Lipschitz properties of vector- and set-valued functions with applications

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Chapter 1

Introduction

In our daily life, we usually want to find the best choice or better solutions for our problems which have many contradictory goals. For example, in logistics, we need to find the shortest way with the cheapest cost to travel from one destination to another. In health care, especially in radiotherapy treatment, the dose delivered to sick organs should be maximal but we also want to keep other organs healthy, etc. Mathematical models of these problems are actually some examples for the application of vector optimization in practical perspectives. Nowadays, optimization theory not only is an interesting field in the mathematical point of view but also affects many areas of life, for instance, economics, energies, politics, culture, computer science. This dissertation is concerned with two of the most important branches of optimization theory, which are *vector optimization* and *set optimization*.

Vector optimization (or *multiobjective optimization*) deals with optimizing problems whose objective functions are vector-valued mappings. This has been studied early in the 19th century, and probably first appeared in publications of Edgeworth and Pareto who introduced some initial definitions of efficient points of vector problems. However, this branch of optimization had already started and grown rapidly since 1951, when Kuhn and Tucker [45] derived the necessary and sufficient conditions for efficient elements of vector optimization problems. There are several important practical applications of vector optimization, for instance, location problems, approximate problems, fractional problems and multiobjective control problems. Concerning solutions of vector optimization problems, there are many different solution concepts, such as (weakly) Pareto-minimal points, properly efficient points, Henig properly minimal points, approximate efficient points, etc. Those definitions have been systematically studied in Ha [27, 28], Khan et al. [44], and Luc [51], etc.

Set optimization has naturally appeared and been investigated as an expansion of vector optimization. It is concerned with problems whose objective functions are set-valued mappings. Recently, this field has attracted a great deal of attention and been developed in many publications; see [6, 7, 23, 24, 27, 37, 39, 40, 47]. We also refer the reader to the survey book by Khan et al. [44] with the references and discussions therein. In the literature, there are three main approaches for the formulation of optimality notions in set-valued optimization, namely the vector approach, the set approach and the lattice approach. For the vector approach, basically, the solution concepts are defined on the graph of a set-valued function. In more detail, a solution defined by the vector approach depends on only a special element in the image of that point and the other elements are ignored. Therefore, though this approach is interesting in the mathematical point of view, it cannot be used often in practice. In order to avoid this drawback, relevant order relations to compare two sets will be contributed. The solution concepts based on these order relations are given by the set approach. In this context, we should not fail to mention publications of Kuroiwa [46, 47], Jahn and Ha [40]. In this dissertation, we are using the primal-space approach as well as the dual-space approach in order to derive optimality conditions for set-valued optimization problems.

In order to show necessary and sufficient conditions for solutions of vector optimization problems as well as set-valued optimization problems, one needs certain structures of the objective function such as convexity and Lipschitz continuity. Both convexity and Lipschitz continuity have various important and interesting properties. The convexity is a natural and powerful property of functions that plays a significant role in many areas of mathematics, not only in theoretical but also in applied problems. It connects notions from topology, algebra, geometry and analysis, and is an important tool in deriving optimality conditions in optimization. In optimization, to get sufficient conditions for optimal solutions, we need either a second order condition or a convexity assumption. The Lipschitz continuity has been also known for a long time in applied sciences and optimization theory. For example, in order to show subdifferential chain rules or the relationships between the coderivative of a vector-valued function and the subdifferential of its scalarization, then this function should be strictly Lipschitzian; see [55, Theorem 3.28]. In particular, the Lipschitz properties for set-valued functions are used for deriving generalized differential calculus and necessary conditions for minimizers of the set-valued optimization problem; see Bao and Mordukhovich [3, 4, 5, 6].

Concerning the relationships between these two properties in finite-dimensional spaces, one of the well-known theorems of convex analysis states that: A proper convex functional $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz; see proofs in [13, 62]. We would also like to know whether the convexity implies to the locally Lipschitz continuity in the case that f is a vector-valued function or a set-valued function acting between general spaces. Let us now briefly decribe how this problem is dealt with in the literature. Initially, Roberts and Varberg [62] observed that this implication holds true for a function $f : X \to \mathbb{R}$, when X is a normed vector space, and f is locally bounded. In addition, the Lipschitz constant can be estimated; see Zălinescu [74]. For investigating a vector-valued function $f : \mathbb{R}^m \to \mathbb{R}^n$, Luc et al. [52] defined the C-convexity of f, where C is a proper, convex cone in $\mathbb{R}^n$. They proved that f is locally Lipschitz when the closure of C is pointed, and f is C-convex. A more general result is given first by Borwein [9] for a C-convex bounded function $f : X \to Y$, where X, Y are normed vector spaces and $C \subset Y$ is a normal cone. In the case of vector lattice spaces, other similar results are known from Papageorgiou [58], Reiland [61], and Thibault [69].

For set-valued maps there are many different definitions of Lipschitz continuity and convexity. The reader can find the *Lipschitz-like* property in [1, 55], and other extended Lipschitz properties in [2]. In [49], Kuroiwa et al. proposed six kinds of convexity for set-valued functions based on corresponding set relations. Consequently, it is possible to get more relationships between the convexity and the Lipschitz continuity of set-valued functions. In [53], Minh and Tan defined the C-Lipschitzianity of set-valued functions and proved that a lower C-convex set-valued function $F : X \rightrightarrows Y$ is C-Lipschitz, where X is a finite-dimensional space, and Y is a Banach space.

In this dissertation, we investigate new relationships between convexity and Lipschitzianity of vector- and set-valued functions, and their applications. Especially, we achieve the following new results:

- We prove Lipschitz properties of a cone-convex vector-valued function, under a boundedness condition of this function which is weaker than that in Borwein [9]. In this thesis, this result is proved by two different methods, in which an accurate Lipschitz constant is derived; see Theorem 4.2.7.

- We study C-Lipschitz properties of cone-convex set-valued functions. Our goal is to extend the results of Minh and Tan in [53, Theorem 2.9] to general normed vector spaces. In addition, some conditions concerning the ordering cone in [53, Theorem 2.9] can be significantly relaxed; see Theorem 5.2.8.

- We use the aforementioned results to derive optimality conditions for solutions of vector- and set-valued optimization problems, in which the objective functions are cone-convex; see Chapter 7, and Chapter 8.

