B). \tag{3.11}$$

3.2.1 Difference of Two Controlled Schrödinger Problems

By choosing $\Theta \in [0,1]$ and $\delta U \in \mathcal{U}$, we obtain a further admissible control $U + \Theta\delta U$ that is supposed to differ slightly from $U \in \mathcal{U}$ if $\Theta > 0$ is sufficiently small. Hence, we regard the pathwise controlled Schrödinger problem corresponding to $U + \Theta\delta U \in \mathcal{U}$ constituted by

$$\begin{aligned}\big(Z^{U+\Theta\delta U}(t), v\big) = \big(\varphi, v\big) - i\int_0^t \big\langle AZ^{U+\Theta\delta U}(s), v\big\rangle\, ds + i\int_0^t \big((U+\Theta\delta U)(s)Z^{U+\Theta\delta U}(s), v\big)\, ds \\ + i\lambda\int_0^t B(s)\big(f(Z^{U+\Theta\delta U}(s)), v\big)\, ds\end{aligned} \tag{3.12}$$

for all $t \in [0,T]$ and all $v \in V$. Analogously to problem (3.8), there exists a unique variational solution $Z^{U+\Theta\delta U} \in C([0,T];H) \cap L^2([0,T];V)$ of problem (3.12) if the related admissible control $(U+\Theta\delta U)(s,x)$ depends only on the time variable $s \in [0,T]$ or only on the space variable $x \in [0,1]$. Now, we investigate $\delta Z := Z^{U+\Theta\delta U} - Z^U$ that denotes the difference of two variational solutions Z^U and $Z^{U+\Theta\delta U}$ corresponding to problems (3.8) and (3.12), respectively.

Theorem 3.2.1. *There exists a unique variational solution $\delta Z \in C([0,T];H) \cap L^2([0,T];V)$ of the pathwise Schrödinger problem*

$$\begin{aligned}\big(\delta Z(t), v\big) = &- i\int_0^t \big\langle A\delta Z(s), v\big\rangle\, ds + i\int_0^t \big((U+\Theta\delta U)(s)\delta Z(s), v\big)\, ds \\ &+ i\int_0^t \big(\Theta\delta U(s)Z^U(s), v\big)\, ds + i\lambda\int_0^t B(s)\big(f(Z^{U+\Theta\delta U}(s)) - f(Z^U(s)), v\big)\, ds\end{aligned} \tag{3.13}$$

for all $t \in [0,T]$ and all $v \in V$.

Proof. Since $Z^U, Z^{U+\Theta\delta U} \in C([0,T];H) \cap L^2([0,T];V)$ are the unique variational solutions of problems (3.8) and (3.12), it follows that the composition $\delta Z = Z^{U+\Theta\delta U} - Z^U$ also belongs to $C([0,T];H) \cap L^2([0,T];V)$ because

$$\begin{aligned}\sup_{t\in[0,T]} \|\delta Z(t)\|^2 &\le 2\sup_{t\in[0,T]} \big\|Z^{U+\Theta\delta U}(t)\big\|^2 + 2\sup_{t\in[0,T]} \big\|Z^U(t)\big\|^2, \\ \int_0^T \|\delta Z(t)\|_V^2\, dt &\le 2\int_0^T \big\|Z^{U+\Theta\delta U}(t)\big\|_V^2\, dt + 2\int_0^T \big\|Z^U(t)\big\|_V^2\, dt.\end{aligned} \tag{3.14}$$

By definition, δZ is the unique variational solution of

$$\begin{aligned}\big(\delta Z(t), v\big) &= \big(Z^{U+\Theta\delta U}(t), v\big) - \big(Z^U(t), v\big) \\ &= \big(\varphi, v\big) - i\int_0^t \big\langle AZ^{U+\Theta\delta U}(s), v\big\rangle\, ds + i\int_0^t \big((U+\Theta\delta U)(s)Z^{U+\Theta\delta U}(s), v\big)\, ds \\ &\quad + i\lambda\int_0^t B(s)\big(f(Z^{U+\Theta\delta U}(s)), v\big)\, ds - \big(\varphi, v\big) + i\int_0^t \big\langle AZ^U(s), v\big\rangle\, ds \\ &\quad - i\int_0^t \big(U(s)Z^U(s), v\big)\, ds - i\lambda\int_0^t B(s)\big(f(Z^U(s)), v\big)\, ds \\ &= - i\int_0^t \big\langle A\delta Z(s), v\big\rangle\, ds + i\int_0^t \big((U+\Theta\delta U)(s)\delta Z(s), v\big)\, ds \\ &\quad + i\int_0^t \big(\Theta\delta U(s)Z^U(s), v\big)\, ds + i\lambda\int_0^t B(s)\big(f(Z^{U+\Theta\delta U}(s)) - f(Z^U(s)), v\big)\, ds\end{aligned}$$

for all $t \in [0,T]$ and all $v \in V$, which verifies equation (3.13). □

Due to the boundedness of the admissible control $\delta U(t,x)$ from above by $\alpha_1 > 0$ P-a.s. for all $t \in [0,T]$ and all $x \in [0,1]$, we conclude the existence of $\|\delta U\|_{L^\infty([0,T]\times[0,1])}$ for a.e. $\omega \in \Omega$. Thus, we have the following continuous dependence of δZ on the control $\delta U \in \mathcal{U}$.

Theorem 3.2.2. *There exists a positive constant C such that*

$$\|\delta Z(t)\|^2 \leq C(\sigma,\lambda,\varphi,\alpha_1,C_B,T)\,\Theta^2\,\|\delta U\|^2_{L^\infty([0,T]\times[0,1])}\,, \qquad \text{for all } t\in[0,T]. \tag{3.15}$$

Proof. Initially, we apply the energy equality to (3.13) and receive

$$\begin{aligned}\|\delta Z(t)\|^2 =&\, 2\operatorname{Im}\int_0^t \big\langle A\delta Z(s),\delta Z(s)\big\rangle\,ds - 2\operatorname{Im}\int_0^t \big((U+\Theta\delta U)(s)\delta Z(s),\delta Z(s)\big)\,ds\\ &- 2\operatorname{Im}\int_0^t \big(\Theta\delta U(s)Z^U(s),\delta Z(s)\big)\,ds\\ &- 2\lambda\operatorname{Im}\int_0^t B(s)\big(f(Z^{U+\Theta\delta U}(s)) - f(Z^U(s)),\delta Z(s)\big)\,ds\end{aligned} \tag{3.16}$$

for all $t\in[0,T]$. The first two terms on the right-hand side vanish by $\operatorname{Im}\big\langle Av,v\big\rangle = 0$ for all $v\in V$ and (3.4). Based on relation (3.9), the third term on the right-hand side in (3.16) obeys

$$\begin{aligned}-2\operatorname{Im}\int_0^t \big(\Theta\delta U(s)Z^U(s),\delta Z(s)\big)\,ds &\leq 2\int_0^t \big\|\Theta\delta U(s)Z^U(s)\big\|\,\|\delta Z(s)\|\,ds\\ &\leq \int_0^t \big\|\Theta\delta U(s)Z^U(s)\big\|^2\,ds + \int_0^t \|\delta Z(s)\|^2\,ds\\ &\leq \Theta^2\,\|\delta U\|^2_{L^\infty([0,T]\times[0,1])}\int_0^T \big\|Z^U(s)\big\|^2\,ds + \int_0^t \|\delta Z(s)\|^2\,ds\\ &= \|\varphi\|^2\,T\,\Theta^2\,\|\delta U\|^2_{L^\infty([0,T]\times[0,1])} + \int_0^t \|\delta Z(s)\|^2\,ds.\end{aligned}$$

Taking into account the power-type nonlinearity $f(v) := |v|^{2\sigma}v$ with $\sigma\in(0,2)$, Lemma D.4 (a) and Lemma D.2, the last term on the right-hand side in (3.16) results in

$$\begin{aligned}&-2\lambda\operatorname{Im}\int_0^t B(s)\big(f(Z^{U+\Theta\delta U}(s)) - f(Z^U(s)),\delta Z(s)\big)\,ds\\ \leq&\, 2\lambda B(T)\int_0^t \big\|f(Z^{U+\Theta\delta U}(s)) - f(Z^U(s))\big\|\,\|\delta Z(s)\|\,ds\\ \leq&\, 5\lambda B(T)\int_0^t \left[\int_0^1 \Big(\big|Z^{U+\Theta\delta U}(s,x)\big|^{2\sigma} + \big|Z^U(s,x)\big|^{2\sigma}\Big)^2 |\delta Z(s,x)|^2\,dx\right]^{\frac{1}{2}} \|\delta Z(s)\|\,ds\\ \leq&\, 5\lambda B(T)\int_0^t \left(\sup_{x\in[0,1]}\big|Z^{U+\Theta\delta U}(s,x)\big|^{2\sigma} + \sup_{x\in[0,1]}\big|Z^U(s,x)\big|^{2\sigma}\right)\|\delta Z(s)\|^2\,ds\\ \leq&\, 5\cdot 2^\sigma\lambda B(T)\int_0^t \Big(\big\|Z^{U+\Theta\delta U}(s)\big\|_V^{2\sigma} + \big\|Z^U(s)\big\|_V^{2\sigma}\Big)\|\delta Z(s)\|^2\,ds\\ \leq&\, 5\cdot 2^\sigma\lambda B(T)\operatorname*{ess\,sup}_{t\in[0,T]}\Big(\big\|Z^{U+\Theta\delta U}(t)\big\|_V^{2\sigma} + \big\|Z^U(t)\big\|_V^{2\sigma}\Big)\int_0^t \|\delta Z(s)\|^2\,ds.\end{aligned}$$

Due to relation (3.11) applied for Z^U and $Z^{U+\Theta\delta U}$, it follows that

$$-2\lambda\operatorname{Im}\int_0^t B(s)\big(f(Z^{U+\Theta\delta U}(s)) - f(Z^U(s)),\delta Z(s)\big)\,ds \leq C(\sigma,\lambda,\varphi,\alpha_1,C_B)\int_0^t \|\delta Z(s)\|^2\,ds.$$

Consequently, it holds for all $t\in[0,T]$ that

$$\|\delta Z(t)\|^2 \leq \|\varphi\|^2\,T\,\Theta^2\,\|\delta U\|^2_{L^\infty([0,T]\times[0,1])} + [1 + C(\sigma,\lambda,\varphi,\alpha_1,C_B)]\int_0^t \|\delta Z(s)\|^2\,ds,$$

and Gronwall's lemma implies

$$\|\delta Z(t)\|^2 \leq C(\sigma,\lambda,\varphi,\alpha_1,C_B,T)\,\Theta^2\,\|\delta U\|^2_{L^\infty([0,T]\times[0,1])}\,, \qquad \text{for all } t\in[0,T]. \qquad \square$$

Introducing the space $L^2([0,T];V') := \{v \in L^2([0,T];V) : \frac{\partial}{\partial t} v \in L^2([0,T];V^*)\}$, we show that $\delta Z \in L^2([0,T];V')$ and an analogue variational formulation to (3.13) for $v \in L^2([0,T];V')$.

Theorem 3.2.3. *It holds that $\delta Z \in L^2([0,T];V')$ and, referring to problem (3.13), we get that*

$$\begin{aligned}\big(\delta Z(t), v(t)\big) = &\int_0^t \overline{\left\langle \frac{\partial}{\partial s} v(s), \delta Z(s) \right\rangle}\, ds - i \int_0^t \langle A\delta Z(s), v(s)\rangle\, ds \\ &+ i\int_0^t \big((U+\Theta\delta U)(s)\delta Z(s), v(s)\big)\, ds + i \int_0^t \big(\Theta \delta U(s) Z^U(s), v(s)\big)\, ds \\ &+ i\lambda \int_0^t B(s)\big(f(Z^{U+\Theta\delta U}(s)) - f(Z^U(s)), v(s)\big)\, ds\end{aligned} \tag{3.17}$$

for all $t \in [0,T]$ and all $v \in L^2([0,T];V')$.

Proof. The pathwise Schrödinger problem (3.13) is equivalent to the integral equation

$$\begin{aligned}\delta Z(t) = &-i\int_0^t A\delta Z(s)\, ds + i\int_0^t (U+\Theta\delta U)(s)\delta Z(s)\, ds + i \int_0^t \Theta\delta U(s) Z^U(s)\, ds \\ &+ i\lambda\int_0^t B(s)\left[f(Z^{U+\Theta\delta U}(s)) - f(Z^U(s))\right] ds\end{aligned}$$

in V^* for all $t \in [0,T]$. Observe that this equation does not possess any noise term. Hence, we formally differentiate with respect to the time t and receive the symbolic initial value problem

$$\begin{aligned}\frac{\partial}{\partial t}\delta Z(t) = &-iA\delta Z(t) + i(U+\Theta\delta U)(t)\delta Z(t) + i\Theta\delta U(t) Z^U(t) \\ &+ i\lambda B(t)\left[f(Z^{U+\Theta\delta U}(t)) - f(Z^U(t))\right], \\ \delta Z(0) = &\,0\end{aligned} \tag{3.18}$$

in V^* for all $t \in [0,T]$. Having already shown that $\delta Z \in L^2([0,T];V)$ (compare Theorem 3.2.1), it suffices to verify that $\frac{\partial}{\partial t}\delta Z \in L^2([0,T];V^*)$ in order to obtain the result $\delta Z \in L^2([0,T];V')$. Therefore, we consider the differential equation in (3.18), take the norm square in V^* and integrate over all $t \in [0,T]$ such that

$$\begin{aligned}\int_0^T \left\| \frac{\partial}{\partial t}\delta Z(t)\right\|_{V^*}^2 dt = &\int_0^T \Big\| -i\Big[A\delta Z(t) - (U+\Theta\delta U)(t)\delta Z(t) - \Theta\delta U(t) Z^U(t) \\ &- \lambda B(t)\left[f(Z^{U+\Theta\delta U}(t)) - f(Z^U(t))\right]\Big]\Big\|_{V^*}^2 dt \\ \leq &\, 8\int_0^T \Big[\|A\delta Z(t)\|_{V^*}^2 + \|(U+\Theta\delta U)(t)\delta Z(t)\|_{V^*}^2 + \left\|\Theta\delta U(t) Z^U(t)\right\|_{V^*}^2 \\ &+ \lambda^2 B^2(t)\left\|f(Z^{U+\Theta\delta U}(t)) - f(Z^U(t))\right\|_{V^*}^2\Big]\, dt.\end{aligned}$$

The continuity of the operator $A : V \to V^*$ (compare Section 2.1) and of the embedding $H \hookrightarrow V^*$ with embedding constant $\tilde{C} = C_{H,V^*}$, the boundedness of the admissible controls $U(t,x)$ and $\delta U(t,x)$ by $\alpha_1 > 0$ P-a.s. for all $t \in [0,T]$ and all $x \in [0,1]$, Lemma D.4 (a) and Lemma D.2 entail

$$\begin{aligned}\int_0^T \left\| \frac{\partial}{\partial t}\delta Z(t)\right\|_{V^*}^2 dt \leq &\, 8\int_0^T \Big[\|\delta Z(t)\|_V^2 + 2\tilde{C}^2\alpha_1^2\|\delta Z(t)\|^2 + \tilde{C}^2\Theta^2\alpha_1^2\left\|Z^U(t)\right\|^2 \\ &+ 5\cdot 2^{2\sigma}\lambda^2 B^2(T)\tilde{C}^2\left(\left\|Z^{U+\Theta\delta U}(t)\right\|_V^{4\sigma} + \left\|Z^U(t)\right\|_V^{4\sigma}\right)\|\delta Z(t)\|^2\Big]\, dt \\ \leq &\, C(\sigma, \lambda, \varphi, \alpha_1, C_B, T, \tilde{C}, \Theta)\end{aligned}$$

due to (3.9), (3.10), (3.11), (3.14) and (3.15). Thus, we conclude that $\frac{\partial}{\partial t}\delta Z \in L^2([0,T];V^*)$ and, therefore, $\delta Z \in L^2([0,T];V')$, which proves the first assertion.

To show the representation (3.17), we replace t by s in the differential equation of (3.18), multiply by $\overline{v} \in L^2([0,T];V')$, use the properties of the Gelfand triple (V,H,V^*) and integrate over all $s \in [0,t]$ such that

$$\begin{aligned}\int_0^t \left\langle \frac{\partial}{\partial s}\delta Z(s), v(s)\right\rangle ds = &- i\int_0^t \langle A\delta Z(s), v(s)\rangle\, ds + i\int_0^t \big((U+\Theta\delta U)(s)\delta Z(s), v(s)\big)\, ds \\ &+ i\int_0^t \big(\Theta\delta U(s) Z^U(s), v(s)\big)\, ds \\ &+ i\lambda\int_0^t B(s)\big(f(Z^{U+\Theta\delta U}(s)) - f(Z^U(s)), v(s)\big)\, ds\end{aligned} \tag{3.19}$$

for all $t \in [0,T]$ and all $v \in L^2([0,T];V')$. Notice that δZ and v are measurable in time and space and that

$$\begin{aligned}\int_0^t \left\langle \frac{\partial}{\partial s}\delta Z(s), v(s)\right\rangle ds &\le \int_0^t \left|\left\langle \frac{\partial}{\partial s}\delta Z(s), v(s)\right\rangle\right| ds \le \int_0^t \left\|\frac{\partial}{\partial s}\delta Z(s)\right\|_{V^*} \|v(s)\|_V\, ds \\ &\le \frac{1}{2}\int_0^t \left\|\frac{\partial}{\partial s}\delta Z(s)\right\|_{V^*}^2 ds + \frac{1}{2}\int_0^t \|v(s)\|_V^2\, ds < \infty\end{aligned}$$

since $\delta Z, v \in L^2([0,T];V')$. Hence, the order of integration can be changed (by Tonelli's theorem) and integration by parts with respect to the time variable yields

$$\int_0^t \left\langle \frac{\partial}{\partial s}\delta Z(s), v(s)\right\rangle ds = \big(\delta Z(t), v(t)\big) - \big(\delta Z(0), v(0)\big) - \int_0^t \overline{\left\langle \frac{\partial}{\partial s} v(s), \delta Z(s)\right\rangle}\, ds.$$

Regarding that $\delta Z(0) = 0$, relation (3.19) implies the representation (3.17) given by

$$\begin{aligned}\big(\delta Z(t), v(t)\big) = &\int_0^t \overline{\left\langle \frac{\partial}{\partial s} v(s), \delta Z(s)\right\rangle}\, ds - i\int_0^t \langle A\delta Z(s), v(s)\rangle\, ds \\ &+ i\int_0^t \big((U+\Theta\delta U)(s)\delta Z(s), v(s)\big)\, ds + i\int_0^t \big(\Theta\delta U(s) Z^U(s), v(s)\big)\, ds \\ &+ i\lambda\int_0^t B(s)\big(f(Z^{U+\Theta\delta U}(s)) - f(Z^U(s)), v(s)\big)\, ds\end{aligned}$$

for all $t \in [0,T]$ and all $v \in L^2([0,T];V')$. Besides, this equation coincides with the variational formulation (3.13) by identifying $v \in L^2([0,T];V')$ with $v \in V$. □

3.2.2 Continuity of the Objective Functional

Since we deal with the pathwise controlled Schrödinger problem (3.8) in this section, it is useful to consider the pathwise analogue of the objective functional (3.5). Due to the transformation formula $Z^U(t,\cdot) = X^U(t,\cdot)Y(t)$, it is given by

$$J(U) = \gamma E\left\|\frac{Z^U(T)}{Y(T)} - y\right\|^2 + \beta E\int_0^T \|U(t) - \Upsilon(t)\|^2\, dt, \qquad \text{for all } U \in \mathcal{U}. \tag{3.20}$$

Referring to Theorem 3.1.2 and its subsequent deliberations, we only have to show the lower semi-continuity of

$$F(U) := \gamma E\left\|\frac{Z^U(T)}{Y(T)} - y\right\|^2, \qquad \text{for all } U \in \mathcal{U}, \tag{3.21}$$

to ensure the unique existence of an optimal control. Thus, by verifying the continuity of F, its lower semi-continuity is especially satisfied.

Theorem 3.2.4. *The functional* $F : \mathcal{U} \to \overline{\mathbb{R}}_+$*, defined by* (3.21)*, is continuous in* $\mathcal{U}$*.*

Proof. Due to definition (3.1) of the set $\mathcal{U}$ of admissible controls, we establish that for $U, \delta U \in \mathcal{U}$ with

$$|\delta U| := \|\delta U\|_{L^\infty(\Omega\times[0,T]\times[0,1])} + \left\|\frac{\partial}{\partial x}\delta U\right\|_{L^\infty(\Omega\times[0,T]\times[0,1])} \tag{3.22}$$

it holds that

$$\lim_{|\delta U|\to 0} |F(U+\delta U) - F(U)| = 0.$$

Therefore, we consider

$$\begin{aligned} F(U+\delta U) - F(U) =& \gamma E \left\| \frac{Z^{U+\delta U}(T)}{Y(T)} - y \right\|^2 - \gamma E \left\| \frac{Z^U(T)}{Y(T)} - y \right\|^2 \\ =& \gamma E \left(\left\| \frac{Z^{U+\delta U}(T)}{Y(T)} - y \right\|^2 - \left\| \frac{Z^U(T)}{Y(T)} - y \right\|^2 \right). \end{aligned}$$

Because of $\|u\|^2 - \|v\|^2 = \operatorname{Re}\big(u-v, u+v\big)$ for all $u, v \in H$ and the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} F(U+\delta U) - F(U) =& \gamma E \operatorname{Re}\left(\frac{Z^{U+\delta U}(T)}{Y(T)} - \frac{Z^U(T)}{Y(T)}, \frac{Z^{U+\delta U}(T)}{Y(T)} + \frac{Z^U(T)}{Y(T)} - 2y \right) \\ \le& \gamma \left(E \left\| \frac{Z^{U+\delta U}(T) - Z^U(T)}{Y(T)} \right\|^2 \right)^{\frac{1}{2}} \left(E \left\| \frac{Z^{U+\delta U}(T)}{Y(T)} + \frac{Z^U(T)}{Y(T)} - 2y \right\|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Choosing $\Theta = 1$ and $\delta U \in \mathcal{U}$ such that $|\delta U| \to 0$, then $U + \delta U \in \mathcal{U}$ and we can state an analogue result as in Theorem 3.2.2 for $\Lambda Z := Z^{U+\delta U} - Z^U$. Thus, it follows that

$$\begin{aligned} E \left\| \frac{Z^{U+\delta U}(T) - Z^U(T)}{Y(T)} \right\|^2 &= E\left(B^{\frac{1}{\sigma}}(T)\, \|\Lambda Z(T)\|^2 \right) \\ &\le C(\sigma, \lambda, \varphi, \alpha_1, C_B, T)\, \|\delta U\|^2_{L^\infty(\Omega\times[0,T]\times[0,1])} \\ &\le C(\sigma, \lambda, \varphi, \alpha_1, C_B, T) |\delta U|^2. \end{aligned}$$

Moreover, equality (3.9) entails

$$\begin{aligned} E \left\| \frac{Z^{U+\delta U}(T)}{Y(T)} + \frac{Z^U(T)}{Y(T)} - 2y \right\|^2 \le& 4E \left\| \frac{Z^{U+\delta U}(T)}{Y(T)} \right\|^2 + 4E \left\| \frac{Z^U(T)}{Y(T)} \right\|^2 + 2E\|2y\|^2 \\ =& 4B^{\frac{1}{\sigma}}(T) \left[E\left\|Z^{U+\delta U}(T)\right\|^2 + E\left\|Z^U(T)\right\|^2 \right] + 8E\|y\|^2 \\ =& 8B^{\frac{1}{\sigma}}(T)\, \|\varphi\|^2 + 8\,\|y\|^2 \le C(\sigma, \varphi, y, C_B). \end{aligned}$$

By taking the square roots of the last two estimates, we deduce

$$0 \le \lim_{|\delta U|\to 0} |F(U+\delta U) - F(U)| \le \lim_{|\delta U|\to 0} C(\sigma, \lambda, \varphi, \gamma, y, \alpha_1, C_B, T)|\delta U| = 0. \qquad \square$$

For the sake of completeness, we further state the continuity of the objective functional (3.20).

Theorem 3.2.5. *The objective functional* $J : \mathcal{U} \to \overline{\mathbb{R}}_+$*, defined by* (3.20)*, is continuous in* $\mathcal{U}$*.*

Proof. Using notations (3.21) and (3.22), we obtain for $U, \delta U \in \mathcal{U}$ that

$$J(U+\delta U) - J(U) = F(U+\delta U) - F(U) + \beta E \int_0^T \left(\|(U+\delta U)(t) - \Upsilon(t)\|^2 - \|U(t) - \Upsilon(t)\|^2 \right) dt.$$

Taking into account that $U, \delta U$ and Υ are real-vaued and due to $\|u\|^2 - \|v\|^2 = \operatorname{Re}\big(u - v, u + v\big)$ for all $u, v \in H$ and the Cauchy-Schwarz inequality, the last term obeys

$$\begin{aligned}
&\beta E \int_0^T \Big(\|(U + \delta U)(t) - \Upsilon(t)\|^2 - \|U(t) - \Upsilon(t)\|^2 \Big)\, dt \\
=&\beta E \int_0^T \|\delta U(t)\|^2\, dt + 2\beta E \int_0^T \big(\delta U(t), U(t) - \Upsilon(t)\big)\, dt \\
\leq &\beta E \int_0^T \|\delta U(t)\|^2\, dt + 2\beta \left(E \int_0^T \|\delta U(t)\|^2\, dt \right)^{\frac{1}{2}} \left(E \int_0^T \|U(t) - \Upsilon(t)\|^2\, dt \right)^{\frac{1}{2}}.
\end{aligned}$$

The triangle inequality, the boundedness of $U(t,x)$ from above by $\alpha_1 > 0$ P-a.s. for all $t \in [0,T]$ and all $x \in [0,1]$ and the fact that $\Upsilon \in L^2(\Omega \times [0,T]; L^2([0,1];\mathbb{R})) \subset L^2(\Omega \times [0,T]; H)$ lead to

$$K_{U,\Upsilon}^2 := E \int_0^T \|U(t) - \Upsilon(t)\|^2\, dt \leq 2E \int_0^T \|U(t)\|^2\, dt + 2E \int_0^T \|\Upsilon(t)\|^2\, dt < \infty.$$

Hence, based on the continuous embedding $L^\infty(\Omega \times [0,T] \times [0,1]) \hookrightarrow L^2(\Omega \times [0,T] \times [0,1])$ with embedding constant $\overline{C}$, it follows that

$$\begin{aligned}
0 \leq &\lim_{|\delta U| \to 0} \left| \beta E \int_0^T \Big(\|(U + \delta U)(t) - \Upsilon(t)\|^2 - \|U(t) - \Upsilon(t)\|^2 \Big)\, dt \right| \\
\leq &\lim_{|\delta U| \to 0} \Big(\beta \|\delta U\|^2_{L^2(\Omega \times [0,T] \times [0,1])} + 2\beta K_{U,\Upsilon} \|\delta U\|_{L^2(\Omega \times [0,T] \times [0,1])} \Big) \\
\leq &\lim_{|\delta U| \to 0} \Big(\beta \overline{C}^2 \|\delta U\|^2_{L^\infty(\Omega \times [0,T] \times [0,1])} + 2\beta K_{U,\Upsilon} \overline{C} \|\delta U\|_{L^\infty(\Omega \times [0,T] \times [0,1])} \Big) \\
\leq &\lim_{|\delta U| \to 0} \Big(\beta \overline{C}^2 |\delta U|^2 + 2\beta K_{U,\Upsilon} \overline{C}\, |\delta U| \Big) = 0.
\end{aligned}$$

Together with the result of Theorem 3.2.4, we get

$$\lim_{|\delta U| \to 0} |J(U + \delta U) - J(U)| = 0. \qquad \square$$

The continuity result in Theorem 3.2.4 particularly implies the lower semi-continuity of the functional F. Thus, all assumptions of Theorem 3.1.2 are fulfilled and we deduce the existence of a unique element $U^* \in \mathcal{U}$ that minimizes the objective functional (3.20). Therefore, it is worth asking for a necessary condition of an optimal control that requires the complex conjugated version of the adjoint Schrödinger problem.

3.2.3 Complex Conjugated Adjoint Schrödinger Problem

Now, we investigate the complex conjugated adjoint problem of the pathwise controlled Schrödinger problem (3.8) and deduce its appropriate concept of solution. Notice that, in general, it is not possible to establish the adjoint equation of each stochastic partial differential equation with the present approach (see Remark I.1). That is the reason why we exclude general multiplicative noise here. Choosing again $\omega \in \Omega$ arbitrarily fixed such that each relation holds for a.e. $\omega \in \Omega$, the variable $\Phi^U : \Omega \times [0,T] \times [0,1] \to \mathbb{C}$ has to fulfill the complex conjugated adjoint Schrödinger problem of (3.8) given by

$$\begin{aligned}
\frac{\partial}{\partial t} \Phi^U(t) =& - iA\Phi^U(t) + iU(t)\Phi^U(t) + i\lambda(\sigma + 1)B(t)\big|Z^U(t)\big|^{2\sigma} \Phi^U(t) \\
&+ i\lambda\sigma B(t)\big|Z^U(t)\big|^{2(\sigma-1)} \big(Z^U(t)\big)^2\, \overline{\Phi^U}(t), \\
\Phi^U(T) =& - 2i\gamma \frac{1}{\overline{Y}(T)} \left[\frac{Z^U(T)}{Y(T)} - y \right]
\end{aligned} \tag{3.23}$$

in V^* for all $t \in [0,T]$ (compare Appendix I). The complex conjugated adjoint Schrödinger equation and its appropriate final condition, which is adjusted to the considered objective functional, can be obtained by the method of Lagrange multipliers (compare for example [95, pp. 96 f.]). Integration by parts motivates the concept of solution of the final value problem (3.23) in the following sense.

Definition 3.2.6. *A process $\Phi^U \in C([0,T];H) \cap L^2([0,T];V)$ is called a variational solution of the complex conjugated adjoint Schrödinger problem* (3.23) *if it fulfills*

$$\begin{aligned}\big(\Phi^U(t),v\big) =& -2i\gamma \frac{1}{\overline{Y}(T)}\bigg(\frac{Z^U(T)}{Y(T)}-y,v\bigg)+i\int_t^T \big\langle A\Phi^U(s),v\big\rangle\,ds\\ &-i\int_t^T \big(U(s)\Phi^U(s),v\big)\,ds - i\lambda(\sigma+1)\int_t^T B(s)\big(\big|Z^U(s)\big|^{2\sigma}\Phi^U(s),v\big)\,ds\\ &-i\lambda\sigma\int_t^T B(s)\big(\big|Z^U(s)\big|^{2(\sigma-1)}\big(Z^U(s)\big)^2\,\overline{\Phi^U}(s),v\big)\,ds\end{aligned}\tag{3.24}$$

for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $v \in V$.

