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Groups of p-rank 2 containing an isolated element of order p

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Abstract. Let p be an odd prime and G be a finite group with $O_{p'}(G) = 1$ of p-rank at most 2 that contains an isolated element of order p. If $x \notin Z(G)$, we show that $F^*(G)$ is simple and we describe the structure of a Sylow p-subgroup P of $F^*(G)$ as well as the fusion system $\mathcal{F}_P(F^*(G))$ without using the classification of finite simple groups.

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1. Introduction. An element x of a finite group G is called isolated if and only if $x^G \cap C_G(x) = \{x\}$. In 1966, George Glauberman [5] showed that $G = O(G) \cdot C_G(x)$ if x is an isolated involution. A generalisation of his result for elements of odd prime order is a consequence of the classification of the finite simple groups. But there is still no proof revealing conceptual reasons or a deeper understanding of finite groups in terms of the subgroup structure. Even for the prime 3 the influence of isolated elements on finite groups is not understood. Together with Rebecca Waldecker, the author studied in [11] isolated elements of order 3. Under specific assumptions, they have been able to prove a version of the so-called Z_3^*-Theorem. One of these assumptions was the existence of an elementary abelian subgroup of order 27.

In this paper, we study a finite group G with $O_{p'}(G) = 1$ for an odd prime p such that G has an isolated element x of order p but does not contain any elementary abelian subgroup of order p^3. We will first show that either $x \in Z(G)$ or that $F^*(G)$ is nonabelian simple and investigate the case where a Sylow p-subgroup of $F^*(G)$ is abelian.

Then we assume that a Sylow p-subgroup P of $F^*(G)$ is not abelian. The restriction on the p-rank of G implies that every subgroup of order p^3 is generated by two elements. This allows to apply a theorem of Norman Blackburn

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[2] to obtain a list of possible isomorphism types for P. Most of the cases can be eliminated by more arguments regarding p-fusion and p-transfer in G. Assuming that P is not extraspecial of order p^3 and exponent p, we deduce that P is a 3-group of maximal class and determine the possible isomorphism types of P with help of Chris Parker's and Jason Semeraro's results in [9]. More precisely we will prove the following theorem:

Theorem. *Let G be a finite group and p be an odd prime such that $O_{p'}(G) = 1$. Suppose that G does not have an elementary abelian subgroup of order p^3 and that there is some isolated element $x \in G$ of order p such that $x \notin Z(G)$.*

Then $F^*(G)$ is nonabelian simple and for every Sylow p-subgroup P of $F^*(G)$, either $x^G \cap P = \varnothing$ and P is cyclic or $x^G \cap P \neq \varnothing$ and one of the following holds:

(a) P is extraspecial of order p^3 and exponent p.

(b) $p = 3$, $P = \langle s, s_1, ..., s_{2k} \mid \forall i \in \{2, ..., 2k\} : s_i = [s_{i-1}, s], [s_1, s_i] = 1, \forall i \in \{2, ..., 2k-1\} : s_{i-1}^3 s_i^3 = s_{i+1}^{-1}$ and $s^3 = s_{2k-1}^3 = s_{2k}^3 = 1\rangle$ is of maximal class and of order 3^{2k+1} for some integer $k \geq 2$.

We further use the classification of fusion systems on finite 3-groups of maximal class and rank 2 of [4] and [9] to see that P is normal in the fusion system induced by $F^*(G)$ on P or that case (b) of the above theorem holds and the fusion system induced by $F^*(G)$ on P is isomorphic to the fusion system of $\mathrm{SL}_3^{\varepsilon}(q)$ for some $\varepsilon \in \{1, -1\}$ and some prime power q such that $3^k \mid q - \varepsilon$ but $3^{k+1} \nmid q - \varepsilon$ on a Sylow 3-subgroup.
The notation used is given in [10] or [3].

2. Preliminaries.

Definition 2.1. Let G be a finite group, p be a prime, and $n \in \mathbb{N}$.

(a) An element $x \in G$ is called isolated in G if and only if $x^G \cap C_G(x) = \{x\}$.

(b) If G has an elementary abelian subgroup of order p^n but does not contain any of order p^{n+1}, then G has p-rank n. We write $r_p(G) = n$. We further say that G has rank n and write $r(G) = n$ if G is a p-group.

(c) A proper subgroup U of G is strongly p-embedded in G if and only if $p \mid |U|$ but $p \nmid |U \cap U^g|$ for all $g \in G \setminus U$.

Lemma 2.2. *Let G be a finite group and p be a prime. Suppose that $x \in G$ is an isolated element of order p in G. Then x is isolated in H for every subgroup H of G such that $x \in H$. If further $x \in P \in \mathrm{Syl}_p(G)$, then $x \in Z(P)$ and if P is cyclic, then G has a normal p-complement.*

Proof. The assertions follow from [11, Lemma 3.2 and 3.4]. □

Lemma 2.3. *Let P be a p-group for an odd prime p and α be an automorphism of P of coprime order. Then the following hold:*

(a) $P/P' = [P, \alpha]P'/P' \times C_P(\alpha)P'/P'$.