This study is organized as follows: The basic framework of vector optimization and variational analysis is given in Chapter 2. We investigate binary relations on a nonempty set and ordering cones in topological vector spaces. These binary relations are the basic tools to define the convexity of vector-valued functions and set-valued functions. Moreover, these relations are benificial to define the solution concepts for vector optimization problems as well as set optimization problems. In order to provide main scalarization techniques for vector optimization problems, we introduce several scalarizing functionals and corresponding separation theorems for not necessarily convex sets. We also study set differences, which will be used in the sequel to derive new concepts of Lipschitz continuity of set-valued functions.

In Chapter 3, we define the Lipschitz continuity of vector-valued functions as well as set-valued functions. We recall the *strictly Lipschitzianity* of a vector-valued function, and the *equi-Lipschitzianity* of a family of functionals. For set-valued functions, we study the concepts *Lipschitz-like, epigraphically Lipschitz-like (ELL), upper (lower) C-Lipschitzianity*, and all Lipschitz properties which are generated by set differences given in Section 2.4.

The aim of Chapter 4 is to prove the Lipschitz continuity of C-convex vector-valued functions. As indicated above, Borwein [9] proved the Lipschitz continuity in the case that C is a normal cone, we will present new proofs for this result, and provide a more precise Lipschitz constant in Theorem 4.2.7.

In Chapter 5, we derive new results concerning Lipschitz properties for C-convex set-valued mappings. To do this, we recall the notations of C-convexity of set-valued functions firstly introduced by Kuroiwa et al. [49]. We study the proofs of C-Lipschitzianity for C-convex functions given by Kuwano and Tanaka in [50], and obtain stronger results in comparison with the results in [50].

In the first section of Chapter 6, we present some basic definitions of derivatives and directional derivatives for vector-valued functions. In Section 6.2, we investigate definitions and several properties of subdifferentials in the sense of convex analysis and subdifferentials of convex vector-valued functions. We also introduce normal cones as well as subdifferentials in the senses of Clarke, Mordukhovich and Ioffe. In the last sections, we present the derivatives and directional derivatives for set-valued functions.

In Chapter 7, we study optimality conditions for vector optimization problems. We begin this chapter with collecting some techniques to scalarize the vector optimization problem by an appropriate scalar optimization problem whose solutions are also solutions of the given problem. In the second section, we derive the necessary conditions for (weakly) Pareto efficient solutions in both solid and non-solid cases. In the last section, we use the previous results to derive necessary conditions for solutions of vector-valued approximation problems.

In Chapter 8, we use both the primal-space approach and the dual-space approach to establish optimality conditions for solutions of set-valued optimization problems. In section 8.1.1, we deal with solutions of set-valued optimization problems based on vector approach as well as set approach. By using contigent cones, contigent derivatives and contigent epiderivatives, we get optimality conditions for the solutions of the set optimization problems. For the dual-space approach, we use the Mordukhovich coderivatives in Asplund spaces to obtain necessary conditions of set optimization problems in Section 8.2.

Chapter 2

Background

In this chapter, we will present some necessary background related to vector optimization and variational analysis. This chapter is organized as follows. In Section 2.2, we introduce several properties of functionals. In particular, we recall Lipschitz properties, convex properties, and then we investigate the relationships between them, which play a significant role in this dissertation. In order to prove the Lipschitzianity of scalar convex functions, all techniques used in [62, 63, 74] are presented in this section. The relationships between the Lipschitz continuity and the convexity of scalar functions will be extended to vector-valued functions, set-valued functions as well as functions in infinite-dimensional spaces in Chapter 4 and 5.

Section 2.3.2 introduces some types of cones which are related to topologies and order structures of linear vector spaces. We refer the reader to [26, 42, 44, 66] for a survey and additional materials on ordering cones. In this section, we especially emphasize a notion, namely, a normal cone and some characterizations, which will be used in Chapter 4 to prove the Lipschitz properties of cone-convex vector-valued functions. In Section 2.3.3 we present some ordering relations between two nonempty sets in order to define the convex properties of set-valued functions in Chapter 5, and to study solutions of set-valued optimization problems in Chapter 8. These relations have been investigated by many authors, such as Kuroiwa [46, 47], Kuroiwa, Tanaka and Ha [49], Jahn and Ha [40].

In Section 2.4, following Baier and Farkhi [2], we introduce notions of differences of two sets, which will be used to define several Lipschitz properties for set-valued functions in Chapter 3. We concentrate on Demyanov differences which are studied in detail in [2, 64], and recently be modified by Dempe and Pilecka [14] and by Jahn [39].

Section 2.5 is devoted to scalarizing functionals and separation theorems, which provide main tools for deriving optimality conditions for vector optimization problems in Chapter 7, and set-valued optimization problems in Chapter 8. Section 2.6 and Section 2.7 introduce solution concepts for vector-valued optimization problems and set-valued optimization problems.

2.1 Topological vector spaces

This section mentions some basic concepts of linear spaces or vector spaces, and topological spaces. We will consider only real vector spaces throughout this dissertation, so the term *vector space* will refer to a vector space over the real field $\mathbb{R}$.

Definition 2.1.1. *Let X be a nonempty set. X is called to be a* ***vector space*** *if an addition (that is, a mapping $+ : X \times X \to X$) and a multiplication by scalars (that is, a mapping $\cdot : \mathbb{R} \times X \to X$) are defined satisfying the following conditions:*

(i) $\forall\ x, y, z \in X\ :\ (x + y) + z = x + (y + z)$ ***(associativity)***,

(ii) $\forall\ x, y \in X\ :\ x + y = y + x$ ***(commutativity)***,

(iii) $\exists\ 0 \in X,\ \forall\ x \in X\ :\ x + 0 = x$ ***(null element)***,

(iv) $\forall\ x \in X,\ \exists\ x' \in X\ :\ x + x' = 0$*; we write* $x' = -x$,

(v) $\forall\ x, y \in X,\ \forall\ \lambda \in \mathbb{R}\ :\ \lambda(x + y) = \lambda x + \lambda y$,

(vi) $\forall\ x \in X,\ \forall\ \lambda, \mu \in \mathbb{R}\ :\ (\lambda + \mu)x = \lambda x + \mu x$,

(vii) $\forall\ x \in X,\ \forall\ \lambda, \mu \in \mathbb{R}\ :\ \lambda(\mu x) = (\lambda\mu)x$,

(viii) $\forall\ x \in X\ :\ 1x = x$ ***(unity element)***.

