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An efficient semi-analytical solution of the Reynolds equation

Simon Pfeil[1,*], **Hauke Gravenkamp**[2], **Fabian Duvigneau**[1], and **Elmar Woschke**[1]

[1] Otto von Guericke University Magdeburg, Institute of Mechanics, Universitätsplatz 2, 39106 Magdeburg, Germany
[2] University of Duisburg-Essen, Dept. of Civil Engineering, Universitätsstr. 15, 45141 Essen, Germany

A numerically efficient, semi-analytical solution of the Reynolds equation for hydrodynamic journal bearings is developed based on the Scaled Boundary Finite Element Method. The pressure field is discretized along the circumferential coordinate of the lubrication gap, while an analytical formulation is used in the axial direction. A system of inhomogeneous ordinary differential equations is obtained, which is solved under consideration of the boundary conditions. The solution is verified, and its numerical efficiency is investigated in comparison to an FEM solution.

1 Introduction

The dynamic properties of rotor systems with hydrodynamic journal bearings crucially depend on the nonlinear bearing characteristics. The bearing forces acting on the shaft result from the hydrodynamic pressure generation in the lubrication gap, which is described by the Reynolds equation (RE) [1]. In transient rotordynamic simulations, the bearing forces need to be calculated in every time step, which is performed either by solving the RE or by use of look-up tables. A disadvantage of the look-up tables is that the interpolation effort increases with every considered physical effect if the grid of data points is maintained sufficiently fine to guarantee a low interpolation error. Hence, the solution of the RE is becoming more practically relevant. Since closed-form analytical solutions are known for special cases only [2, 3], numerical methods such as the Finite Element Method (FEM) [4] or the Finite Volume Method (FVM) [5] are used. The numerical effort to solve the RE dominates the computational cost of the rotordynamic simulation, which is why in this study, an efficient solution based on the Scaled Boundary Finite Element Method (SBFEM) [6] is developed. The SBFEM is a semi-analytical approach, which was designed to model wave propagation in unbounded domains [7] and has been extended to solve problems in various fields. The SBFEM formulation of the RE, which is later referred to as the SBFEM equation, is derived and solved in Section 2. In Section 3, the solution is verified and its numerical efficiency is investigated in comparison to an FEM solution.

2 Derivation and solution of the SBFEM equation

The RE for hydrodynamic journal bearings with a rotating shaft and a fixed shell can be written as

$$\frac{4}{D^2}\frac{\partial}{\partial\theta}\left(\frac{h^3}{\mu}\frac{\partial p}{\partial\theta}\right)+\frac{\partial}{\partial y}\left(\frac{h^3}{\mu}\frac{\partial p}{\partial y}\right)-6\omega\frac{\partial h}{\partial\theta}-12\frac{\partial h}{\partial t}=0\,, \tag{1}$$

in which p is the pressure, h is the gap function, ω is the rotational velocity, μ is the dynamic viscosity, and D is the bearing diameter. The bearing geometry and the coordinates θ and y are defined in Figure 1. The SBFEM requires a weak formulation of the governing equilibrium equation, which is achieved by applying the Galerkin method to Equation (1). The residual is multiplied by a test function w, integrated over the solution domain and set equal to zero

$$\int\limits_{-L/2}^{0}\int\limits_{0}^{2\pi} w\left[\frac{4}{D^2}\frac{\partial}{\partial\theta}\left(\frac{h^3}{\mu}\frac{\partial p}{\partial\theta}\right)+\frac{\partial}{\partial y}\left(\frac{h^3}{\mu}\frac{\partial p}{\partial y}\right)-6\omega\frac{\partial h}{\partial\theta}-12\frac{\partial h}{\partial t}\right]\mathrm{d}\theta\,\mathrm{d}y=0\,. \tag{2}$$

The solution domain is represented by the unwinded lubrication gap depicted in Figure 2. It is assumed that the pressure distribution is symmetric in the axial direction y, which means that only half of the bearing needs to be considered. This assumption requires that the shaft tilting is neglected, i.e., $h_{,y}=0$, which moreover facilitates the SBFEM equation. On the considered bearing half, a superelement [6] is defined which consists of rectangular sectors (grey) and finite line elements (green) on its axial boundaries. A set of dimensionless local coordinates (blue), η and ξ, is defined for each sector. Equation (2) is transformed into the local coordinate system, and the pressure distribution within a sector e is described by the semi-analytical ansatz $p_e(\eta,\xi)=\underline{N}^{\mathrm{T}}(\eta)\cdot\underline{p}_e(\xi)$. In this ansatz, $\underline{N}(\eta)$ is a vector of linear shape functions, and $\underline{p}_e(\xi)$ is a vector of nodal pressure solutions in the axial direction, which is determined later by solving the SBFEM equation. An analogous ansatz is defined for the test function $w_e(\eta,\xi)=\underline{w}_e^{\mathrm{T}}(\xi)\cdot\underline{N}(\eta)$. Since the derivation of the SBFEM equation is performed

* Corresponding author: e-mail simon.pfeil@ovgu.de, phone +49 391 67 57402

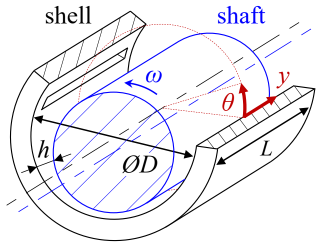

Fig. 1: Draft of a hydrodynamic radial journal bearing.