(b) If α centralises $\Omega_1(P)$, then α centralises P.

Proof. Part (a) follows from [10, 8.2.2 and 8.4.2] and (b) is [6, 5.3.10]. $\square$

Lemma 2.4. *Suppose that G is a finite group, p is an odd prime, P is a Sylow p-subgroup of G of rank 2 containing an isolated element x of order p. Then for all $R \leq P$ the group $(N_G(R)/C_G(R))/O_p(N_G(R)/C_G(R))$ is cyclic of order dividing $p-1$ or R has a subgroup that is extraspecial of order p^3 and exponent p.*

Proof. Let $R \leq P$ such that R does not have an extraspecial subgroup of order p^3 and exponent p and set

$$\mathcal{OUT}_G(R) := (N_G(R)/C_G(R))/O_p(N_G(R)/C_G(R)).$$

[1, Corollary 14.4] provides a characteristic subgroup T of R of exponent p and class at most 2 such that every nontrivial p'-automorphism of R induces a nontrivial automorphism of T. Our assumption on R and $r(T) \leq r(P) = 2$ imply that T does not have a subgroup of order p^3 and so T is elementary abelian of order at most p^2.

If T is cyclic, then [10, 2.2.5] yields that $\mathcal{OUT}_G(T)$ is cyclic of order dividing $p-1$. In the other case, if T is not cyclic, then from $r(P) \leq 2$ and $x \in C_P(T)$, by Lemma 2.2, we see that $x \in T$. Hence we deduce that $N_G(T)/C_G(T)$ is isomorphic to a subgroup of $C_p \rtimes C_{p-1}$, as x is isolated in G. So in both cases $\mathcal{OUT}_G(T)$ is cyclic of order dividing $p-1$. Since every nontrivial p'-automorphism of R induces a nontrivial automorphism of T, we conclude that also $\mathcal{OUT}_G(R)$ is cyclic of order dividing $p-1$. $\square$

Proposition 2.5. *Let G be a finite group and p be an odd prime such that $O_{p'}(G) = 1$. Suppose that $r_p(G) \leq 2$ and that $x \in G$ is isolated of order p such that $x \notin Z(G)$.*
Then $F^*(G)$ is nonabelian simple.

Proof. First Lemma 2.2 yields that x centralises a Sylow p-subgroup P of G and so x centralises $O_p(G)$.

Suppose for a contradiction that x centralises $F^*(G)$. Then $x \in Z(F^*(G))$ and so $x^G \subseteq Z(F^*(G)) \leq C_G(x)$. From $x^G \cap C_G(x) = \{x\}$, we see that $x \in Z(G)$, a contradiction showing that x does not centralise $F^*(G)$.

By $O_{p'}(G) = 1$, we obtain a component K of G such that $[K, x] \neq 1$ and $p \mid |K|$. Let $R := P \cap K \in \mathrm{Syl}_p(K)$, then $1 \neq R \leq P \leq C_G(x)$ and Lemma 2.2 gives that $R\langle x\rangle$ is not cyclic.

Suppose for a contradiction that $Z(K) \neq 1$ and set $Z_0 := \Omega_1(Z(K))$. Then $Z(K) = O_p(G) \cap K \leq R$ and $x \notin Z(K) \leq Z(F^*(G))$. From $r(R\langle x\rangle) \leq r_p(G) \leq 2$ and $x \in Z(R\langle x\rangle)$, we deduce that $\Omega_1(R\langle x\rangle) = Z_0 \times \langle x\rangle$ is elementary abelian of order p^2. Moreover, Frobenius's p-complement theorem [10, 7.2.4] provides a subgroup $T \leq R$ such that $N_K(T)/C_K(T)$ is not a p-group. It follows that $N_K(\Omega_1(T))/C_K(\Omega_1(T))$ is not a p-group by Lemma 2.3 (b). On the other hand, $N_K(\Omega_1(T)) \leq K \leq C_G(Z_0)$. We conclude that $\Omega_1(T)$ is not equal to Z_0 but contained in $\Omega_1(R\langle x\rangle) = Z_0 \times \langle x\rangle$. Let $g \in N_K(\Omega_1(T)) \setminus C_K(\Omega_1(T)) \subseteq C_G(Z_0)$. Then there is some $t \in \Omega_1(T) \leq Z_0 \times \langle x\rangle$ such that $t^g \neq t$ and $t \notin Z_0$. We may choose t such that $t = yx$ for some $y \in Z_0$. Then $yx \neq (yx)^g = yx^g$

implies that $x \neq x^g$. Since $x^g = y^{-1}t \in \Omega_1(T)Z_0 \leq Z_0 \times \langle x \rangle \leq C_G(x)$, this is a contradiction showing that K is simple.