Let A, B be nonempty subsets of a vector space X. The multiplication of a set with a scalar $\alpha \in \mathbb{R}$ and the sum of sets are given by

$$\alpha A := \{\alpha a \mid a \in A\}, \qquad A + B := \{a + b \mid a \in A, b \in B\}.$$

In particular, $A - B := A + (-1)B = \{a - b \mid a \in A, b \in B\}$. We use the following conventions for any real number α, and a set A,

$$\alpha \cdot \emptyset = \emptyset, \quad \emptyset + A = A + \emptyset = \emptyset.$$

We consider now the topological structure on the family of subsets of a nonempty set X.

Definition 2.1.2. *Let X be a nonempty set, and τ be a family of subsets of X. We say that (X, τ) is a* ***topological space*** *if τ satisfies the following conditions:*

(i) every union of sets of τ belongs to τ,

(ii) every finite intersection of sets of τ belongs to τ,

(iii) *the empty set* $\emptyset$ *and the whole set* X *belong to* τ.

The elements of τ *are called* ***open*** *sets. A subset of* X *is* ***closed*** *if and only if its complement is open.*

The following definitions present some of the standard vocabulary that will be used.

Definition 2.1.3. *Let* (X, τ) *be a topological space,* A *be a nonempty subset of* X*, and* $x \in X$*. The* ***closure*** $\operatorname{cl} A$ *of* A *is the intersection of all closed sets that contain* A*. The* ***interior*** $\operatorname{int} A$ *of* A *is the union of all open sets that are subsets of* A*. The subset* U *of* X *is a* ***neighborhood*** *of* x *(relative to* τ*) if there exists an open* $U_x \in \tau$ *such that* $x \in U_x \subset U$*. The class of all neighborhoods of* x *will be denoted by* $\mathcal{N}_\tau(x)$*. A subset* $\mathcal{B}(x)$ *of* $\mathcal{N}_\tau(x)$ *is called a* ***neighborhood base*** *of* x *relative to* τ *if for every* $U \in \mathcal{N}_\tau(x)$ *there exists* $V \in \mathcal{B}(x)$ *such that* $V \subseteq U$*.*

Now we give one of the basic structures investigated in functional analysis which is a combination of a topological space and the algebraic structure of a vector space.

Definition 2.1.4. *Let* X *be a vector space, and* τ *be a topology on* X*. We say that* (X, τ) *is a* ***topological vector space*** *if the following conditions hold:*

(i) *every point of* X *is a closed set,*

(ii) *the vector space operations are continuous w.r.t.* τ.

2.2 Topological and algebraic properties of functionals

For the convenience of the reader we collect some basic concepts in topological and some properties of functionals. These concepts and properties are presented in many classical references, so we will omit their proofs. We shall be working in topological vector spaces X whose elements are either vectors or points. The element 0_X is the origin of X. To simplify notation, we use the same symbol 0 for origin elements of all topological vector spaces if no confusion arises. In the case X is a normed vector space (nvs for short), we shall denote the norm of x by $\|x\|_X$, and if there is no confusion, we omit the subscript X for brevity. We denote by X^* its dual space equipped with the weak* topology ω^*, while its dual norm is denoted by $\|\cdot\|_*$. We denote the closed unit ball and the unit sphere in X by U_X and S_X, respectively. The closed ball centered at $x_0 \in X$ with radius $r > 0$ is defined as

$$B(x_0, r) := x + rU_X = \{x \in X \mid \|x - x_0\|_X \leq r\}.$$

The symbol * is used to indicate relations to dual spaces (dual elements, adjoint operators, dual cone etc.).

Furthermore, we use the notations $\mathbb{R}^n$ for n-dimensional Euclidean space, $\mathbb{R}^n_+$ for non-negative orthant of $\mathbb{R}^n$, and $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$. The set of positive integers is denoted by $\mathbb{N}^* := \{1, 2, \ldots\}$.

For a scalar function $f : X \to \overline{\mathbb{R}}$, the ***domain*** of f is given by

$$\operatorname{dom} f := \{x \in X \mid f(x) < +\infty\},$$

while its ***graph*** and ***epigraph*** are given, respectively, by

$$\operatorname{gph} f := \{(x, t) \in X \times \mathbb{R} \mid f(x) = t\},$$

$$\operatorname{epi} f := \{(x, t) \in X \times \mathbb{R} \mid f(x) \le t\}.$$

Definition 2.2.1. *The function $f : X \to \overline{\mathbb{R}}$ is called:*

(i) ***proper*** *if* $\operatorname{dom} f \neq \emptyset$ *and $f(x) > -\infty$ for all $x \in X$;*

(ii) ***positively homogeneous*** *if $f(\lambda x) = \lambda f(x)$ for all $x \in X$ and $\lambda \in \mathbb{R}^+$;*

(iii) ***symmetric*** *if $f(-x) = f(x)$ for all $x \in X$;*

(iv) ***lower semi-continuous*** *(lsc for short) if* $\operatorname{epi} f$ *is closed;*

(v) ***subadditive*** *if $f(x + y) \le f(x) + f(y)$ for all $x, y \in X$;*

(vi) ***sublinear*** *if f is subadditive and positively homogeneous.*

In the following definitions, we define the Lipschitz continuity and the convexity of scalar functions. Then we will recall the relationships between the Lipschitz continuity and convexity of scalar functions. These properties will be generated in Chapters 3, 4 and 5 for vector-valued and set-valued mappings in general spaces.

Definition 2.2.2. *([13]) Let X be a normed vector space, $f : X \to \overline{\mathbb{R}}$ be a function, $A \subseteq X$. Then, f is said to be* ***Lipschitz*** *on A with a nonnegative constant ℓ provided that f is finite on A and*

$$|f(x) - f(x')| \le \ell \left\|x - x'\right\|_X$$

for all points x, x' in A. This is also referred to as a Lipschitz condition of rank ℓ. We shall say that f is ***Lipschitz around*** *x if there is a neighborhood U of x such that f is Lipschitz on U (in particular $x \in \operatorname{int}(\operatorname{dom} f)$). In addition, f is said to be* ***locally Lipschitz*** *on A, if f is Lipschitz around every point $x \in A$. Hence, $A \subseteq \operatorname{int}(\operatorname{dom} f)$.*

Definition 2.2.3. *([74]) Let X be a real topological vector space, $f : X \to \overline{\mathbb{R}}$ be a function. We say that f is* ***convex*** *on a convex subset A of X if for all $x, x' \in A, \lambda \in (0, 1)$, one has*

$$\lambda f(x) + (1 - \lambda) f(x') \ge f(\lambda x + (1 - \lambda) x').$$

If the equality always holds, f is said to be ***affine****.*

For convenience we will say that the function $f : X \to \overline{\mathbb{R}}$ is convex if f is convex on whole space X. Obviously, if f is convex, then f is convex on every convex subset of X.