The next aim is to show the unique existence of the variational solution of (3.24) (for a.e. $\omega \in \Omega$ arbitrarily fixed). At first, we are concerned with its uniqueness.

Theorem 3.2.7. *If $\Phi^U \in C([0,T];H) \cap L^2([0,T];V)$ is a variational solution of the complex conjugated adjoint Schrödinger problem* (3.24)*, then it is unique.*

Proof. Assume that there are two variational solutions $\Phi_1^U, \Phi_2^U \in C([0,T];H) \cap L^2([0,T];V)$ of problem (3.24). Thus, by denoting $\delta\Phi^U := \Phi_1^U - \Phi_2^U$ and regarding that $\delta\Phi^U(T) = 0$, we get

$$\begin{aligned}\big(\delta\Phi^U(t),v\big) =& \, i\int_t^T \big\langle A\delta\Phi^U(s),v\big\rangle\,ds - i\int_t^T \big(U(s)\delta\Phi^U(s),v\big)\,ds\\ &-i\lambda(\sigma+1)\int_t^T B(s)\big(\big|Z^U(s)\big|^{2\sigma}\delta\Phi^U(s),v\big)\,ds\\ &-i\lambda\sigma\int_t^T B(s)\big(\big|Z^U(s)\big|^{2(\sigma-1)}\big(Z^U(s)\big)^2\,\overline{\delta\Phi^U}(s),v\big)\,ds\end{aligned}$$

for all $t \in [0,T]$ and all $v \in V$. The application of the energy equality yields

$$\begin{aligned}\big\|\delta\Phi^U(t)\big\|^2 =& -2\,\mathrm{Im}\int_t^T \big\langle A\delta\Phi^U(s),\delta\Phi^U(s)\big\rangle\,ds + 2\,\mathrm{Im}\int_t^T \big(U(s)\delta\Phi^U(s),\delta\Phi^U(s)\big)\,ds\\ &+2\lambda(\sigma+1)\,\mathrm{Im}\int_t^T B(s)\big(\big|Z^U(s)\big|^{2\sigma}\delta\Phi^U(s),\delta\Phi^U(s)\big)\,ds\\ &+2\lambda\sigma\,\mathrm{Im}\int_t^T B(s)\big(\big|Z^U(s)\big|^{2(\sigma-1)}\big(Z^U(s)\big)^2\,\overline{\delta\Phi^U}(s),\delta\Phi^U(s)\big)\,ds\end{aligned}$$

for all $t \in [0,T]$. Writing the imaginary parts under the integral signs and observing that $\mathrm{Im}\big\langle Av(s),v(s)\big\rangle = 0$ for all $v(s) \in V$ and $\mathrm{Im}\big\{h(s)\overline{h}(s)\big\} = \mathrm{Im}\,|h(s)|^2 = 0$ for all $h(s) \in H$, the first three imaginary parts on the right-hand side vanish and it only remains

$$\begin{aligned}\big\|\delta\Phi^U(t)\big\|^2 =& \,2\lambda\sigma\int_t^T B(s)\,\mathrm{Im}\Big\{\big(\big|Z^U(s)\big|^{2(\sigma-1)}\big(Z^U(s)\big)^2\,\overline{\delta\Phi^U}(s),\delta\Phi^U(s)\big)\Big\}\,ds\\ \le& \,2\lambda\sigma\int_t^T B(s)\,\Big\|\big|Z^U(s)\big|^{2(\sigma-1)}\big(Z^U(s)\big)^2\,\overline{\delta\Phi^U}(s)\Big\|\,\big\|\delta\Phi^U(s)\big\|\,ds\\ \le& \,\lambda^2\sigma^2\int_t^T B^2(s)\,\Big\|\big|Z^U(s)\big|^{2(\sigma-1)}\big(Z^U(s)\big)^2\,\overline{\delta\Phi^U}(s)\Big\|^2\,ds + \int_t^T \big\|\delta\Phi^U(s)\big\|^2\,ds\end{aligned}$$

for all $t \in [0,T]$. Using Lemma D.2, it follows that

$$\begin{aligned} \left\| \left|Z^U(s)\right|^{2(\sigma-1)} \left(Z^U(s)\right)^2 \overline{\delta\Phi^U}(s) \right\|^2 &= \int_0^1 \left|Z^U(s,x)\right|^{4\sigma} \left|\delta\Phi^U(s,x)\right|^2 dx \\ \leq \left(\sup_{x\in[0,1]} \left|Z^U(s,x)\right|^{4\sigma} \right) \left\|\delta\Phi^U(s)\right\|^2 &\leq 2^{2\sigma} \left\|Z^U(s)\right\|_V^{4\sigma} \left\|\delta\Phi^U(s)\right\|^2, \end{aligned}$$

which results in

$$\left\|\delta\Phi^U(t)\right\|^2 \leq \int_t^T \left(1 + 2^{2\sigma}\lambda^2\sigma^2 B^2(s) \left\|Z^U(s)\right\|_V^{4\sigma}\right) \left\|\delta\Phi^U(s)\right\|^2 ds, \qquad \text{for all } t \in [0,T].$$

Assuming that the variational solutions $\Phi_1^U, \Phi_2^U \in C([0,T];H) \cap L^2([0,T];V)$, it also holds that $\delta\Phi^U \in C([0,T];H) \cap L^2([0,T];V)$ and, therefore,

$$\left\|\delta\Phi^U(t)\right\|^2 \leq \int_0^T \left(1 + 2^{2\sigma}\lambda^2\sigma^2 B^2(s) \left\|Z^U(s)\right\|_V^{4\sigma}\right) \sup_{r\in[0,s]} \left\|\delta\Phi^U(r)\right\|^2 ds$$

such that the right-hand side is independent of t. Hence, we take the supremum over all $t \in [0,T]$ and obtain

$$\sup_{t\in[0,T]} \left\|\delta\Phi^U(t)\right\|^2 \leq \int_0^T \left(1 + 2^{2\sigma}\lambda^2\sigma^2 B^2(s) \left\|Z^U(s)\right\|_V^{4\sigma}\right) \sup_{r\in[0,s]} \left\|\delta\Phi^U(r)\right\|^2 ds.$$

Finally, Gronwall's lemma implies that

$$\left\|\delta\Phi^U(t)\right\|^2 \leq \sup_{t\in[0,T]} \left\|\delta\Phi^U(t)\right\|^2 = 0, \qquad \text{for all } t \in [0,T],$$

and, consequently, $\delta\Phi^U(t) = \Phi_1^U(t) - \Phi_2^U(t) = 0$ for all $t \in [0,T]$. □

To show the existence of the variational solution $\Phi^U \in C([0,T];H) \cap L^2([0,T];V)$ of the complex conjugated adjoint Schrödinger problem (3.24), we proceed with the corresponding Galerkin equations for each $n \in \mathbb{N}$ which are given by

$$\begin{aligned} \left(\Phi_n^U(t), h_k\right) = &- 2i\gamma \frac{1}{\overline{Y}(T)} \left(\pi_n \left\{ \frac{Z^U(T)}{Y(T)} - y \right\}, h_k \right) + i \int_t^T \left\langle A\Phi_n^U(s), h_k \right\rangle ds \\ &- i \int_t^T \left(\pi_n \left\{ U(s)\Phi_n^U(s) \right\}, h_k \right) ds \\ &- i\lambda(\sigma+1) \int_t^T B(s) \left(\pi_n \left\{ \left|Z^U(s)\right|^{2\sigma} \Phi_n^U(s) \right\}, h_k \right) ds \\ &- i\lambda\sigma \int_t^T B(s) \left(\pi_n \left\{ \left|Z^U(s)\right|^{2(\sigma-1)} \left(Z^U(s)\right)^2 \overline{\Phi_n^U}(s) \right\}, h_k \right) ds \end{aligned} \tag{3.25}$$

for all $t \in [0,T]$ and all $k \in \{1,2,\ldots,n\}$. Observe that this finite-dimensional system is equivalent to the integral equation

$$\begin{aligned} \Phi_n^U(t) = &- 2i\gamma \frac{1}{\overline{Y}(T)} \pi_n \left\{ \frac{Z^U(T)}{Y(T)} - y \right\} + i \int_t^T A\Phi_n^U(s)\, ds - i \int_t^T \pi_n \left\{ U(s)\Phi_n^U(s) \right\} ds \\ &- i\lambda(\sigma+1) \int_t^T B(s)\, \pi_n \left\{ \left|Z^U(s)\right|^{2\sigma} \Phi_n^U(s) \right\} ds \\ &- i\lambda\sigma \int_t^T B(s)\, \pi_n \left\{ \left|Z^U(s)\right|^{2(\sigma-1)} \left(Z^U(s)\right)^2 \overline{\Phi_n^U}(s) \right\} ds \end{aligned}$$

in V^* for all $t \in [0,T]$. Since it does not include any noise term, we differentiate this equation with respect to the time variable t and obtain

$$
\begin{aligned}
\frac{\partial}{\partial t}\Phi_n^U(t) =& - iA\Phi_n^U(t) + i\,\pi_n\left\{U(t)\Phi_n^U(t)\right\} + i\lambda(\sigma+1)B(t)\,\pi_n\left\{\left|Z^U(t)\right|^{2\sigma}\Phi_n^U(t)\right\}\\
&+ i\lambda\sigma B(t)\,\pi_n\left\{\left|Z^U(t)\right|^{2(\sigma-1)}\left(Z^U(t)\right)^2\overline{\Phi_n^U}(t)\right\},\\
\Phi_n^U(T) =& - 2i\gamma\,\frac{1}{\overline{Y}(T)}\,\pi_n\left\{\frac{Z^U(T)}{Y(T)} - y\right\}
\end{aligned}
\tag{3.26}
$$

in V^* for all $t \in [0,T]$. Plugging in the series representations of the orthogonal projection π_n (see (2.2)) and of the Galerkin approximations

$$
\Phi_n^U(t) := \sum_{k=1}^{n} c_{nk}(t)\,h_k \in H_n, \qquad \text{for all } t \in [0,T] \text{ and all } n \in \mathbb{N}, \tag{3.27}
$$

where $c_{nk}(t) := \left(\Phi_n^U(t), h_k\right)$ for all $k = 1,2,\dots,n$, and splitting $(c_{nk})_{k=1,2,\dots,n}$ in real and imaginary part, problem (3.26) represents a $2n$-dimensional linear homogeneous system of ordinary differential equations with bounded coefficients and appropriate final value conditions. Such a system possesses exactly one $2n$-dimensional solution in $C^1([0,T];H)$ that are the real and imaginary parts of $(c_{nk})_{k=1,2,\dots,n}$. These coefficients

$$
c_{nk}(t) := \mathrm{Re}\{c_{nk}(t)\} + i\,\mathrm{Im}\{c_{nk}(t)\}, \qquad \text{for all } k = 1,2,\dots,n,
$$

can finally be composed in form of (3.27) to the unique solution $\Phi_n^U \in C^1([0,T];H)$ of the Galerkin equations (3.25). Now, we state uniform a priori estimates of the Galerkin approximations Φ_n^U in H and V to deduce that the solution is a variational one.

Theorem 3.2.8. *Let $n \in \mathbb{N}$ be arbitrarily fixed. Then there exists a positive constant C such that*

$$
\left\|\Phi_n^U(t)\right\|^2 \le C(\sigma,\lambda,\varphi,\gamma,y,\alpha_1,C_B,T), \qquad \textit{for all } t \in [0,T]. \tag{3.28}
$$

Proof. Initially, observe that problem (3.26) (given in differential form) and problem (3.25) (stated in integral form) are equivalent. Thus, we multiply the differential equation in (3.26) by $\overline{\Phi_n^U}(t)$ and use the properties of the Gelfand triple (V,H,V^*) such that

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}\Phi_n^U(t), \Phi_n^U(t)\right) =& - i\langle A\Phi_n^U(t), \Phi_n^U(t)\rangle + i\left(\pi_n\left\{U(t)\Phi_n^U(t)\right\}, \Phi_n^U(t)\right)\\
&+ i\lambda(\sigma+1)B(t)\left(\pi_n\left\{\left|Z^U(t)\right|^{2\sigma}\Phi_n^U(t)\right\}, \Phi_n^U(t)\right)\\
&+ i\lambda\sigma B(t)\left(\pi_n\left\{\left|Z^U(t)\right|^{2(\sigma-1)}\left(Z^U(t)\right)^2\overline{\Phi_n^U}(t)\right\}, \Phi_n^U(t)\right).
\end{aligned}
$$

Notice that each relation in this proof holds for all $t \in [0,T]$. By the first equality in (2.3), we get

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}\Phi_n^U(t), \Phi_n^U(t)\right) =& - i\langle A\Phi_n^U(t), \Phi_n^U(t)\rangle + i\left(U(t)\Phi_n^U(t), \Phi_n^U(t)\right)\\
&+ i\lambda(\sigma+1)B(t)\left(\left|Z^U(t)\right|^{2\sigma}\Phi_n^U(t), \Phi_n^U(t)\right)\\
&+ i\lambda\sigma B(t)\left(\left|Z^U(t)\right|^{2(\sigma-1)}\left(Z^U(t)\right)^2\overline{\Phi_n^U}(t), \Phi_n^U(t)\right).
\end{aligned}
$$

Taking the real part of this equation and observe that $\mathrm{Re}\{iz\} = \mathrm{Im}\{z\}$ for all $z \in \mathbb{C}$ and

$$
\mathrm{Re}\left(\frac{\partial}{\partial t}\Phi_n^U(t), \Phi_n^U(t)\right) = \frac{1}{2}\left[\left(\frac{\partial}{\partial t}\Phi_n^U(t), \Phi_n^U(t)\right) + \left(\Phi_n^U(t), \frac{\partial}{\partial t}\Phi_n^U(t)\right)\right] = \frac{1}{2}\frac{d}{dt}\left\|\Phi_n^U(t)\right\|^2
$$

since the representation of $\Phi_n^U(t)$ is separated in time and space, it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\Phi_n^U(t)\right\|^2 &= -\operatorname{Im}\left\langle A\Phi_n^U(t),\Phi_n^U(t)\right\rangle + \operatorname{Im}\left(U(t)\Phi_n^U(t),\Phi_n^U(t)\right)\\ &\quad + \lambda(\sigma+1)B(t)\operatorname{Im}\left(\left|Z^U(t)\right|^{2\sigma}\Phi_n^U(t),\Phi_n^U(t)\right)\\ &\quad + \lambda\sigma B(t)\operatorname{Im}\left(\left|Z^U(t)\right|^{2(\sigma-1)}\left(Z^U(t)\right)^2\overline{\Phi_n^U}(t),\Phi_n^U(t)\right).\end{aligned}$$

Like in the proof of uniqueness, the first three imaginary parts on the right-hand side vanish and

$$\begin{aligned}\frac{d}{dt}\left\|\Phi_n^U(t)\right\|^2 &= 2\lambda\sigma B(t)\operatorname{Im}\int_0^1\left|Z^U(t,x)\right|^{2(\sigma-1)}\left(Z^U(t,x)\right)^2\left(\overline{\Phi_n^U}(t,x)\right)^2dx\\ &= 2\lambda\sigma B(t)\int_0^1\left|Z^U(t,x)\right|^{2(\sigma-1)}\operatorname{Im}\left\{\left(Z^U(t,x)\right)^2\left(\overline{\Phi_n^U}(t,x)\right)^2\right\}dx\\ &=: 2\lambda\sigma B(t)\int_0^1\left|Z^U(t,x)\right|^{2(\sigma-1)}\operatorname{Im}\left\{\Gamma(t,x)\right\}dx.\end{aligned}$$

Renaming t by s and integrating over all $s\in[t,T]$, we deduce

$$\left\|\Phi_n^U(T)\right\|^2-\left\|\Phi_n^U(t)\right\|^2 = 2\lambda\sigma\int_t^T B(s)\int_0^1\left|Z^U(s,x)\right|^{2(\sigma-1)}\operatorname{Im}\left\{\Gamma(s,x)\right\}dx\,ds.$$

With the help of the substitutions

$$s := T-r \qquad \text{and} \qquad \tilde{g}(T-t) := g(T-(T-t)) = g(t) \tag{3.29}$$

for functions $g:[0,T]\to\mathbb{C}$ such that

$$\int_t^T g(s)\,ds = -\int_{T-t}^0 g(T-r)\,dr = \int_0^{T-t} g(T-r)\,dr = \int_0^{T-t}\tilde{g}(r)\,dr, \tag{3.30}$$

we rearrange the final value problem

$$\begin{aligned}\left\|\Phi_n^U(t)\right\|^2 &= \left\|\Phi_n^U(T)\right\|^2 - 2\lambda\sigma\int_t^T B(s)\int_0^1\left|Z^U(s,x)\right|^{2(\sigma-1)}\operatorname{Im}\left\{\Gamma(s,x)\right\}dx\,ds,\\ \left\|\Phi_n^U(T)\right\|^2 &= 4\gamma^2\frac{1}{|Y(T)|^2}\left\|\pi_n\left\{\frac{Z^U(T)}{Y(T)}-y\right\}\right\|^2\end{aligned}$$

into the initial value problem

$$\begin{aligned}\left\|\tilde{\Phi}_n^U(T-t)\right\|^2 &= \left\|\tilde{\Phi}_n^U(0)\right\|^2 - 2\lambda\sigma\int_0^{T-t}\tilde{B}(r)\int_0^1\left|\tilde{Z}^U(r,x)\right|^{2(\sigma-1)}\operatorname{Im}\left\{\tilde{\Gamma}(r,x)\right\}dx\,dr,\\ \left\|\tilde{\Phi}_n^U(0)\right\|^2 &= 4\gamma^2\frac{1}{\left|\tilde{Y}(0)\right|^2}\left\|\pi_n\left\{\frac{\tilde{Z}^U(0)}{\tilde{Y}(0)}-y\right\}\right\|^2.\end{aligned}$$

Remembering that $\tilde{\Gamma}(r,x) = \left(\tilde{Z}^U(r,x)\right)^2\left(\overline{\tilde{\Phi}_n^U}(r,x)\right)^2$ and referring to Lemma D.2, we estimate from above by

$$\begin{aligned}\left\|\tilde{\Phi}_n^U(T-t)\right\|^2 &\le \left\|\tilde{\Phi}_n^U(0)\right\|^2 + 2\lambda\sigma\int_0^{T-t}\tilde{B}(r)\int_0^1\left|\tilde{Z}^U(r,x)\right|^{2\sigma}\left|\tilde{\Phi}_n^U(r,x)\right|^2dx\,dr\\ &\le \left\|\tilde{\Phi}_n^U(0)\right\|^2 + 2\lambda\sigma\tilde{B}(0)\int_0^{T-t}\left(\sup_{x\in[0,1]}\left|\tilde{Z}^U(r,x)\right|^{2\sigma}\right)\left\|\tilde{\Phi}_n^U(r)\right\|^2dr\\ &\le \left\|\tilde{\Phi}_n^U(0)\right\|^2 + 2^{\sigma+1}\lambda\sigma\tilde{B}(0)\int_0^{T-t}\left\|\tilde{Z}^U(r)\right\|_V^{2\sigma}\left\|\tilde{\Phi}_n^U(r)\right\|^2dr.\end{aligned}$$

Because of $\tilde{B}(0) = B(T) \le C_B$ (compare (3.7)) and relation (3.11), which particularly holds for the time transformed variable $\tilde{Z}^U$ since

$$\operatorname*{ess\,sup}_{r\in[0,T-t]} \left\|\tilde{Z}^U(r)\right\|_V^{2\sigma} = \operatorname*{ess\,sup}_{s\in[t,T]} \left\|Z^U(s)\right\|_V^{2\sigma} \le \operatorname*{ess\,sup}_{s\in[0,T]} \left\|Z^U(s)\right\|_V^{2\sigma} \le C(\sigma,\lambda,\varphi,\alpha_1,C_B), \tag{3.31}$$

we further get

$$\begin{aligned}\left\|\tilde{\Phi}_n^U(T-t)\right\|^2 &\le \left\|\tilde{\Phi}_n^U(0)\right\|^2 + 2^{\sigma+1}\lambda\sigma\tilde{B}(0)\left(\operatorname*{ess\,sup}_{r\in[0,T-t]} \left\|\tilde{Z}^U(r)\right\|_V^{2\sigma}\right)\int_0^{T-t} \left\|\tilde{\Phi}_n^U(r)\right\|^2 dr\\ &\le \left\|\tilde{\Phi}_n^U(0)\right\|^2 + C(\sigma,\lambda,\varphi,\alpha_1,C_B)\int_0^{T-t} \left\|\tilde{\Phi}_n^U(r)\right\|^2 dr.\end{aligned}$$

With Gronwall's lemma we conclude

$$\left\|\tilde{\Phi}_n^U(T-t)\right\|^2 \le \left\|\tilde{\Phi}_n^U(0)\right\|^2 C(\sigma,\lambda,\varphi,\alpha_1,C_B,T)$$

such that resubstitution yields

$$\left\|\Phi_n^U(t)\right\|^2 \le \left\|\Phi_n^U(T)\right\|^2 C(\sigma,\lambda,\varphi,\alpha_1,C_B,T).$$

Due to the second relation in (2.3) and equality (3.9), it follows that

$$\begin{aligned}\left\|\Phi_n^U(T)\right\|^2 &= 4\gamma^2\frac{1}{|Y(T)|^2}\left\|\pi_n\left\{\frac{Z^U(T)}{Y(T)} - y\right\}\right\|^2 \le 4\gamma^2 B^{\frac{1}{\sigma}}(T)\left\|\frac{Z^U(T)}{Y(T)} - y\right\|^2\\ &\le 8\gamma^2 B^{\frac{1}{\sigma}}(T)\left(\left\|\frac{Z^U(T)}{Y(T)}\right\|^2 + \|y\|^2\right) = 8\gamma^2 B^{\frac{1}{\sigma}}(T)\left(B^{\frac{1}{\sigma}}(T)\left\|\varphi\right\|^2 + \|y\|^2\right)\\ &\le C(\sigma,\varphi,\gamma,y,C_B),\end{aligned} \tag{3.32}$$

and we finally obtain

$$\left\|\Phi_n^U(t)\right\|^2 \le C(\sigma,\lambda,\varphi,\gamma,y,\alpha_1,C_B,T), \qquad \text{for all } t\in[0,T]. \qquad \square$$

Corollary 3.2.9. *Since the constant on the right-hand side of* (3.28) *is independent of* $t \in [0,T]$, *we especially deduce the uniform boundedness of the solution* Φ_n^U *of the Galerkin equations* (3.25) *in* $C([0,T];H)$ *and* $L^2([0,T];H)$. *Moreover, the independence of the constant in* (3.28) *of* $\omega \in \Omega$ *induces the uniform boundedness of* Φ_n^U *in* $L^2(\Omega; C([0,T];H))$ *and* $L^2(\Omega\times[0,T];H)$ *as well.*

Next, we show the uniform a priori estimate of the variational solution Φ_n^U of the Galerkin equations (3.25) in V.

Theorem 3.2.10. *Let* $n \in \mathbb{N}$ *be arbitrarily fixed. Then there exists a positive constant* C *such that*

$$\left\|\Phi_n^U(t)\right\|_V^2 \le C(\sigma,\lambda,\varphi,\gamma,y,\alpha_1,\alpha_2,C_B,T), \qquad \textit{for all } t\in[0,T]. \tag{3.33}$$

Proof. By starting with the Galerkin equations (3.25) and regarding the first property of the orthogonal projection in (2.3), it results that

$$\begin{aligned}\left(\Phi_n^U(t),h_k\right) =& \left(\Phi_n^U(T),h_k\right) + i\int_t^T \left\langle A\Phi_n^U(s),h_k\right\rangle ds - i\int_t^T \left(U(s)\Phi_n^U(s),h_k\right) ds\\ &- i\lambda(\sigma+1)\int_t^T B(s)\left(\left|Z^U(s)\right|^{2\sigma}\Phi_n^U(s),h_k\right) ds\\ &- i\lambda\sigma\int_t^T B(s)\left(\left|Z^U(s)\right|^{2(\sigma-1)}\left(Z^U(s)\right)^2\overline{\Phi_n^U}(s),h_k\right) ds,\\ \left(\Phi_n^U(T),h_k\right) =& -2i\gamma\frac{1}{\overline{Y}(T)}\left(\frac{Z^U(T)}{Y(T)} - y, h_k\right).\end{aligned}$$

Notice that each relation in this proof holds for all $t \in [0,T]$ and all $k \in \{1,2,\dots,n\}$. Due to the substitutions (3.29) and relation (3.30) from the proof of Theorem 3.2.8, we transform this final value problem into the initial value problem

$$\begin{aligned}
\big(\tilde{\Phi}_n^U(T-t), h_k\big) = {} & \big(\tilde{\Phi}_n^U(0), h_k\big) + i\int_0^{T-t} \big\langle A\tilde{\Phi}_n^U(r), h_k\big\rangle\,dr - i\int_0^{T-t} \big(\tilde{U}(r)\tilde{\Phi}_n^U(r), h_k\big)\,dr \\
& - i\lambda(\sigma+1)\int_0^{T-t} \tilde{B}(r)\big(\big|\tilde{Z}^U(r)\big|^{2\sigma}\tilde{\Phi}_n^U(r), h_k\big)\,dr \\
& - i\lambda\sigma\int_0^{T-t} \tilde{B}(r)\big(\big|\tilde{Z}^U(r)\big|^{2(\sigma-1)}\big(\tilde{Z}^U(r)\big)^2\,\overline{\tilde{\Phi}_n^U}(r), h_k\big)\,dr, \\
\big(\tilde{\Phi}_n^U(0), h_k\big) = {} & -2i\gamma\,\frac{1}{\overline{\tilde{Y}(0)}}\left(\frac{\tilde{Z}^U(0)}{\tilde{Y}(0)} - y, h_k\right).
\end{aligned}$$

Now, we apply the energy equality and then we multiply the obtained equation with μ_k and sum up over all $k = 1,2,\dots,n$ such that

$$\begin{aligned}
& \sum_{k=1}^n \mu_k \left|\big(\tilde{\Phi}_n^U(T-t), h_k\big)\right|^2 \\
= & \sum_{k=1}^n \mu_k \left|\big(\tilde{\Phi}_n^U(0), h_k\big)\right|^2 - 2\sum_{k=1}^n \mu_k \operatorname{Im}\int_0^{T-t} \big\langle A\tilde{\Phi}_n^U(r), \big(\tilde{\Phi}_n^U(r), h_k\big)\,h_k\big\rangle\,dr \\
& + 2\sum_{k=1}^n \mu_k \operatorname{Im}\int_0^{T-t} \big(\tilde{U}(r)\tilde{\Phi}_n^U(r), \big(\tilde{\Phi}_n^U(r), h_k\big)\,h_k\big)\,dr \\
& + 2\lambda(\sigma+1)\sum_{k=1}^n \mu_k \operatorname{Im}\int_0^{T-t} \tilde{B}(r)\big(\big|\tilde{Z}^U(r)\big|^{2\sigma}\tilde{\Phi}_n^U(r), \big(\tilde{\Phi}_n^U(r), h_k\big)\,h_k\big)\,dr \\
& + 2\lambda\sigma\sum_{k=1}^n \mu_k \operatorname{Im}\int_0^{T-t} \tilde{B}(r)\big(\big|\tilde{Z}^U(r)\big|^{2(\sigma-1)}\big(\tilde{Z}^U(r)\big)^2\,\overline{\tilde{\Phi}_n^U}(r), \big(\tilde{\Phi}_n^U(r), h_k\big)\,h_k\big)\,dr.
\end{aligned} \tag{3.34}$$

With the help of the second equality in (2.5), the term on the left-hand side and the first one on the right-hand side obey

$$\sum_{k=1}^n \mu_k \left|\big(\tilde{\Phi}_n^U(T-t), h_k\big)\right|^2 = \left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(T-t)\right\|^2, \qquad \sum_{k=1}^n \mu_k \left|\big(\tilde{\Phi}_n^U(0), h_k\big)\right|^2 = \left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(0)\right\|^2.$$

In the following, all the other terms on the right-hand side are successively regarded in detail. At first, we investigate the second term on the right-hand side of (3.34) including the operator A. Because of the first equality in (2.5), we get

$$\begin{aligned}
& -2\sum_{k=1}^n \mu_k \operatorname{Im}\int_0^{T-t} \big\langle A\tilde{\Phi}_n^U(r), \big(\tilde{\Phi}_n^U(r), h_k\big)\,h_k\big\rangle\,dr \\
= & -2\operatorname{Im}\int_0^{T-t} \left(\sum_{j=1}^n \mu_j\big(\tilde{\Phi}_n^U(r), h_j\big)\,h_j, \sum_{k=1}^n \mu_k\big(\tilde{\Phi}_n^U(r), h_k\big)\,h_k\right)dr \\
= & -2\operatorname{Im}\int_0^{T-t} \sum_{j=1}^n \mu_j\big(\tilde{\Phi}_n^U(r), h_j\big)\sum_{k=1}^n \mu_k\,\overline{\big(\tilde{\Phi}_n^U(r), h_k\big)}\,\big(h_j, h_k\big)\,dr \\
= & -2\sum_{j=1}^n \mu_j^2\int_0^{T-t} \operatorname{Im}\left|\big(\tilde{\Phi}_n^U(r), h_j\big)\right|^2 = 0.
\end{aligned}$$

Based on the first property in (2.5), property (2.6) and definition (2.1) of the operator A and the application of the product rule for derivatives, it results for the third term on the right-hand side of (3.34) that

$$
\begin{aligned}
&2\sum_{k=1}^{n}\mu_k \operatorname{Im}\int_0^{T-t} \left(\tilde{U}(r)\tilde{\Phi}_n^U(r), \left(\tilde{\Phi}_n^U(r), h_k\right) h_k\right) dr \\
&= 2\operatorname{Im}\int_0^{T-t} \left(\tilde{U}(r)\tilde{\Phi}_n^U(r), \sum_{k=1}^{n}\mu_k \left(\tilde{\Phi}_n^U(r), h_k\right) h_k\right) dr = 2\operatorname{Im}\int_0^{T-t} \left(\tilde{U}(r)\tilde{\Phi}_n^U(r), A\tilde{\Phi}_n^U(r)\right) dr \\
&= 2\operatorname{Im}\int_0^{T-t} \overline{\langle A\tilde{\Phi}_n^U(r), \tilde{U}(r)\tilde{\Phi}_n^U(r)\rangle}\, dr = 2\operatorname{Im}\int_0^{T-t} \left(\frac{\partial}{\partial x}\left[\tilde{U}(r)\tilde{\Phi}_n^U(r)\right], \frac{\partial}{\partial x}\tilde{\Phi}_n^U(r)\right) dr \\
&= 2\operatorname{Im}\int_0^{T-t} \left(\left[\frac{\partial}{\partial x}\tilde{U}(r)\right]\tilde{\Phi}_n^U(r), \frac{\partial}{\partial x}\tilde{\Phi}_n^U(r)\right) dr + 2\operatorname{Im}\int_0^{T-t} \left(\tilde{U}(r)\left[\frac{\partial}{\partial x}\tilde{\Phi}_n^U(r)\right], \frac{\partial}{\partial x}\tilde{\Phi}_n^U(r)\right) dr.
\end{aligned}
$$

Due to relation (3.4), the second term vanishes such that it only remains

$$
2\sum_{k=1}^{n}\mu_k \operatorname{Im}\int_0^{T-t} \left(\tilde{U}(r)\tilde{\Phi}_n^U(r), \left(\tilde{\Phi}_n^U(r), h_k\right) h_k\right) dr = 2\operatorname{Im}\int_0^{T-t} \left(\left[\frac{\partial}{\partial x}\tilde{U}(r)\right]\tilde{\Phi}_n^U(r), \frac{\partial}{\partial x}\tilde{\Phi}_n^U(r)\right) dr.
$$

Now, we have to remember the case differentiation for $\tilde{U} \in \mathcal{U}$ (compare Lemma 3.1.1).