Suppose for a final contradiction that $K \neq F^*(G)$. Then there is a subgroup $E \leq F^*(G)$ such that $E \times K = F^*(G)$ and $E \neq 1$ is subnormal in G. We set $S := E \cap P$ and observe $S \neq 1$. Additionally $SR = S \times R$ and from $r_p(G) \leq 2$, we deduce that S and R have rank 1 and so R and S are cyclic, as p is odd and by [10, 5.3.7]. Moreover, $x \in C_G(R \times S)$ and so $r_p(G) \leq 2$ gives that $x \in R \times S$. Hence there are elements $a \in R$ and $b \in S$ such that $x = ab$. Since $x \notin C_G(K) \geq S$, we see that $a \neq 1$. As R is cyclic and a Sylow p-subgroup of the simple group K, Burnside's p-complement theorem (see [10, 7.2.1]) provides an element $g \in N_K(R) \leq C_G(E)$ that has order coprime to p and does not centralise R. Lemma 2.3 (b) then gives that g does not centralise a. Thus $x^g = a^g b \in RS \leq C_G(x)$ but $x^g \neq x$. This is a contradiction. □

Definition 2.6. Let p be a prime, G a finite group, $P \in \mathrm{Syl}_p(G)$, and $\mathcal{F} := \mathcal{F}_P(G)$ the fusion system of G on P. Let further Q be a subgroup of P.

(a) Q is called fully $\mathcal{F}$-normalised if and only if $N_P(Q) \in \mathrm{Syl}_p(N_G(Q))$.
(b) Q is called $\mathcal{F}$-centric if and only if $C_G(Q) = Z(Q) \times O_{p'}(C_G(Q))$.
(c) Q is called $\mathcal{F}$-essential if and only if $N_G(Q)/QC_G(Q)$ contains a strongly p-embedded subgroup and Q is $\mathcal{F}$-centric.
(d) Q is called $\mathcal{F}$-radical if and only if $O_p(N_G(Q)/QC_G(Q)) = 1$.
(e) Q is called $\mathcal{F}$-Alperin if and only if Q is fully $\mathcal{F}$-normalised, $\mathcal{F}$-centric, and $\mathcal{F}$-radical.

Lemma 2.7. *Let G be a finite group, p be a prime, and P be a Sylow p-subgroup of G. If $[D, N_G(D)] \leq [P, N_G(P)]$ for all $\mathcal{F}_P(G)$-essential subgroups D of G, then we have $P \cap G' = [P, N_G(P)]$.*

Proof. By Alperin's fusion theorem [3, 4.51] and the focal subgroup theorem [10, 7.1.3], the group $P \cap G'$ is generated by $[P, N_G(P)]$ and $[D, N_G(D)]$ where D runs through all $\mathcal{F}_P(G)$-essential subgroups. Thus $P \cap G' = [P, N_G(P)]$ by the assumptions. □

3. The structure of a Sylow p-subgroup.

Proposition 3.1. *Let G be a finite group and p be an odd prime such that $O_{p'}(G) = 1$. Suppose that $r_p(G) \leq 2$ and that $x \in G$ is isolated of order p such that $x \notin Z(G)$. Then $x \notin F^*(G)$ if and only if $F^*(G)$ has abelian Sylow p-subgroups if and only if $F^*(G)$ has cyclic Sylow p-subgroups.*

Proof. We set $K := F^*(G)$. Then K is nonabelian simple by Proposition 2.5. Let further S be a Sylow p-subgroup of $K\langle x \rangle$ containing x. Then $P := S \cap K$ is a Sylow p-subgroup of K and $P\langle x \rangle = S$.

Suppose first that $x \notin K$. Then $S = P \times \langle x \rangle$, as x centralises every Sylow p-subgroup of G that contains x by Lemma 2.2. Since $r_p(G) \leq 2$, we see that P does not contain an elementary abelian subgroup of order p^2 and so P is cyclic. If P is cyclic, then P is abelian.

Let finally P be abelian. Then again Lemma 2.2 shows that S is abelian. We choose an element $u \in S$ of maximal order. Then [10, Lemma 2.1.2] provides a subgroup R of S such that $S = \langle u \rangle \times R$. As $r(P\langle x \rangle) \leq 2$, we see that R is cyclic. In particular, $S/\langle u \rangle$ and S/R are cyclic groups. Since $\langle u \rangle \cap R = 1$, there is some $Q \in \{\langle u \rangle, R\}$ such that $x \notin Q$. Then x is an element of $S \setminus Q$ of order p. In addition every G-conjugate of x in S is equal to x since x is isolated in G. We apply [7, Lemma 15.18] to conclude that $x \notin O^p(K\langle x \rangle) = K$. □

Lemma 3.2. *Let P be a finite p-group for an odd prime p such that $r(P) \leq 2$. Then one of the following holds:*

(a) P is a 3-group of maximal class,
(b) $\Omega_1(P)$ is extraspecial of order p^3 and exponent p and $P/\Omega_1(P)$ is cyclic,
(c) P is metacyclic, or
(d) $|P| \leq p^4$.