Now we give several simple examples of Lipschitz functions and convex functions.

Example 2.2.4. *Let $(X, \|\cdot\|)$ be a normed vector space.*

*(i) It is simple to verify that the **norm function** $f(x) = \|x\|$ is convex, and Lipschitz on X with Lipschitz constant $\ell = 1$.*

*(ii) If A is a nonempty convex subset of X, then the **distance function***

$$d(x, A) := \inf_{y \in A} \|x - y\|, \quad x \in X,$$

is convex on X. It is also Lipschitz with rank $\ell = 1$.

The following maps are often used in the literature.

Example 2.2.5. ***(Indicator function)*** *Given a nonempty subset $A \subset X$, the indicator function $\delta_A : X \to \overline{\mathbb{R}}$ is defined by*

$$\delta_A(x) := \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{otherwise} . \end{cases} \tag{2.1}$$

Obviously, if A is convex, then δ_A is proper and convex.

Example 2.2.6. ***(Support function)*** *Given a nonempty subset A of a normed space X, the support function $\sigma(\cdot, A) : X^* \to \overline{\mathbb{R}}$ w.r.t. A is defined by*

$$\sigma(x^*, A) := \sup_{a \in A} \langle x^*, a \rangle , \qquad (x^* \in X^*). \tag{2.2}$$

It is easy to verify that $\sigma(\cdot, A)$ is a positively homogeneous, closed and convex function.

In the next proposition, we present some properties of convex functions, which can be considered as equivalent concepts of convex functions.

Proposition 2.2.7. *([74]) Let X be a topological vector space, $f : X \to \overline{\mathbb{R}}$ be a function. The following statements are equivalent:*

(i) f is convex,

(ii) $\operatorname{dom} f$ is a convex set and

$$\forall x, y \in \operatorname{dom} f, \forall \lambda \in (0, 1) : \lambda f(x) + (1 - \lambda) f(x') \geq f(\lambda x + (1 - \lambda) x'),$$

(iii) $\operatorname{epi} f$ is a convex subset of $X \times \mathbb{R}$.

The following lemma shows that every proper scalar convex function is locally bounded in a finite-dimensional space. For the proof, we refer the reader to [62, Lemma A].

Lemma 2.2.8. *(Lemma A [62]) A proper convex function $f : \mathbb{R}^m \to \overline{\mathbb{R}}$ is bounded in a neighborhood of each point $x_0 \in \operatorname{int}(\operatorname{dom} f)$.*

By using the lemma above, Roberts and Varberg obtained the locally Lipschitz property of a proper convex functional. They also estimated the Lipschitz constant which plays an important role in proving the Lipschitz properties of cone-convex set-valued functions. For convenience we recall the proof in this thesis.

Lemma 2.2.9. *([62, Theorem A]) Let $f : \mathbb{R}^m \to \overline{\mathbb{R}}$ be a proper convex function. Then, f is Lipschitz on a neighborhood of each point x_0 of $\operatorname{int}(\operatorname{dom} f)$.*

Proof. Let x_0 be a given point in $\operatorname{int}(\operatorname{dom} f)$. Taking into account Lemma 2.2.8, we see that there exist $\epsilon > 0$ and $M > 0$ such that $B(x_0, 2\epsilon) \subseteq \operatorname{int}(\operatorname{dom} f)$ and

$$|f(x)| \le M, \quad \text{for all} \quad x \in B(x_0, 2\epsilon).$$

Take $x, y \in B(x_0, \epsilon), x \neq y$, and set

$$z = x + (x - y)\frac{\epsilon}{\|x - y\|}.$$

This implies that

$$z \in B(x_0, 2\epsilon) \quad \text{and} \quad x = ty + (1 - t)z,$$

where $t = \frac{\epsilon}{\epsilon + \|x-y\|} \in (0, 1)$. Since f is convex, we have

$$f(x) \le tf(y) + (1 - t)f(z).$$

Therefore,

$$\begin{aligned} f(x) - f(y) &\le (1 - t)(f(z) - f(y)) \\ &\le \frac{\|x - y\|}{\epsilon}\|f(z) - f(y)\| \le \frac{2M}{\epsilon}\|x - y\|, \end{aligned}$$

which leads to the conclusion. □

However, a scalar convex function on an infinite-dimensional normed space may be locally unbounded (for example, see [62]). Therefore, we need a mild additional condition on f, for example, the boundedness from above of the function on a nonempty open set; see also Roberts and Varberg [62, Theorem B] and Zălinescu [74, Corollary 2.2.12].

Lemma 2.2.10. ([62, Lemma B]) *Let $(X, \|\cdot\|_X)$ be a normed vector space, and $f : X \to \overline{\mathbb{R}}$ be a proper convex function. If f is bounded from above in a neighborhood of just one point x_0 of $\operatorname{int}(\operatorname{dom} f)$, then f is locally bounded on $\operatorname{int}(\operatorname{dom} f)$.*

It implies from Lemma 2.2.10 that f is locally bounded on $\operatorname{int}(\operatorname{dom} f)$. Therefore, the Lipschitzianity of f follows the same lines of argument in the proof of Lemma 2.2.9. Moreover, if f is bounded by $M > 0$ on a neighborhood $B(x_0, 2\epsilon)$ of x_0, we can estimate the Lipschitz constant $\ell = 2M/\epsilon$.

Lemma 2.2.11. ([62, Theorem B]) *Let $(X, \|\cdot\|_X)$ be a normed vector space, and $f : X \to \overline{\mathbb{R}}$ be a proper convex function. If f is bounded from above in a neighborhood of just one point x_0 of $\operatorname{int}(\operatorname{dom} f)$, then f is Lipschitz around x_0. Moreover, f is locally Lipschitz on $\operatorname{int}(\operatorname{dom} f)$.*

In the following lemma, Zălinescu utilized another technique to show the Lipschitz property of a proper convex function f, and to estimate the Lipschitz constant. For the proof, we refer to [74, Corollary 2.2.12].