(i) If $\tilde{U}(r,x) = \tilde{U}_1(r)$, then $\frac{\partial}{\partial x}\tilde{U}_1(r) = 0$ and, therefore, it holds that

$$
2\operatorname{Im}\int_0^{T-t} \left[\frac{\partial}{\partial x}\tilde{U}_1(r)\right]\left(\tilde{\Phi}_n^U(r), \frac{\partial}{\partial x}\tilde{\Phi}_n^U(r)\right) dr = 0.
$$

(ii) If $\tilde{U}(r,x) = \tilde{U}_2(x) = U_2(x)$, the boundedness $\left\|\frac{\partial}{\partial x}U_2\right\| \leq \alpha_2$ and Lemma D.2 yield

$$
\begin{aligned}
&2\operatorname{Im}\int_0^{T-t} \left(\left[\frac{\partial}{\partial x}U_2\right]\tilde{\Phi}_n^U(r), \frac{\partial}{\partial x}\tilde{\Phi}_n^U(r)\right) dr \leq 2\int_0^{T-t} \left\|\left[\frac{\partial}{\partial x}U_2\right]\tilde{\Phi}_n^U(r)\right\| \left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(r)\right\| dr \\
&\leq \int_0^{T-t} \left\|\left[\frac{\partial}{\partial x}U_2\right]\tilde{\Phi}_n^U(r)\right\|^2 dr + \int_0^{T-t} \left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(r)\right\|^2 dr \\
&\leq \int_0^{T-t} \left\|\frac{\partial}{\partial x}U_2\right\|^2 \left(\sup_{x\in[0,1]}\left|\tilde{\Phi}_n^U(r,x)\right|^2\right) dr + \int_0^{T-t} \left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(r)\right\|^2 dr \\
&\leq 2\alpha_2^2 \int_0^{T-t} \left\|\tilde{\Phi}_n^U(r)\right\|_V^2 dr + \int_0^{T-t} \left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(r)\right\|^2 dr \\
&= 2\alpha_2^2 \int_0^{T-t} \left\|\tilde{\Phi}_n^U(r)\right\|^2 dr + [1+2\alpha_2^2]\int_0^{T-t} \left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(r)\right\|^2 dr.
\end{aligned}
$$

The estimates of the last two terms on the right-hand side of (3.34) are similar but of considerable length. Thus, we present the approach for one term and only state the result for the other. Due to the second equality in (2.5), the fourth term on the right-hand side of (3.34) fulfills

$$
\begin{aligned}
&2\lambda(\sigma+1)\sum_{k=1}^{n}\mu_k \operatorname{Im}\int_0^{T-t} \tilde{B}(r)\left(\left|\tilde{Z}^U(r)\right|^{2\sigma}\tilde{\Phi}_n^U(r), \left(\tilde{\Phi}_n^U(r), h_k\right) h_k\right) dr \\
&\leq 2\lambda(\sigma+1)\tilde{B}(0)\sum_{k=1}^{n}\mu_k \int_0^{T-t} \left|\left(\left|\tilde{Z}^U(r)\right|^{2\sigma}\tilde{\Phi}_n^U(r), h_k\right)\right| \left|\overline{\left(\tilde{\Phi}_n^U(r), h_k\right)}\right| dr \\
&\leq \lambda(\sigma+1)\tilde{B}(0)\int_0^{T-t} \left[\sum_{k=1}^{n}\mu_k \left|\left(\left|\tilde{Z}^U(r)\right|^{2\sigma}\tilde{\Phi}_n^U(r), h_k\right)\right|^2 + \sum_{k=1}^{n}\mu_k \left|\left(\tilde{\Phi}_n^U(r), h_k\right)\right|^2\right] dr \\
&= \lambda(\sigma+1)\tilde{B}(0)\int_0^{T-t} \left\|\frac{\partial}{\partial x}\left[\left|\tilde{Z}^U(r)\right|^{2\sigma}\tilde{\Phi}_n^U(r)\right]\right\|^2 dr + \lambda(\sigma+1)\tilde{B}(0)\int_0^{T-t} \left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(r)\right\|^2 dr.
\end{aligned}
$$

The first integrand satisfies by Lemma D.2

$$\begin{aligned}
&\left\| \frac{\partial}{\partial x} \left[\left| \tilde{Z}^U(r) \right|^{2\sigma} \tilde{\Phi}_n^U(r) \right] \right\|^2 \\
&= \int_0^1 \left| \frac{\partial}{\partial x} \left| \tilde{Z}^U(r,x) \right|^{2\sigma} \right|^2 \left| \tilde{\Phi}_n^U(r,x) \right|^2 dx + \int_0^1 \left| \tilde{Z}^U(r,x) \right|^{4\sigma} \left| \frac{\partial}{\partial x} \tilde{\Phi}_n^U(r,x) \right|^2 dx \\
&\leq \int_0^1 \left| \frac{\partial}{\partial x} \left| \tilde{Z}^U(r,x) \right|^{2\sigma} \right|^2 \left| \tilde{\Phi}_n^U(r,x) \right|^2 dx + \left(\sup_{x\in[0,1]} \left| \tilde{Z}^U(r,x) \right|^{4\sigma} \right) \left\| \frac{\partial}{\partial x} \tilde{\Phi}_n^U(r) \right\|^2 \\
&\leq \int_0^1 \left| \frac{\partial}{\partial x} \left| \tilde{Z}^U(r,x) \right|^{2\sigma} \right|^2 \left| \tilde{\Phi}_n^U(r,x) \right|^2 dx + 2^{2\sigma} \| \tilde{Z}^U(r) \|_V^{4\sigma} \left\| \frac{\partial}{\partial x} \tilde{\Phi}_n^U(r) \right\|^2 .
\end{aligned} \tag{3.35}$$

Denoting $v_x := \frac{d}{dx} v$, we get for all $v \in V$ that

$$\left| \frac{d}{dx} |v|^{2\sigma} \right|^2 = \left| \frac{d}{dx} (v^\sigma \overline{v}^\sigma) \right|^2 = \left| \sigma |v|^{2(\sigma-1)} \left(v_x \overline{v} + v \overline{v}_x \right) \right|^2 = 4\sigma^2 |v|^{4(\sigma-1)} \left| \mathrm{Re}\{ v \overline{v}_x \} \right|^2 .$$

Therefore, we deduce (again by Lemma D.2)

$$\begin{aligned}
&\int_0^1 \left| \frac{\partial}{\partial x} \left| \tilde{Z}^U(r,x) \right|^{2\sigma} \right|^2 \left| \tilde{\Phi}_n^U(r,x) \right|^2 dx \\
&= 4\sigma^2 \int_0^1 \left| \tilde{Z}^U(r,x) \right|^{4(\sigma-1)} \left| \mathrm{Re} \left\{ \tilde{Z}^U(r,x) \left[\frac{\partial}{\partial x} \overline{\tilde{Z}^U}(r,x) \right] \right\} \right|^2 \left| \tilde{\Phi}_n^U(r,x) \right|^2 dx \\
&\leq 4\sigma^2 \int_0^1 \left| \tilde{Z}^U(r,x) \right|^{4\sigma-2} \left| \frac{\partial}{\partial x} \tilde{Z}^U(r,x) \right|^2 \left| \tilde{\Phi}_n^U(r,x) \right|^2 dx \\
&\leq 4\sigma^2 \left(\sup_{x\in[0,1]} \left| \tilde{Z}^U(r,x) \right|^{2(2\sigma-1)} \right) \int_0^1 \left| \frac{\partial}{\partial x} \tilde{Z}^U(r,x) \right|^2 \left| \tilde{\Phi}_n^U(r,x) \right|^2 dx \\
&\leq 4 \cdot 2^{2\sigma-1} \sigma^2 \left\| \tilde{Z}^U(r) \right\|_V^{2(2\sigma-1)} \int_0^1 \left| \frac{\partial}{\partial x} \tilde{Z}^U(r,x) \right|^2 \left| \tilde{\Phi}_n^U(r,x) \right|^2 dx
\end{aligned}$$

with

$$\begin{aligned}
\tilde{\mathcal{I}}(r) &:= \int_0^1 \left| \frac{\partial}{\partial x} \tilde{Z}^U(r,x) \right|^2 \left| \tilde{\Phi}_n^U(r,x) \right|^2 dx \leq \left\| \frac{\partial}{\partial x} \tilde{Z}^U(r) \right\|^2 \left(\sup_{x\in[0,1]} \left| \tilde{\Phi}_n^U(r,x) \right|^2 \right) \\
&\leq 2 \left\| \frac{\partial}{\partial x} \tilde{Z}^U(r) \right\|^2 \left\| \tilde{\Phi}_n^U(r) \right\|_V^2 = 2 \left\| \tilde{Z}^U(r) \right\|_V^2 \left[\left\| \tilde{\Phi}_n^U(r) \right\|^2 + \left\| \frac{\partial}{\partial x} \tilde{\Phi}_n^U(r) \right\|^2 \right]
\end{aligned} \tag{3.36}$$

such that relation (3.35) results in

$$\left\| \frac{\partial}{\partial x} \left[\left| \tilde{Z}^U(r) \right|^{2\sigma} \tilde{\Phi}_n^U(r) \right] \right\|^2 \leq 4 \cdot 2^{2\sigma} \sigma^2 \left\| \tilde{Z}^U(r) \right\|_V^{4\sigma} \left\| \tilde{\Phi}_n^U(r) \right\|^2 + 2^{2\sigma} \left[1 + 4\sigma^2 \right] \left\| \tilde{Z}^U(r) \right\|_V^{4\sigma} \left\| \frac{\partial}{\partial x} \tilde{\Phi}_n^U(r) \right\|^2 .$$

Hence, the fourth term on the right-hand side of relation (3.34) satisfies

$$\begin{aligned}
&2\lambda(\sigma+1) \sum_{k=1}^n \mu_k \, \mathrm{Im} \int_0^{T-t} \tilde{B}(r) \left(\left| \tilde{Z}^U(r) \right|^{2\sigma} \tilde{\Phi}_n^U(r), \left(\tilde{\Phi}_n^U(r), h_k \right) h_k \right) dr \\
&\leq 4 \cdot 2^{2\sigma} \lambda(\sigma+1) \sigma^2 \tilde{B}(0) \int_0^{T-t} \left\| \tilde{Z}^U(r) \right\|_V^{4\sigma} \left\| \tilde{\Phi}_n^U(r) \right\|^2 dr \\
&\quad + \lambda(\sigma+1) \tilde{B}(0) \int_0^{T-t} \left(1 + 2^{2\sigma} \left[1 + 4\sigma^2 \right] \left\| \tilde{Z}^U(r) \right\|_V^{4\sigma} \right) \left\| \frac{\partial}{\partial x} \tilde{\Phi}_n^U(r) \right\|^2 dr.
\end{aligned}$$

Finally, the fifth term on the right-hand side of (3.34) is estimated analogously to the fourth term. We obtain

$$\begin{aligned}&2\lambda\sigma\sum_{k=1}^{n}\mu_k \operatorname{Im}\int_0^{T-t}\tilde{B}(r)\big(\big|\tilde{Z}^U(r)\big|^{2(\sigma-1)}\big(\tilde{Z}^U(r)\big)^2\,\overline{\tilde{\Phi}_n^U}(r),\big(\tilde{\Phi}_n^U(r),h_k\big)\,h_k\big)\,dr\\&\le\lambda\sigma\tilde{B}(0)\int_0^{T-t}\left\|\frac{\partial}{\partial x}\left[\big|\tilde{Z}^U(r)\big|^{2(\sigma-1)}\big(\tilde{Z}^U(r)\big)^2\,\overline{\tilde{\Phi}_n^U}(r)\right]\right\|^2dr+\lambda\sigma\tilde{B}(0)\int_0^{T-t}\left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(r)\right\|^2dr.\end{aligned}$$

The product rule for derivatives and relation (3.36) imply

$$\begin{aligned}&\left\|\frac{\partial}{\partial x}\left[\big|\tilde{Z}^U(r)\big|^{2(\sigma-1)}\big(\tilde{Z}^U(r)\big)^2\,\overline{\tilde{\Phi}_n^U}(r)\right]\right\|^2\\&=\int_0^1\left|\frac{\partial}{\partial x}\big|\tilde{Z}^U(r,x)\big|^{2(\sigma-1)}\right|^2\big|\tilde{Z}^U(r,x)\big|^4\big|\tilde{\Phi}_n^U(r,x)\big|^2\,dx\\&\quad+\int_0^1\big|\tilde{Z}^U(r,x)\big|^{4(\sigma-1)}\left|\frac{\partial}{\partial x}\big(\tilde{Z}^U(r,x)\big)^2\right|^2\big|\tilde{\Phi}_n^U(r,x)\big|^2\,dx\\&\quad+\int_0^1\big|\tilde{Z}^U(r,x)\big|^{4(\sigma-1)}\big|\tilde{Z}^U(r,x)\big|^4\left|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(r,x)\right|^2dx\\&\le4\cdot2^{2\sigma-1}\left[1+|\sigma-1|^2\right]\left\|\tilde{Z}^U(r)\right\|_V^{2(2\sigma-1)}\tilde{\mathcal{I}}(r)+2^{2\sigma}\left\|\tilde{Z}^U(r)\right\|_V^{4\sigma}\left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(r)\right\|^2\\&\le4\cdot2^{2\sigma}\left[1+|\sigma-1|^2\right]\left\|\tilde{Z}^U(r)\right\|_V^{4\sigma}\left\|\tilde{\Phi}_n^U(r)\right\|^2+2^{2\sigma}\left[5+4|\sigma-1|^2\right]\left\|\tilde{Z}^U(r)\right\|_V^{4\sigma}\left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(r)\right\|^2,\end{aligned}$$

and it follows that

$$\begin{aligned}&2\lambda\sigma\sum_{k=1}^{n}\mu_k \operatorname{Im}\int_0^{T-t}\tilde{B}(r)\big(\big|\tilde{Z}^U(r)\big|^{2(\sigma-1)}\big(\tilde{Z}^U(r)\big)^2\,\overline{\tilde{\Phi}_n^U}(r),\big(\tilde{\Phi}_n^U(r),h_k\big)\,h_k\big)\,dr\\&\le4\cdot2^{2\sigma}\lambda\sigma\left[1+|\sigma-1|^2\right]\tilde{B}(0)\int_0^{T-t}\left\|\tilde{Z}^U(r)\right\|_V^{4\sigma}\left\|\tilde{\Phi}_n^U(r)\right\|^2dr\\&\quad+\lambda\sigma\tilde{B}(0)\int_0^{T-t}\left(1+2^{2\sigma}\left[5+4|\sigma-1|^2\right]\left\|\tilde{Z}^U(r)\right\|_V^{4\sigma}\right)\left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(r)\right\|^2dr.\end{aligned}$$

Summarized, equation (3.34) results in

$$\begin{aligned}\left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(T-t)\right\|^2\le&\left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(0)\right\|^2+2\alpha_2^2\int_0^{T-t}\left\|\tilde{\Phi}_n^U(r)\right\|^2dr+[1+2\alpha_2^2]\int_0^{T-t}\left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(r)\right\|^2dr\\&+4\cdot2^{2\sigma}\lambda(\sigma+1)\sigma^2\tilde{B}(0)\int_0^{T-t}\left\|\tilde{Z}^U(r)\right\|_V^{4\sigma}\left\|\tilde{\Phi}_n^U(r)\right\|^2dr\\&+\lambda(\sigma+1)\tilde{B}(0)\int_0^{T-t}\left(1+2^{2\sigma}\left[1+4\sigma^2\right]\left\|\tilde{Z}^U(r)\right\|_V^{4\sigma}\right)\left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(r)\right\|^2dr\\&+4\cdot2^{2\sigma}\lambda\sigma\left[1+|\sigma-1|^2\right]\tilde{B}(0)\int_0^{T-t}\left\|\tilde{Z}^U(r)\right\|_V^{4\sigma}\left\|\tilde{\Phi}_n^U(r)\right\|^2dr\\&+\lambda\sigma\tilde{B}(0)\int_0^{T-t}\left(1+2^{2\sigma}\left[5+4|\sigma-1|^2\right]\left\|\tilde{Z}^U(r)\right\|_V^{4\sigma}\right)\left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(r)\right\|^2dr.\end{aligned}$$

This equation is simplified to

$$\left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(T-t)\right\|^2\le\left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(0)\right\|^2+\int_0^{T-t}\tilde{K}_1(r)\left\|\tilde{\Phi}_n^U(r)\right\|^2dr+\int_0^{T-t}\tilde{K}_2(r)\left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(r)\right\|^2dr,$$

where the coefficients are given by

$$\tilde{K}_1(r) := 2\alpha_2^2 + 4\cdot 2^{2\sigma}\lambda\sigma\tilde{B}(0)\left[\sigma(\sigma+1)+1+|\sigma-1|^2\right]\left\|\tilde{Z}^U(r)\right\|_V^{4\sigma},$$
$$\tilde{K}_2(r) := 1 + 2\alpha_2^2 + \lambda\tilde{B}(0)\left[2\sigma+1+2^{2\sigma}\Big((\sigma+1)\left[1+4\sigma^2\right]+\sigma\left[5+4|\sigma-1|^2\right]\Big)\left\|\tilde{Z}^U(r)\right\|_V^{4\sigma}\right].$$

Keeping in mind that $\tilde{B}(0) = B(T) \leq C_B$ (compare (3.7)) and relation (3.31), it holds that

$$\tilde{K}_1(r) \leq C(\sigma,\lambda,\varphi,\alpha_1,\alpha_2,C_B) =: K_1, \qquad \tilde{K}_2(r) \leq C(\sigma,\lambda,\varphi,\alpha_1,\alpha_2,C_B) =: K_2 \tag{3.37}$$

such that

$$\left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(T-t)\right\|^2 \leq \left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(0)\right\|^2 + K_1\int_0^{T-t}\left\|\tilde{\Phi}_n^U(r)\right\|^2 dr + K_2\int_0^{T-t}\left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(r)\right\|^2 dr.$$

Now, Gronwall's lemma implies that

$$\left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(T-t)\right\|^2 \leq \left(\left\|\frac{\partial}{\partial x}\tilde{\Phi}_n^U(0)\right\|^2 + K_1\int_0^{T-t}\left\|\tilde{\Phi}_n^U(r)\right\|^2 dr\right)\exp\left\{K_2T\right\},$$

and resubstitution yields

$$\begin{aligned}\left\|\frac{\partial}{\partial x}\Phi_n^U(t)\right\|^2 &\leq \left(\left\|\frac{\partial}{\partial x}\Phi_n^U(T)\right\|^2 + K_1\int_t^T\left\|\Phi_n^U(s)\right\|^2 ds\right)\exp\left\{K_2T\right\}\\ &\leq \left(\left\|\frac{\partial}{\partial x}\Phi_n^U(T)\right\|^2 + K_1\int_0^T\left\|\Phi_n^U(s)\right\|^2 ds\right)\exp\left\{K_2T\right\}\\ &\leq C(\sigma,\lambda,\varphi,\gamma,y,\alpha_1,\alpha_2,C_B,T)\end{aligned}$$

due to the estimates (3.28), (3.37) and since (by relations (3.32) and (3.11))

$$\begin{aligned}\left\|\frac{\partial}{\partial x}\Phi_n^U(T)\right\|^2 &\leq 8\gamma^2B^{\frac{1}{\sigma}}(T)\left(B^{\frac{1}{\sigma}}(T)\left\|\frac{\partial}{\partial x}Z^U(T)\right\|^2 + \left\|\frac{d}{dx}y\right\|^2\right)\\ &\leq 8\gamma^2B^{\frac{1}{\sigma}}(T)\left(B^{\frac{1}{\sigma}}(T)\left(\operatorname*{ess\,sup}_{t\in[0,T]}\left\|Z^U(t)\right\|_V^2\right) + \|y\|_V^2\right)\\ &\leq C(\sigma,\lambda,\varphi,\gamma,y,\alpha_1,C_B).\end{aligned}$$

Consequently, it holds that

$$\left\|\Phi_n^U(t)\right\|_V^2 = \left\|\Phi_n^U(t)\right\|^2 + \left\|\frac{\partial}{\partial x}\Phi_n^U(t)\right\|^2 \leq C(\sigma,\lambda,\varphi,\gamma,y,\alpha_1,\alpha_2,C_B,T), \qquad \text{for all } t\in[0,T]. \quad \square$$

Corollary 3.2.11. *Since the constant on the right-hand side of relation* (3.33) *is again independent of $t\in[0,T]$ and $\omega\in\Omega$, we generalize the result of Theorem 3.2.10 to the uniform boundedness of the solution Φ_n^U of the Galerkin equations* (3.25) *in $C([0,T];V)$ and $L^2([0,T];V)$ as well as in $L^2(\Omega;C([0,T];V))$ and $L^2(\Omega\times[0,T];V)$.*

Knowing that the Galerkin approximations Φ_n^U of the complex conjugated adjoint Schrödinger problem (3.25) are uniformly bounded in H and V, we receive the same boundedness and convergence results as for the pathwise Schrödinger problem (see Subsection 2.3.1, especially Corollary 2.3.7 and Theorem 2.3.8). That is why we will not reproduce the proofs here and instead only state some remarks.

Theorem 3.2.12. *The sequence of variational solutions* $(\Phi_n^U)_n$ *of the Galerkin equations* (3.25) *is bounded in* $C([0,T];H), L^2([0,T];H)$ *and* $L^2([0,T];V)$ *and relatively compact in* $L^2([0,T];H)$. *Furthermore, it holds that*

(a) $(\Phi_n^U)_n$ *converges strongly in* $L^2([0,T];H)$ *and weakly in* $L^2([0,T];V)$ *to the variational solution* Φ^U *of the complex conjugated adjoint Schrödinger problem* (3.24),

(b) $\Phi^U \in L^\infty([0,T];V)$ *and especially*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|\Phi^U(t)\|_V^2 \le C(\sigma,\lambda,\varphi,\gamma,y,\alpha_1,\alpha_2,C_B,T),$$

(c) $(\Phi_n^U)_n$ *also converges to* Φ^U *in* $C([0,T];H)$.

The proof is nearly identical to the proofs of Corollary 2.3.7 and Theorem 2.3.8. Since the complex conjugated adjoint Schrödinger problem is linear, we obtain the weak convergences in $L^2([0,T];H)$ and $L^2([0,T];V)$ by the appropriate boundedness results. Nevertheless, we need the relative compactness in $L^2([0,T];H)$ to show the strong convergence in $L^2([0,T];H)$ in part (a) and to verify part (c) of the preceding theorem.

Finally, we state a similar result to Theorem 3.2.3 for the complex conjugated adjoint variable Φ^U containing an equivalent variational formulation of (3.24) for $v \in L^2([0,T];V')$. Observing that this is a final value problem, the proof proceeds in the same way as the proof of Theorem 3.2.3 and is, therefore, omitted too.

Theorem 3.2.13. *It holds that* $\Phi^U \in L^2([0,T];V')$ *and we obtain from equation* (3.24) *that*

$$\begin{aligned}
\big(\Phi^U(t), v(t)\big) =& -2i\gamma \frac{1}{\overline{Y(T)}}\left(\frac{Z^U(T)}{Y(T)} - y, v(T)\right) - \int_t^T \overline{\left\langle \frac{\partial}{\partial s} v(s), \Phi^U(s)\right\rangle}\, ds \\
&+ i\int_t^T \langle A\Phi^U(s), v(s)\rangle\, ds - i\int_t^T \big(U(s)\Phi^U(s), v(s)\big)\, ds \\
&- i\lambda(\sigma+1)\int_t^T B(s)\big(\big|Z^U(s)\big|^{2\sigma}\Phi^U(s), v(s)\big)\, ds \\
&- i\lambda\sigma\int_t^T B(s)\big(\big|Z^U(s)\big|^{2(\sigma-1)}\big(Z^U(s)\big)^2\,\overline{\Phi^U}(s), v(s)\big)\, ds
\end{aligned} \tag{3.38}$$

for all $t \in [0,T]$ *and all* $v \in L^2([0,T];V')$.

This representation coincides with the variational formulation (3.24) of the complex conjugated adjoint Schrödinger problem by identifying $v \in L^2([0,T];V')$ with $v \in V$.

3.2.4 Gradient Formula

As known from the finite-dimensional analysis, a gradient formula in the sense of Gâteaux or Fréchet represents an opportunity to calculate an extreme value. In this case we search for an optimal control $U^* \in \mathcal{U}$ such that the objective functional (3.20) will be minimized. Referring to the approaches in the deterministic articles [3, 33, 102], we deduce a gradient formula by combining the variational solutions of $\delta Z = Z^{U+\Theta\delta U} - Z^U$ and of the corresponding complex conjugated adjoint variable Φ^U.

Theorem 3.2.14. *Let the above assumptions be satisfied and* $\sigma \in [\frac{1}{2}, 2)$. *Then the Gâteaux differential of the objective functional* (3.20) *is given by*

$$\delta J(U;\delta U) = E\,\mathrm{Re}\int_0^T \big(\Phi^U(t)\,\overline{Z^U}(t), \delta U(t)\big)\, dt + 2\beta E\int_0^T \big(U(t) - \Upsilon(t), \delta U(t)\big)\, dt$$

for all $\delta U \in \mathcal{U}$.