Proof. We may suppose that $|P| \geq p^5$ and let R be a normal subgroup of P of order p^3. Then R is not elementary abelian and so $\Phi(R) \neq 1$. Thus $|R/\Phi(R)| < |R| = p^3$ and we deduce from [10, 5.2.5] that R is generated by two elements.

Altogether we may apply [2, Theorem 4.1]. □

Hypothesis 3.3. Let G be a finite nonabelian simple group and p be an odd prime. Suppose that $r_p(G) \leq 2$ and that $x \in G$ of order p is isolated in G. Let further $P \in \mathrm{Syl}_p(G)$ be such that $x \in P$ and assume that P is not extraspecial of order p^3 and exponent p.

Lemma 3.4. *Assume Hypothesis 3.3 and let Q be an extraspecial normal subgroup of P of order p^3 and exponent p. Then P/Q is not cyclic.*

Proof. From Hypothesis 3.3, we see that $P \neq Q$ and Lemma 2.2 as well as $r(P) \leq 2$ imply that $\langle x \rangle = Z(Q)$. Moreover, [1, Lemma 1.4] provides a normal subgroup V of P that is contained in Q and is elementary abelian of order p^2. Then $Q \nleq C_P(V)$ and from $|\mathrm{Aut}(V)|_p = p$, we see that $C_P(V)$ is a maximal subgroup of $P = QC_P(V)$.

We suppose for a contradiction that P/Q is cyclic and choose $w \in C_P(V) \setminus V$ such that $w^p \in Q$. From $r(P) \leq 2$ and $w \in C_P(V) \setminus V$, we get that w does not have order p. We conclude that $1 \neq w^p \in C_P(V) \cap Q = V$. Moreover, $\langle w \rangle V$ is a maximal and hence normal subgroup of $\langle w \rangle Q$. So $\Phi(\langle w \rangle V) = \langle w^p \rangle$ is $\langle w \rangle Q$-invariant. Now $w^p \in Q$ implies that $\langle w^p \rangle$ is normal in Q and so $\langle w^p \rangle = Z(Q) = \langle x \rangle$.

We may assume that $w^p = x$. Since x is isolated in G, no G-conjugate of a power of w^p lies in $P \setminus Q$. Further $w^g \notin Q$ for all $g \in G$, as w does not have order p and Q has exponent p. From $w \in G = O^p(G)$ and [7, Proposition 15.15], we obtain an element $g \in G$ such that $w^g = w^k z$ for some $z \in Q$ and $k \in \mathbb{N}$ such that p does divide neither k nor $k-1$ and such that $C_P(w^g) \in \mathrm{Syl}_p(C_G(w^g))$.

If we had $[w^k, z] = 1$, then $x^k = (w^p)^k = w^{kp} z^p = (w^k z)^p = x^g \in C_G(x) \cap x^G = \{x\}$ would be a contradiction. Hence $w \in C_P(V) \setminus C_P(z)$ and so $Q = \langle z, V \rangle$. We further deduce that $C_P(z) = C_Q(z)$ from $\langle w \rangle Q/Q = \Omega_1(P/Q)$ and so $C_P(Q) = C_P(z) \cap C_P(V) \leq C_Q(z) \cap C_P(V) = Z(Q)$. Now [12] gives that $|N_G(Q)/QC_G(Q)|$ divides $p(p^2-1)$, as $Z(Q) = \langle x \rangle$ is centralised by $N_G(Q)$. Consequently $P/Q = N_P(Q)/QC_P(Q)$ has order p and so $P = Q\langle w \rangle$ has order p^4. By [12], the element w does not normalise $\langle x, z \rangle$. Altogether we have $\langle w^g, w \rangle = \langle z, w \rangle = Q\langle w \rangle = P$, implying that $C_P(w) \cap C_P(w^g) \leq Z(P) = \langle x \rangle$. This shows that $|P : C_P(w^g)| \geq |C_P(w) : C_P(w) \cap C_P(w^g)| = |V\langle w \rangle : \langle x \rangle| = p^2$, entailing that $C_P(w^g)$ has order p^2. But $C_P(w^g) \in \mathrm{Syl}_p(C_G(w^g))$ and so $|C_P(w^g)| \geq |(C_P(w))^g| \geq |(V\langle w \rangle)^g| = p^3$. This is a contradiction. □