Lemma 2.2.12. ([74, Corollary 2.2.12]) *Let $(X, \|\cdot\|_X)$ be a normed vector space, and $f : X \to \overline{\mathbb{R}}$ be a proper convex function. Suppose that $x_0 \in \operatorname{dom} f$ and there exist $\theta > 0$, $m \geq 0$ such that*

$$\forall x \in B(x_0, \theta) : f(x) \leq f(x_0) + m.$$

Then

$$\forall \theta' \in (0, \theta), \forall x, x' \in B(x_0, \theta') : \left|f(x) - f(x')\right| \leq \frac{m}{\theta} \cdot \frac{\theta + \theta'}{\theta - \theta'} \cdot \left\|x - x'\right\|_X .$$

In the next chapters, we will study the Lipschitzianity of vector-valued convex functions and set-valued convex functions arising in a natural way from the results of the aforementioned Lemmas 2.2.11, and 2.2.12. We investigate these problems not only in finite-dimensional spaces but also in infinite-dimensional spaces.

2.3 Binary relations, Ordering cones and Set relations

In this section, order relationships w.r.t. a given convex cone C, between two vectors and between two nonempty sets are considered. Based on these order relationships, we are able to derive some solution concepts for vector-valued optimization problems in Section 2.6, and solution concepts for set-valued optimization problems in Section 2.7.

2.3.1 Binary relations

We begin with binary relations and some of their properties which are the basis for the definition of ordering cones and of optimal elements.

Definition 2.3.1. *Let M be a nonempty set, $M \times M$ is the set of ordered pairs of elements of M:*

$$M \times M := \big\{(x, y) \mid x, y \in M\big\}.$$

*If $\mathfrak{R}$ is a nonempty subset of $M \times M$, then $\mathfrak{R}$ is called a **binary relation** on M and we write $x\mathfrak{R}y$ for $(x, y) \in \mathfrak{R}$. The pair $(M, \mathfrak{R})$ is called a set M with binary relation $\mathfrak{R}$. Two elements $x, y \in M$ are said to be **comparable** if $x\mathfrak{R}y$ or $y\mathfrak{R}x$ holds.*
The binary relation $\mathfrak{R}$ is called:

*(i) **reflexive** if $x\mathfrak{R}x$ for every $x \in M$;*

*(ii) **transitive** if for all $x, y, z \in M$: $x\mathfrak{R}y$ and $y\mathfrak{R}z$ imply that $x\mathfrak{R}z$;*

*(iii) **symmetric** if for all $x, y \in M$: $x\mathfrak{R}y$ implies that $y\mathfrak{R}x$;*

*(iv) **antisymmetric** if for all $x, y \in M$: $x\mathfrak{R}y$ and $y\mathfrak{R}x$ imply that $x = y$;*

*(v) **complete** if any two elements of M are comparable.*

*(vi) a **preorder** if $\mathfrak{R}$ is reflexive and transitive;*

*(vii) a **partial order** if $\mathfrak{R}$ is reflexive, transitive and antisymmetric.*

Example 2.3.2. *Let $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ be the set of real numbers, integers, and nonnegative integers, respectively. Take*

$$\mathfrak{R}_1 := \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x - y \in \mathbb{Z}\},$$

$$\mathfrak{R}_2 := \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x - y \in \mathbb{N}\},$$

$$\mathfrak{R}_3 := \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x - y \in \mathbb{N}^*\}.$$

Then,

(i) $\mathfrak{R}_1$ is a preorder on $\mathbb{R}$. It is also symmetric but not antisymmetric or complete on $\mathbb{R}$.

(ii) $\mathfrak{R}_2$ is a partial order but it is neither symmetric nor complete on $\mathbb{Z}$.

(iii) $\mathfrak{R}_3$ is only transitive on $\mathbb{Z}$.

Definition 2.3.3. *Let $\mathfrak{R}$ be a binary relation on a nonempty set M, and take $M_0 \subseteq M$. An element $\bar{x} \in M_0$ is called a **maximal** or a **minimal** element of M_0 w.r.t. $\mathfrak{R}$ if for every $x \in M_0$:*

$$\bar{x}\mathfrak{R}x \Longrightarrow x\mathfrak{R}\bar{x} \quad \textit{or}$$

$$x\mathfrak{R}\bar{x} \Longrightarrow \bar{x}\mathfrak{R}x, \textit{ respectively.}$$

We denote by $\mathrm{Max}(M_0; \mathfrak{R})$ the set of all maximal elements of M_0, and by $\mathrm{Min}(M_0; \mathfrak{R})$ the set of all minimal elements of M_0.

If $\mathfrak{R}$ is a partial order on M, then a subset $M_0 \subseteq M$ can have no, one or several minimal (maximal) elements.

Definition 2.3.4. *Let $\mathfrak{R}$ be a binary relation on a nonempty set M, and M_0 be a subset of M. We call that M_0 is* ***bounded below*** *(or* ***bounded above****) w.r.t. $\mathfrak{R}$ if there exists some $a \in M$ such that $a\mathfrak{R}x$ ($x\mathfrak{R}a$, respectively) for every $x \in M_0$. In this case, the element a is called a* ***lower bound*** *(****upper bound****, respectively) of M_0.*

If $\mathfrak{R}$ is a partial order, an element $a \in M$ is called the ***infimum*** *(or* ***supremum****) of M if a is a lower bound (upper bound, respectively) of M_0 and for any lower bound (upper bound, respectively) a' of M_0 we have $a'\mathfrak{R}a$ ($a\mathfrak{R}a'$, respectively).*

We consider $M := \mathbb{Z}$, $M_0 := \mathbb{N}^*$ and the binary relation $\mathfrak{R}_2$ given in Example 2.3.2. Observe that the unit element 1 is a unique minimal element and also an infimum M_0.

2.3.2 Ordering cones

In this section, we will list some basic notions of cones of a topological vector space Y, which can be found, for instance, in [26, 42, 44, 51, 66]. These cones induce the class of binary relations, which are compatible with the linear structure of Y.

Definition 2.3.5. *A nonempty set $C \subseteq Y$ is said to be a cone if $tc \in C$ for every $c \in C$ and every $t \geq 0$. The cone C is called:*

(i) ***convex*** *if $\forall \lambda \in (0, 1), \forall x_1, x_2 \in C$: $\lambda x_1 + (1 - \lambda)x_2 \in C$,*

(ii) ***proper*** *if $C \neq \{0\}$ and $C \neq Y$,*

(iii) ***reproducing*** *if $C - C = Y$,*

(iv) ***pointed*** *if $C \cap (-C) = \{0\}$.*

Obviously, if C is a cone, then $0 \in C$. We will give some examples of cone.

Example 2.3.6. *(i) The nonnegative orthant of the n-dimensional Euclidean space is given by $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \quad \forall i = 1, \ldots, n\}$. Clearly, $\mathbb{R}^n_+$ is a proper, convex and reproducing cone.*

(ii) $C := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \forall i = 1, \ldots, n : x_i > 0\} \cup \{0\}$ is a convex and pointed cone.