Proof. The Gâteaux differential of the objective functional (3.20) at $U \in \mathcal{U}$ in the direction $\delta U \in \mathcal{U}$ is defined by

$$\delta J(U;\delta U) := \lim_{\Theta\to 0} \frac{J(U+\Theta\delta U) - J(U)}{\Theta}.$$

If the limit exists for all $\delta U \in \mathcal{U}$, we say that J is Gâteaux differentiable at $U \in \mathcal{U}$. At first, we calculate the numerator that is given by

$$\begin{aligned}
J(U+\Theta\delta U) - J(U) =& \gamma E \left\| \frac{Z^{U+\Theta\delta U}(T)}{Y(T)} - y \right\|^2 + \beta E \int_0^T \|(U+\Theta\delta U)(t) - \Upsilon(t)\|^2 \, dt \\
& - \gamma E \left\| \frac{Z^U(T)}{Y(T)} - y \right\|^2 - \beta E \int_0^T \|U(t) - \Upsilon(t)\|^2 \, dt \\
=& \gamma E \left(\left\| \frac{Z^{U+\Theta\delta U}(T)}{Y(T)} - y \right\|^2 - \left\| \frac{Z^U(T)}{Y(T)} - y \right\|^2 \right) \\
& + \beta E \int_0^T \left(\|(U+\Theta\delta U)(t) - \Upsilon(t)\|^2 - \|U(t) - \Upsilon(t)\|^2 \right) dt.
\end{aligned}$$

Disregarding the time variable, for the time being, and using the notation $Z^{U+\Theta\delta U} = \delta Z + Z^U$, the first expression in parentheses is rewritten as

$$\begin{aligned}
\left\| \frac{\delta Z}{Y} + \frac{Z^U}{Y} - y \right\|^2 - \left\| \frac{Z^U}{Y} - y \right\|^2 &= \left(\frac{\delta Z}{Y} + \frac{Z^U}{Y} - y, \frac{\delta Z}{Y} + \frac{Z^U}{Y} - y \right) - \left(\frac{Z^U}{Y} - y, \frac{Z^U}{Y} - y \right) \\
&= \left(\frac{\delta Z}{Y}, \frac{\delta Z}{Y} \right) + \left(\frac{Z^U}{Y} - y, \frac{\delta Z}{Y} \right) + \left(\frac{\delta Z}{Y}, \frac{Z^U}{Y} - y \right) \\
&= \left\| \frac{\delta Z}{Y} \right\|^2 + 2 \operatorname{Re} \left(\frac{Z^U}{Y} - y, \frac{\delta Z}{Y} \right)
\end{aligned}$$

because $(u,v) + (v,u) = (u,v) + \overline{(u,v)} = 2 \operatorname{Re}(u,v)$ for all $u, v \in H$. Furthermore, by remembering that the admissible controls are real-valued, it holds that

$$\begin{aligned}
\|(U+\Theta\delta U) - \Upsilon\|^2 - \|U - \Upsilon\|^2 &= \big(\Theta\delta U + U - \Upsilon, \Theta\delta U + U - \Upsilon\big) - \big(U - \Upsilon, U - \Upsilon\big) \\
&= \Theta^2 \|\delta U\|^2 + 2\Theta\big(U - \Upsilon, \delta U\big).
\end{aligned}$$

Thus, it results that

$$\begin{aligned}
J(U+\Theta\delta U) - J(U) =& \gamma E \left(B^{\frac{1}{\sigma}}(T) \|\delta Z(T)\|^2 \right) + 2\gamma E \operatorname{Re} \left(\frac{Z^U(T)}{Y(T)} - y, \frac{\delta Z(T)}{Y(T)} \right) \\
& + \beta\Theta^2 E \int_0^T \|\delta U(t)\|^2 \, dt + 2\beta\Theta E \int_0^T \big(U(t) - \Upsilon(t), \delta U(t)\big) \, dt.
\end{aligned} \tag{3.39}$$

Being interested in the Gâteaux differential, we have to divide this expression by Θ and take the limit for $\Theta \to 0$. Since the second term on the right-hand side would cause problems, we do some calculations to reformulate this term. Therefore, we replace v by δZ in the variational formulation (3.38) of the complex conjugated adjoint Schrödinger problem and obtain for all $t \in [0,T]$

$$\begin{aligned}
\big(\Phi^U(t), \delta Z(t)\big) =& -2i\gamma \frac{1}{\overline{Y(T)}} \left(\frac{Z^U(T)}{Y(T)} - y, \delta Z(T) \right) - \int_t^T \overline{\left\langle \frac{\partial}{\partial s} \delta Z(s), \Phi^U(s) \right\rangle} \, ds \\
& + i \int_t^T \langle A\Phi^U(s), \delta Z(s) \rangle \, ds - i \int_t^T \big(U(s)\Phi^U(s), \delta Z(s)\big) \, ds \\
& - i\lambda(\sigma+1) \int_t^T B(s) \big(|Z^U(s)|^{2\sigma} \Phi^U(s), \delta Z(s) \big) \, ds \\
& - i\lambda\sigma \int_t^T B(s) \big(|Z^U(s)|^{2(\sigma-1)} \big(Z^U(s)\big)^2 \, \overline{\Phi^U}(s), \delta Z(s) \big) \, ds.
\end{aligned}$$

Choosing $t = 0$ and using the properties of the inner product in H, it holds that

$$\begin{aligned} 0 = &- 2i\gamma\Big(\frac{Z^U(T)}{Y(T)} - y, \frac{\delta Z(T)}{Y(T)}\Big) - \int_0^T \overline{\Big\langle \frac{\partial}{\partial s}\delta Z(s), \Phi^U(s)\Big\rangle}\, ds \\ &+ i\int_0^T \big\langle A\Phi^U(s), \delta Z(s)\big\rangle\, ds - i\int_0^T \big(\Phi^U(s), U(s)\delta Z(s)\big)\, ds \\ &- i\lambda(\sigma+1)\int_0^T B(s)\big(\Phi^U(s), \big|Z^U(s)\big|^{2\sigma}\delta Z(s)\big)\, ds \\ &- i\lambda\sigma\int_0^T B(s)\big(\overline{\Phi^U}(s), \big|Z^U(s)\big|^{2(\sigma-1)}\big(\overline{Z^U}(s)\big)^2\delta Z(s)\big)\, ds. \end{aligned} \tag{3.40}$$

Its complex conjugated equation is given by

$$\begin{aligned} 0 = &2i\gamma\overline{\Big(\frac{Z^U(T)}{Y(T)} - y, \frac{\delta Z(T)}{Y(T)}\Big)} - \int_0^T \Big\langle \frac{\partial}{\partial s}\delta Z(s), \Phi^U(s)\Big\rangle\, ds \\ &- i\int_0^T \overline{\big\langle A\Phi^U(s), \delta Z(s)\big\rangle}\, ds + i\int_0^T \overline{\big(\Phi^U(s), U(s)\delta Z(s)\big)}\, ds \\ &+ i\lambda(\sigma+1)\int_0^T B(s)\overline{\big(\Phi^U(s), \big|Z^U(s)\big|^{2\sigma}\delta Z(s)\big)}\, ds \\ &+ i\lambda\sigma\int_0^T B(s)\overline{\big(\overline{\Phi^U}(s), \big|Z^U(s)\big|^{2(\sigma-1)}\big(\overline{Z^U}(s)\big)^2\delta Z(s)\big)}\, ds. \end{aligned} \tag{3.41}$$

Regarding that $\big(u,v\big) + \overline{\big(u,v\big)} = 2\,\mathrm{Re}\,\big(u,v\big) = 2\,\mathrm{Re}\,\overline{\big(u,v\big)}$ and $\big(u,v\big) - \overline{\big(u,v\big)} = 2i\,\mathrm{Im}\,\big(u,v\big)$ for all $u, v \in H$ and

$$\overline{\big(\overline{\Phi^U}(s), \big|Z^U(s)\big|^{2(\sigma-1)}\big(\overline{Z^U}(s)\big)^2\delta Z(s)\big)} = \big(\Phi^U(s), \big|Z^U(s)\big|^{2(\sigma-1)}\big(Z^U(s)\big)^2\,\overline{\delta Z}(s)\big),$$

we get by subtracting equation (3.40) from equation (3.41) that

$$\begin{aligned} 0 = &4i\gamma\,\mathrm{Re}\,\Big(\frac{Z^U(T)}{Y(T)} - y, \frac{\delta Z(T)}{Y(T)}\Big) + 2i\,\mathrm{Im}\int_0^T \overline{\Big\langle \frac{\partial}{\partial s}\delta Z(s), \Phi^U(s)\Big\rangle}\, ds \\ &- 2i\,\mathrm{Re}\int_0^T \big\langle A\Phi^U(s), \delta Z(s)\big\rangle\, ds + 2i\,\mathrm{Re}\int_0^T \big(\Phi^U(s), U(s)\delta Z(s)\big)\, ds \\ &+ 2i\lambda(\sigma+1)\,\mathrm{Re}\int_0^T B(s)\big(\Phi^U(s), \big|Z^U(s)\big|^{2\sigma}\delta Z(s)\big)\, ds \\ &+ 2i\lambda\sigma\,\mathrm{Re}\int_0^T B(s)\big(\Phi^U(s), \big|Z^U(s)\big|^{2(\sigma-1)}\big(Z^U(s)\big)^2\,\overline{\delta Z}(s)\big)\, ds. \end{aligned} \tag{3.42}$$

Next, we substitute v with Φ^U in the variational formulation (3.17) of the solution δZ of the difference of two controlled Schrödinger problems, which entails for all $t \in [0,T]$

$$\begin{aligned} \big(\delta Z(t), \Phi^U(t)\big) = &\int_0^t \overline{\Big\langle \frac{\partial}{\partial s}\Phi^U(s), \delta Z(s)\Big\rangle}\, ds - i\int_0^t \big\langle A\delta Z(s), \Phi^U(s)\big\rangle\, ds \\ &+ i\int_0^t \big((U + \Theta\delta U)(s)\delta Z(s), \Phi^U(s)\big)\, ds \\ &+ i\int_0^t \big(\Theta\delta U(s)Z^U(s), \Phi^U(s)\big)\, ds \\ &+ i\lambda\int_0^t B(s)\big(f(Z^{U+\Theta\delta U}(s)) - f(Z^U(s)), \Phi^U(s)\big)\, ds. \end{aligned}$$

Observe $\big(u,v\big) = \overline{\big(v,u\big)}$ for all $u,v \in H$ such that for $t=T$ this relation is equivalent to

$$\begin{aligned} 0 =& -2i\gamma\overline{\left(\frac{Z^U(T)}{Y(T)} - y, \frac{\delta Z(T)}{Y(T)}\right)} + \int_0^T \overline{\left\langle \frac{\partial}{\partial s}\Phi^U(s), \delta Z(s)\right\rangle}\, ds \\ &- i\int_0^T \overline{\big\langle A\Phi^U(s), \delta Z(s)\big\rangle}\, ds + i\int_0^T \overline{\big(\Phi^U(s), (U+\Theta\delta U)(s)\delta Z(s)\big)}\, ds \\ &+ i\int_0^T \overline{\big(\Phi^U(s), \Theta\delta U(s) Z^U(s)\big)}\, ds \\ &+ i\lambda\int_0^T B(s)\overline{\big(\Phi^U(s), f(Z^{U+\Theta\delta U}(s)) - f(Z^U(s))\big)}\, ds \end{aligned} \tag{3.43}$$

and its complex conjugated equation is constituted by

$$\begin{aligned} 0 =& \,2i\gamma\left(\frac{Z^U(T)}{Y(T)} - y, \frac{\delta Z(T)}{Y(T)}\right) + \int_0^T \left\langle \frac{\partial}{\partial s}\Phi^U(s), \delta Z(s)\right\rangle ds \\ &+ i\int_0^T \big\langle A\Phi^U(s), \delta Z(s)\big\rangle\, ds - i\int_0^T \big(\Phi^U(s), (U+\Theta\delta U)(s)\delta Z(s)\big)\, ds \\ &- i\int_0^T \big(\Phi^U(s), \Theta\delta U(s) Z^U(s)\big)\, ds \\ &- i\lambda\int_0^T B(s)\big(\Phi^U(s), f(Z^{U+\Theta\delta U}(s)) - f(Z^U(s))\big)\, ds. \end{aligned} \tag{3.44}$$

Subtracting equation (3.43) from equation (3.44) leads to

$$\begin{aligned} 0 =& \,4i\gamma\,\mathrm{Re}\left(\frac{Z^U(T)}{Y(T)} - y, \frac{\delta Z(T)}{Y(T)}\right) + 2i\,\mathrm{Im}\int_0^T \left\langle \frac{\partial}{\partial s}\Phi^U(s), \delta Z(s)\right\rangle ds \\ &+ 2i\,\mathrm{Re}\int_0^T \big\langle A\Phi^U(s), \delta Z(s)\big\rangle\, ds - 2i\,\mathrm{Re}\int_0^T \big(\Phi^U(s), (U+\Theta\delta U)(s)\delta Z(s)\big)\, ds \\ &- 2i\,\mathrm{Re}\int_0^T \big(\Phi^U(s), \Theta\delta U(s) Z^U(s)\big)\, ds \\ &- 2i\lambda\,\mathrm{Re}\int_0^T B(s)\big(\Phi^U(s), f(Z^{U+\Theta\delta U}(s)) - f(Z^U(s))\big)\, ds. \end{aligned} \tag{3.45}$$

Finally, adding equations (3.42) and (3.45) yields

$$\begin{aligned} 0 =& \,8i\gamma\,\mathrm{Re}\left(\frac{Z^U(T)}{Y(T)} - y, \frac{\delta Z(T)}{Y(T)}\right) \\ &+ 2i\,\mathrm{Im}\int_0^T \left\langle \frac{\partial}{\partial s}\Phi^U(s), \delta Z(s)\right\rangle ds + 2i\,\mathrm{Im}\int_0^T \overline{\left\langle \frac{\partial}{\partial s}\delta Z(s), \Phi^U(s)\right\rangle}\, ds \\ &- 2i\,\mathrm{Re}\int_0^T \big(\Phi^U(s), \Theta\delta U(s)\delta Z(s)\big)\, ds - 2i\,\mathrm{Re}\int_0^T \big(\Phi^U(s), \Theta\delta U(s) Z^U(s)\big)\, ds \\ &- 2i\lambda\,\mathrm{Re}\int_0^T B(s)\big(\Phi^U(s), f(Z^{U+\Theta\delta U}(s)) - f(Z^U(s))\big)\, ds \\ &+ 2i\lambda(\sigma+1)\,\mathrm{Re}\int_0^T B(s)\big(\Phi^U(s), \big|Z^U(s)\big|^{2\sigma}\delta Z(s)\big)\, ds \\ &+ 2i\lambda\sigma\,\mathrm{Re}\int_0^T B(s)\big(\Phi^U(s), \big|Z^U(s)\big|^{2(\sigma-1)}\big(Z^U(s)\big)^2\,\overline{\delta Z}(s)\big)\, ds. \end{aligned} \tag{3.46}$$

We reverse one partial integration with respect to the time by

$$\begin{aligned}
&2i\,\mathrm{Im}\int_0^T \left\langle \frac{\partial}{\partial s}\Phi^U(s), \delta Z(s)\right\rangle ds + 2i\,\mathrm{Im}\int_0^T \overline{\left\langle \frac{\partial}{\partial s}\delta Z(s), \Phi^U(s)\right\rangle}\, ds \\
&= 2i\,\mathrm{Im}\left(\Phi^U(T), \delta Z(T)\right) - 2i\,\mathrm{Im}\left(\Phi^U(0), \delta Z(0)\right) \\
&\quad - 2i\,\mathrm{Im}\int_0^T \overline{\left\langle \frac{\partial}{\partial s}\delta Z(s), \Phi^U(s)\right\rangle}\, ds + 2i\,\mathrm{Im}\int_0^T \overline{\left\langle \frac{\partial}{\partial s}\delta Z(s), \Phi^U(s)\right\rangle}\, ds \\
&= -4i\gamma\,\mathrm{Re}\left(\frac{Z^U(T)}{Y(T)} - y, \frac{\delta Z(T)}{Y(T)}\right).
\end{aligned}$$

Hence, taking expectation of equation (3.46) divided by $2i$ results in

$$\begin{aligned}
&2\gamma E\,\mathrm{Re}\left(\frac{Z^U(T)}{Y(T)} - y, \frac{\delta Z(T)}{Y(T)}\right) \\
&= E\,\mathrm{Re}\int_0^T \left(\Phi^U(s), \Theta\delta U(s)\delta Z(s)\right) ds + E\,\mathrm{Re}\int_0^T \left(\Phi^U(s), \Theta\delta U(s) Z^U(s)\right) ds \\
&\quad + \lambda E\,\mathrm{Re}\int_0^T B(s)\left(\Phi^U(s), f(Z^{U+\Theta\delta U}(s)) - f(Z^U(s))\right) ds \\
&\quad - \lambda(\sigma+1)E\,\mathrm{Re}\int_0^T B(s)\left(\Phi^U(s), \left|Z^U(s)\right|^{2\sigma}\delta Z(s)\right) ds \\
&\quad - \lambda\sigma E\,\mathrm{Re}\int_0^T B(s)\left(\Phi^U(s), \left|Z^U(s)\right|^{2(\sigma-1)}\left(Z^U(s)\right)^2\overline{\delta Z}(s)\right) ds.
\end{aligned}$$

Now, the numerator of the Gâteaux differential (3.39) is rewritten as

$$\begin{aligned}
J(U+\Theta\delta U) - J(U) = &\gamma E\left(B^{\frac{1}{\sigma}}(T)\left\|\delta Z(T)\right\|^2\right) + \beta\Theta^2 E\int_0^T \left\|\delta U(t)\right\|^2 dt \\
&+ E\,\mathrm{Re}\int_0^T \left(\Phi^U(t), \Theta\delta U(t) Z^U(t)\right) dt \\
&+ 2\beta\Theta E\int_0^T \left(U(t) - \Upsilon(t), \delta U(t)\right) dt + R_1 + R_2
\end{aligned} \tag{3.47}$$

with

$$\begin{aligned}
R_1 :=\, &E\,\mathrm{Re}\int_0^T \left(\Phi^U(t), \Theta\delta U(t)\delta Z(t)\right) dt, \\
R_2 :=\, &\lambda E\,\mathrm{Re}\int_0^T B(t)\left(\Phi^U(t), f(Z^{U+\Theta\delta U}(t)) - f(Z^U(t))\right) dt \\
&- \lambda(\sigma+1)E\,\mathrm{Re}\int_0^T B(t)\left(\Phi^U(t), \left|Z^U(t)\right|^{2\sigma}\delta Z(t)\right) dt \\
&- \lambda\sigma E\,\mathrm{Re}\int_0^T B(t)\left(\Phi^U(t), \left|Z^U(t)\right|^{2(\sigma-1)}\left(Z^U(t)\right)^2\overline{\delta Z}(t)\right) dt.
\end{aligned}$$

To calculate the Gâteaux differential, we have to divide the terms on the right-hand side of (3.47) by Θ and investigate its limit with respect to $\Theta \to 0$. At first, we state by Theorem 3.2.2 that

$$0 \le \lim_{\Theta\to 0}\frac{1}{\Theta}\gamma E\left(B^{\frac{1}{\sigma}}(T)\left\|\delta Z(T)\right\|^2\right) \le \lim_{\Theta\to 0} C(\sigma,\lambda,\varphi,\gamma,\alpha_1,C_B,T)\,\Theta\left\|\delta U\right\|^2_{L^\infty(\Omega\times[0,T]\times[0,1])} = 0.$$

Furthermore, we obtain

$$\lim_{\Theta\to 0}\frac{1}{\Theta}\beta\Theta^2 E\int_0^T \left\|\delta U(t)\right\|^2 dt = \lim_{\Theta\to 0}\beta\Theta E\int_0^T \left\|\delta U(t)\right\|^2 dt = 0,$$

and it obviously holds that

$$\lim_{\Theta\to 0}\frac{1}{\Theta}E\,\mathrm{Re}\int_0^T \big(\Phi^U(t),\Theta\delta U(t)Z^U(t)\big)\,dt = E\,\mathrm{Re}\int_0^T \big(\Phi^U(t),\delta U(t)Z^U(t)\big)\,dt$$

and

$$\lim_{\Theta\to 0}\frac{1}{\Theta}2\beta\Theta E\int_0^T \big(U(t)-\Upsilon(t),\delta U(t)\big)\,dt = 2\beta E\int_0^T \big(U(t)-\Upsilon(t),\delta U(t)\big)\,dt.$$

Therefore, the Gâteaux differential for all $\delta U\in\mathcal{U}$ reduces to

$$\begin{aligned}\delta J(U;\delta U) =& E\,\mathrm{Re}\int_0^T \big(\Phi^U(t),\delta U(t)Z^U(t)\big)\,dt\\ &+2\beta E\int_0^T \big(U(t)-\Upsilon(t),\delta U(t)\big)\,dt+\lim_{\Theta\to 0}\frac{R_1+R_2}{\Theta}.\end{aligned}$$

Based on Theorem 3.2.2 and estimate (3.28) combined with the result of Theorem 3.2.12 (a), the term R_1 is estimated from above by

$$\begin{aligned}R_1 \le& E\int_0^T \big|\big(\Phi^U(t),\Theta\delta U(t)\delta Z(t)\big)\big|\,dt\\ \le& \Theta\,\|\delta U\|_{L^\infty(\Omega\times[0,T]\times[0,1])}\,E\int_0^T \big|\big(\Phi^U(t),\delta Z(t)\big)\big|\,dt\\ \le& \Theta\,\|\delta U\|_{L^\infty(\Omega\times[0,T]\times[0,1])}\left(E\int_0^T \big\|\Phi^U(t)\big\|^2\,dt\right)^{\frac12}\left(E\int_0^T \|\delta Z(t)\|^2\,dt\right)^{\frac12}\\ \le& C(\sigma,\lambda,\varphi,\gamma,y,\alpha_1,C_B,T)\,\Theta^2\,\|\delta U\|^2_{L^\infty(\Omega\times[0,T]\times[0,1])}.\end{aligned}$$

Thus, it follows that

$$0\le \lim_{\Theta\to 0}\frac{|R_1|}{\Theta}\le \lim_{\Theta\to 0}C(\sigma,\lambda,\varphi,\gamma,y,\alpha_1,C_B,T)\,\Theta\,\|\delta U\|^2_{L^\infty(\Omega\times[0,T]\times[0,1])}=0,$$

which implies

$$\lim_{\Theta\to 0}\frac{R_1}{\Theta}=0.$$

The last term R_2 obeys

$$\begin{aligned}R_2 =& \lambda E\,\mathrm{Re}\int_0^T B(t)\Big(\Phi^U(t),f(Z^{U+\Theta\delta U}(t))-f(Z^U(t))-(\sigma+1)\big|Z^U(t)\big|^{2\sigma}\delta Z(t)\\ &-\sigma\big|Z^U(t)\big|^{2(\sigma-1)}\big(Z^U(t)\big)^2\,\overline{\delta Z}(t)\Big)\,dt.\end{aligned}\tag{3.48}$$

Taking into account that $f(v)=f(v,\overline{v})=|v|^{2\sigma}v=v^{\sigma+1}\overline{v}^{\sigma}$ for all $v\in\mathbb{C}$ and applying the linear Taylor series expansion of $f(Z^{U+\Theta\delta U}(t))$ at the point $Z^U(t)$ in Lagrange form, we get for $\vartheta\in(0,1)$ and based on the Wirtinger derivatives (see Appendix H) that

$$\begin{aligned}f(Z^{U+\Theta\delta U}(t)) =& f(Z^U(t))+(\sigma+1)\big|Z^U(t)\big|^{2\sigma}\delta Z(t)+\sigma\big|Z^U(t)\big|^{2(\sigma-1)}\big(Z^U(t)\big)^2\,\overline{\delta Z}(t)\\ &+\frac12\sigma(\sigma+1)\big|Z^U(t)+\vartheta\delta Z(t)\big|^{2(\sigma-1)}\big(\overline{Z^U}(t)+\vartheta\overline{\delta Z}(t)\big)\,(\delta Z(t))^2\\ &+\frac12 2\sigma(\sigma+1)\big|Z^U(t)+\vartheta\delta Z(t)\big|^{2(\sigma-1)}\big(Z^U(t)+\vartheta\delta Z(t)\big)\,\delta Z(t)\,\overline{\delta Z}(t)\\ &+\frac12\sigma(\sigma-1)\big|Z^U(t)+\vartheta\delta Z(t)\big|^{2(\sigma-2)}\big(Z^U(t)+\vartheta\delta Z(t)\big)^3\,(\overline{\delta Z}(t))^2.\end{aligned}$$

Plugging this expansion in (3.48), only the quadratic remainders are left such that, by denoting $\mathcal{Z}(t) := Z^U(t) + \vartheta \delta Z(t)$, it results that

$$\begin{aligned} R_2 = \frac{1}{2}\lambda\sigma E \operatorname{Re} \int_0^T B(t)\Big(\Phi^U(t), (\sigma+1)\big|\mathcal{Z}(t)\big|^{2(\sigma-1)}\,\overline{\mathcal{Z}}(t)\,\big(\delta Z(t)\big)^2 \\ + 2(\sigma+1)\big|\mathcal{Z}(t)\big|^{2(\sigma-1)}\mathcal{Z}(t)\,\delta Z(t)\,\overline{\delta Z}(t) \\ + (\sigma-1)\big|\mathcal{Z}(t)\big|^{2(\sigma-2)}\big(\mathcal{Z}(t)\big)^3\,\big(\overline{\delta Z}(t)\big)^2\Big)\,dt. \end{aligned}$$

Writing the inner product as an integral, enlarging the real part by the absolute value and regarding that $|\sigma-1| \le \sigma+1$ for $\sigma \in [\frac{1}{2}, 2)$, we entail

$$\begin{aligned} R_2 \le & \frac{1}{2}\lambda\sigma B(T)\, E \int_0^T \int_0^1 \big|\Phi^U(t,x)\big| \Big[(\sigma+1)\big|\mathcal{Z}(t,x)\big|^{2\sigma-1}\,|\delta Z(t,x)|^2 \\ & + 2(\sigma+1)\big|\mathcal{Z}(t,x)\big|^{2\sigma-1}\,|\delta Z(t,x)|^2 \\ & + |\sigma-1|\big|\mathcal{Z}(t,x)\big|^{2\sigma-1}\,|\delta Z(t,x)|^2\Big]\,dx\,dt \\ \le & 2\lambda\sigma(\sigma+1)B(T)\,E\int_0^T\int_0^1 \big|\Phi^U(t,x)\big|\,\big|\mathcal{Z}(t,x)\big|^{2\sigma-1}\,|\delta Z(t,x)|^2\,dx\,dt. \end{aligned}$$

For $\sigma = \frac{1}{2}$ it holds that $\big|\mathcal{Z}(t,x)\big|^{2\sigma-1} = 1$. Moreover, if $\sigma \in (\frac{1}{2}, 2)$, the application of Lemma D.2 leads to

$$\begin{aligned} R_2 \le & 2\lambda\sigma(\sigma+1)B(T)\,E\int_0^T \left(\sup_{x\in[0,1]}\big|\Phi^U(t,x)\big|\right)\left(\sup_{x\in[0,1]}\big|\mathcal{Z}(t,x)\big|^{2\sigma-1}\right)\|\delta Z(t)\|^2\,dt \\ \le & 2\sqrt{2}\lambda\sigma(\sigma+1)B(T)\,E\int_0^T \big\|\Phi^U(t)\big\|_V \left(\sup_{x\in[0,1]}\big|\mathcal{Z}(t,x)\big|^{2\sigma-1}\right)\|\delta Z(t)\|^2\,dt. \end{aligned}$$

Since $\vartheta \in (0,1)$, it especially follows by Lemma D.1 for $\sigma \in (\frac{1}{2}, 2)$ that

$$\begin{aligned} \big|\mathcal{Z}(t,x)\big|^{2\sigma-1} = & \big|Z^U(t,x) + \vartheta\delta Z(t,x)\big|^{2\sigma-1} \le \Big(\big|Z^U(t,x)\big| + \vartheta\big|\delta Z(t,x)\big|\Big)^{2\sigma-1} \\ \le & 2^{2\sigma-1}\Big(\big|Z^U(t,x)\big|^{2\sigma-1} + \big|\delta Z(t,x)\big|^{2\sigma-1}\Big) \\ = & 2^{2\sigma-1}\Big(\big|Z^U(t,x)\big|^{2\sigma-1} + \big|Z^{U+\Theta\delta U}(t,x) - Z^U(t,x)\big|^{2\sigma-1}\Big) \\ \le & 2^{2\sigma-1}\Big(\big|Z^U(t,x)\big|^{2\sigma-1} + \Big(\big|Z^{U+\Theta\delta U}(t,x)\big| + \big|Z^U(t,x)\big|\Big)^{2\sigma-1}\Big) \\ \le & 2^{2\sigma-1}\Big(\big|Z^U(t,x)\big|^{2\sigma-1} + 2^{2\sigma-1}\Big(\big|Z^{U+\Theta\delta U}(t,x)\big|^{2\sigma-1} + \big|Z^U(t,x)\big|^{2\sigma-1}\Big)\Big). \end{aligned}$$

We resume by Lemma D.2 that

$$\begin{aligned} \sup_{x\in[0,1]}\big|\mathcal{Z}(t,x)\big|^{2\sigma-1} \le & 2^{2(2\sigma-1)}\sup_{x\in[0,1]}\big|Z^{U+\Theta\delta U}(t,x)\big|^{2\sigma-1} + 2^{2\sigma-1}\big[1+2^{2\sigma-1}\big]\sup_{x\in[0,1]}\big|Z^U(t,x)\big|^{2\sigma-1} \\ \le & 2^{\frac{5}{2}(2\sigma-1)}\big\|Z^{U+\Theta\delta U}(t)\big\|_V^{2\sigma-1} + 2^{\frac{3}{2}(2\sigma-1)}\big[1+2^{2\sigma-1}\big]\big\|Z^U(t)\big\|_V^{2\sigma-1} \\ \le & C(\sigma)\Big(\big\|Z^{U+\Theta\delta U}(t)\big\|_V^{2\sigma-1} + \big\|Z^U(t)\big\|_V^{2\sigma-1}\Big). \end{aligned}$$

Hence, with relation (3.11), Theorem 3.2.12 (b) and finally Theorem 3.2.2 we deduce

$$\begin{aligned} R_2 \le & C(\sigma,\lambda,C_B)\,E\int_0^T \big\|\Phi^U(t)\big\|_V\Big(\big\|Z^{U+\Theta\delta U}(t)\big\|_V^{2\sigma-1} + \big\|Z^U(t)\big\|_V^{2\sigma-1}\Big)\|\delta Z(t)\|^2\,dt \\ \le & C(\sigma,\lambda,\varphi,\gamma,y,\alpha_1,\alpha_2,C_B,T)\,E\int_0^T \|\delta Z(t)\|^2\,dt \\ \le & C(\sigma,\lambda,\varphi,\gamma,y,\alpha_1,\alpha_2,C_B,T)\,\Theta^2\,\|\delta U\|^2_{L^\infty(\Omega\times[0,T]\times[0,1])}. \end{aligned}$$

We see that

$$0 \leq \lim_{\Theta \to 0} \frac{|R_2|}{\Theta} \leq \lim_{\Theta \to 0} C(\sigma, \lambda, \varphi, \gamma, y, \alpha_1, \alpha_2, C_B, T)\, \Theta\, \|\delta U\|^2_{L^\infty(\Omega \times [0,T] \times [0,1])} = 0$$

and, therefore,

$$\lim_{\Theta \to 0} \frac{R_2}{\Theta} = 0.$$

Thus, the Gâteaux differential of the objective functional (3.20) has the form

$$\begin{aligned} \delta J(U; \delta U) &= E\,\mathrm{Re} \int_0^T \big(\Phi^U(t), \delta U(t) Z^U(t)\big)\, dt + 2\beta E \int_0^T \big(U(t) - \Upsilon(t), \delta U(t)\big)\, dt \\ &= E\,\mathrm{Re} \int_0^T \big(\Phi^U(t)\, \overline{Z^U}(t), \delta U(t)\big)\, dt + 2\beta E \int_0^T \big(U(t) - \Upsilon(t), \delta U(t)\big)\, dt \end{aligned}$$

for all $\delta U \in \mathcal{U}$. □

Now, we generalize the result of Theorem 3.2.14 for the objective functional (3.5) corresponding to the stochastic controlled Schrödinger problem (3.6) by applying the transformation formula $Z^U(t,\cdot) = X^U(t,\cdot) Y(t)$.

Corollary 3.2.15. *Under the above assumptions and the restriction of $\sigma \in [\frac{1}{2}, 2)$, the Gâteaux differential of the objective functional* (3.5) *is constituted by*

$$\delta J(U; \delta U) = E\,\mathrm{Re} \int_0^T \big(\Phi^U(t)\, \overline{X^U}(t)\, \overline{Y}(t), \delta U(t)\big)\, dt + 2\beta E \int_0^T \big(U(t) - \Upsilon(t), \delta U(t)\big)\, dt \tag{3.49}$$

for all $\delta U \in \mathcal{U}$.

Summarized, Theorem 3.1.2 entails that the optimal control problem (3.5) based on the controlled Schrödinger problem (3.6) has a unique solution for a.e. $\Upsilon \in L^2(\Omega \times [0,T]; H)$. Using the pathwise approach and the variational solutions of the difference of two controlled Schrödinger problems (3.17) and of the complex conjugated adjoint Schödinger problem (3.38), we obtain the Gâteaux differential (3.49) of the objective functional (3.5) in Corollary 3.2.15. Since relation (3.49) holds for all $\delta U \in \mathcal{U}$, it is especially true for arbitrary controls $U \in \mathcal{U}$ and for the optimal control $U^* \in \mathcal{U}$ that minimizes the objective functional (3.5). Hence, it follows that

$$\delta J(U^*, U - U^*) \geq 0, \qquad \text{for all } U \in \mathcal{U}.$$

This inequality is equivalent to the following necessary condition of optimality that we indicate in form of a stochastic variational inequality (compare [66, 95]), which can only be solved numerically.

Corollary 3.2.16. *Denoting $X^* := X^{U^*}$ and $\Phi^* := \Phi^{U^*}$ and observing that $\Phi^*(t)$ is only $\mathcal{F}_T$-measurable, the necessary optimality condition for the optimal control problem* (3.5), (3.6) *is constituted by*

$$E \int_0^T \bigg(\mathrm{Re} \Big\{ E\left[\Phi^*(t) | \mathcal{F}_t\right] \overline{X^*}(t)\, \overline{Y}(t) \Big\} + 2\beta \Big(U^*(t) - E\left[\Upsilon(t) | \mathcal{F}_t\right]\Big), U(t) - U^*(t) \bigg)\, dt \geq 0 \tag{3.50}$$

for all $U \in \mathcal{U}$.

