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Enhanced integration scheme for unfitted polygonal elements

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In this contribution a novel integration scheme, extending the conventional quadtree-decomposition-based approach by image compression techniques, is investigated for unfitted polygonal meshes with a particular focus on the rational Wachspress shape functions. It is shown that a meaningful reduction of integration points can be achieved without a significant loss in accuracy. However, the full potential of the method in terms of time savings can only be leveraged when applied to higher order polynomial elements. For more information on this topic see Enhanced Numerical Integration Scheme Based on Image Compression Techniques: Application to Rational Polygonal Interpolants by Petö et al. [1].

1 Fictitious domain approach with polygonal elements

Consider a fictitious domain setting where the original boundary value problem over a physical domain Ω_{phys} is solved on a domain $\Omega = \Omega_{\text{phys}} \cup \Omega_{\text{fict}}$, where Ω_{fict} is the fictitious domain such that $\Omega_{\text{phys}} \cap \Omega_{\text{fict}} = \emptyset$. In this contribution, the domain Ω is discretized by polygonal elements which are occasionally intersected by the boundary $\partial\Omega_{\text{phys}}$. Then, for computation of the element matrices, this formulation typically requires the evaluation of integrals of the form

$$\int\limits_{\Omega^{\text{poly}}} \alpha\mathcal{R}\,\mathrm{d}\Omega = \sum_{i=1}^{n_{\text{v}}}\sum_{j=1}^{n_{\text{sc}}}\int_{-1}^{1}\int_{-1}^{1}\alpha\mathcal{R}\,\underbrace{\det(\boldsymbol{J}^{(j)}_{\zeta\to\eta})}_{\text{const.}}\underbrace{\det(\boldsymbol{J}^{(i)}_{\eta\to\xi})}_{\text{linear}}\underbrace{\det(\boldsymbol{J}_{\xi\to x})}_{\text{rational}}\,\mathrm{d}\zeta_1\mathrm{d}\zeta_2. \tag{1}$$

On the left hand side of Eq. (1), Ω^{poly} is the polygonal element domain with n_{v} vertices, and α is a step function (indicator function) with value 0 in the fictitious domain and 1 in the physical domain [2]. Furthermore $\mathcal{R}$ is a rational function due to the typically rational nature of the polygonal shape functions which are based on *generalized barycentric coordinates*. In this contribution we investigate the Wachspress poylgonal basis functions in particular [3,4]. Although it is possible to derive special quadrature rules for polygonal domains, it is common practice to decompose Ω^{poly} into n_{v} quadrilateral domains for which high-order quadrature rules are readily available. The discontinuity $\partial\Omega_{\text{phys}}$ is considered by further partitioning each quadrilateral domain into n_{sc} sub-cells based on a quadtree-decomposition (QTD) [1,5]. The discontinuous integral in Eq. (1) is then solved by integration over the individual sub-cells, where the determinants of Jacobian matrices $\boldsymbol{J}_{\zeta\to\eta}$, $\boldsymbol{J}_{\eta\to\xi}$, and $\boldsymbol{J}_{\xi\to x}$ consider the change in integration parameters due to the geometry mappings $\boldsymbol{Q}_{\zeta\to\eta}$, $\boldsymbol{Q}_{\eta\to\xi}$ and $\boldsymbol{Q}_{\xi\to x}$ between the different domains (see Fig. 1) [1].

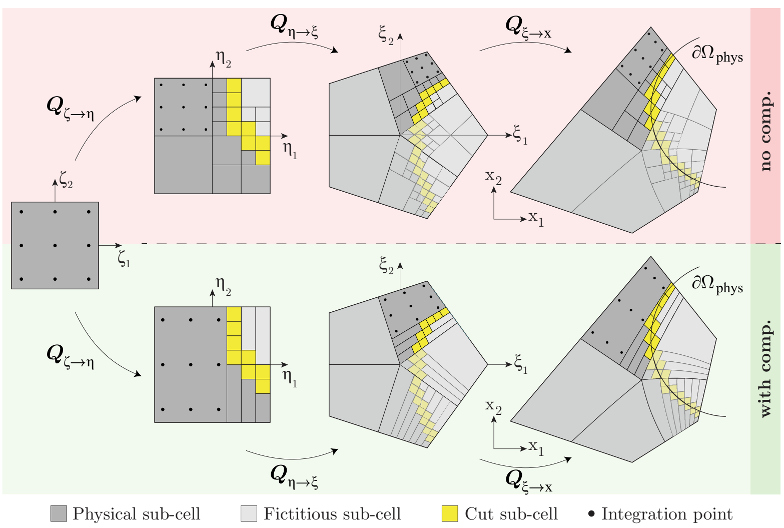

Fig. 1: Comparison of the integration sub-cells in cut a polygonal element with and without compression (refinement level $k = 3$).