Definition 2.3.7. *Let Y be a topological vector space and C be a proper, convex cone in Y. A nonempty set B of C is called a base for C if each nonzero element $y \in C$ has a unique representation of the form $y = \lambda b$ with $\lambda > 0$ and $b \in B$.*

On the topological vector space Y, we consider an ordering relation $\geq_C$ generated by a proper, convex cone $C \subseteq Y$. This ordering relation is given by

$$y \geq_C y' \quad \text{if and only if} \quad y - y' \in C \quad \text{for all} \quad y, y' \in Y. \tag{2.3}$$

In several cases, if it causes no confusion, we will use the notation $\leq_C$ as an ordering relation on Y, i.e. $y \leq_C y' \Leftrightarrow y' \geq_C y$.

We present some properties of $\geq_C$ in the following proposition.

Proposition 2.3.8. *Let Y be a topological vector space, and C be a convex cone. Then, the ordering relation $\geq_C$ given by (2.3) has the following properties:*

(i) $y \geq_C y$ for all $y \in Y$ (reflexive),

(ii) $y \geq_C y'$, $y' \geq_C y''$ implies $y \geq_C y''$ for all $y, y', y'' \in Y$ (transitive),

(iii) $y \geq_C y'$ implies $y + z \geq_C y' + z$ for all $y, y', z \in Y$,

(iv) $y \geq_C y'$ implies $\lambda y \geq_C \lambda y'$ for all $\lambda \geq 0$ and $y, y' \in Y$.

(v) If C is pointed, then $\geq_C$ is antisymmetric. Moreover, $\geq_C$ is called a partial order.

Now we study some cone properties which show the connection between the topology and the order of the space Y. Before giving the definition of a normal cone, we recall that the nonempty set A of the topological vector space Y is ***full*** w.r.t. the convex cone $C \subset Y$ if $A = [A]_C$, where

$$[A]_C := (A + C) \cap (A - C);$$

note that $[A]_C$ is full w.r.t. C for every nonempty subset A of Y.

Definition 2.3.9. *Let Y be a topological vector space, and let $C \subset Y$ be a proper, convex cone. Then C is called **normal** if there exists a neighborhood base of the origin $0 \in Y$ formed by full sets w.r.t. C.*

Remark 2.3.10. *If the neighborhood base of the origin in Definition 2.3.9 is taken in the weak topology of Y, then C is called **weakly normal** (w-normal, for short).*

Example 2.3.11. *We give an example of a normal cone. In the 2-dimensional Euclidean space $\mathbb{R}^2$, we consider the nonnegative orthant $C := \mathbb{R}^2_+$, and a neighborhood base formed by sets $A_n := \{|x_i| < 1, i = 1, 2\}$ for every $n \in N^*$. Clearly, A_n is full w.r.t. C for every $n \in N^*$, and therefore $\mathbb{R}^n_+$ is a normal cone.*

Although the concepts given in Definition 2.3.9 and Remark 2.3.9 are defined for the general topological vector spaces, in this section we consider them in normed vector spaces. Until the end of this section, unless otherwise stated, by Y we mean a normed vector space over the field $\mathbb{R}$ with the norm $\|\cdot\|_Y$. The topological dual of Y is denoted by Y^*.

The continuous dual cone of C and its quasi-interior are respectively given by

$$C^+ := \{y^* \in Y^* \mid \forall c \in C : \; y^*(c) \geq 0\},$$

and

$$C^{\#} := \{y^* \in Y^* \mid \forall c \in C : \ y^*(c) > 0\}.$$

We use the following convention, $y^*(\emptyset) := \{+\infty\}$, for all $y^* \in C^+$.

We adjoin a maximal element $+\infty$ to Y ($+\infty \notin Y$) such that $+\infty \geq_C y$ for all $y \in Y$, and we use the notation $Y^\bullet := Y \cup \{+\infty\}$. The infinity element satisfies

$$\alpha \cdot (+\infty) = +\infty, \quad y + (+\infty) = +\infty, \quad 0 \cdot (+\infty) = 0, \quad y^*(+\infty) = +\infty$$

for any positive real α, any y in Y and any $y^* \in C^+$.

In the next results we give several characterizations of normal cones in a normed vector space.

Lemma 2.3.12. *([26, Theorem 2.1.22]) Let Y be a normed vector space, and $C \subset Y$ be a convex cone. The following statements are equivalent:*

(i) *C is normal;*

(ii) *There exists $\rho > 0$ such that $\rho[U_Y]_C \subset U_Y$;*

(iii) $\operatorname{cl} C$ *is normal;*

Taking into account of Lemma 2.3.12 $(i), (ii)$, then (i) is equivalent to the boundedness of $[U_Y]_C$ (compare to [26, Theorem 2.2.10]).

Lemma 2.3.13. *([26, Corollary 2.1.23]) Let $(Y, \|.\|_Y)$ be a normed vector space, and $C \subset Y$ be a convex cone. If C is normal, then C is pointed, and so* $\operatorname{cl} C$ *is pointed, too.*

Proof. Indeed, if $y \in C \cap (-C)$, then $y \in (\{0\}+C) \cap (\{0\}-C) \subseteq (\rho U_Y + C) \cap (\rho U_Y - C) = [\rho U_Y]_C$ for every $\rho > 0$. Since the family $\{\rho[U_Y]_C, \rho > 0\}$ is a neighborhood base of 0, $y = 0$. From Lemma 2.3.12, it follows that $\operatorname{cl} C$ is a normal cone, and thus $\operatorname{cl} C$ is pointed. $\square$

Lemma 2.3.14. *([26, Corollary 2.2.11]) Let $(Y, \|.\|_Y)$ be a finite-dimensional normed vector space, and $C \subset Y$ be a convex cone. Then, C is normal if and only if* $\operatorname{cl} C$ *is pointed.*

The next result is a particular case of [42, 3.4.8]; see also the remark from [66, p. 220].