The left argument of the inner product in (3.50) defines a linear continuous functional of the $\mathcal{F}_t$-adapted processes in $L^2(\Omega \times [0,T]; H)$. By Riesz' representation theorem, we can identify this functional with the gradient of the form

$$\mathrm{Re} \Big\{ E\left[\Phi^*(t) | \mathcal{F}_t\right] \overline{X^*}(t)\, \overline{Y}(t) \Big\} + 2\beta \Big(U^*(t) - E\left[\Upsilon(t) | \mathcal{F}_t\right]\Big)$$

for all $\omega \in \Omega$, all $t \in [0,T]$ and all $x \in [0,1]$.

In Lemma 3.1.1 we presumed that the control $U \in \mathcal{U}$ depends either on the time variable $t \in [0,T]$ or on the space variable $x \in [0,1]$. That is the reason why we indicate how inequality (3.50) simplifies for these two special cases.

(i) Firstly, let $U(t,x) = U_1(t) =: U(t)$ and, therefore, $\Upsilon(t,x) = \Upsilon(t)$, then (3.50) is equivalent to

$$\begin{aligned}
&E\int_0^T\int_0^1 \mathrm{Re}\left\{E\left[\Phi^*(t,x)|\,\mathcal{F}_t\right]\overline{X^*}(t,x)\overline{Y}(t)\right\}\big(U(t)-U^*(t)\big)\,dx\,dt\\
&+2\beta E\int_0^T\int_0^1 \Big(U^*(t)-E\left[\Upsilon(t)|\,\mathcal{F}_t\right]\Big)\big(U(t)-U^*(t)\big)\,dx\,dt\\
=&E\int_0^T \mathrm{Re}\left\{\int_0^1 E\left[\Phi^*(t,x)|\,\mathcal{F}_t\right]\overline{X^*}(t,x)\,dx\,\overline{Y}(t)\right\}\big(U(t)-U^*(t)\big)\,dt\\
&+2\beta\, E\int_0^T \Big(U^*(t)-E\left[\Upsilon(t)|\,\mathcal{F}_t\right]\Big)\big(U(t)-U^*(t)\big)\,dt \geq 0.
\end{aligned}$$

Thus, it follows the necessary optimality condition

$$\left[\mathrm{Re}\left\{\int_0^1 E\left[\Phi^*(t,x)|\,\mathcal{F}_t\right]\overline{X^*}(t,x)\,dx\,\overline{Y}(t)\right\}+2\beta\Big(U^*(t)-E\left[\Upsilon(t)|\,\mathcal{F}_t\right]\Big)\right]\big(U(t)-U^*(t)\big)\geq 0$$

for a.e. $\omega \in \Omega$ and Lebesgue almost all $t \in [0,T]$.

(ii) Secondly, let $U(t,x) = U_2(x) =: U(x)$ and $\Upsilon(t,x) = \Upsilon(x)$ which are deterministic now (see Lemma 3.1.1). Then we conclude by reasons of measurability that relation (3.50) reduces to

$$\begin{aligned}
&E\int_0^T\int_0^1 \mathrm{Re}\left\{E\left[\Phi^*(t,x)|\,\mathcal{F}_t\right]\overline{X^*}(t,x)\overline{Y}(t)\right\}\big(U(x)-U^*(x)\big)\,dx\,dt\\
&+2\beta E\int_0^T\int_0^1 \big(U^*(x)-\Upsilon(x)\big)\big(U(x)-U^*(x)\big)\,dx\,dt\\
=&\int_0^1\int_0^T \mathrm{Re}\left\{E\left(E\left[\Phi^*(t,x)\overline{X^*}(t,x)\overline{Y}(t)\big|\,\mathcal{F}_t\right]\right)\right\}dt\,\big(U(x)-U^*(x)\big)\,dx\\
&+2\beta T\int_0^1 \big(U^*(x)-\Upsilon(x)\big)\big(U(x)-U^*(x)\big)\,dx\\
=&\left(\int_0^T \mathrm{Re}\left\{E\left(\Phi^*(t)\overline{X^*}(t)\overline{Y}(t)\right)\right\}dt+2\beta T\left(U^*-\Upsilon\right),U-U^*\right)\geq 0.
\end{aligned}$$

3.2.5 Further Remarks

In Section 3.2, we considered the controlled Schrödinger problem (3.6) and the corresponding pathwise controlled Schrödinger problem (3.8) for $\lambda \in \mathbb{R}_+$. Without loss of generality, we can vary this prefactor of the nonlinear term (compare Subsection 2.3.3). Thus, it is possible to treat the case $\lambda < 0$ in the controlled Schrödinger problem (3.6) analogously to Section 3.2. The only thing we have to take care of is to proceed to $|\lambda|$ after enlarging the real and imaginary parts by the absolute values. However, remember the additional assumptions in Corollary 2.3.10 ensuring the unique existence of the variational solution of problem (3.6).

Furthermore, we can take into account the case $\lambda = i\tilde{\lambda}$ with $\tilde{\lambda} \in \mathbb{R}_+$ as an appropriate prefactor of the nonlinear term in the controlled Schrödinger problem (3.6) and (3.8), respectively. Although we follow the approach of Section 3.2, we refer to Corollary 2.3.11 and choose $\sigma \in [1,2)$ to apply Lemma D.6 and exploit the ideas of the uniform a priori estimate in V from Section 2.2. Therefore, we do not need the case differentiation for the control term stated in Lemma 3.1.1 and the a priori estimate in V simplifies for each $n \in \mathbb{N}$ (arbitrarily fixed) to

$$\left\|Z_n^U(t)\right\|_V^2 \leq C(\alpha_2,T)\left\|\varphi\right\|_V^2 \leq C(\varphi,\alpha_2,T), \qquad \text{for all } t \in [0,T].$$

Observe that the real and imaginary parts swap after replacing λ by $i\tilde{\lambda}$ with $\tilde{\lambda} > 0$ in problem (3.6) and the analogue of the complex conjugated adjoint Schrödinger problem (3.38) is given by

$$\begin{aligned}\big(\Phi^U(t), v(t)\big) =& -2i\gamma \frac{1}{\overline{Y}(T)}\left(\frac{Z^U(T)}{Y(T)} - y, v(T)\right) - \int_t^T \overline{\left\langle \frac{\partial}{\partial s} v(s), \Phi^U(s)\right\rangle}\, ds \\ &+ i\int_t^T \langle A\Phi^U(s), v(s)\rangle\, ds - i\int_t^T \big(U(s)\Phi^U(s), v(s)\big)\, ds \\ &- \tilde{\lambda}(\sigma+1)\int_t^T B(s)\big(\left|Z^U(s)\right|^{2\sigma}\Phi^U(s), v(s)\big)\, ds \\ &+ \tilde{\lambda}\sigma\int_t^T B(s)\big(\left|Z^U(s)\right|^{2(\sigma-1)}\big(Z^U(s)\big)^2\,\overline{\Phi^U}(s), v(s)\big)\, ds\end{aligned} \tag{3.51}$$

for all $t \in [0, T]$ and all $v \in L^2([0, T]; V')$.

Consequently, in both cases ($\lambda < 0$ and $\lambda = i\tilde{\lambda}$ with $\tilde{\lambda} > 0$), the optimal control problem (3.5), (3.6) possesses a unique solution $U^* \in \mathcal{U}$ that minimizes the objective functional (3.5). By combining the variational formulation (3.17) of the difference $\delta Z = Z^{U+\Theta\delta U} - Z^U$ of the two controlled Schrödinger problems (3.8) and (3.12) and the variational formulation (3.38) and (3.51), respectively, of the complex conjugated adjoint Schrödinger problem, we obtain the same gradient formula (3.49) in the sense of Gâteaux as in the case $\lambda \in \mathbb{R}_+$.

Corollary 3.2.17. *Let the above assumptions be satisfied and regard the controlled Schrödinger problem* (3.6) *and* (3.8)*, respectively, for the case $\lambda < 0$ and $\sigma \in [\frac{1}{2}, 2)$ or the case $\lambda = i\tilde{\lambda}$ with $\tilde{\lambda} \in \mathbb{R}_+$ and $\sigma \in [1, 2)$. Then the Gâteaux differential of the objective functional* (3.5) *has the form*

$$\delta J(U; \delta U) = E\,\mathrm{Re}\int_0^T \big(\Phi^U(t)\,\overline{X^U}(t)\,\overline{Y}(t), \delta U(t)\big)\, dt + 2\beta E\int_0^T \big(U(t) - \Upsilon(t), \delta U(t)\big)\, dt$$

for all $\delta U \in \mathcal{U}$.

From the pure mathematical point of view, we can think of one more case of admissible controls $U \in \mathcal{U}$ for the pathwise controlled Schrödinger problem (3.6). Having investigated the two cases (i) and (ii) in Lemma 3.1.1, there arises the idea of combining them in the way that $U(s, x) = U_1(s)U_2(x)$. Analogously to the calculation in case (ii) and due to integration by parts with respect to the time variable, it holds for all $v \in L^2([0, T]; H')$ with a representation that is separated in time and space and for all $t \in [0, T]$ that

$$\begin{aligned}&2\,\mathrm{Re}\int_0^t \left(U(s)v(s), \frac{\partial}{\partial s} v(s)\right) ds \\ &= 2\int_0^t U_1(s)\int_0^1 U_2(x)\,\mathrm{Re}\left\{v(s,x)\,\frac{\partial}{\partial s}\overline{v}(s,x)\right\} dx\, ds = \int_0^t U_1(s)\frac{d}{ds}\left(\int_0^1 U_2(x)\,|v(s,x)|^2\, dx\right) ds \\ &= U_1(t)\left(\int_0^1 U_2(x)\,|v(t,x)|^2\, dx\right) - U_1(0)\left(\int_0^1 U_2(x)\,|v(0,x)|^2\, dx\right) \\ &\quad - \int_0^t \left(\frac{d}{ds}U_1(s)\right)\left(\int_0^1 U_2(x)\,|v(s,x)|^2\, dx\right) ds.\end{aligned}$$

By definition, $U(s, x) = U_1(s)U_2(x) \geq 0$ P-a.s. for all $s \in [0, t]$ and all $x \in [0, 1]$, which implies that $U_1(s)$ and $U_2(x)$ have the same signs. Thus, assuming that $\frac{d}{ds}U_1(s)$ also has the same sign as $U_2(x)$ for all $s \in [0, t]$ and all $x \in [0, 1]$, we can neglect the last two terms on the right-hand side since they are negative ones. We conclude for all $t \in [0, T]$ that

$$\begin{aligned}2\,\mathrm{Re}\int_0^t \left(U(s)v(s), \frac{\partial}{\partial s} v(s)\right) ds &\leq U_1(t)\left(\int_0^1 U_2(x)\,|v(t,x)|^2\, dx\right) \\ &= \int_0^1 U(t,x)\,|v(t,x)|^2\, dx \leq \alpha_1 \left\|v(t)\right\|^2.\end{aligned}$$

Corollary 3.2.18. *We can add a third case of admissible controls $U \in \mathcal{U}$ in Lemma 3.1.1. Presuming that $U(t,x) := U_1(t)U_2(x)$ and its differentiability with respect to the time variable $t \in [0,T]$ such that $\frac{\partial}{\partial t}U(t,x) = \left[\frac{d}{dt}U_1(t)\right]U_2(x) \geq 0$, the methods of Lemma 3.1.1 also work. For all $v \in L^2([0,T];H')$ with a representation that is separated in time and space it follows that*

$$2\operatorname{Re}\int_0^t \left(U(s)v(s), \frac{\partial}{\partial s}v(s)\right) ds \leq \alpha_1 \|v(t)\|^2, \qquad \text{for all } t \in [0,T].$$

In this case, the necessary optimality condition for the optimal control problem (3.5), (3.6) results immediately from (3.50) and is given by

$$\left[\operatorname{Re}\left\{E\left[\Phi^*(t,x)|\,\mathcal{F}_t\right]\overline{X^*}(t,x)\overline{Y}(t)\right\} + 2\beta\Big(U^*(t,x) - E\left[\Upsilon(t,x)|\,\mathcal{F}_t\right]\Big)\right]\big(U(t,x) - U^*(t,x)\big) \geq 0$$

for a.e. $\omega \in \Omega$, Lebesgue almost all $t \in [0,T]$ and $x \in [0,1]$ and for all $U \in \mathcal{U}$.

As indicated at the beginning of Subsection 3.2.3, there are problems in generally establishing the complex conjugated adjoint Schrödinger equation with the present approach in Appendix I. Section 3.2 illustrates that the method applies for the controlled Schrödinger problem with linear multiplicative noise and power-type nonlinearity since it can be transformed into a pathwise controlled Schrödinger problem such that we can use the corresponding a priori estimates (compare Section 2.3). Thus, it arises the question if there is a chance to treat the controlled Schrödinger problem with Lipschitz continuous nonlinear drift term of bounded growth that will be answered in the following section.

3.3 Pathwise Problem with Lipschitz Continuous Drift Term

The following deliberations are based on Section 2.4 including the existence and uniqueness results of the stochastic nonlinear Schrödinger problem with a drift function f that is Lipschitz continuous and of bounded growth, while its prefactor λ is assumed to be complex-valued. Due to Section 3.1, we already know that there exists a unique variational solution of this Schrödinger problem including an additional control term of the form (3.3). Moreover, Theorem 3.1.2 ensures that there exists a unique optimal control $U^* \in \mathcal{U}$ that minimizes the objective functional (3.5) given by

$$J(U) = \gamma E\left\|X^U(T) - y\right\|^2 + \beta E\int_0^T \|U(t) - \Upsilon(t)\|^2\,dt, \qquad \text{for all } U \in \mathcal{U},$$

if we can show the lower semi-continuity of $F(U) = \gamma E\left\|X^U(T) - y\right\|^2$ for all $U \in \mathcal{U}$. In order to exploit the pathwise approach of Section 3.2, we consider the controlled Schrödinger problem with linear multiplicative noise (compare Subsection 2.4.2). We remember Lemma 2.4.1 implying the Lipschitz constant c_f and the growth constant k_f, which appear as parameters in the constants while estimating the function $f(t,z) = f(t,z,\overline{z})$ for all $t \in [0,T]$ and all $z \in \mathbb{C}$. Notice that we only indicate the results and emphasize the changes to Section 3.2 to prevent redundancy.

Referring to Section 3.2 and the relation $Z^U(t,\cdot) = X^U(t,\cdot)Y(t)$, we transfer the controlled Schrödinger problem with linear multiplicative noise

$$\begin{aligned}\big(X^U(t),v\big) = \big(\varphi,v\big) &- i\int_0^t \big\langle AX^U(s),v\big\rangle\,ds + i\int_0^t \big(U(s)X^U(s),v\big)\,ds \\ &+ i\lambda\int_0^t \big(f(s,X^U(s)),v\big)\,ds + i\sum_{j=1}^{\infty}\int_0^t b_j(s)\big(X^U(s),v\big)\,d\beta_j(s)\end{aligned} \tag{3.52}$$

for a.e. $\omega \in \Omega$, all $t \in [0,T]$ and all $v \in V$, into the pathwise controlled Schrödinger problem

$$\begin{aligned}\big(Z^U(t),v\big) = \big(\varphi,v\big) &- i\int_0^t \big\langle AZ^U(s),v\big\rangle\,ds + i\int_0^t \big(U(s)Z^U(s),v\big)\,ds \\ &+ i\lambda\int_0^t \left(f\left(s,\frac{Z^U(s)}{Y(s)}\right),v\right) ds\end{aligned} \tag{3.53}$$

for a.e. $\omega \in \Omega$ (arbitrarily fixed), all $t \in [0,T]$ and all $v \in V$. Due to the uniform a priori estimates (2.85), it results for all $t \in [0,T]$ that

$$\left\|Z^U(t)\right\|^2 \leq C(|\lambda|, \varphi, k_f, T) \qquad \text{and} \qquad \operatorname*{ess\,sup}_{t\in[0,T]} \left\|Z^U(t)\right\|_V^2 \leq C(|\lambda|, \varphi, \alpha_1, k_f, T). \tag{3.54}$$

Analogously to Subsection 3.2.1, there exists a unique variational solution $\delta Z = Z^{U+\Theta\delta U} - Z^U$ with $\delta Z \in C([0,T];H) \cap L^2([0,T];V)$ of the pathwise Schrödinger problem

$$\begin{aligned} \big(\delta Z(t), v\big) = &- i \int_0^t \big\langle A\delta Z(s), v\big\rangle\, ds + i \int_0^t \big((U+\Theta\delta U)(s)\delta Z(s), v\big)\, ds \\ &+ i \int_0^t \big(\Theta\delta U(s) Z^U(s), v\big)\, ds \\ &+ i\lambda \int_0^t \left(f\left(s, \frac{Z^{U+\Theta\delta U}(s)}{Y(s)}\right) - f\left(s, \frac{Z^U(s)}{Y(s)}\right), v\right) ds \end{aligned} \tag{3.55}$$

for all $t \in [0,T]$ and all $v \in V$. This solution depends continuously on the control $\delta U \in \mathcal{U}$ in the way that there exists a positive constant C such that

$$\|\delta Z(t)\|^2 \leq C(\sigma, |\lambda|, \varphi, c_f, k_f, C_B, T)\, \Theta^2 \, \|\delta U\|^2_{L^\infty([0,T]\times[0,1])}, \qquad \text{for all } t \in [0,T], \tag{3.56}$$

where we exploit the first estimate in (3.54), the Lipschitz continuity of f and take $\lambda \in \mathbb{C}$ into account (that is reflected in the energy equality containing the real and the imaginary part of the nonlinear term). Now, it is possible to show that $\delta Z \in L^2([0,T];V')$ and the generalization of the variational formulation (3.55) is given by

$$\begin{aligned} \big(\delta Z(t), v(t)\big) = &\int_0^t \overline{\left\langle \frac{\partial}{\partial s} v(s), \delta Z(s)\right\rangle}\, ds - i \int_0^t \big\langle A\delta Z(s), v(s)\big\rangle\, ds \\ &+ i \int_0^t \big((U+\Theta\delta U)(s)\delta Z(s), v(s)\big)\, ds + i \int_0^t \big(\Theta\delta U(s) Z^U(s), v(s)\big)\, ds \\ &+ i\lambda \int_0^t \left(f\left(s, \frac{Z^{U+\Theta\delta U}(s)}{Y(s)}\right) - f\left(s, \frac{Z^U(s)}{Y(s)}\right), v(s)\right) ds \end{aligned} \tag{3.57}$$

for all $t \in [0,T]$ and all $v \in L^2([0,T];V')$. On the one hand, we need these results (especially inequality (3.56)) to show the continuity of the objective functional (3.20) given by

$$J(U) = \gamma E \left\| \frac{Z^U(T)}{Y(T)} - y \right\|^2 + \beta E \int_0^T \|U(t) - \Upsilon(t)\|^2\, dt, \qquad \text{for all } U \in \mathcal{U}.$$

This particularly implies the continuity and, therefore, also the lower semi-continuity of

$$F(U) = \gamma E \left\| \frac{Z^U(T)}{Y(T)} - y \right\|^2, \qquad \text{for all } U \in \mathcal{U}.$$

Hence, we can apply Theorem 3.1.2 that ensures the unique existence of the optimal control $U^* \in \mathcal{U}$, which minimizes the objective functional (see Subsection 3.2.2). On the other hand, we require the representation (3.57) to calculate a gradient formula of the objective functional (3.20) in order to obtain a necessary optimality condition.

Therefore, we analyze the complex conjugated adjoint Schrödinger problem (see Subsection 3.2.3). For simplified spelling, we introduce the notation of the Wirtinger derivatives by

$$\overline{f}_v\left(t, \frac{Z^U(t)}{Y(t)}\right) := \frac{\partial}{\partial v}\overline{f}(t, v(t))\Big|_{v(t):=\frac{Z^U(t)}{Y(t)}} := \frac{\partial}{\partial v}\overline{f}(t, v(t), \overline{v}(t))\Big|_{v(t):=\frac{Z^U(t)}{Y(t)},\, \overline{v}(t):=\frac{\overline{Z^U(t)}}{\overline{Y(t)}}},$$

$$f_{\overline{v}}\left(t, \frac{Z^U(t)}{Y(t)}\right) := \frac{\partial}{\partial \overline{v}} f(t, v(t))\Big|_{v(t):=\frac{Z^U(t)}{Y(t)}} := \frac{\partial}{\partial \overline{v}} f(t, v(t), \overline{v}(t))\Big|_{v(t):=\frac{Z^U(t)}{Y(t)},\, \overline{v}(t):=\frac{\overline{Z^U(t)}}{\overline{Y(t)}}}.$$

Thus, the complex conjugated adjoint problem of the pathwise controlled Schrödinger problem (3.53) is constituted by

$$\begin{aligned}
\frac{\partial}{\partial t}\Phi^U(t) =& - iA\Phi^U(t) + iU(t)\Phi^U(t) + i\overline{\lambda}\,\overline{f}_v\left(t, \frac{Z^U(t)}{Y(t)}\right)\frac{1}{\overline{Y}(t)}\,\Phi^U(t) \\
&+ i\lambda\, f_{\bar{v}}\left(t, \frac{Z^U(t)}{Y(t)}\right)\frac{1}{\overline{Y}(t)}\,\overline{\Phi^U}(t), \\
\Phi^U(T) =& - 2i\gamma\,\frac{1}{\overline{Y}(T)}\left[\frac{Z^U(T)}{Y(T)} - y\right]
\end{aligned}$$

in V^* for all $t \in [0,T]$ (for further details we refer to Appendix I). We say that a process $\Phi^U \in C([0,T];H) \cap L^2([0,T];V)$ is a variational solution of this complex conjugated adjoint Schrödinger problem, if it fulfills

$$\begin{aligned}
\left(\Phi^U(t), v\right) =& - 2i\gamma\,\frac{1}{\overline{Y}(T)}\left(\frac{Z^U(T)}{Y(T)} - y, v\right) + i\int_t^T \left\langle A\Phi^U(s), v\right\rangle ds \\
&- i\int_t^T \left(U(s)\Phi^U(s), v\right) ds - i\overline{\lambda}\int_t^T \left(\overline{f}_v\left(s, \frac{Z^U(s)}{Y(s)}\right)\frac{1}{\overline{Y}(s)}\,\Phi^U(s), v\right) ds \\
&- i\lambda\int_t^T \left(f_{\bar{v}}\left(s, \frac{Z^U(s)}{Y(s)}\right)\frac{1}{\overline{Y}(s)}\,\overline{\Phi^U}(s), v\right) ds
\end{aligned} \tag{3.58}$$

for all $t \in [0,T]$ and all $v \in V$. Regarding that $\lambda \in \mathbb{C}$, with the consequence that we get the real and imaginary parts after applying the energy equality, and using the fact that the absolute values of the first Wirtinger derivatives are assumed to be bounded by

$$|f_v(t,v)| = \left|\frac{\partial}{\partial v} f(t,v)\right| \le C', \qquad |f_{\bar{v}}(t,v)| = \left|\frac{\partial}{\partial \bar{v}} f(t,v)\right| \le C'$$

for all $t \in [0,T]$ and all $v \in \mathbb{C}$ (compare (2.79)), we deduce the results of Subsection 3.2.3 for the complex conjugated adjoint problem (3.58) of the pathwise controlled Schrödinger problem with Lipschitz continuous drift term of bounded growth. Thus, we obtain that if there exists a variational solution of (3.58), then it is unique. With the help of the Galerkin method, we reduce the infinite-dimensional complex conjugated adjoint Schrödinger problem to a finite-dimensional system of ordinary differential equations that possesses a unique solution and yields $\Phi_n^U \in C^1([0,T];H)$ for each $n \in \mathbb{N}$. Furthermore, we verify for arbitrarily fixed $n \in \mathbb{N}$ that

$$\left\|\Phi_n^U(t)\right\|^2 \le C(\sigma, |\lambda|, \varphi, \gamma, y, k_f, C', C_B, T), \qquad \text{for all } t \in [0,T].$$

To state a uniform a priori estimate in V as well, we have to assume that the drift function f is twice continuously differentiable and that the absolute values of the second derivatives of f in the sense of Wirtinger are also bounded, which means that

$$\begin{aligned}
|f_{vv}(t,v)| &= \left|\frac{\partial^2}{\partial v^2} f(t,v)\right| \le C'', \qquad |f_{v\bar{v}}(t,v)| = \left|\frac{\partial}{\partial \bar{v}}\frac{\partial}{\partial v} f(t,v)\right| \le C'', \\
|f_{\bar{v}v}(t,v)| &= \left|\frac{\partial}{\partial v}\frac{\partial}{\partial \bar{v}} f(t,v)\right| \le C'', \qquad |f_{\bar{v}\bar{v}}(t,v)| = \left|\frac{\partial^2}{\partial \bar{v}^2} f(t,v)\right| \le C''
\end{aligned} \tag{3.59}$$

for all $t \in [0,T]$ and all $v \in \mathbb{C}$. Observe that the two examples of functions in Section 2.4 satisfy these assumptions. Now, we get for (arbitrarily fixed) $n \in \mathbb{N}$ that

$$\left\|\Phi_n^U(t)\right\|_V^2 \le C(\sigma, |\lambda|, \varphi, \gamma, y, \alpha_1, \alpha_2, k_f, C', C'', C_B, T), \qquad \text{for all } t \in [0,T].$$

Analogously to Theorem 3.2.12 the sequence of variational solutions $(\Phi_n^U)_n$ of the corresponding Galerkin equations of (3.58) is bounded in $C([0,T];H), L^2([0,T];H)$ and $L^2([0,T];V)$. Moreover, this sequence is relatively compact in $L^2([0,T];H)$ and, therefore, converges strongly in

$L^2([0,T];H)$ and weakly in $L^2([0,T];V)$ to the variational solution Φ^U of the complex conjugated adjoint Schrödinger problem (3.58). We also receive that $\Phi^U \in L^\infty([0,T];V)$ with

$$\operatorname*{ess\,sup}_{t\in[0,T]} \left\|\Phi^U(t)\right\|_V^2 \le C(\sigma,|\lambda|,\varphi,\gamma,y,\alpha_1,\alpha_2,k_f,C',C'',C_B,T) \tag{3.60}$$

and that $(\Phi_n^U)_n$ converges to Φ^U in $C([0,T];H)$. To calculate the gradient formula, we prove that $\Phi^U \in L^2([0,T];V')$ and the generalization of the variational formulation (3.58) is given by

$$\begin{aligned}\big(\Phi^U(t),v(t)\big) =& -2i\gamma\frac{1}{\overline{Y}(T)}\left(\frac{Z^U(T)}{Y(T)}-y,v(T)\right)-\int_t^T\overline{\left\langle\frac{\partial}{\partial s}v(s),\Phi^U(s)\right\rangle}\,ds\\ &+i\int_t^T\langle A\Phi^U(s),v(s)\rangle\,ds-i\int_t^T\big(U(s)\Phi^U(s),v(s)\big)\,ds\\ &-i\overline{\lambda}\int_t^T\left(\overline{f}_v\left(s,\frac{Z^U(s)}{Y(s)}\right)\frac{1}{\overline{Y}(s)}\Phi^U(s),v(s)\right)ds\\ &-i\lambda\int_t^T\left(f_{\overline{v}}\left(s,\frac{Z^U(s)}{Y(s)}\right)\frac{1}{\overline{Y}(s)}\overline{\Phi^U}(s),v(s)\right)ds\end{aligned}\tag{3.61}$$

for all $t\in[0,T]$ and all $v\in L^2([0,T];V')$ (compare Theorem 3.2.13).

By skillfully combining the variational formulations (3.57) and (3.61), it is possible to establish the Gâteaux derivative

$$\delta J(U;\delta U)=\lim_{\Theta\to 0}\frac{J(U+\Theta\delta U)-J(U)}{\Theta}$$

of the objective functional (3.20) at $U\in\mathcal{U}$ in the direction $\delta U\in\mathcal{U}$. With the same approach as in Subsection 3.2.4, the numerator has the form

$$\begin{aligned}J(U+\Theta\delta U)-J(U)=&\gamma E\left(B^{\frac{1}{\sigma}}(T)\left\|\delta Z(T)\right\|^2\right)+\beta\Theta^2E\int_0^T\left\|\delta U(t)\right\|^2dt\\ &+E\operatorname{Re}\int_0^T\big(\Phi^U(t),\Theta\delta U(t)Z^U(t)\big)\,dt\\ &+2\beta\Theta E\int_0^T\big(U(t)-\Upsilon(t),\delta U(t)\big)\,dt+R_1+R_2,\end{aligned}$$

where

$$R_1=E\operatorname{Re}\int_0^T\big(\Phi^U(t),\Theta\delta U(t)\delta Z(t)\big)\,dt$$

and (based on $\lambda:=\lambda_1+i\lambda_2$) the varied expression

$$\begin{aligned}R_2:=\lambda_1E\operatorname{Re}\int_0^T&\left(\Phi^U(t),f\left(t,\frac{Z^{U+\Theta\delta U}(t)}{Y(t)}\right)-f\left(t,\frac{Z^U(t)}{Y(t)}\right)\right.\\ &\left.-f_v\left(t,\frac{Z^U(t)}{Y(t)}\right)\frac{\delta Z(t)}{Y(t)}-f_{\overline{v}}\left(t,\frac{Z^U(t)}{Y(t)}\right)\frac{\overline{\delta Z}(t)}{\overline{Y}(t)}\right)dt\\ +\lambda_2E\operatorname{Im}\int_0^T&\left(\Phi^U(t),f\left(t,\frac{Z^{U+\Theta\delta U}(t)}{Y(t)}\right)-f\left(t,\frac{Z^U(t)}{Y(t)}\right)\right.\\ &\left.-f_v\left(t,\frac{Z^U(t)}{Y(t)}\right)\frac{\delta Z(t)}{Y(t)}-f_{\overline{v}}\left(t,\frac{Z^U(t)}{Y(t)}\right)\frac{\overline{\delta Z}(t)}{\overline{Y}(t)}\right)dt.\end{aligned}$$

Dividing each term by Θ and taking the limit with respect to $\Theta\to 0$, we do the same calculations as before. Thus, we only emphasize the approach for the new term R_2. The linear Taylor series expansion of the drift function $f(t,v(t))=f(t,v(t),\overline{v}(t))$ for all $t\in[0,T]$ and $v(t)=Z^{U+\Theta\delta U}(t)Y^{-1}(t)$

at the point $Z^U(t)Y^{-1}(t)$ in Lagrange form for $\vartheta \in (0,1)$ is constituted by

$$\begin{aligned} f\left(t, \frac{Z^{U+\Theta\delta U}(t)}{Y(t)}\right) &= f\left(t, \frac{Z^U(t)}{Y(t)}\right) + f_v\left(t, \frac{Z^U(t)}{Y(t)}\right)\frac{\delta Z(t)}{Y(t)} + f_{\bar v}\left(t, \frac{Z^U(t)}{Y(t)}\right)\frac{\overline{\delta Z}(t)}{\overline{Y}(t)} \\ &+ \frac{1}{2} f_{vv}\left(t, \frac{Z^U(t)+\vartheta\delta Z(t)}{Y(t)}\right)\frac{(\delta Z(t))^2}{(Y(t))^2} \\ &+ \frac{1}{2} 2 f_{v\bar v}\left(t, \frac{Z^U(t)+\vartheta\delta Z(t)}{Y(t)}\right)\frac{\delta Z(t)}{Y(t)}\frac{\overline{\delta Z}(t)}{\overline{Y}(t)} \\ &+ \frac{1}{2} f_{\bar v\bar v}\left(t, \frac{Z^U(t)+\vartheta\delta Z(t)}{Y(t)}\right)\frac{(\overline{\delta Z}(t))^2}{(\overline{Y}(t))^2}. \end{aligned}$$

Using this representation in R_2, only the quadratic remainders are left. For all $t \in [0,T]$ and all complex-valued functions $v(t)$ we denote $\nabla^2 f(t,v(t)) := f_{vv}(t,v(t)) + 2f_{v\bar v}(t,v(t)) + f_{\bar v\bar v}(t,v(t))$ and regard that the absolute values of the second Wirtinger derivatives are bounded (see (3.59)) such that

$$|\nabla^2 f(t,v(t))| = |f_{vv}(t,v(t)) + 2f_{v\bar v}(t,v(t)) + f_{\bar v\bar v}(t,v(t))| \le 4C''.$$

Hence, enlarging the real and imaginary part in R_2 by the absolute value, it holds that

$$R_2 \le C(\sigma, |\lambda|, C'', C_B)\, E\int_0^T \int_0^1 \left|\Phi^U(t,x)\right| |\delta Z(t,x)|^2 \,dx\,dt.$$

Now, we take the supremum of $\Phi^U(t,x)$ over all $x \in [0,1]$, apply Lemma D.2 and use inequalities (3.60) and (3.56) such that

$$R_2 \le C(\sigma, |\lambda|, \varphi, \gamma, y, \alpha_1, \alpha_2, c_f, k_f, C', C'', C_B, T)\,\Theta^2 \left\|\delta U\right\|^2_{L^\infty(\Omega\times[0,T]\times[0,1])}.$$

This yields

$$\lim_{\Theta\to 0}\frac{R_2}{\Theta} = 0,$$

and we conclude the same results as in Subsection 3.2.4.