Lemma 3.5. *Assume Hypothesis 3.3. Then P has a unique elementary abelian normal subgroup of order p^2.*

Proof. As in the lemma before, [1, Lemma 1.4] provides a normal elementary abelian subgroup V of P of order p^2. Let $W \trianglelefteq P$ be elementary abelian of order p^2 and suppose for a contradiction that $V \neq W$. From $r(P) \leq 2$ and $[V, W] \leq V \cap W$, we see that $V \gneq V \cap W \neq 1$ and so $|VW| = \frac{|V|\cdot|W|}{|V\cap W|} = p^3$ by [10, 1.1.6]. We set $Q := VW$. Then $Q = \Omega_1(VW)$ and so $r(Q) \leq r(P) = 2$ implies that Q is not abelian. Thus Q is extraspecial of order p^3 by [8, I.14.10] and of exponent p, as $Q = \Omega_1(VW)$. Every $y \in P$ normalises Q, V, and W and so y normalises every maximal subgroup of Q. In particular, y centralises the elementary abelian group $Q/\Phi(Q) = Q/Z(Q)$ and hence $[P, Q] \leq Z(Q)$. Thus [12] implies that $P = Q \cdot C_P(Q)$.

For all $z \in C_P(Q)$ of order p, we see that $\langle V, z \rangle$ and $\langle W, z \rangle$ are elementary abelian. Then $r(P) = 2$ yields that $z \in V \cap W \leq Z(Q)$. Thus $C_P(Q)$ has a unique minimal subgroup and is consequently cyclic, as p is odd. This contradicts Lemma 3.4. □

Lemma 3.6. *Suppose that Hypothesis 3.3 holds and that P does not have an extraspecial subgroup of order p^3 and exponent p. Then there is some element $a \in N_G(P)$ with $a^{p-1} \in O_{p'}(C_G(P))$ that induces a fixed-point-free automorphism on P/P'; moreover $C_P(a)$ is cyclic.*

Proof. We investigate the fusion system $\mathcal{F}_P(G)$ and let D be an $\mathcal{F}_P(G)$-radical subgroup. Then $O_p(N_G(D)/C_G(D)) = DC_G(D)/C_G(D)$, our assumption and Lemma 2.4 imply that $|N_G(D)/DC_G(D)|$ divides $p-1$.

If D is $\mathcal{F}_P(G)$-essential subgroup of P, then D is also $\mathcal{F}_P(G)$-radical (see [3, p. 119]) and $\mathcal{F}_P(G)$-centric. It follows that $N_P(D) \leq DC_G(D) = DZ(D)O_{p'}(C_G(D)) = DO_{p'}(C_G(D))$ and hence $D = P$. This is a contradiction. In particular, there does not exists any $\mathcal{F}_P(G)$-essential subgroup of P. From Lemma 2.7, we deduce that $P = P \cap G' = [P, N_G(P)]$ and so $H := N_G(P) \neq C_G(P)P$.

As P is $\mathcal{F}_P(G)$-radical, the above argument provides some $a \in H \setminus O_{p'}(H)P$ of order prime to p such that $H = O_{p'}(H)P\langle a \rangle$ and $a^{p-1} \in O^p(PC_G(P)) = O_{p'}(H)$. We conclude that $P = [P, H] = P' \cdot [P, a]$ and hence that a acts fixed-point-freely on the abelian group P/P' by Lemma 2.3 (a).

Let now V be the unique normal elementary abelian subgroup of order p^2 of P which exists by Lemma 3.5. Then $\Omega_1(C_P(V)) = V$, as $r(P) \leq 2$, and $C_P(V)$ is a maximal subgroup of P. We conclude that $C_P(a) \leq P' \lneq C_P(V)$

and so $\Omega_1(C_P(a)) \leq \Omega_1(C_P(V)) = V$. If $C_P(a)$ was not cyclic, then a would centralise $\Omega_1(C_P(V)) = V$ and from Lemma 2.3 (b), we would deduce that $C_P(V) \leq C_P(a)$, a contradiction. $\square$

Lemma 3.7. *Suppose that Hypothesis 3.3 holds. Then P is not metacyclic.*

Proof. Suppose for a contradiction that P is metacyclic. Then P does not have an extraspecial subgroup of order p^3 and exponent p and so Lemma 3.6 provides some $a \in N_G(P)$ with $a^{p-1} \in C_G(P)$ that acts fixed-point-freely on P/P' and hence on $\Omega_1(P/P')$ of order p^2. Thus [8, Satz II 3.10] and $o(a) \mid p-1$ provide at least one a-invariant subgroup of order p of $\Omega_1(P/P')$. In addition Maschke's theorem (see for example [10, 8.4.6]) gives a second one. The full pre-images of these a-invariant subgroups are maximal subgroups of the full pre-image U of $\Omega_1(P/P')$. As P is metacyclic, U has a maximal cyclic subgroup. Hence by [1, Theorem 1.2 (a)], all but at most one maximal subgroups of U are cyclic, as p is odd. Altogether U has a maximal subgroup R, that is a-invariant, cyclic, and contains P'.