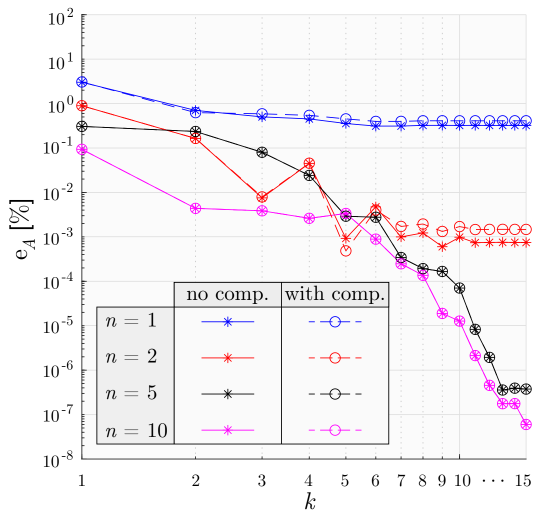

Fig. 2: Error in computing the area of an intersected element for various quadrature orders n.

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2 Compression of the sub-cells

Although the QTD-based integration scheme enables a highly robust and fully automatic integration within the intersected elements, the number of required integration points increases exponentially with the refinement level k. For tackling this negative side effect, Petö et al. [6] proposed a novel approach using image compression techniques for reducing the number of sub-cells and thus the integration time. In case of polynomial shape functions and Cartesian meshes, the compression does not influence the integration quality, assuming, that the cut sub-cells (see yellow quadrilaterals in Fig. 1) are not compressed. Due to its modular and simple structure, the compression algorithm can be straightforwardly embedded between the QTD and numerical integration stages. This is also valid for intersected polygonal elements [1]. A visual comparison of the sub-cells with and without compression is provided in Fig. 1. While the achievable reduction in integration points is the same as in Ref. [6], due to the generally rational shape functions and distorted shape of the polygonal elements, the decreased integration point density may lead to a deterioration of the accuracy of the Gaussian quadrature applied to non-polynomial integrands. This loss of accuracy is depicted in Fig. 2 for the error in the area approximation e_{A} of a given cut element for various refinement levels k and quadrature orders using $n \times n$ integration points per sub-cell. For lower quadrature orders the inaccurate integration of the rational integrand dominates the solution regardless of k and the results are indeed affected by the compression. Once n of the Gaussian quadrature is high enough for the rational integrand, no loss of accuracy is introduced by the compression and the results are dominated mainly by the error caused by the discontinuous nature of the integrand.

3 Numerical example

When embedded into an entire simulation, the compression in the polygonal elements results in negligible deterioration of the solution quality, as demonstrated on a benchmark problem of linear elasticity depicted in Fig. 3. Due to the rather poor convergence of the low order polygonal elements the approximation error dominates the solution and the accurate capturing of the discontinuity plays a less significant role. Consequently, the theoretical convergence rate of the relative error in the energy norm e_{E} is obtained already for a refinement level of $k = 3$ and $n = 2$ in the entire range of investigated discretizations (see Fig. 4). For these settings in the current example, only 55% of the original integration points were used when compared to the QTD and only 5% of the computational time was saved during the generation of element matrices. While higher values of k and n result in more significant time savings $[1, 6]$, such settings are commonly associated with high-order shape functions which are not yet robustly available for polygonal elements.

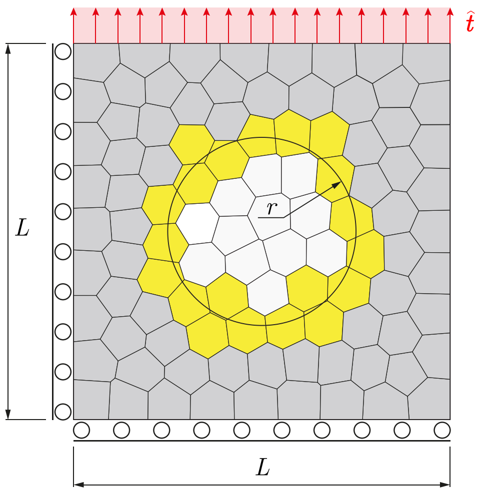

Fig. 3: Problem setup with unfitted polygonal mesh. $L = 4$ mm, $r = 1$ mm, $E = 206900$ MPa, $\nu = 0.29$, $\hat{\boldsymbol{t}} = [0, 100]$ MPa.

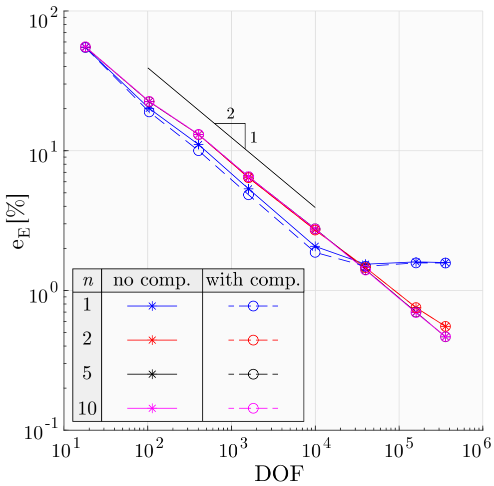

Fig. 4: Relative error in the energy norm during h-refinement using $n \times n$ integration points per sub-cell and $k = 3$.

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