Theorem 3.3.1. *Under the given assumptions and the condition that the Lipschitz continuous drift function f of bounded growth is twice continuously differentiable with bounded absolute values of the first and second Wirtinger derivatives, the Gâteaux differential of the objective functional* (3.20) *with respect to the pathwise controlled Schrödinger problem* (3.53) *is given by*

$$\delta J(U;\delta U) = E\,\mathrm{Re}\int_0^T \left(\Phi^U(t)\,\overline{Z^U}(t), \delta U(t)\right)dt + 2\beta E\int_0^T \left(U(t) - \Upsilon(t), \delta U(t)\right)dt$$

for all $\delta U \in \mathcal{U}$. Based on the transformation formula $Z^U(t,\cdot) = X^U(t,\cdot)Y(t)$, the Gâteaux differential of the objective functional (3.5) *with respect to the controlled Schrödinger problem* (3.52) *is constituted by*

$$\delta J(U;\delta U) = E\,\mathrm{Re}\int_0^T \left(\Phi^U(t)\,\overline{X^U}(t)\,\overline{Y}(t), \delta U(t)\right)dt + 2\beta E\int_0^T \left(U(t) - \Upsilon(t), \delta U(t)\right)dt$$

for all $\delta U \in \mathcal{U}$. Thus, with the notation $X^* := X^{U^*}$ and $\Phi^* := \Phi^{U^*}$, the necessary optimality condition for the optimal control problem (3.5), (3.52) *can be stated by*

$$E\int_0^T \left(\mathrm{Re}\left\{E\left[\Phi^*(t)\middle|\mathcal{F}_t\right]\overline{X^*}(t)\,\overline{Y}(t)\right\} + 2\beta\Big(U^*(t) - E\left[\Upsilon(t)\middle|\mathcal{F}_t\right]\Big), U(t) - U^*(t)\right)dt \ge 0$$

for all $U \in \mathcal{U}$.

4 Conclusion and Outlook

Nonlinear stochastic differential equations are current objects of mathematical research. Hence, there is a multitude of scientific papers containing such equations with Lipschitz continuous nonlinearities of bounded growth whose solution follows immediately from the classical existence and uniqueness theory of stochastic partial differential equations. Here, we considered the stochastic Schrödinger equation with drift term in form of the power-type nonlinearity $f(v) = |v|^{2\sigma}v$ for all $v \in \mathbb{C}$ and $\sigma > 0$ that does not fulfill these assumptions. Nevertheless, this nonlinear equation of Schrödinger type is realistic and motivated by physical applications. Due to the mathematical point of view, we also took Lipschitz continuous drift functions of bounded growth into account.

In the first part of this thesis, we proved the unique existence of the variational solution of the stochastic nonlinear Schrödinger problem over a finite time horizon and a bounded one-dimensional domain. Based on Galerkin approximations, truncation techniques, stopping times, useful inequalities of the nonlinear terms etc., we gained the unique existence of the variational solution of the finite-dimensional equations and corresponding uniform a priori estimates. Thereafter, these results were extended to the variational solution of the stochastic nonlinear Schrödinger equation by a combination of further estimates and useful embedding and convergence results.

In the special case of linear multiplicative Gaussian noise, we established an equivalent pathwise nonlinear Schrödinger problem. The corresponding smoothness results, which are distinctive for the pathwise problem, were transferred to the stochastic case and led to properties we required to treat an optimal control problem in the second main part. We searched for an optimal control that minimizes a given objective functional depending on the control and the solution of the controlled Schrödinger problem. Whether there exists such an optimal control is a common question in optimal control theory. Referring to the difference process of two controlled Schrödinger problems and the complex conjugated adjoint Schrödinger problem, whose unique existence of the variational solution we have also proved, we calculated a gradient formula in the sense of Gâteaux and inferred a necessary optimality condition.

Summarized, we successfully treated the unique existence of the variational solution of the stochastic nonlinear Schrödinger problem and investigated a corresponding problem of optimal control for some convenient cases. Observe that this is the first work which is concerned with the variational solution of stochastic Schrödinger equations including a non-Lipschitz continuous drift term. Moreover, we do not know any article that deals with optimal control problems for the stochastic nonlinear Schrödinger equation. Hence, this dissertation broadens the mathematical horizon in the field of stochastic analysis, especially stochastic differential equations, and establishes a basis for future research works, for example, on the following subjects:

- Additive Gaussian noise: Unfortunately, it is not possible to handle the controlled Schrödinger problem with additive Gaussian noise with the present approach. Although the method in Appendix I yields the appropriate complex conjugated adjoint Schrödinger equation, there are difficulties in deriving a priori estimates for the solution of its Galerkin equations. However, under sufficient conditions, it is also possible to transfer the nonlinear Schrödinger problem with additive noise into a pathwise problem by introducing a new variable defined by the state variable minus the noise term (see [38] and a special case of [43]). Investigating the characteristic smoothness properties of this pathwise Schrödinger problem and extending them to the stochastic case, the control problem should be solvable.

- Fractional noise: We suggest to investigate the nonlinear Schrödinger problem perturbed by linear fractional white noise. Introducing an appropriate definition of the stochastic integral and using the methods of [50, pp. 179–182], it is also possible to transfer this fractional Schrödinger problem into a pathwise one (compare [45]). Then the present ideas for the problem of optimal control should be applicable.
- More general nonlinearities: Although including many types of nonlinearities, the discussion of more general nonlinear drift terms is another research perspective. For example, the discussion of nonlinearities like in the deterministic works [13], [14, Section 3.2], [15, 73, 96, 97] etc. would be interesting.
- Multi-dimensional case: After the generalization from the bounded domain $x \in [0,1]$ to the unbounded domain $x \in \mathbb{R}$ (while ensuring a discrete spectrum of eigenvalues and a complete set of eigenfunctions), one could also think of multi-dimensional domains with respect to the space variable. However, observe that the continuous embedding $H^1(G) \hookrightarrow L^p(G)$ for $G \subset \mathbb{R}^n$ depends on p and the dimension n of the domain (see [2, pp. 97–99, Theorem 5.4 and Remark 5.5]). Referring to the semigroup approach, the nonlinear Schrödinger equation on multi-dimensional domains is treated, for example, in the deterministic works [49, 51, 55] and in the stochastic articles [20, 21, 32].
- Other solution concepts: In order to obtain the complex conjugated adjoint Schrödinger equation, the present approach in Appendix I only works for the controlled Schrödinger problem with additive or linear multiplicative noise. For general multiplicative noise, the stochastic nonlinear Schrödinger problem has to be regarded with respect to other concepts of solutions (compare Section 1.2 and Appendix B). Proceeding, for example, to the mild solution, auxiliary results like Strichartz estimates (for $x \in \mathbb{R}^n$) and properties of strongly continuous groups can be applied. Moreover, the complex conjugated adjoint Schrödinger equation can be established by another procedure as a forward-backward stochastic Volterra integral equation (see [4]) which may be a starting point to solve corresponding problems of optimal control.
- Numerical implementation: Since our stochastic nonlinear Schrödinger equations cannot be solved analytically, it is necessary to approximate them or particularly the nonlinearity. First deterministic approaches were conducted in [36, 37] in the 1970s and deepened by numerical simulations for the deterministic Schrödinger equation in [33, 90, 91] and for the stochastic case in [22, 23, 25, 26] at the beginning of the 21st century. They often used truncation methods to show the unique existence of the solution. Applying, for example, the cut-off function
$$P_r v(t) := \begin{cases} v(t) & : |v(t)| \leq r, \\ r\frac{v(t)}{|v(t)|} & : |v(t)| > r \end{cases}$$
for $r \in \mathbb{R}_+$ and $v(t) \in V$ to the Kerr-nonlinearity (power-type nonlinearity for $\sigma = 1$), we get a Lipschitz continuous function of bounded growth we can work with. In order to model and simulate the solution of the Galerkin equations of the stochastic nonlinear Schrödinger problem and the corresponding problem of optimal control, we propose discretization schemes, truncation techniques, linearization procedures, splitting methods or other types of approximations (we refer to the ideas in the above mentioned articles and to [10, 44, 67]). Notice that, for example, the Wong-Zakai approximation (see [98, 100]) represents a method to discretize the Wiener process such that it results a differentiable version, which allows to solve the problem as in the deterministic case.
- Other types of control: Here, we considered a bilinear control term (linear in U and X^U), which physically modifies the external potential. We can also imagine control terms that are not linear in both variables anymore, some kind of additive control (which appears as an inhomogeneity on the right-hand side of the state equation), control on the boundary (like in [101, 106]) etc.

- Other objective functionals: Within the framework of this thesis, there arises the question of other objective functionals aiming to minimize or maximize its value over a given set. Due to the unique existence of the optimal control, we considered a regularized objective functional such that the solution of the stochastic nonlinear Schrödinger problem represents a best possible approximation of a given function at the final time T. To control the Schrödinger equation such that the solution X^U approximates a given function at the final point $t = T$ physically means to control the matter and, therefore, the electron flux. In one-dimensional bounded domains, one can think of potential wells where the electrons are captured. To obtain a special final state, one can apply electric or magnetic fields which are reflected as additional potentials in the Schrödinger equation. Other objective functionals with physical motivation are, for example, functionals including running costs, a best possible approximation of the solution of the stochastic nonlinear Schrödinger problem to given functions on the boundary (compare [57, 58]), functions of the variational solution of the stochastic nonlinear Schrödinger problem and/or the control or even a combination of them.

Although the stochastic nonlinear Schrödinger equation is extensively studied during the last years, there is still a multitude of open problems. Maybe one or another of these questions can be solved based on this thesis.

A Cylindrical Wiener Process

All the following results are extracted from [82, Chapter 2 and Appendix B], where one can find more details and the corresponding proofs. Let K and H be two real separable Hilbert spaces with inner products $(\cdot\,,\cdot)_K$ and $(\cdot\,,\cdot)$ and norms $\|\cdot\|_K$ and $\|\cdot\|$, respectively. Furthermore, let $L(K,H)$ be the space of all linear and bounded operators from K into H, $L(K) := L(K,K)$ and $(e_k)_{k\in\mathbb{N}}$ an orthonormal basis of K. Then the covariance operator of a K-valued Gaussian random variable is a non-negative, symmetric operator $Q \in L(K)$ with finite trace, which means that for all $u,v \in K$ it holds that

$$\big(Qu,u\big)_K \geq 0, \qquad \big(Qu,v\big)_K = \big(u,Qv\big)_K, \qquad \operatorname{tr} Q := \sum_{k=1}^{\infty} \big(Qe_k,e_k\big)_K < \infty$$

(compare [82, p. 6 and p. 109]). For each covariance operator $Q \in L(K)$ there exists an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of K representing the eigenfunctions of Q and a corresponding sequence of eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$ such that $Qe_k = \lambda_k e_k$ for all $k \in \mathbb{N}$ with $\lambda_k \geq 0$, and zero is the only accumulation point of the sequence $(\lambda_k)_{k\in\mathbb{N}}$ (see [82, p. 9, Proposition 2.1.5]). Then a K-valued Q-Wiener process can be represented by

$$\tilde{W}(t) := \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, e_k \beta_k(t), \qquad \text{for all } t \in [0,T],$$

where $(\beta_k(t))_{t\in[0,T]}$ with $k \in \{n \in \mathbb{N} :\ \lambda_n > 0\}$ are independent real-valued Wiener processes. This series even converges in $L^2(\Omega; C([0,T];K))$, and thus always has a P-a.s. continuous modification (see [19, pp. 86–89] and [82, p. 13, Proposition 2.1.10]).

Next, (referring to [82, pp. 110 f.]) an operator $S \in L(K,H)$ is called Hilbert-Schmidt operator from K into H, in the following $S \in L_2(K,H)$ for short, if its Hilbert-Schmidt norm satisfies

$$\|S\|^2_{L_2(K,H)} := \sum_{k=1}^{\infty} \|Se_k\|^2 < \infty.$$

If $Q \in L(K)$ is non-negative and symmetric, then there exists exactly one element $Q^{\frac{1}{2}} \in L(K)$, which is also non-negative and symmetric, such that $Q^{\frac{1}{2}} \circ Q^{\frac{1}{2}} = Q$. Additionally, if $\operatorname{tr} Q < \infty$, it results that $Q^{\frac{1}{2}} \in L_2(K) := L_2(K,K)$ with $\|Q^{\frac{1}{2}}\|^2_{L_2(K)} = \operatorname{tr} Q$ and $S \circ Q^{\frac{1}{2}} \in L_2(K,H)$ for all $S \in L(K,H)$ (see [82, p. 25, Proposition 2.3.4]). Thus, the Q-Wiener process can be equivalently stated by

$$\tilde{W}(t) := \sum_{k=1}^{\infty} f_k \beta_k(t), \qquad \text{for all } t \in [0,T], \tag{A.1}$$

where $(f_k)_{k\in\mathbb{N}}$ is an orthonormal basis of (the separable Hilbert space) $Q^{\frac{1}{2}}(K) =: K_0$ with inner product $\big(u_0,v_0\big)_{K_0} := \big(Q^{-\frac{1}{2}}u_0, Q^{-\frac{1}{2}}v_0\big)_K$ for all $u_0, v_0 \in K_0$ ($Q^{-\frac{1}{2}}$ is the pseudo inverse of $Q^{\frac{1}{2}}$ in the case that Q is not one-to-one). Since the inclusion $K_0 \subset K$ defines a Hilbert-Schmidt embedding from K_0 to K, which means that the mapping from K_0 to K is a Hilbert-Schmidt operator, the series (A.1) converges in $L^2(\Omega;K)$ (compare [82, p. 27 and p. 39]).

Now, if Q has no finite trace any longer and is no covariance operator anymore, this convergence gets lost. However, we can introduce another Wiener process, the so-called cylindrical one. Therefore, let K_1 be a further Hilbert space and $J : K_0 \to K_1$ another Hilbert-Schmidt embedding, which always exist, for example, by choosing $K_1 := K$, a sequence $(\alpha_j)_{j\in\mathbb{N}}$ of positive real-valued numbers such that $\sum_{j=1}^{\infty} \alpha_j^2 < \infty$ and $J : K_0 \to K$ by

$$J(u_0) := \sum_{j=1}^{\infty} \alpha_j \big(u_0, f_j\big)_{K_0} f_j, \qquad \text{for all } u_0 \in K_0$$

(see [82, p. 39, Remark 2.5.1]). Then the process

$$W(t) := \sum_{k=1}^{\infty} J f_k \beta_k(t), \qquad \text{for all } t \in [0,T], \tag{A.2}$$

is called a cylindrical Wiener process in K. This series converges in the space of all K_1-valued continuous, square integrable martingales on $[0,T]$. Defining $Q_1 := JJ^*$, we obtain $Q_1 \in L_2(K_1)$ and, in particular, (A.2) is a Q_1-Wiener process on K_1. Moreover, it holds that

$$Q_1^{\frac{1}{2}}(K_1) = J(K_0) \qquad \text{and} \qquad \|u_0\|_{K_0} = \left\|Q_1^{-\frac{1}{2}} J u_0\right\|_{K_1} = \|Ju_0\|_{Q_1^{\frac{1}{2}} K_1}, \qquad \text{for all } u_0 \in K_0,$$

which implies that $J : K_0 \to Q_1^{\frac{1}{2}} K_1$ is an isometry (see [82, p. 40, Proposition 2.5.2]).

Based on [19, pp. 90–96], [39, pp. 23–61] or [82, pp. 21–34], a process $(\Phi(t))_{t\in[0,T]}$ is integrable with respect to a Q-Wiener process $(\tilde{W}(t))_{t\in[0,T]}$ on K if $\Phi : \Omega \times [0,T] \to L_2(K_0, H)$ is predictable and satisfies

$$P\left(\int_0^T \|\Phi(s)\|^2_{L_2(K_0,H)}\, ds < \infty\right) = 1.$$

Here, a predictable process is an arbitrary measurable mapping from $(\Omega\times[0,T], \mathcal{P}_T)$ into $(H, \mathcal{B}(H))$ with $\mathcal{P}_T := \sigma\left(\{(s,t] \times F : 0 \le s < t \le T, F \in \mathcal{F}_s\} \cup \{\{0\} \times F_0 : F_0 \in \mathcal{F}_0\}\right)$. Then the stochastic integral is defined by

$$\int_0^t \Phi(s)\, d\tilde{W}(s) := \int_0^t \mathbb{1}_{(0,\tau_n]}(s)\Phi(s)\, d\tilde{W}(s), \qquad \text{for all } t \in [0,T],$$

where $(\tau_n)_{n\in\mathbb{N}}$ is an increasing sequence of stopping times with respect to $(\mathcal{F}_t)_{t\in[0,T]}$ given by

$$\tau_n := \inf\left\{t \in [0,T] : \int_0^t \|\Phi(s)\|^2_{L_2(K_0,H)}\, ds > n\right\} \wedge T.$$

Since the cylindrical Wiener process $(W(t))_{t\in[0,T]}$ is a Q_1-Wiener process on K_1, we conclude that $(\Phi(t))_{t\in[0,T]}$ is integrable with respect to $(W(t))_{t\in[0,T]}$ by replacing $K_0 = Q^{\frac{1}{2}}(K)$ by $Q_1^{\frac{1}{2}}(K_1)$. The fact that $(Jf_k)_{k\in\mathbb{N}}$ is an orthonormal basis of $J(K_0) = Q_1^{\frac{1}{2}}(K_1)$ yields that $\Phi \in L_2(Q^{\frac{1}{2}}(K), H)$ if and only if $\Phi \circ J^{-1} \in L^2(Q_1^{\frac{1}{2}}(K_1), H)$. Thus, (see [82, p. 42]) we can define

$$\int_0^t \Phi(s)\, dW(s) := \int_0^t \Phi(s) \circ J^{-1}\, dW(s), \qquad \text{for all } t \in [0,T]. \tag{A.3}$$

Remark A.1. [1]*If Q is a covariance operator on K, the standard Q-Wiener process can also be considered as a cylindrical Wiener process by setting $J = I : K_0 \to K$ where I is the identity map. In this case both definitions of the stochastic integral coincide.*

Remark A.2. *Combining the representation* (A.2) *of a cylindrical Wiener process and the definition* (A.3) *of its stochastic integral, we get for all $t \in [0,T]$ that*

$$\int_0^t \Phi(s)\, dW(s) := \int_0^t \Phi(s) \circ J^{-1}\, dW(s) = \sum_{k=1}^{\infty} \int_0^t \Phi(s) \circ J^{-1} J f_k\, d\beta_k(s) = \sum_{k=1}^{\infty} \int_0^t \Phi(s) f_k\, d\beta_k(s).$$

[1] Prévôt & Röckner [82], p. 42, Remark 2.5.3 (2)

B Concepts of Solutions

There are several types of solutions for stochastic differential equations. In the following, some of them are introduced and can be found, for example, in [19, 42, 46, 62, 82]. Therefore, let K and H be two real separable Hilbert spaces with inner products $(\,\cdot\,,\,\cdot\,)_K$ and $(\,\cdot\,,\,\cdot\,)$ and corresponding norms $\|\cdot\|_K$ and $\|\cdot\|$, respectively. Let $T \in \mathbb{R}_+$ be fixed and $(W(t))_{t\in[0,T]}$ a K-valued cylindrical Q-Wiener process (see Appendix A) on a complete filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P\big)$. Then we consider the following problem of a stochastic differential equation

$$dX(t) = [CX(t) + F(X(t))]\,dt + B(X(t))\,dW(t), \qquad X(0) = \xi \tag{B.1}$$

in H for all $t \in [0,T]$, where $C : D(C) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup $(S_t)_{t\in[0,T]}$ of linear operators on H, $F : H \to H$ is $\mathcal{B}(H)$-measurable, $B : H \to L(K,H)$ and ξ is an H-valued $\mathcal{F}_0$-measurable random variable. Assuming that all the following integrals are well-defined, we introduce the concepts of solutions.

Definition B.1. [1]*A $D(C)$-valued predictable process $(X(t))_{t\in[0,T]}$ is said to be an analytically strong solution of problem* (B.1) *if*

$$X(t) = \xi + \int_0^t [CX(s) + F(X(s))]\,ds + \int_0^t B(X(s))\,dW(s)$$

P-a.s. for each $t \in [0,T]$.

Definition B.2. [2]*An H-valued predictable process $(X(t))_{t\in[0,T]}$ is called an analytically (or generalized) weak solution of problem* (B.1) *if*

$$\big(X(t),\zeta\big) = \big(\xi,\zeta\big) + \int_0^t \big[\big(X(s), C^*\zeta\big) + \big(F(X(s)),\zeta\big)\big]\,ds + \int_0^t \big(B(X(s))\,dW(s),\zeta\big)$$

P-a.s. for each $t \in [0,T]$ and $\zeta \in D(C^*) = \{v \in H : C^*v \in H\}$.

Definition B.3. [3]*An H-valued predictable process $(X(t))_{t\in[0,T]}$ is called a mild solution of problem* (B.1) *if*

$$X(t) = S_t\xi + \int_0^t S_{t-s}F(X(s))\,ds + \int_0^t S_{t-s}B(X(s))\,dW(s)$$

P-a.s. for each $t \in [0,T]$.

Finally, the variational solution is introduced. On that account, let (V, H, V^*) be a triple of rigged Hilbert spaces with compact embeddings each (see [82, p. 55] or [104, pp. 416 f.]). Moreover, let C be a linear continuous operator $A : V \to V^*$ which defines the bilinear form $\langle Au, v\rangle$ for all $u, v \in V$.

[1] Prévôt & Röckner [82], pp. 133 f., Definition F.0.2
[2] Prévôt & Röckner [82], p. 134, Definition F.0.3
[3] Prévôt & Röckner [82], p. 133, Definition F.0.1

Definition B.4. [4,5]*A continuous H-valued predictable process $(X(t))_{t\in[0,T]}$, which is also measurable with respect to time and takes values in V, is said to be a variational solution of problem* (B.1) *if*

$$\big(X(t),\zeta\big) = \big(\xi,\zeta\big) + \int_0^t \big[\langle AX(s),\zeta\rangle + \big(F(X(s)),\zeta\big)\big]\,ds + \int_0^t \big(B(X(s))\,dW(s),\zeta\big)$$

P-a.s. for each $t\in[0,T]$ and $\zeta\in V$.

Notice that every analytically strong solution of problem (B.1) is also an analytically weak solution and the concepts of a mild and an analytically weak solution are equivalent. Furthermore, every variational solution is a mild solution, but not vice versa. However, if the operator $C = A$ and $D(C) = \{v\in V : Cv\in H\}$, then the variational solution is an analytically weak solution as well. For more relations between the different types of solutions we recommend [19, p. 115-149], [46, p. 63–73] or [62, Chapter 2].

Remark B.5. **(a)** *Without loss of generality, we can also regard complex Hilbert spaces which can be handled by the partition in real and imaginary part.*
(b) *In this thesis we consider the variational solution of a Schrödinger equation. Hence, the operator A changes into $iA := -i\Delta$, which is the infinitesimal generator of a strongly continuous group $(S_t)_{t\in[0,T]}$ of linear operators on H. The first three solution concepts can be transferred for $C := iA$ and the last one for $A := iA$.*

[4]Prévôt & Röckner [82], p. 73, Definition 4.2.1
[5]Grecksch & Lisei [42], p. 634, Formula (2)

C Norm Square Itô Formula

Initially, the definition of a progressively measurable process is stated.

Definition C.1. [1] *Let E be a separable Banach space. Then a stochastic process $(X(t))_{t\in[0,T]}$ is called progressively measurable if the mapping*

$$\Omega \times [0,t] \;\to\; (E, \mathcal{B}(E)), \qquad (\omega, s) \;\mapsto\; X(\omega, s)$$

is $(\mathcal{F}_t \times \mathcal{B}([0,t]))$-measurable for each $t \in [0,T]$, which means that

$$\{(\omega, s) : X(\omega, s) \in A, s \le t\} \in \mathcal{F}_t \times \mathcal{B}([0,t]), \qquad \text{for all } A \in \mathcal{B}(E).$$

Now, we introduce the norm square Itô formula for a Hilbert space H, which is used several times throughout this work.

Lemma C.2. [2] *Let H be a real separable Hilbert space with inner product $(\,\cdot\,,\,\cdot\,)$, V a reflexive Banach space and V^* the dual space of V such that (V, H, V^*) defines a Gelfand triple and $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the duality pairing of V^* and V. Moreover, let K also be a real separable Hilbert space, $X_0 \in L^2(\Omega; H)$ be $\mathcal{F}_0$-measurable, $Y \in L^2(\Omega \times [0,T]; V^*)$ and $Z \in L^2(\Omega \times [0,T]; L_2(K,H))$ be progressively measurable processes and $(W(t))_{t\in[0,T]}$ a K-valued $\mathcal{F}_t$-adapted cylindrical Wiener process. We define*

$$X(t) := X_0 + \int_0^t Y(s)\, ds + \int_0^t Z(s)\, dW(s)$$

in V^* for a.e. $\omega \in \Omega$ and all $t \in [0,T]$, and we choose the progressively measurable version of $X \in L^2(\Omega \times [0,T]; V)$. Then $(X(t))_{t\in[0,T]}$ is an H-valued continuous $\mathcal{F}_t$-adapted process with

$$E \sup_{t\in[0,T]} \|X(t)\|^2 < \infty,$$

and it holds the following Itô formula for the square of its H-norm

$$\|X(t)\|^2 = \|X_0\|^2 + 2\int_0^t \big\langle Y(s), X(s)\big\rangle\, ds + 2\int_0^t \big(Z(s)\, dW(s), X(s)\big) + \int_0^t \|Z(s)\|^2_{L_2(K,H)}\, ds \quad \text{(C.1)}$$

P-a.s. for all $t \in [0,T]$.

Regarding the definition of a cylindrical Wiener process with an orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of K and a sequence of independent real-valued Wiener processes $((\beta_j(t))_{t\in[0,T]})_{j\in\mathbb{N}}$ (compare Appendix A), the stochastic integral is given by

$$\int_0^t \big(Z(s)\, dW(s), X(s)\big) := \sum_{j=1}^{\infty} \int_0^t \big(Z(s)e_j, X(s)\big)\, d\beta_j(s), \qquad \text{for all } t \in [0,T].$$

Since we are dealing with complex separable Hilbert spaces, we are interested how the norm square Itô formula changes in this case.

[1] Da Prato & Zabczyk [19], p. 75

[2] Prévôt & Röckner [82], p. 75, Theorem 4.2.5

Remark C.3. *Replace the real separable Hilbert spaces H and V in Lemma C.2 by complex separable Hilbert spaces, let all the other assumptions of Lemma C.2 be satisfied (let especially K be a real separable Hilbert space) and consider the problem*

$$X(t) = X_0 + \int_0^t \big[Y_1(s) + iY_2(s)\big]\, ds + \int_0^t \big[Z_1(s) + iZ_2(s)\big]\, dW(s)$$

in V^* for a.e. $\omega \in \Omega$ and all $t \in [0,T]$. Then all the assertions of Lemma C.2 are still true and (by choosing the progressively measurable version of $X \in L^2(\Omega \times [0,T]; V)$) the norm square Itô formula (C.1) *changes to*

$$\begin{aligned} \|X(t)\|^2 = \|X_0\|^2 &+ 2\,\mathrm{Re} \int_0^t \big\langle Y_1(s), X(s)\big\rangle\, ds - 2\,\mathrm{Im} \int_0^t \big\langle Y_2(s), X(s)\big\rangle\, ds \\ &+ 2\,\mathrm{Re} \int_0^t \big(Z_1(s)\, dW(s), X(s)\big) - 2\,\mathrm{Im} \int_0^t \big(Z_2(s)\, dW(s), X(s)\big) \\ &+ \int_0^t \|Z_1(s) + iZ_2(s)\|^2_{L_2(K,H)}\, ds \end{aligned} \tag{C.2}$$

P-a.s. for all $t \in [0,T]$.

D Important Inequalities

Here, we provide some results used throughout the thesis.

Lemma D.1. *Let $x, y \geq 0$ and $q > 0$, then $(x+y)^q \leq 2^q\,(x^q + y^q)$.*

Proof. The relation $0 \leq x + y \leq 2\max\{x, y\}$ implies for $q > 0$ that

$$(x+y)^q \leq 2^q\,(\max\{x,y\})^q \leq 2^q\,(x^q + y^q)\,. \qquad \square$$

Lemma D.2. *For $v \in V$ it holds that*

$$\sup_{x\in[0,1]} |v(x)|^2 \leq \|v\| \left(\|v\| + 2 \left\| \frac{dv}{dx} \right\| \right) \leq 2\|v\|_V^2.$$

Proof. The first inequality follows from [36, Lemma 1.1] and the second one is obvious. $\square$

Lemma D.3. *For $\sigma \in (0, 2)$ and $Z \in C([0,T]; H) \cap L^2([0,T]; V)$ we have*

$$\int_0^T \sup_{x\in[0,1]} |Z(s,x)|^{2\sigma}\, ds \leq C(\sigma, T) \left[\sup_{s\in[0,T]} \|Z(s)\|^{2\sigma} + \left(\int_0^T \|Z(s)\|_V^2\, ds \right)^{\sigma} \right],$$

where $C(\sigma, T) := 2^\sigma T \max\{2^{\sigma-1} + 1; 2^{\sigma-1}T^{-\sigma}\}$.