Since a acts fixed-point-freely on P/P', a does not centralise R/P' and hence not R. From Lemma 2.3 (b), we see that a does not centralise $\Omega_1(R) = \Omega_1(P')$ and so $C_P(a) \leq C_{P'}(a) = 1$. This contradicts $x \in C_P(a)$, as x is isolated in G. $\square$

Lemma 3.8. *Suppose that Hypothesis 3.3 holds. Then $|P| \geq p^5$.*

Proof. From Lemma 3.7 and Hypothesis 3.3, it firstly follows that $|P| \geq p^4$. Suppose for a contradiction that $|P| = p^4$. If P contained an extraspecial subgroup Q of order p^3 and exponent p, then Q would be a maximal subgroup of P and hence normal in P with cyclic factor group, contradicting Lemma 3.4. We conclude that every proper subgroup R of P is metacyclic. It follows from [2, Theorem 3.2] that P is a 3-group of class 3 and order 3^4, as P itself is not metacyclic by Lemma 3.7. In particular, we have $p = 3$. If V is a normal elementary abelian subgroup of P of order 9, that exists by Lemma 3.5, then P/V has order 9 and so $P' \leq V$. It follows that P' is elementary abelian of order 9, as P has class 3.

In addition Lemma 3.6 provides an element $a \in N_G(P)$ that induces a fixed-point-free automorphism of order 2 on P/P' and such that $C_P(a)$ is cyclic. Since x is isolated in P, we see that $x \in C_P(a) \leq P' = V$ and hence $C_P(a) = \langle x \rangle = Z(P) \trianglelefteq P$. Altogether a induces a fixed-point-free automorphism of order 2 on the nonabelian group $P/Z(P) = P/C_P(a)$. This contradicts [10, 8.1.10]. $\square$

Lemma 3.9. *Suppose that Hypothesis 3.3 holds and that P is a 3-group of maximal class. Then $N_G(P) = C_G(P)P\langle a \rangle$ where $a^2 \in C_G(P)$ and a acts fixed-point-freely on P/P' and $|P| = 3^{2k+1}$ for some integer $k \geq 2$.*

Proof. Let $|P| = 3^n$. Then Lemma 3.8 implies that that $n \geq 5$. As P has maximal class, we see that $\langle x \rangle = Z(P)$ by Lemma 2.2. Now, [9, Proposition 3.3] describes $\mathrm{Aut}(P)$. Let $a \in N_G(P)$, then the proposition provides $e, f \in \{1, 2\}$ such that $x^a = x^{e^{n-2}f}$ by [9, Lemma 3.4]. As x is isolated in G, we see

$1 \equiv e^{n-2} \cdot f \bmod 3$. So if n is even, then $e^{n-2} \equiv 1 \bmod 3$ and so $f = 1$. If n is odd, then $e^{n-2} \equiv e \bmod 3$ and so $f = e$. In both cases, we conclude again from [9, Proposition 3.3] that $|N_G(P)/C_G(P)|_2 \leq 2$ and hence that $N_G(P)/PC_G(P)$ is cyclic of order at most 2. Let $a \in N_G(P)$ be a possibly trivial 2-element such that $N_G(P) = C_G(P)P\langle a\rangle$ and $a^2 \in C_G(P)$. We want to apply Lemma 2.7. Therefore we first observe that $[P, N_G(P)] = [P, C_G(P)P\langle a\rangle] = P'[P, a]$.

Suppose now that D is an $\mathcal{F}_P(G)$-essential subgroup. Then $N_G(D)/DC_G(D)$ contains a strongly 3-embedded subgroup and so D has a subgroup that is extraspecial of order 27 and exponent 3 by Lemma 2.4. Thus [9, Lemma 4.1 and Lemma 4.2] yield that D itself is extraspecial of order 3^3 and exponent 3 and determine $N_G(D)/DC_G(D)$. From $r(P) \leq 2$ and $x \in Z(P)$, by Lemma 2.2, we have $x \in Z(D)$ and so $N_G(D)/DC_G(D)$ is isomorphic to $\mathrm{Sp}_2(3) = \mathrm{SL}_2(3)$, as x is isolated in G. By Lemma 3.5, the group P has a unique normal elementary abelian subgroup V of order 9. Then $C_P(V)$ is a maximal subgroup of P and from $V = \Omega_1(C_P(V))$, we deduce that $V \leq D$.