Proposition 2.3.15. *Let $(Y, \|\cdot\|_Y)$ be a normed vector space, and $C \subset Y$ be a convex cone. Then,*

$$C \text{ is normal} \Leftrightarrow C \text{ is weakly normal} \Leftrightarrow C^+ - C^+ = Y^*.$$

Proof. The following implications are well known in locally convex spaces; see, e.g., [26].

$$C \text{ is normal } \Rightarrow\ C \text{ is weakly normal } \Leftrightarrow\ C^{+} - C^{+} = Y^{*}$$

Assume that C is weakly normal. Obviously, U_Y is weakly bounded. Since C is weakly normal, by [42, Section 3.2.6], $[U_Y]_C$ is weakly bounded. By [65, Corollary 3.18], $[U_Y]_C$ is bounded. This shows that C is normal. □

Lemma 2.3.16. *Let $(Y, \|\cdot\|_Y)$ be a normed vector space, and $C \subset Y$ be a normal cone. Then,*

$$\rho := \sup\{\|y\| \mid y \in [U_Y]_C\} \in [1, +\infty) \text{ and } \rho^{-1} U_{Y^*} \subseteq C_1^{+} - C_1^{+},$$

where $C_1^{+} := U_{Y^*} \cap C^{+}$.

This result can be deduced from Jameson's book [42]. We provide its proof for the reader's convenience. In this proof we are dealing with the polar set of a nonempty set $A \subseteq Y$ defined by

$$A^0 := \{y^* \in Y^* \mid \forall y \in A : y^*(y) \geq -1\}.$$

Proof. Since C is normal, there exists $r > 0$ such that $[U_Y]_C \subset rU_Y$. It follows that $1 \leq \rho \leq r < +\infty$. Since $[U_Y]_C = (U_Y + C) \cap (U_Y - C) \subseteq \rho U_Y$ and $(U_Y + C)^0 = C^{+} \cap U_{Y^*} = C_1^{+}$ is convex and w^*-compact, we get

$$\begin{aligned}
\rho^{-1} U_{Y^*} &= (\rho U_Y)^0 \subseteq [(U_Y + C) \cap (U_Y - C)]^0 \\
&= \overline{\operatorname{conv}}^{w^*} \left[(U_Y + C)^0 \cup (U_Y - C)^0 \right] = \overline{\operatorname{conv}}^{w^*} \left[C_1^{+} \cup (-C_1^{+}) \right] \\
&\subseteq \overline{\operatorname{conv}}^{w^*} \left[C_1^{+} - C_1^{+} \right] = C_1^{+} - C_1^{+},
\end{aligned}$$

where $\overline{\operatorname{conv}}^{w^*} E$ is the closed convex hull of the subset E of the vector space Y^* with respect to the weak* topology. This completes the proof. □

Before giving some useful notions of cones, we recall that a net $(x_i)_{i \in I} \subset X$ is nonincreasing if

$$\forall i, j \in I : j \succeq i \Rightarrow x_i \geq_C x_j,$$

where C is a convex cone.

Given a nonempty set $A \subseteq X$, we say that A is **lower bounded** w.r.t. C if there is an element $a \in X$ such that $x \geq_C a$ for every $x \in A$.

Definition 2.3.17. *Let Y be a normed vector space, and $C \subset Y$ a proper, convex cone. We say that*

*(i) C is **based** if there exists a convex set B such that $C = \mathbb{R}_+ B$ and $0 \notin \operatorname{cl} B$.*

*(ii) C is **well-based** if there exists a bounded convex set B such that $C = \mathbb{R}_+ B$ and $0 \notin \operatorname{cl} B$.*

Table 2.1: The relationships among different kinds of cones

$$
\begin{array}{ccccc}
 & & C \text{ has compact base} & & \\
 & & \Downarrow \ \Uparrow {\scriptstyle C=\operatorname{cl} C,\ Y=\mathbb{R}^n} & & \\
 & & C \text{ is well-based} & \Longleftrightarrow & \operatorname{int} C^+ \neq \emptyset \\
 & & \Downarrow & & \\
\exists \text{ proper cone } K\colon\ C \setminus \{0\} \subseteq \operatorname{int} K & \Longleftrightarrow & C \text{ is based} & \Longleftrightarrow & C^{\#} \neq \emptyset \\
 & & \Downarrow \ \Uparrow {\scriptstyle C=\operatorname{cl} C,\ Y \text{ separable}} & & \\
C \text{ well-based} & \overset{C=\operatorname{cl} C,\ Y=\mathbb{R}^n}{\Longleftarrow} & C \text{ pointed} & \Longleftarrow & C^+ - C^+ = Y^* \\
\Downarrow & & \Uparrow & & \Updownarrow \\
\operatorname{cl} C \text{ is normal} & \Longleftrightarrow & C \text{ is normal} & \overset{Y \text{ nvs}}{\Longleftrightarrow} & C \text{ is } w\text{-normal.}
\end{array}
$$

(iii) C has a ***compact base*** *if there exists a compact convex set B such that $C = \mathbb{R}_+ B$ and $0 \notin \operatorname{cl} B$.*

(iv) C is said to be ***Daniell*** *if any nonincreasing net which has a lower bound, converges to its infimum.*

Remark 2.3.18. *(i) Table 2.1 describes the relationships among different kinds of cones.*

(ii) Obviously, if a proper, convex cone C is well-based, then C is also based. It is clear that the nonnegative orthant $\mathbb{R}^2_+$ is well-based in $\mathbb{R}^2$ with a bounded convex set $B := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0\}$.

(iii) A convex cone with a weakly compact base is Daniell.

2.3.3 Set relations

In what follows X and Y are normed spaces, C is a proper, convex cone in Y. We take two arbitrary nonempty subsets A, B of Y, and consider the ordered relationship between them w.r.t. the cone C. We begin this part with the ordering relation $\geq_C$ to compare two vectors $x, y \in Y$ given by (2.3) as in the previous section. It is clear that the ordering relation between two vectors does not imply the ordering relation between two sets:

$$A \subset B + C \quad \not\Leftrightarrow \quad B \subset A - C.$$

In order to avoid this drawback it is necessary to derive other set relations to compare two arbitrary nonempty sets in Y.

First, we recall a *set less order relation*, which plays an important role in set optimization, and was first independently introduced by Young [72] and Nishnianidze [57].

Definition 2.3.19. *([57, 72]) Let A, B be nonempty subsets of Y, and C be a proper, convex cone; then the* ***set less order relation*** *$\preceq^s_C$ is defined by*

$$A \preceq^s_C B \iff B \subseteq A + C \text{ and } A \subseteq B - C.$$

We follow the lines of Kuroiwa, Tanaka, and Ha [49] and define six kinds of set relations between two nonempty sets as follows

Definition 2.3.20. *([49]) For two nonempty sets $A, B \subseteq Y$ and a proper, convex cone C in Y, we introduce the following set relations*

(i) $A \preceq^{(i)}_C B \iff B \subseteq \bigcap\limits_{a \in A} (a + C)$;

(ii) $A \preceq^{(ii)}_C B \iff A \bigcap \big(\bigcap\limits_{b \in B} b - C \big) \neq \emptyset$;

(iii) $A \preceq^{(iii)}_C B \iff B \subseteq A + C$;

(iv) $A \preceq^{(iv)}_C B \iff B \bigcap \big(\bigcap\limits_{a \in A} a + C \big) \neq \emptyset$;

(v) $A \preceq^{(v)}_C B \iff A \subseteq B - C$;

(vi) $A \preceq^{(vi)}_C B \iff B \bigcap (A + C) \neq \emptyset$.