Proof. Taking into account that $\sigma \in (0, 2)$ and using Lemma D.2, Lemma D.1, the property that $\|\frac{\partial}{\partial x} Z(s)\|^\sigma \leq \|Z(s)\|_V^\sigma$ and Hölder's inequality for integrable functions, we write

$$\begin{aligned}
&\int_0^T \sup_{x\in[0,1]} |Z(s,x)|^{2\sigma}\, ds \\
&\leq \int_0^T \left(\|Z(s)\|^2 + 2\|Z(s)\| \left\| \frac{\partial}{\partial x} Z(s) \right\| \right)^{\sigma} ds \\
&\leq 2^\sigma \left[\int_0^T \|Z(s)\|^{2\sigma}\, ds + 2^\sigma \int_0^T \|Z(s)\|^\sigma \left\| \frac{\partial}{\partial x} Z(s) \right\|^\sigma ds \right] \\
&\leq 2^\sigma \left[T \sup_{s\in[0,T]} \|Z(s)\|^{2\sigma} + 2^\sigma \sup_{s\in[0,T]} \|Z(s)\|^\sigma\, T^{1-\frac{\sigma}{2}} \left(\int_0^T \|Z(s)\|_V^2\, ds \right)^{\frac{\sigma}{2}} \right].
\end{aligned}$$

Now, we apply the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ for $a, b \in \mathbb{R}$ and obtain

$$\begin{aligned}
&\int_0^T \sup_{x\in[0,1]} |Z(s,x)|^{2\sigma}\, ds \\
&\leq 2^\sigma T \left[\sup_{s\in[0,T]} \|Z(s)\|^{2\sigma} + 2^\sigma \left(\frac{1}{2} \sup_{s\in[0,T]} \|Z(s)\|^{2\sigma} + \frac{1}{2}\, T^{-\sigma} \left(\int_0^T \|Z(s)\|_V^2\, ds \right)^{\sigma} \right) \right] \\
&\leq 2^\sigma T \max\left\{2^{\sigma-1} + 1; 2^{\sigma-1} T^{-\sigma}\right\} \left[\sup_{s\in[0,T]} \|Z(s)\|^{2\sigma} + \left(\int_0^T \|Z(s)\|_V^2\, ds \right)^{\sigma} \right].
\end{aligned}$$

By choosing $C(\sigma, T) := 2^\sigma T \max\{2^{\sigma-1} + 1; 2^{\sigma-1}T^{-\sigma}\}$, the assertion ensues. $\square$

Lemma D.4. *For $z_1, z_2 \in \mathbb{C}$ the following inequalities are fulfilled*

(a) $\left||z_1|^{2\sigma}z_1 - |z_2|^{2\sigma}z_2\right| \le \frac{5}{2}\left(|z_1|^{2\sigma} + |z_2|^{2\sigma}\right)|z_1 - z_2|,$ *for all* $\sigma \in (0,2)$,

(b) $\left||z_1|^{2\sigma}z_1 - |z_2|^{2\sigma}z_2\right| \le 2\sigma\left(|z_1|^{2\sigma} + |z_2|^{2\sigma}\right)|z_1 - z_2|,$ *for all* $\sigma \ge \frac{1}{2}$.

Proof. **(a)** In the case $|z_1| = |z_2|$ the inequality is obvious. Hence, without loss of generality, we consider $|z_1| < |z_2|$ that entails $1 - \left|\frac{z_1}{z_2}\right|^{\frac{\sigma}{2}} \le 1 - \left|\frac{z_1}{z_2}\right|$ for all $\sigma \in (0,2)$. Then we have

$$\begin{aligned}
\left||z_1|^{2\sigma}z_1 - |z_2|^{2\sigma}z_2\right| &= \left||z_1|^{2\sigma}(z_1 - z_2) + z_2\left(|z_1|^{2\sigma} - |z_2|^{2\sigma}\right)\right| \\
&= \left||z_1|^{2\sigma}(z_1 - z_2) + z_2\left(|z_1|^{\sigma} + |z_2|^{\sigma}\right)\left(|z_1|^{\sigma} - |z_2|^{\sigma}\right)\right| \\
&\le |z_1|^{2\sigma}|z_1 - z_2| + |z_2|^{\frac{\sigma}{2}+1}\left(|z_1|^{\sigma} + |z_2|^{\sigma}\right)\left|\frac{|z_1|^{\sigma}}{|z_2|^{\frac{\sigma}{2}}} - |z_2|^{\frac{\sigma}{2}}\right| \\
&= |z_1|^{2\sigma}|z_1 - z_2| + |z_2|^{\frac{\sigma}{2}+1}\left(|z_1|^{\sigma} + |z_2|^{\sigma}\right)\left(|z_1|^{\frac{\sigma}{2}} + |z_2|^{\frac{\sigma}{2}}\right)\left(1 - \left|\frac{z_1}{z_2}\right|^{\frac{\sigma}{2}}\right) \\
&\le |z_1|^{2\sigma}|z_1 - z_2| + |z_2|^{\frac{\sigma}{2}+1}\left(|z_1|^{\sigma} + |z_2|^{\sigma}\right)\left(|z_1|^{\frac{\sigma}{2}} + |z_2|^{\frac{\sigma}{2}}\right)\left(1 - \left|\frac{z_1}{z_2}\right|\right) \\
&= |z_1|^{2\sigma}|z_1 - z_2| + |z_2|^{\frac{\sigma}{2}}\left(|z_1|^{\sigma} + |z_2|^{\sigma}\right)\left(|z_1|^{\frac{\sigma}{2}} + |z_2|^{\frac{\sigma}{2}}\right)\left(|z_2| - |z_1|\right) \\
&\le |z_1|^{2\sigma}|z_1 - z_2| + |z_2|^{\frac{\sigma}{2}}\left(|z_1|^{\sigma} + |z_2|^{\sigma}\right)\left(|z_1|^{\frac{\sigma}{2}} + |z_2|^{\frac{\sigma}{2}}\right)|z_2 - z_1| \\
&= \left(|z_1|^{2\sigma} + |z_1|^{\frac{3}{2}\sigma}|z_2|^{\frac{\sigma}{2}} + |z_1|^{\sigma}|z_2|^{\sigma} + |z_1|^{\frac{\sigma}{2}}|z_2|^{\frac{3}{2}\sigma} + |z_2|^{2\sigma}\right)|z_1 - z_2| \\
&\le \frac{5}{2}\left(|z_1|^{2\sigma} + |z_2|^{2\sigma}\right)|z_1 - z_2|,
\end{aligned}$$

where we used Young's inequality at the final step.

(b) Initially, we prove the auxiliary inequality

$$x^s - 1 \le s(x-1)x^{s-1}, \qquad \text{for all } x \ge 1 \text{ and all } s \ge 1. \tag{D.1}$$

Therefore, we regard $F : [1,\infty) \to \mathbb{R}$ defined by $F(x) := (s-1)x^s - sx^{s-1} + 1$ with $F(1) = 0$. Calculating its first derivative, we get $F'(x) = s(s-1)x^{s-2}(x-1)$, which is non-negative for all $x \ge 1$ and all $s \ge 1$. This implies that F is a monotonically increasing real-valued function on $[1,\infty)$ and $F(x) \ge F(1)$ for all $x \ge 1$. Thus, we obtain inequality (D.1) by rearranging the relation

$$F(x) = (s-1)x^s - sx^{s-1} + 1 \ge 0 = F(1), \qquad \text{for all } x \ge 1 \text{ and all } s \ge 1.$$

Now, we face the assertion of our lemma and assume that $|z_1| > |z_2|$ and $z_2 \ne 0$ since in the case $|z_1| = |z_2|$ or $|z_2| = 0$ the inequality is obvious. Due to

$$\left||z_1|^{2\sigma}z_1 - |z_2|^{2\sigma}z_2\right| \le |z_1|^{2\sigma}|z_1 - z_2| + \left(|z_1|^{2\sigma} - |z_2|^{2\sigma}\right)|z_2|$$

and inequality (D.1) applied for $x = \left|\frac{z_1}{z_2}\right|$ and $s = 2\sigma$ that yields

$$|z_1|^{2\sigma} - |z_2|^{2\sigma} \le 2\sigma\left(|z_1| - |z_2|\right)|z_1|^{2\sigma-1} \le 2\sigma|z_1 - z_2||z_1|^{2\sigma-1}, \qquad \text{for all } \sigma \ge \frac{1}{2},$$

we receive with Young's inequality that

$$\begin{aligned}
\left||z_1|^{2\sigma}z_1 - |z_2|^{2\sigma}z_2\right| &\le \left(|z_1|^{2\sigma} + 2\sigma|z_1|^{2\sigma-1}|z_2|\right)|z_1 - z_2| \\
&\le \left(|z_1|^{2\sigma} + 2\sigma\left[\frac{2\sigma-1}{2\sigma}|z_1|^{2\sigma} + \frac{1}{2\sigma}|z_2|^{2\sigma}\right]\right)|z_1 - z_2| \\
&\le 2\sigma\left(|z_1|^{2\sigma} + |z_2|^{2\sigma}\right)|z_1 - z_2|.
\end{aligned}$$

□

Lemma D.5. *Let z_1 and z_2 be two complex-valued numbers and $\sigma > 0$, then*

$$\operatorname{Re}\left\{\left(|z_1|^{2\sigma} z_1 - |z_2|^{2\sigma} z_2\right)\left(\overline{z_1} - \overline{z_2}\right)\right\} \geq 0.$$

Proof. Expressing $z_1, z_2 \in \mathbb{C}$ in polar coordinates, it results that $z_1 := r_1[\cos\alpha_1 + i\sin\alpha_1]$ and $z_2 := r_2[\cos\alpha_2 + i\sin\alpha_2]$ with $r_1, r_2 \geq 0$ and $\alpha_1, \alpha_2 \in [0, 2\pi)$. By using the Pythagorean identity, trigonometric formulas and taking into account that the codomain of the cosine function is $[-1, 1]$, we compute

$$\begin{aligned}
&\operatorname{Re}\left\{\left(|z_1|^{2\sigma} z_1 - |z_2|^{2\sigma} z_2\right)\left(\overline{z_1} - \overline{z_2}\right)\right\} \\
&= \operatorname{Re}\left\{\left(r_1^{2\sigma+1}[\cos\alpha_1 + i\sin\alpha_1] - r_2^{2\sigma+1}[\cos\alpha_2 + i\sin\alpha_2]\right)\cdot\right. \\
&\qquad \left.\left(r_1[\cos\alpha_1 - i\sin\alpha_1] - r_2[\cos\alpha_2 - i\sin\alpha_2]\right)\right\} \\
&= \operatorname{Re}\left\{r_1^{2\sigma+2} - r_1^{2\sigma+1} r_2\left[\cos(\alpha_1 - \alpha_2) + i\sin(\alpha_1 - \alpha_2)\right]\right. \\
&\qquad \left. - r_1 r_2^{2\sigma+1}\left[\cos(\alpha_1 - \alpha_2) - i\sin(\alpha_1 - \alpha_2)\right] + r_2^{2\sigma+2}\right\} \\
&= r_1^{2\sigma+2} - r_1^{2\sigma+1} r_2 \cos(\alpha_1 - \alpha_2) - r_1 r_2^{2\sigma+1}\cos(\alpha_1 - \alpha_2) + r_2^{2\sigma+2} \\
&\geq r_1^{2\sigma+2} - r_1^{2\sigma+1} r_2 - r_1 r_2^{2\sigma+1} + r_2^{2\sigma+2} = \left(r_1^{2\sigma+1} - r_2^{2\sigma+1}\right)(r_1 - r_2) \geq 0.
\end{aligned}$$

□

Lemma D.6. *Let $v \in V$ such that $Av \in H$ and let $\sigma \geq 1$, then* $\operatorname{Re}\left\{\left(|v|^{2\sigma} v, Av\right)\right\} \geq 0$.

Proof. Observe that $\operatorname{Re}\left\{\left(|v|^{2\sigma} v, Av\right)\right\} = \operatorname{Re}\left\{\overline{\left(Av, |v|^{2\sigma} v\right)}\right\} = \operatorname{Re}\left\{\left(Av, |v|^{2\sigma} v\right)\right\}$. For the sake of simplicity, we omit to write the dependence of $v \in V$ on the space variable x in this proof. It holds that $|v|^{2\sigma} v \in V$ for all $v \in V$ such that the definition of the operator A and the relation $\frac{d}{dx}|v|^2 = \left(\frac{d}{dx}v\right)\overline{v} + v\left(\frac{d}{dx}\overline{v}\right)$ entail for $\sigma > 1$ that

$$\begin{aligned}
\left(Av, |v|^{2\sigma} v\right) = \left\langle Av, |v|^{2\sigma} v\right\rangle &= \int_0^1 \left(\frac{d}{dx}v\right)\left(\frac{d}{dx}(v^{\sigma}\overline{v}^{\sigma+1})\right) dx \\
&= \int_0^1 \left[\sigma |v|^{2(\sigma-1)}\overline{v}^2\left(\frac{d}{dx}v\right)^2 + (\sigma+1)|v|^{2\sigma}\left(\frac{d}{dx}v\right)\left(\frac{d}{dx}\overline{v}\right)\right] dx \\
&= \sigma\int_0^1 |v|^{2(\sigma-1)}\,\overline{v}\left(\frac{d}{dx}v\right)\left[\overline{v}\left(\frac{d}{dx}v\right) + v\left(\frac{d}{dx}\overline{v}\right)\right] dx + \int_0^1 |v|^{2\sigma}\left|\frac{d}{dx}v\right|^2 dx \\
&= \sigma\int_0^1 |v|^{2(\sigma-1)}\,\overline{v}\left(\frac{d}{dx}v\right)\left(\frac{d}{dx}|v|^2\right) dx + \int_0^1 |v|^{2\sigma}\left|\frac{d}{dx}v\right|^2 dx.
\end{aligned}$$

Because of

$$\operatorname{Re}\left\{\overline{v}\left(\frac{d}{dx}v\right)\right\} = \frac{1}{2}\left[\overline{v}\left(\frac{d}{dx}v\right) + v\left(\frac{d}{dx}\overline{v}\right)\right] = \frac{1}{2}\left(\frac{d}{dx}|v|^2\right),$$

one obtains by taking the real part that

$$\begin{aligned}
\operatorname{Re}\left\{\left(Av, |v|^{2\sigma} v\right)\right\} &= \sigma\int_0^1 |v|^{2(\sigma-1)}\operatorname{Re}\left\{\overline{v}\left(\frac{d}{dx}v\right)\right\}\left(\frac{d}{dx}|v|^2\right) dx + \int_0^1 |v|^{2\sigma}\left|\frac{d}{dx}v\right|^2 dx \\
&= \frac{1}{2}\sigma\int_0^1 |v|^{2(\sigma-1)}\left(\frac{d}{dx}|v|^2\right)^2 dx + \int_0^1 |v|^{2\sigma}\left|\frac{d}{dx}v\right|^2 dx,
\end{aligned}$$

which is non-negative. In the special case $\sigma = 1$ the term $|v|^{2(\sigma-1)}$ does not appear such that the same calculations yield

$$\operatorname{Re}\left\{\left(Av, |v|^2 v\right)\right\} = \frac{1}{2}\int_0^1 \left(\frac{d}{dx}|v|^2\right)^2 dx + \int_0^1 |v|^2\left|\frac{d}{dx}v\right|^2 dx \geq 0.$$

□

E Details of Generalized Drift Function

We can choose $f : V \to H$ defined by $f(v) := F(|v|^2)v$ for the nonlinear drift term in the stochastic Schrödinger problem (2.17). Here, $F : [0,\infty) \to [0,\infty)$ is once continuously differentiable with $F'(x) \geq 0$ for all $x \geq 0$, and there exist $C > 0$ and $\sigma > 1$ such that

$$|F(x_1) - F(x_2)| \leq C\left(1 + |x_1|^{\sigma-1} + |x_2|^{\sigma-1}\right)|x_1 - x_2|, \qquad \text{for all } x_1, x_2 \geq 0. \tag{E.1}$$

Under the assumption of globally Lipschitz continuity, the case $\sigma = 1$ may also be included. Now, we indicate how to derive the necessary inequalities stated in the proof of Corollary 2.2.12.

Lemma E.1. *Under the above conditions it holds that*

$$\begin{aligned} \|f(v)\| &\leq C(\sigma)\left(1 + \|v\|_V^{2\sigma+1}\right), && \textit{for all } v \in V, \\ \|f(u) - f(v)\| &\leq C(\sigma)\left(1 + \|u\|_V^{2\sigma} + \|v\|_V^{2\sigma}\right)\|u - v\|, && \textit{for all } u, v \in V. \end{aligned}$$

Proof. We begin with an auxiliary statement. Based on relation (E.1), Young's inequality and Lemma D.2, it results for all $v \in V$ that

$$\begin{aligned} \left\|F(|v|^2)\right\|^2 &= \int_0^1 \left|F(|v(x)|^2) - F(0) + F(0)\right|^2 dx \\ &\leq 2\int_0^1 \left|F(|v(x)|^2) - F(0)\right|^2 dx + 2\int_0^1 |F(0)|^2\, dx \\ &\leq 2C^2 \int_0^1 \left(1 + |v(x)|^{2(\sigma-1)}\right)^2 |v(x)|^4\, dx + 2F^2(0) \\ &\leq 4C^2 \int_0^1 \left(|v(x)|^4 + |v(x)|^{4\sigma}\right) dx + 2F^2(0) \\ &\leq 4C^2 \sup_{x\in[0,1]} \left(|v(x)|^4 + |v(x)|^{4\sigma}\right) + 2F^2(0) \\ &\leq 4C^2 \sup_{x\in[0,1]} \left(\frac{\sigma-1}{\sigma} + \frac{\sigma+1}{\sigma}|v(x)|^{4\sigma}\right) + 2F^2(0) \\ &\leq 2\left(F^2(0) + 2C^2\frac{\sigma-1}{\sigma} + 2^{2\sigma+1}C^2\frac{\sigma+1}{\sigma}\|v\|_V^{4\sigma}\right) \\ &\leq C(\sigma)\left(1 + \|v\|_V^{4\sigma}\right). \end{aligned}$$

Then Lemma D.2, the last estimate and Young's inequality are used to derive

$$\begin{aligned} \|f(v)\|^2 &= \int_0^1 \left|F(|v(x)|^2)\right|^2 |v(x)|^2\, dx \leq \sup_{x\in[0,1]} |v(x)|^2 \int_0^1 \left|F(|v(x)|^2)\right|^2 dx \leq 2\|v\|_V^2 \left\|F(|v|^2)\right\|^2 \\ &\leq 2C(\sigma)\left(\|v\|_V^2 + \|v\|_V^{4\sigma+2}\right) \leq 2C(\sigma)\left(\frac{2\sigma}{2\sigma+1} + \frac{2\sigma+2}{2\sigma+1}\|v\|_V^{4\sigma+2}\right) \leq C(\sigma)\left(1 + \|v\|_V^{4\sigma+2}\right) \end{aligned}$$

for all $v \in V$, which implies the analogue of (2.18) given by

$$\|f(v)\| \leq C(\sigma)\left(1 + \|v\|_V^{2\sigma+1}\right), \qquad \text{for all } v \in V.$$

Alternatively, we can proceed like in the auxiliary inequality such that

$$\begin{aligned}
\|f(v)\|^2 &= \int_0^1 \left|F(|v(x)|^2)\right|^2 |v(x)|^2\,dx = \int_0^1 \left|F(|v(x)|^2) - F(0) + F(0)\right|^2 |v(x)|^2\,dx \\
&\leq 2\int_0^1 \left|F(|v(x)|^2) - F(0)\right|^2 |v(x)|^2\,dx + 2\int_0^1 |F(0)|^2 |v(x)|^2\,dx \\
&\leq 2C^2 \int_0^1 \left(1 + |v(x)|^{2(\sigma-1)}\right)^2 |v(x)|^4 |v(x)|^2\,dx + 2F^2(0)\|v\|^2 \\
&\leq 4C^2 \int_0^1 \left(|v(x)|^4 + |v(x)|^{4\sigma}\right) |v(x)|^2\,dx + 2F^2(0)\|v\|^2 \\
&\leq 4C^2 \sup_{x\in[0,1]} \left(|v(x)|^4 + |v(x)|^{4\sigma}\right) \|v\|^2 + 2F^2(0)\|v\|^2 \\
&\leq 4C^2 \sup_{x\in[0,1]} \left(\frac{\sigma-1}{\sigma} + \frac{\sigma+1}{\sigma}|v(x)|^{4\sigma}\right) \|v\|^2 + 2F^2(0)\|v\|^2 \\
&\leq 2\left(F^2(0) + 2C^2\frac{\sigma-1}{\sigma} + 2^{2\sigma+1}C^2\frac{\sigma+1}{\sigma}\|v\|_V^{4\sigma}\right)\|v\|^2 \\
&\leq C(\sigma)\left(1 + \|v\|_V^{4\sigma}\right)\|v\|^2, \qquad \text{for all } v \in V.
\end{aligned}$$

This inequality and relation (E.1) are necessary to calculate for all $u, v \in V$ that

$$\begin{aligned}
\|f(u) - f(v)\|^2 &= \int_0^1 \left|\left[F(|u(x)|^2) - F(|v(x)|^2)\right] u(x) + F(|v(x)|^2)\left[u(x) - v(x)\right]\right|^2 dx \\
&\leq 2\int_0^1 \left|F(|u(x)|^2) - F(|v(x)|^2)\right|^2 |u(x)|^2\,dx + 2\int_0^1 \left|F(|v(x)|^2)\right|^2 |u(x) - v(x)|^2\,dx \\
&\leq 2\sup_{x\in[0,1]} |u(x)|^2 \left\|F(|u|^2) - F(|v|^2)\right\|^2 + 2C(\sigma)\left(1 + \|v\|_V^{4\sigma}\right)\|u - v\|^2 \\
&\leq 4C^2\|u\|_V^2 \int_0^1 \left(1 + |u(x)|^{2(\sigma-1)} + |v(x)|^{2(\sigma-1)}\right)^2 \left||u(x)|^2 - |v(x)|^2\right|^2 dx \\
&\quad + 2C(\sigma)\left(1 + \|v\|_V^{4\sigma}\right)\|u - v\|^2.
\end{aligned}$$

Having a closer look at the first term and using Lemma D.2 and Young's inequality, it holds for all $u, v \in V$ that

$$\begin{aligned}
&4C^2\|u\|_V^2 \int_0^1 \left(1 + |u(x)|^{2(\sigma-1)} + |v(x)|^{2(\sigma-1)}\right)^2 \left||u(x)|^2 - |v(x)|^2\right|^2 dx \\
&\leq 16C^2\|u\|_V^2 \int_0^1 \left(1 + |u(x)|^{4(\sigma-1)} + |v(x)|^{4(\sigma-1)}\right) \left(|u(x)| + |v(x)|\right)^2 \left||u(x)| - |v(x)|\right|^2 dx \\
&\leq 16C^2\|u\|_V^2 \int_0^1 \left(1 + |u(x)|^{4(\sigma-1)} + |v(x)|^{4(\sigma-1)}\right) 2\left(|u(x)|^2 + |v(x)|^2\right) |u(x) - v(x)|^2\,dx \\
&\leq 32C^2\|u\|_V^2 \sup_{x\in[0,1]} \left(1 + |u(x)|^{4(\sigma-1)} + |v(x)|^{4(\sigma-1)}\right) \sup_{x\in[0,1]} \left(|u(x)|^2 + |v(x)|^2\right)\|u - v\|^2 \\
&\leq 16\cdot 2^{2\sigma}C^2\|u\|_V^2 \left(1 + \|u\|_V^{4(\sigma-1)} + \|v\|_V^{4(\sigma-1)}\right)\left(\|u\|_V^2 + \|v\|_V^2\right)\|u - v\|^2 \\
&\leq 2^{2\sigma+4}C^2\left(\|u\|_V^4 + \|u\|_V^2\|v\|_V^2 + \|u\|_V^{4\sigma} + \|u\|_V^{4\sigma-2}\|v\|_V^2 + \|u\|_V^4\|v\|_V^{4\sigma-4} + \|u\|_V^2\|v\|_V^{4\sigma-2}\right)\|u - v\|^2 \\
&\leq C(\sigma)\left(1 + \|u\|_V^{4\sigma} + \|v\|_V^{4\sigma}\right)\|u - v\|^2.
\end{aligned}$$

Thus, we get

$$\|f(u) - f(v)\|^2 \leq C(\sigma)\left(1 + \|u\|_V^{4\sigma} + \|v\|_V^{4\sigma}\right)\|u - v\|^2, \qquad \text{for all } u, v, \in V,$$

and, therefore, the analogue of (2.19) stated by

$$\|f(u) - f(v)\| \leq C(\sigma)\left(1 + \|u\|_V^{2\sigma} + \|v\|_V^{2\sigma}\right)\|u - v\|, \qquad \text{for all } u, v \in V. \qquad \square$$

Lemma E.2. *Moreover, we get results analogous to Lemma D.5 and Lemma D.6 constituted by*

$$\operatorname{Re}\left\{\left(F(|z_1|^2)z_1 - F(|z_2|^2)z_2\right)(\overline{z_1}-\overline{z_2})\right\} \geq 0, \qquad \textit{for all } z_1, z_2 \in \mathbb{C},$$
$$\operatorname{Re}\left\{\left(F(|v|^2)v, Av\right)\right\} \geq 0, \qquad \textit{for each } v \in V \textit{ such that } Av \in H.$$

Proof. To show the first result, we regard the trigonometric representation of two complex-valued numbers $z_1 := r_1[\cos\alpha_1 + i\sin\alpha_2]$ and $z_2 := r_2[\cos\alpha_2 + i\sin\alpha_2]$ with $r_1, r_2 \geq 0$ and $\alpha_1, \alpha_2 \in [0, 2\pi)$, follow the ideas of Lemma D.5 and use the fact that F is an increasing and positive function. Then we obtain

$$\begin{aligned}
&\operatorname{Re}\left\{\left(F(|z_1|^2)z_1 - F(|z_2|^2)z_2\right)(\overline{z_1}-\overline{z_2})\right\}\\
&= \operatorname{Re}\left\{\left(F(r_1^2)r_1[\cos\alpha_1 + i\sin\alpha_1] - F(r_2^2)r_2[\cos\alpha_2 + i\sin\alpha_2]\right)\cdot\right.\\
&\qquad \left.\left(r_1[\cos\alpha_1 - i\sin\alpha_1] - r_2[\cos\alpha_2 - i\sin\alpha_2]\right)\right\}\\
&= \operatorname{Re}\left\{F(r_1^2)r_1^2 - F(r_1^2)r_1r_2\left[\cos(\alpha_1-\alpha_2) - i\sin(\alpha_1-\alpha_2)\right]\right.\\
&\qquad \left.- F(r_2^2)r_1r_2\left[\cos(\alpha_1-\alpha_2) - i\sin(\alpha_1-\alpha_2)\right] + F(r_2^2)r_2^2\right\}\\
&= F(r_1^2)r_1^2 - F(r_1^2)r_1r_2\cos(\alpha_1-\alpha_2) - F(r_2^2)r_1r_2\cos(\alpha_1-\alpha_2) + F(r_2^2)r_2^2\\
&\geq F(r_1^1)r_1^2 - F(r_1^2)r_1r_2 - F(r_2^2)r_1r_2 + F(r_2^2)r_2^2 = \left(F(r_1^2)r_1 - F(r_2^2)r_2\right)(r_1 - r_2) \geq 0.
\end{aligned}$$

Furthermore, since F and F' are positive functions and due to

$$\operatorname{Re}\left\{\left(\frac{d}{dx}v(x)\right)\overline{v}(x)\right\} = \frac{1}{2}\left[\left(\frac{d}{dx}v(x)\right)\overline{v}(x) + v(x)\left(\frac{d}{dx}\overline{v}(x)\right)\right] = \frac{1}{2}\left(\frac{d}{dx}|v(x)|^2\right),$$

one proves similarly to Lemma D.6 that

$$\begin{aligned}
&\operatorname{Re}\left\{\left(F(|v|^2)v, Av\right)\right\} = \operatorname{Re}\left\{\overline{\left(Av, F(|v|^2)v\right)}\right\} = \operatorname{Re}\left\{\left(Av, F(|v|^2)v\right)\right\}\\
&= \operatorname{Re}\left\{\langle Av, F(|v|^2)v\rangle\right\} = \operatorname{Re}\left\{\int_0^1 \left(\frac{d}{dx}v(x)\right)\left(\frac{d}{dx}\left[F(|v(x)|^2)\,\overline{v}(x)\right]\right)dx\right\}\\
&= \operatorname{Re}\left\{\int_0^1 \left(\frac{d}{dx}v(x)\right)\left[\left(\frac{d}{dy}F(y)\Big|_{y=|v(x)|^2}\right)\left(\frac{d}{dx}|v(x)|^2\right)\overline{v}(x) + F(|v(x)|^2)\left(\frac{d}{dx}\overline{v}(x)\right)\right]dx\right\}\\
&= \int_0^1 \left(\frac{d}{dy}F(y)\Big|_{y=|v(x)|^2}\right)\left(\frac{d}{dx}|v(x)|^2\right)\operatorname{Re}\left\{\left(\frac{d}{dx}v(x)\right)\overline{v}(x)\right\}dx + \int_0^1 F(|v(x)|^2)\left|\frac{d}{dx}v(x)\right|^2 dx\\
&= \frac{1}{2}\int_0^1 \left(\frac{d}{dy}F(y)\Big|_{y=|v(x)|^2}\right)\left(\frac{d}{dx}|v(x)|^2\right)^2 dx + \int_0^1 F(|v(x)|^2)\left|\frac{d}{dx}v(x)\right|^2 dx \geq 0
\end{aligned}$$

for all $v \in V$ such that $Av \in H$. □

F Basic Convergence Results

Lemma F.1. *Let $(u_n)_n$ be a bounded sequence in $L^{2p}(\Omega \times [0,T]; V)$ with $p \geq 1$. Then there exist a subsequence $(u_{n'})_{n'}$ of $(u_n)_n$ and a function $u \in L^{2p}(\Omega \times [0,T]; V)$ such that $(u_{n'})_{n'}$ converges weakly to u in $L^2(\Omega \times [0,T]; H)$, $L^2(\Omega \times [0,T]; V)$ and $L^{2p}(\Omega \times [0,T]; V)$.*