Let $b \in N_G(D)$ be such that b induces an automorphism of order 2 on D. Then b acts on D such that it normalises but does not centralise any elementary abelian subgroup of order 9. It follows that $b \in N_G(V) \setminus C_G(V)$ and so that $N_G(V)/C_G(V) \cong S_3$. By a Frattini argument, we have $N_G(V) = C_G(V)P \cdot N_G(P)$, so we get that $C_G(P)P\langle a\rangle = N_G(P) \not\leq C_G(V)P$. In particular, a does not centralise but normalise the characteristic subgroup V of P. Since $|N_G(V)/C_G(V)P| = 2$, there is some $c \in C_G(V)P$ such that $b = ca$.

From the Frobenius normal p-complement theorem [10, 7.2.4], we get $V = \Omega_1(C_P(V))$, and Lemma 2.3 (b), we see that $C_G(V)$ has a normal 3-complement. So $C_G(V)P \leq O_{3'}(C_G(V))P$ and it follows that $[P, c] \leq [P, PO_{3'}(C_G(V))] \leq P'O_{3'}(C_G(V))$. As $a \in N_G(P) \leq N_G(V)$, we further see that a normalises $P'O_{3'}(C_G(V))$ and hence $[P, b] = [P, ca] = [P, a][P, c]^a \leq [P, a]P'O_{3'}(C_G(V))$. By [10, 1.1.11], we get

$$D = [D, b] \leq [P, b] \cap P \leq [P, a]P'O_{3'}(C_G(V)) \cap P = [P, a]P'(O_{3'}(C_G(V)) \cap P) = [P, a]P'.$$

We summarise that $[D, N_G(D)] \leq [P, a]P' \leq [P, N_G(P)]$ for every $\mathcal{F}$-essential subgroup D of P. Hence Lemma 2.7 shows that $P = G' \cap P = [P, N_G(P)] = P'[P, a]$. Since a acts coprimely, we infer that a acts fixed-point-freely on P/P'.

Altogether a inverts P/P' and the parameters e and f in [9, Proposition 3.2] for a are both equal to 2. Finally our argument at the beginning of this proof shows that n is odd. $\square$

Lemma 3.10. *If Hypothesis 3.3 holds and P is a 3-group of maximal class, then $P = \langle s, s_1, ..., s_{2k} \mid \forall i \in \{2, ..., 2k\} : s_i = [s_{i-1}, s], [s_1, s_i] = 1, \forall i \in \{2, ..., 2k-1\} : s_{i-1}^3 s_i^3 = s_{i+1}^{-1}$ and $s^3 = s_{2k-1}^3 = s_{2k}^3 = 1\rangle$ has order 3^{2k+1} for some $k \geq 2$.*

Proof. Let $|P| = 3^n$. Then Lemma 3.9 implies that $n = 2k+1$ for some integer $k \geq 2$. From [9], we obtain some $\beta, \gamma, \delta \in \{0, 1, 2\}$ such that $P := \langle s, s_1, ..., s_{2k} \mid R1, R2, R3, R4, R5, R6\rangle$, where the relations are as follows:

R1: $\forall i \in \{2, ..., 2k\} : s_i = [s_{i-1}, s]$;
R2: $\forall i \in \{3, ..., 2k\} : [s_1, s_i] = 1$;
R3: $\forall i \in \{2, ..., 2k\} : s_i^3 s_{i+1}^3 s_{i+2} = 1$, where $s_{2k+1} = s_{2k+2} = 1$ by definition;
R4: $[s_1, s_2] = s_{2k}^{\beta}$; **R5:** $s_1^3 s_2^3 s_3 = s_{2k}^{\gamma}$; and **R6:** $s^3 = s_{2k}^{\delta}$.

To prove the assertion, we need to verify that $\beta = \gamma = \delta = 0$.

With regard to Lemma 3.9, let $a \in N_G(P)$ be such that $a^2 \in C_G(P)$ and such that a induces a fixed-point-free automorphism on P/P'. In particular, a inverts P/P'. Then [9, Lemma 2.1] states that $P' = [\langle s_1, ..., s_{2k}\rangle, P]$ and $Z(P) = \langle s_{2k}\rangle = \langle x\rangle$.