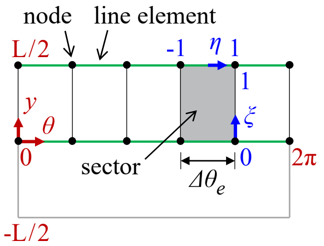

Fig. 2: Definition of a superelement on the unwinded gap.

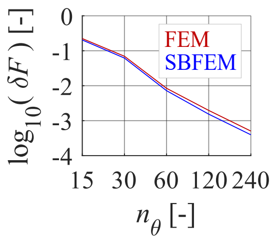

Fig. 3: Convergence of the SBFEM and the FEM.

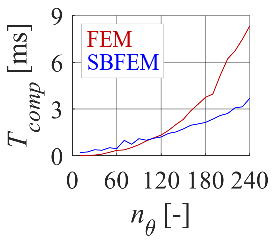

Fig. 4: Computational time of the SBFEM and the FEM.

analogously to the SBFEM literature [6], the subsequent steps are not presented in this proceeding. A system of ordinary differential equations is obtained, which is then assembled over all sectors to yield the SBFEM equation

$$\underline{\underline{E}}_0 \cdot \underline{p}_{,\xi\xi}(\xi) - \underline{\underline{E}}_2 \cdot \underline{p}(\xi) = \underline{x}\,. \tag{3}$$

In this, $\underline{\underline{E}}_2$ and $\underline{\underline{E}}_0$ are the coefficient matrices resulting from the first and the second term of the RE (1), while $\underline{x}$ is the right-hand side vector resulting from the third and the fourth term. The solution of Equation (3) can be written as the sum of a homogeneous solution $\underline{p}_{hom}(\xi)$ and a particular solution $\underline{p}_{par}$

$$\underline{p}_{hom}(\xi) = \sum_{i=1}^{m} \frac{a_i}{\cosh\left(\sqrt{\lambda_i}\right)}\, \underline{q}_i \cosh\left(\sqrt{\lambda_i}\,\xi\right)\,, \quad \text{(a)} \qquad \underline{p}_{par} = -\underline{\underline{E}}_2^{-1} \cdot \underline{x}\,, \quad \text{(b)} \tag{4}$$

which satisfy the symmetry boundary condition (BC) at the axial bearing center. λ_i and $\underline{q}_i$ are the eigenvalues and eigenvectors obtained from the generalized eigenvalue problem given in Equation (5a), and $\underline{a} = [a_1\ a_2\ \ldots\ a_m]^{\mathrm{T}}$ is a set of integration constants that depends on the ambient pressure BC $\underline{p}(\xi = 1) = \underline{p}_{amb}$ and is calculated using Equation (5b)

$$\underline{\underline{E}}_2 \cdot \underline{q} = \lambda\, \underline{\underline{E}}_0 \cdot \underline{q}\,, \quad \text{(a)} \qquad \underline{a} = \underline{\underline{Q}}^{-1} \cdot \left(\underline{p}_{amb} - \underline{p}_{par}\right) \ \text{ with } \ \underline{\underline{Q}} = [\underline{q}_1\ \underline{q}_2\ \ldots\ \underline{q}_m]\,. \quad \text{(b)} \tag{5}$$

The supply pressure BCs are enforced by eliminating the nodes within the circumferential range of the oil supply groove from Equation (3) before the equation is solved. This enforces the supply pressure BCs across the full axial bearing length L, which is a simplification of the groove geometry.

3 Verification and investigation of the numerical efficiency

To verify the SBFEM solution, the bearing forces obtained from the calculated pressure field are compared to those of a converged FEM solution, which results in a relative error δF. This error is calculated for different circumferential node numbers n_θ so that a convergence curve is obtained (Figure 3). This convergence study is performed also for an FEM solution with linear shape functions and an approximately square element geometry, which is one of the standard numerical solutions of the RE. It is observed that both solutions converge to the same result at an almost identical convergence rate.

To evaluate the efficiency of the SBFEM solution, the computational time T_{comp} required to solve the RE once is compared to that of the FEM solution for different node numbers. The results are illustrated in Figure 4. It is observed that for fine discretizations, the SBFEM is significantly faster than the FEM. However, it should be noted that the current SBFEM model has some disadvantages over the numerical solutions, since these solutions are able to consider shaft tilting and an oil supply groove of arbitrary axial length. In the SBFEM solution, an arbitrary groove length requires an additional superelement, which will be investigated in detail in further studies.

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