Proof. Notice that $L^{2p}(\Omega \times [0,T]; V)$ is a reflexive Banach space (see [29, p. 100, Corollary 2]). Hence, (referring to [104, p. 258, Proposition 21.23 (i)]) there exist a subsequence $(u_{n'})_{n'}$ of $(u_n)_n$ and a function $u \in L^{2p}(\Omega \times [0,T]; V)$ such that $(u_{n'})_{n'}$ converges weakly to u in $L^{2p}(\Omega \times [0,T]; V)$. Using the continuity of the embeddings

$$L^{2p}(\Omega \times [0,T]; V) \hookrightarrow L^2(\Omega \times [0,T]; V) \hookrightarrow L^2(\Omega \times [0,T]; H),$$

we obtain the weak convergences of $(u_{n'})_{n'}$ to u in $L^2(\Omega \times [0,T]; V)$ and $L^2(\Omega \times [0,T]; H)$ as well (compare [104, p. 265, Proposition 21.35 (c)]). □

Lemma F.2. *Let $\mathcal{H}$ be a complex separable Hilbert space with appropriate norm $\|\cdot\|_{\mathcal{H}}$ and let $(u_n)_n$ be a bounded sequence in $L^\infty([0,T]; \mathcal{H})$. Then there exist a subsequence $(u_{n'})_{n'}$ of $(u_n)_n$ and a function $u \in L^\infty([0,T]; \mathcal{H})$ such that*

$$\int_0^T \big(u_{n'}(t), h\big)_{\mathcal{H}}\, dt \to \int_0^T \big(u(t), h\big)_{\mathcal{H}}\, dt, \qquad \text{for all } h \in L^1([0,T]; \mathcal{H}) \text{ as } n' \to \infty.$$

Moreover,

$$\operatorname*{ess\,sup}_{t \in [0,T]} \|u(t)\|_{\mathcal{H}}^2 \leq \liminf_{n' \to \infty} \operatorname*{ess\,sup}_{t \in [0,T]} \|u_{n'}(t)\|_{\mathcal{H}}^2.$$

Proof. The proof relies on the duality $(L^1([0,T]; \mathcal{H}))^* = L^\infty([0,T]; \mathcal{H}^*)$ (stated in [104, p. 449, Problem 23.12 d]). By Riesz' representation theorem, there exists a unique correspondence between the elements $u \in \mathcal{H}$ and $u^* \in \mathcal{H}^*$ given by

$$u^*(h) = \big(u, h\big)_{\mathcal{H}}, \qquad \text{for all } h \in \mathcal{H}, \tag{F.1}$$

and

$$\|u^*\|_{\mathcal{H}^*} = \|u\|_{\mathcal{H}}.$$

Hence, there exists a bounded sequence $(u_n^*)_n$ in $L^\infty([0,T]; \mathcal{H}^*)$ corresponding to the bounded sequence $(u_n)_n$ from $L^\infty([0,T]; \mathcal{H})$ such that

$$u_n^*(t)(h) = \big(u_n(t), h\big)_{\mathcal{H}}, \qquad \text{for a.e. } t \in [0,T], \text{ all } h \in \mathcal{H} \text{ and all } n \in \mathbb{N}, \tag{F.2}$$

and

$$\|u_n^*(t)\|_{\mathcal{H}^*} = \|u_n(t)\|_{\mathcal{H}}, \qquad \text{for a.e. } t \in [0,T] \text{ and all } n \in \mathbb{N}. \tag{F.3}$$

Regarding the weak* sequential compactness of $L^\infty([0,T]; \mathcal{H}^*) = (L^1([0,T]; \mathcal{H}))^*$ (see [104, p. 449, Problem 23.12 e]), we derive the existence of a subsequence $(u_{n'}^*)_{n'}$ of $(u_n^*)_n$ and a function $u^* \in L^\infty([0,T]; \mathcal{H}^*)$ such that

$$u_{n'}^* \overset{*}{\rightharpoonup} u^* \qquad \text{in } L^\infty([0,T]; \mathcal{H}^*) \text{ as } n' \to \infty,$$

which means that

$$\int_0^T u_{n'}^*(t)(h)\,dt \to \int_0^T u^*(t)(h)\,dt, \qquad \text{for all } h \in L^1([0,T];\mathcal{H}) \text{ as } n' \to \infty. \tag{F.4}$$

We apply again the correspondence (F.1) between the elements of $\mathcal{H}$ and $\mathcal{H}^*$ to obtain the existence of a function $u \in L^\infty([0,T];\mathcal{H})$ such that

$$u^*(t)(h) = \big(u(t), h\big), \qquad \text{for a.e. } t \in [0,T] \text{ and all } h \in \mathcal{H}, \tag{F.5}$$

and

$$\|u^*(t)\|_{\mathcal{H}^*} = \|u(t)\|_{\mathcal{H}}, \qquad \text{for a.e. } t \in [0,T]. \tag{F.6}$$

Thus, plugging (F.2) and (F.5) in (F.4), the first assertion ensues

$$\int_0^T \big(u_{n'}(t), h\big)_{\mathcal{H}}\,dt \to \int_0^T \big(u(t), h\big)_{\mathcal{H}}\,dt, \qquad \text{for all } h \in L^1([0,T];\mathcal{H}) \text{ as } n' \to \infty.$$

Moreover, (by [104, p. 261, Proposition 21.26 (b)]) we have

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|u^*(t)\|^2_{\mathcal{H}^*} = \|u^*\|^2_{L^\infty([0,T];\mathcal{H}^*)} \le \liminf_{n'\to\infty} \|u^*_{n'}\|^2_{L^\infty([0,T];\mathcal{H}^*)} = \liminf_{n'\to\infty} \operatorname*{ess\,sup}_{t\in[0,T]} \|u^*_{n'}(t)\|^2_{\mathcal{H}^*}.$$

Here, the application of (F.3) and (F.6) yields

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|u(t)\|^2_{\mathcal{H}} \le \liminf_{n'\to\infty} \operatorname*{ess\,sup}_{t\in[0,T]} \|u_{n'}(t)\|^2_{\mathcal{H}}. \qquad \square$$

G Local Martingale Property

To prove the results of Remark 2.3.1, let $b_j : \Omega \times [0,T] \to \mathbb{R}$ be $\mathcal{F}_t$-adapted processes for all $j \geq 1$ that satisfy

$$\sum_{j=1}^{\infty} \int_0^T b_j^2(s)\, ds < \infty, \qquad \text{for a.e. } \omega \in \Omega.$$

Furthermore, let the increasing sequence of stopping times $(\mathcal{T}_M)_{M\in\mathbb{N}}$ be defined by

$$\mathcal{T}_M := \begin{cases} T & : \sum_{j=1}^{\infty} \int_0^T b_j^2(s)\, ds < M, \\ \inf\left\{ t \in [0,T] : \sum_{j=1}^{\infty} \int_0^t b_j^2(s)\, ds \geq M \right\} & : \sum_{j=1}^{\infty} \int_0^T b_j^2(s)\, ds \geq M. \end{cases}$$

Lemma G.1. *The sequence of stopping times $(\mathcal{T}_M)_{M\in\mathbb{N}}$ converges P-a.s. to T and*

$$P\left(\bigcup_{M=1}^{\infty} \{\mathcal{T}_M = T\} \right) = 1.$$

Proof. The increasing sequence of stopping times obeys

$$\begin{aligned} \lim_{M\to\infty} P\left(\mathcal{T}_M < T\right) &\leq \lim_{M\to\infty} P\left(\sum_{j=1}^{\infty} \int_0^T b_j^2(s)\, ds \geq M \right) \\ &= P\left(\bigcap_{M=1}^{\infty} \left\{ \sum_{j=1}^{\infty} \int_0^T b_j^2(s)\, ds \geq M \right\} \right) = 0 \end{aligned}$$

due to the sequential continuity of the probability. Thus, the monotonically decreasing and non-negative sequence $(T - \mathcal{T}_M)_{M\in\mathbb{N}}$ converges in probability to zero since

$$\lim_{M\to\infty} P\left(T - \mathcal{T}_M > 0\right) = \lim_{M\to\infty} P\left(|T - \mathcal{T}_M| > 0\right) = 0$$

includes that

$$\lim_{M\to\infty} P\left(|T - \mathcal{T}_M| \geq \varepsilon\right) = 0, \qquad \text{for all } \varepsilon > 0.$$

The monotony of the sequence yields

$$\lim_{M\to\infty} P\left(|T - \mathcal{T}_M| \geq \varepsilon\right) = \lim_{M\to\infty} P\left(\sup_{n\geq M} |T - \mathcal{T}_n| \geq \varepsilon \right) = 0, \qquad \text{for all } \varepsilon > 0,$$

such that $(\mathcal{T}_M)_{M\in\mathbb{N}}$ converges P-a.s. to T because of [88, p. 333, Lemma 14.1.2]. Moreover, the sequential continuity of the probability is used to obtain

$$P\left(\bigcup_{M=1}^{\infty} \{\mathcal{T}_M = T\} \right) = \lim_{M\to\infty} P\left(\mathcal{T}_M = T\right) = 1 - \lim_{M\to\infty} P\left(\mathcal{T}_M < T\right) = 1. \qquad \square$$

Lemma G.2. *If $X \in L^2(\Omega; C([0,T]; H))$, it holds that*

$$\tilde{I}(t) := \sum_{j=1}^{\infty} \int_0^t b_j(s) X(s)\, d\beta_j(s), \qquad \text{for all } t \in [0,T],$$

is a local martingale with respect to the sequence of stopping times $(\mathcal{T}_M)_{M\in\mathbb{N}}$.

Proof. To show that the stochastic process $(\tilde{I}(t))_{t\in[0,T]}$ is a local martingale, we have to ensure that the corresponding sequence of stopping times $(\mathcal{T}_M)_{M\in\mathbb{N}}$ converges P-a.s. to T (compare Lemma G.1) and $\tilde{I}(t)$ is $\mathcal{F}_t$-adapted such that the stopped process $\tilde{I}(t\wedge\mathcal{T}_M)$ is a martingale for all $M \in \mathbb{N}$, which means that the $\mathcal{F}_t$-adapted process $\tilde{I}(t\wedge\mathcal{T}_M)$ is integrable in the sense that $E\|\tilde{I}(t\wedge\mathcal{T}_M)\| < \infty$ for all $t \in [0,T]$ and fulfills $E(\tilde{I}(t\wedge\mathcal{T}_M)|\mathcal{F}_r) = \tilde{I}(r\wedge\mathcal{T}_M)$ P-a.s. for all $0 \le r \le t \le T$ (see [87, p. 40, Definition 3 and 4]).

Being an Itô integral, the stochastic process $\tilde{I}(t)$ is $\mathcal{F}_t$-adapted. Due to the Cauchy-Schwarz inequality, the independence of the Wiener processes, the Itô isometry and the assumptions on the sequence of stopping times $(\mathcal{T}_M)_{M\in\mathbb{N}}$ and the process $(X(t))_{t\in[0,T]}$, it holds for all $M \in \mathbb{N}$ that

$$\begin{aligned}
E\left\|\tilde{I}(t\wedge\mathcal{T}_M)\right\| &= E\left\|\sum_{j=1}^{\infty}\int_0^{t\wedge\mathcal{T}_M} b_j(s)X(s)\,d\beta_j(s)\right\| \le \left(E\left\|\sum_{j=1}^{\infty}\int_0^{t\wedge\mathcal{T}_M} b_j(s)X(s)\,d\beta_j(s)\right\|^2\right)^{\frac{1}{2}}\\
&= \left(E\int_0^1\left|\sum_{j=1}^{\infty}\int_0^t \mathbb{1}_{[0,\mathcal{T}_M]}(s)\, b_j(s)X(s)\,d\beta_j(s)\right|^2 dx\right)^{\frac{1}{2}}\\
&= \left(\int_0^1\sum_{j=1}^{\infty}E\left|\int_0^t \mathbb{1}_{[0,\mathcal{T}_M]}(s)\, b_j(s)X(s)\,d\beta_j(s)\right|^2 dx\right)^{\frac{1}{2}}\\
&= \left(\int_0^1\sum_{j=1}^{\infty}E\int_0^t \mathbb{1}_{[0,\mathcal{T}_M]}(s)\, b_j^2(s)|X(s)|^2\,ds\,dx\right)^{\frac{1}{2}}\\
&\le \left(E\sum_{j=1}^{\infty}\int_0^T \mathbb{1}_{[0,\mathcal{T}_M]}(s)\, b_j^2(s)\|X(s)\|^2\,ds\right)^{\frac{1}{2}}\\
&\le \left(E\left[\sup_{t\in[0,T]}\|X(t)\|^2\sum_{j=1}^{\infty}\int_0^{\mathcal{T}_M} b_j^2(s)\,ds\right]\right)^{\frac{1}{2}} \le \sqrt{M}\left(E\sup_{t\in[0,T]}\|X(t)\|^2\right)^{\frac{1}{2}} < \infty.
\end{aligned}$$

Since the Itô integral itself is a martingale, we finally show with probability one for $0 \le r \le t \le T$ the martingale condition

$$\begin{aligned}
E(\tilde{I}(t\wedge\mathcal{T}_M)|\mathcal{F}_r) &= E\left(\sum_{j=1}^{\infty}\int_0^{t\wedge\mathcal{T}_M} b_j(s)X(s)\,d\beta_j(s)\,\middle|\,\mathcal{F}_r\right)\\
&= E\left(\sum_{j=1}^{\infty}\int_0^{t} \mathbb{1}_{[0,\mathcal{T}_M]}(s)\, b_j(s)X(s)\,d\beta_j(s)\,\middle|\,\mathcal{F}_r\right)\\
&= \sum_{j=1}^{\infty}\int_0^{r} \mathbb{1}_{[0,\mathcal{T}_M]}(s)\, b_j(s)X(s)\,d\beta_j(s)\\
&= \sum_{j=1}^{\infty}\int_0^{r\wedge\mathcal{T}_M} b_j(s)X(s)\,d\beta_j(s) = \tilde{I}(r\wedge\mathcal{T}_M).
\end{aligned}$$

□

H Wirtinger Derivatives

The Wirtinger calculus is mostly applied in the theory of functions and can be found, among others, in [35, 64, 85]. This concept of real differentiability of a complex-valued function is closely connected with complex differentiability and the Cauchy-Riemann equations. First, we define a real differentiable function (based on [85, pp. 57–59]). Thus, let D be an open subset of $\mathbb{C}$ and let $f : D \to \mathbb{C}$ be a complex-valued function, meaning that $f := u + iv$ with $\mathrm{Re}(f) = u$ and $\mathrm{Im}(f) = v$.

Definition H.1. *The function f is real differentiable in $z_0 = x_0 + iy_0 \in D$ if there exist in z_0 continuous functions $f_1,\ f_2 : D \to \mathbb{C}$ such that*

$$f(z) = f(z_0) + (x - x_0)f_1(z) + (y - y_0)f_2(z), \qquad \textit{for all } z = x + iy \in D.$$

f_1 and f_2 are not uniquely determined, however their values in z_0 are constituted by

$$f_1(z_0) = \frac{\partial f}{\partial x}(z_0) =: f_x(z_0) \qquad \text{and} \qquad f_2(z_0) = \frac{\partial f}{\partial y}(z_0) =: f_y(z_0). \tag{H.1}$$

That is why we understand Definition H.1 in the following way (compare [85, pp. 57–59, Satz 1.4.1 and Satz 1.4.2]). A function f is real differentiable in $z_0 \in D$ if there exist a (uniquely determined) $\mathbb{R}$-linear mapping $T : \mathbb{C} \to \mathbb{C}$ and a function $R : D \to \mathbb{C}$ that is continuous in z_0 such that

$$f(z) = f(z_0) + T(z - z_0) + R(z)(z - z_0), \qquad \text{for all } z = x + iy \in D,$$

where $T(z - z_0) := (x - x_0)f_1(z_0) + (y - y_0)f_2(z_0) = (x - x_0)f_x(z_0) + (y - y_0)f_y(z_0)$ and $R(z_0) := 0$,

$$R(z) := \frac{(x - x_0)(f_1(z) - f_1(z_0)) + (y - y_0)(f_2(z) - f_2(z_0))}{z - z_0}, \qquad \text{for all } z \neq z_0.$$

Now, the relations in (H.1) are equivalent to the differentiability of the real-valued functions u and v with $f_x = u_x + iv_x$ and $f_y = u_y + iv_y$. To avoid the partition in real and imaginary part u and v and in the real coordinates x and y, we write $x - x_0 = \frac{1}{2}(z - z_0 + \overline{z} - \overline{z_0})$ and $y - y_0 = \frac{1}{2i}(z - z_0 - (\overline{z} - \overline{z_0}))$. Plugging this in Definition H.1 yields an equivalent formulation

$$\begin{aligned} f(z) &= f(z_0) + (x - x_0)f_1(z) + (y - y_0)f_2(z) \\ &= f(z_0) + \frac{1}{2}(z - z_0 + \overline{z} - \overline{z_0})f_1(z) + \frac{1}{2i}(z - z_0 - (\overline{z} - \overline{z_0}))f_2(z) \\ &= f(z_0) + (z - z_0)\frac{1}{2}\left[f_1(z) - if_2(z)\right] + (\overline{z} - \overline{z_0})\frac{1}{2}\left[f_1(z) + if_2(z)\right]. \end{aligned}$$

Remark H.2. *The function f is real differentiable in $z_0 \in D$ if there exist in z_0 continuous functions $\hat{f}_1,\ \hat{f}_2 : D \to \mathbb{C}$ such that*

$$f(z) = f(z_0) + (z - z_0)\hat{f}_1(z) + (\overline{z} - \overline{z_0})\hat{f}_2(z), \qquad \textit{for all } z \in D.$$

Here, the values $\hat{f}_1(z_0)$ and $\hat{f}_2(z_0)$ are uniquely determined by the function f and given by

$$\hat{f}_1(z_0) = \frac{1}{2}(f_1(z_0) - if_2(z_0)) \qquad \text{and} \qquad \hat{f}_2(z_0) = \frac{1}{2}(f_1(z_0) + if_2(z_0)). \tag{H.2}$$

Definition H.3. [1] *The values $\hat{f}_1(z_0)$ and $\hat{f}_2(z_0)$ of a function f that is in z_0 real differentiable are called Wirtinger derivatives of f in z_0 and are denoted by*

$$\hat{f}_1(z_0) = \frac{\partial f}{\partial z}(z_0) =: f_z(z_0) \qquad \textit{and} \qquad \hat{f}_2(z_0) = \frac{\partial f}{\partial \overline{z}}(z_0) =: f_{\overline{z}}(z_0).$$

Consequently, we call a function $f : D \to \mathbb{C}$ that is real differentiable in the way of Remark H.2 to be differentiable in the sense of Wirtinger. Referring to (H.1), the relations (H.2) imply that

$$f_z = \frac{1}{2}(f_x - if_y) \qquad \text{and} \qquad f_{\overline{z}} = \frac{1}{2}(f_x + if_y).$$

From Remark H.2, we further deduce the connection between real and complex differentiability.

Definition H.4. [2] *The function $f : D \to \mathbb{C}$ is in $z_0 \in D$ complex differentiable if f is real differentiable in z_0 and*

$$\frac{\partial f}{\partial \overline{z}}(z_0) = 0. \tag{H.3}$$

Then $f'(z_0) = \frac{\partial f}{\partial z}(z_0)$ and the differential operator $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ is called operator of the Cauchy-Riemann equation.

For the sake of completeness, the Cauchy-Riemann equations are stated. They follow from assumption (H.3) since

$$0 = \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}(f_x + if_y) = \frac{1}{2}(u_x + iv_x + i(u_y + iv_y)) = \frac{1}{2}(u_x - v_y + i(u_y + v_x)).$$

This equals zero if and only if $u_x = v_y$ and $u_y = -v_x$, which are called the Cauchy-Riemann equations. The following result concerning complex differentiable functions is Liouville's theorem.

Theorem H.5. [3,4] *If $f : D \to \mathbb{C}$ is complex differentiable in each point of D and bounded, then f is constant.*

Remark H.6. *A function $f : D \to \mathbb{C}$ which is complex differentiable in each point of D is called holomorphic. The assumption of a bounded function f is equivalent to $f' \equiv 0$.*

Summarized, the concept of Wirtinger derivatives is a useful tool to represent the real differentiability of a complex-valued function in complex coordinates. Besides basic properties of the Wirtinger derivatives like linearity, satisfaction of the sum, product, quotient and chain rule and the behavior under conjugation, there is an important result used multiple times in this work.

Corollary H.7. [5] *It is allowed to differentiate the function f with respect to the complex conjugated variables z and $\overline{z}$ as if they were independent of each other.*

[1] Lieb & Fischer [64], p. 20, Definition 5.1
[2] Lieb & Fischer [64], pp. 20 f., Theorem 5.1 and Definition 5.2
[3] Lieb & Fischer [64], p. 22, Satz 5.3
[4] Remmert & Schumacher [85], p. 218, Satz 8.3.5
[5] Remmert & Schumacher [85], p. 61

I Complex Conjugated Adjoint Equation

Based on [3, 49, 52, 86] and [95, pp. 216 f.], we establish the complex conjugated adjoint Schrödinger equation in the following way. Choosing $g \equiv 0$, which means that we only take the deterministic part of the controlled Schrödinger equation (3.2), and denoting the solution with X^U as well, we start our considerations with the symbolic form

$$dX^U(t) = -\,iAX^U(t)\,dt + iU(t)X^U(t)\,dt + i\lambda f\big(t, X^U(t)\big)\,dt$$

for a.e. $\omega \in \Omega$ and all $t \in [0,T]$. Remembering the notations from Sections 2.1 and 3.1, we equivalently write

$$\frac{\partial}{\partial t}X^U(t,x) = i\frac{\partial^2}{\partial x^2}X^U(t,x) + iU(t,x)X^U(t,x) + i\lambda f\big(t, X^U(t,x), \overline{X^U}(t,x)\big)$$

as an equation in V^*. To be as general as possible, we choose $\lambda \in \mathbb{C}$ that includes all considered cases. Disregarding the time and space arguments, renaming $y := X^U$ and multiplying with the imaginary unit, we obtain

$$F(t,x,y,y_t,y_{xx},\overline{y}) := iy_t + y_{xx} + Uy + \lambda f(\,\cdot\,,y,\overline{y}) = 0.$$

Now, the adjoint equation is calculated by

$$0 = \frac{\delta}{\delta y}(F\eta^U) = \left(-D_t\frac{\partial}{\partial y_t} + D_x^2\frac{\partial}{\partial y_{xx}} + \frac{\partial}{\partial y} + \overline{D}\frac{\partial}{\partial \overline{y}}\right)\left(\left[iy_t + y_{xx} + Uy + \lambda f(\,\cdot\,,y,\overline{y})\right]\eta^U\right),$$

where D_t and D_x are total derivatives with respect to the time t and the space x and the operator $\overline{D}$ obeys $\overline{D}[z] = \overline{z}$ for all $z \in \mathbb{C}$. Simplification yields

$$\begin{aligned} 0 &= -D_t[i\eta^U] + D_x^2[\eta^U] + U\eta^U + \lambda\left(\frac{\partial}{\partial y}f(\,\cdot\,,y,\overline{y})\right)\eta^U + \overline{D}\left[\lambda\left(\frac{\partial}{\partial \overline{y}}f(\,\cdot\,,y,\overline{y})\right)\eta^U\right] \\ &= -\,i\eta^U_t + \eta^U_{xx} + U\eta^U + \lambda\left(\frac{\partial}{\partial y}f(\,\cdot\,,y,\overline{y})\right)\eta^U + \overline{\lambda}\,\overline{\left(\frac{\partial}{\partial \overline{y}}f(\,\cdot\,,y,\overline{y})\right)}\,\overline{\eta^U}. \end{aligned} \tag{I.1}$$

Multiplying this equation with i and writing X^U instead of y, it results that

$$\eta^U_t = -\,i\eta^U_{xx} - iU\eta^U - i\lambda\left(\frac{\partial}{\partial X^U}f\big(\,\cdot\,,X^U,\overline{X^U}\big)\right)\eta^U - i\overline{\lambda}\,\overline{\left(\frac{\partial}{\partial \overline{X^U}}f\big(\,\cdot\,,X^U,\overline{X^U}\big)\right)}\,\overline{\eta^U},$$

where we use the symbolic notation

$$\frac{\partial}{\partial X^U}f\big(\,\cdot\,,X^U,\overline{X^U}\big) := \frac{\partial}{\partial v}f(\,\cdot\,,v,\overline{v})\Big|_{\substack{v=X^U\\ \overline{v}=\overline{X^U}}}, \qquad \frac{\partial}{\partial \overline{X^U}}f\big(\,\cdot\,,X^U,\overline{X^U}\big) := \frac{\partial}{\partial \overline{v}}f(\,\cdot\,,v,\overline{v})\Big|_{\substack{v=X^U\\ \overline{v}=\overline{X^U}}}.$$

Regarding the time variable again, we receive the adjoint Schrödinger equation

$$\begin{aligned} \frac{\partial}{\partial t}\eta^U(t) =&\; iA\eta^U(t) - iU(t)\eta^U(t) - i\lambda\left(\frac{\partial}{\partial X^U}f\big(t,X^U(t),\overline{X^U}(t)\big)\right)\eta^U(t) \\ &- i\overline{\lambda}\,\overline{\left(\frac{\partial}{\partial \overline{X^U}}f\big(t,X^U(t),\overline{X^U}(t)\big)\right)}\,\overline{\eta^U}(t) \end{aligned} \tag{I.2}$$

in V^* for a.e. $\omega \in \Omega$ and all $t \in [0,T]$.

Remark I.1. *Without loss of generality, it is allowed to start with the deterministic part of the controlled Schrödinger equation in order to calculate its adjoint equation because*

- *additive noise vanishes by considering differences,*
- *for linear multiplicative noise we deal with the pathwise problem having no noise term at all,*
- *general multiplicative noise is excluded (since it requires another approach based on forward-backward stochastic differential equations).*

Here, we investigate the complex conjugated adjoint Schrödinger equation. Therefore, we create the complex conjugated equation of (I.1)

$$0 = i\overline{\eta_t^U} + \overline{\eta_{xx}^U} + U\overline{\eta^U} + \overline{\lambda}\,\overline{\left(\frac{\partial}{\partial y} f(\,\cdot\,, y, \overline{y})\right)}\,\overline{\eta^U} + \lambda\left(\frac{\partial}{\partial \overline{y}} f(\,\cdot\,, y, \overline{y})\right)\eta^U.$$

Introducing the complex conjugated adjoint variable $\Phi^U := \overline{\eta^U}$, renaming $X^U = y$ and multiplying with the imaginary unit, it follows that

$$\Phi_t^U = i\Phi_{xx}^U + iU\Phi^U + i\overline{\lambda}\,\overline{\left(\frac{\partial}{\partial X^U} f(\,\cdot\,, X^U, \overline{X^U})\right)}\,\Phi^U + i\lambda\left(\frac{\partial}{\partial \overline{X^U}} f(\,\cdot\,, X^U, \overline{X^U})\right)\overline{\Phi^U}.$$

Finally, we take into account the dependence on the time variable and state the complex conjugated adjoint Schrödinger equation

$$\begin{aligned}\frac{\partial}{\partial t}\Phi^U(t) = &- iA\Phi^U(t) + iU(t)\Phi^U(t) + i\overline{\lambda}\,\overline{\left(\frac{\partial}{\partial X^U} f(t, X^U(t), \overline{X^U}(t))\right)}\,\Phi^U(t) \\ &+ i\lambda\left(\frac{\partial}{\partial \overline{X^U}} f(t, X^U(t), \overline{X^U}(t))\right)\overline{\Phi^U}(t)\end{aligned} \tag{I.3}$$

in V^* for a.e. $\omega \in \Omega$ and all $t \in [0, T]$, which obviously is the complex conjugated equation of (I.2). Being especially interested in the power-type nonlinearity

$$f\big(X^U(t), \overline{X^U}(t)\big) = |X^U(t)|^{2\sigma} X^U(t) = (X^U(t))^{\sigma+1}\big(\overline{X^U}(t)\big)^{\sigma}$$

with the Wirtinger derivatives

$$\begin{aligned}\frac{\partial}{\partial X^U} f\big(X^U(t), \overline{X^U}(t)\big) &= (\sigma+1)\big(X^U(t)\big)^{\sigma}\big(\overline{X^U}(t)\big)^{\sigma} = (\sigma+1)\big|X^U(t)\big|^{2\sigma}, \\ \frac{\partial}{\partial \overline{X^U}} f\big(X^U(t), \overline{X^U}(t)\big) &= \sigma\big(X^U(t)\big)^{\sigma+1}\big(\overline{X^U}(t)\big)^{\sigma-1} = \sigma\big|X^U(t)\big|^{2(\sigma-1)}\big(X^U(t)\big)^2,\end{aligned}$$

the complex conjugated adjoint Schrödinger equation (I.3) results in

$$\begin{aligned}\frac{\partial}{\partial t}\Phi^U(t) = &- iA\Phi^U(t) + iU(t)\Phi^U(t) + i\overline{\lambda}(\sigma+1)\big|X^U(t)\big|^{2\sigma}\Phi^U(t) \\ &+ i\lambda\sigma\big|X^U(t)\big|^{2(\sigma-1)}\big(X^U(t)\big)^2\,\overline{\Phi^U}(t)\end{aligned}$$

in V^* for a.e. $\omega \in \Omega$ and all $t \in [0, T]$. Moreover, starting from the pathwise controlled Schrödinger equation

$$dZ^U(t) = -\,iAZ^U(t)\,dt + iU(t)Z^U(t)\,dt + i\lambda B(t) f\big(Z^U(t), \overline{Z^U}(t)\big)\,dt, \qquad \text{for all } t \in [0, T],$$

with $Z^U(t, \cdot\,) = X^U(t, \cdot\,)Y(t)$ and $f\big(Z^U(t), \overline{Z^U}(t)\big) = |Z^U(t)|^{2\sigma} Z^U(t)$, the corresponding complex conjugated adjoint Schrödinger equation satisfies

$$\begin{aligned}\frac{\partial}{\partial t}\Phi^U(t) = &- iA\Phi^U(t) + iU(t)\Phi^U(t) + i\overline{\lambda}(\sigma+1)B(t)\big|Z^U(t)\big|^{2\sigma}\Phi^U(t) \\ &+ i\lambda\sigma B(t)\big|Z^U(t)\big|^{2(\sigma-1)}\big(Z^U(t)\big)^2\,\overline{\Phi^U}(t)\end{aligned}$$

in V^* for all $t \in [0, T]$, while $\omega \in \Omega$ is chosen arbitrarily fixed (see Section 2.3).

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