For all $d \in \{0, 1, 2\}$, we see that $ss_1^d \notin P'$ and so there is some $z \in P'$ such that $(ss_1^d)^a = (ss_1^d)^{-1}z$. As $(ss_1^d)^{-1} \notin \langle s_1, ..., s_{2k}\rangle$, [9, Lemma 2.4] provides some $y \in P$ such that $((ss_1^d)^{-1}z)^y = (ss_1^d)^{-1}$. A calculation before [9, equation (3.3)] gives that $(ss_1^d)^3 = s_{2k}^{d^2\beta+\delta+d\gamma}$. Altogether we obtain that

$$\begin{aligned}(s_{2k}^{d^2\beta+\delta+d\gamma})^{ay} &= ((ss_1^d)^3)^{ay} = ((ss_1^d)^{ay})^3 = (((ss_1^d)^{-1}z)^y)^3 = ((ss_1^d)^{-1})^3\\ &= ((ss_1^d)^3)^{-1} = (s_{2k}^{d^2\beta+\delta+d\gamma})^{-1}.\end{aligned}$$

But $s_{2k} \in \langle x\rangle$ is isolated in G and so $d^2\beta + \delta + d\gamma$ is divisible by 3. This is true for all $d \in \{0, 1, 2\}$ and so δ is divisible by 3, which gives $\delta = 0$. In addition we get that $\beta + \gamma$ and $\beta - \gamma$ are divisible by 3 and so is their sum 2β and their difference 2γ. From $\beta, \gamma \in \{0, 1, 2\}$, we obtain that $\beta = \gamma = 0$. □

Proof of the main Theorem. We investigate $K := F^*(G)$ and let $P \in \mathrm{Syl}_p(K)$. Then K is nonabelian simple by Proposition 2.5 and does not contain any elementary abelian subgroup of order p^3, as G does not. In particular, P satisfies the hypothesis of Lemma 3.2.

If $x \notin K$, then $x^G \cap P = \varnothing$ and Proposition 3.1 implies that P is cyclic. So we may suppose that $x \in K$. Then x is isolated in K by Lemma 2.2.

If K does not have extraspecial Sylow p-subgroups of order p^3 and exponent p, then Hypothesis 3.3 holds and so Lemma 3.2 yields together with Lemma 3.4, Lemma 3.7, and Lemma 3.8 that P is a 3-group of maximal class. Finally Lemma 3.10 provides the assertion. □

4. The fusion system induced by G.

Theorem 4.1. *Let G be a finite group and p be an odd prime such that $O_{p'}(G) = 1$. Suppose that G does not have an elementary abelian subgroup of order p^3 and that there is some isolated element $x \in G$ of order p such that $x \notin Z(G)$. Then every Sylow p-subgroup P of $F^*(G)$ is normal in $\mathcal{F}_P(F^*(G))$ or part (b) of the main Theorem is true and $\mathcal{F}_P(F^*(G))$ is isomorphic to $\mathcal{F}_P(H)$, where $H \cong \mathrm{SL}_3^{\varepsilon}(q)$ for some $\varepsilon \in \{1, -1\}$ and some prime power q such that $3^k \mid q - \varepsilon$ but $3^{k+1} \nmid q - \varepsilon$.*

Proof. Let $P \in \mathrm{Syl}_p(F^*(G))$ and set $\mathcal{F} := \mathcal{F}_P(F^*(G))$. If P is cyclic, then it follows from a result of Burnside (see [10, 7.1.5]) that P is normal in $\mathcal{F}$.

From now on, we may suppose that $x \in P$. In particular, Proposition 3.1 gives that P is not abelian. Further [3, Theorem 5.39 (v)] implies that P is normal in $\mathcal{F}$ if and only if P is the only $\mathcal{F}$-Alperin subgroup. Let $R \leq P$ be abelian, then $P \neq R$. Using Lemma 2.4, we see that $(N_G(R)/C_G(R))/O_p(N_G(R)/C_G(R))$ is cyclic of order dividing $p-1$. If R is $\mathcal{F}$-radical and $\mathcal{F}$-centric, then we deduce from $O_p(N_G(R)/C_G(R)) = RC_G(R)/C_G(R) = 1$ that $N_P(R) \leq C_P(R) \leq R$, so that $R = P$, which is a contradiction. This shows that R is not $\mathcal{F}$-Alperin.

If case (a) of the main Theorem holds, then P is extraspecial of order p^3 and exponent p. So every proper subgroup of P is abelian. We deduce that P is the only $\mathcal{F}$-Alperin subgroup, whence P is normal in $\mathcal{F}$.
Consequently, if P is not normal in $\mathcal{F}$, then we see that part (b) of the main theorem holds. In addition P has a proper $\mathcal{F}$-Alperin subgroup R. We check the possibilities of $\mathcal{F}$ in [4, Table 6 of Theorem 5.10] which is correct in our case by [9, Theorem 1]. We already saw that abelian subgroups of P are not $\mathcal{F}$-Alperin and so the groups denoted by V_0, V_1, V_{-1}, and γ_1 are not $\mathcal{F}$-Alperin. Finally if $R \leq P$ is extraspecial of order p^3 and exponent p, then $x \in Z(R)$, as R has rank 2. So we see from [12] that $\mathrm{Aut}_{\mathcal{F}}(R)$ is a subgroup of $\mathrm{SL}_2(3)$. Since $\mathcal{F}$ is not exotic, we see that only the third row may occur and so our theorem is true. □

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