The set relations $\preceq^{(iii)}_C$ and $\preceq^{(v)}_C$ will be called the ***lower*** and ***upper set less order relation***, respectively. In several books and articles, the lower (upper) set less order relation is denoted by $\preceq^{(l)}_C$ (*resp.* $\preceq^{(u)}_C$); see [40, 44] and the references therein.

The following proposition is directly verified from Definition 2.3.20.

Proposition 2.3.21. *([49]) Let $A, B \subseteq Y$ be nonempty sets, and C be a proper, convex cone in Y. The following statements hold:*

$$\begin{array}{ccccc}
A \preceq^{(i)}_C B & \Longrightarrow & A \preceq^{(ii)}_C B & \Longrightarrow & A \preceq^{(iii)}_C B \\
\Downarrow & & & & \Downarrow \\
A \preceq^{(iv)}_C B & \Longrightarrow & A \preceq^{(v)}_C B & \Longrightarrow & A \preceq^{(vi)}_C B
\end{array}$$

The set relations given in Definition 2.3.20 are widely used in literature to define solution concepts for set-valued optimization problems. In this dissertation, we also use these relations to define new convexity notions for set-valued functions.

In [40], Jahn and Ha also derived many new set relations. The authors have equipped the space Y with an arbitrary pre-order, without any topological or linear structure, and then after defining new concepts of optimal solutions of set-valued optimization problems, some existence results for these solutions were derived.

2.4 Set differences

In this section, we study several set differences which were investigated by Baier and Farkhi [2], Rubinov and Akhundov [64], Dempe and Pilecka [14] for finite-dimensional spaces, and by Jahn [39] for infinite-dimensional spaces. It is important to mention that these differences motivate the corresponding Lipschitz continuities in Section 3.2 later. In [2], various set differences are considered on $\mathcal{K}(\mathbb{R}^n)$ (the set of nonempty compact subsets of $\mathbb{R}^n$) or on $\mathcal{C}(\mathbb{R}^n)$ (the set of nonempty convex compact subset of $\mathbb{R}^n$). For each set difference, the corresponding distance (or even a metric) is constructed, and the corresponding Lipschitz continuity of a set-valued function with compact values $F : X \rightrightarrows \mathcal{K}(\mathbb{R}^n)$ (or convex compact values $F : X \rightrightarrows \mathcal{C}(\mathbb{R}^n)$) are also derived. In this approach, the Lipschitzianity is related to the distance or the topological structure of the spaces $\mathcal{K}(\mathbb{R}^n)$ and $\mathcal{C}(\mathbb{R}^n)$. In this dissertation, we follow another direction. The Lipschitz continuities will be defined directly from the topological structure of the original space without mentioning the distance. To do that, the algebraic difference and geometric difference of two arbitrary sets are defined in general vector spaces. In contrast, the Demyanov difference, and metric difference of two compact sets are defined in the n-dimensional Euclidean space $\mathbb{R}^n$. In the following definitions, for each set difference concept, we will consider the corresponding vector spaces which can be either finite-dimensional or general spaces.

Definition 2.4.1. *([2]) Let Y be a general topological vector space, and A, B be subsets of Y. We define the*

(i) algebraic difference as

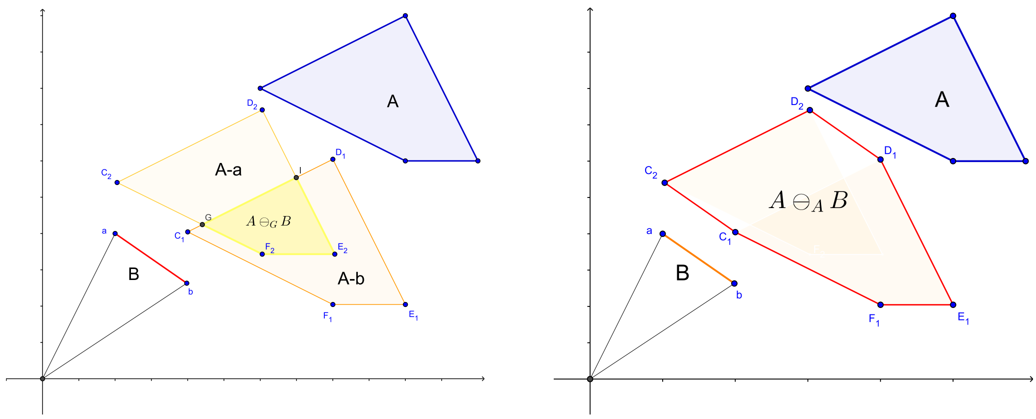
$$A \ominus_A B := A + (-1) \cdot B,$$

(ii) geometric/star-shaped/Hadwiger-Pontryagin difference as

$$A \ominus_G B := \{y \in Y : y + B \subseteq A\}.$$

The algebraic difference and geometric difference can be also presented as

$$A \ominus_A B = \bigcup_{b \in B} A^{-b}, \qquad A \ominus_G B = \bigcap_{b \in B} A^{-b},$$



(a) (b)

Figure 2.1: The geometric difference and the algebraic difference.

where $A^{-b} := \{a - b \mid a \in A\} = A - b$ for $b \in B$. Of course, the second formulae for the algebraic difference and the geometric difference have more geometrical meaning than the first ones given in Definition 2.4.1; see Figure 2.1a and 2.1b.

In the special case that $B = \{b\}$ is a singleton, the algebraic difference and the geometric difference coincide and are exactly the set $A - b$.

Note that the disadvantage of these two concepts is that the cardinality of the algebraic difference set is usually bigger than one of two original sets. Furthermore, the geometric difference sets in several cases can be empty, even if the vector spaces are finite- or infinite-dimensional.

Example 2.4.2. *(i) Let $Y := \mathbb{R}^n$, and two sets $A = B := U_Y$. Then, $A \ominus_A B = 2U_Y$; see Figure 2.2a.*

(ii) Let $A, B \subset Y$ be two balls in a normed vector space Y such that B has a radius bigger than A. Then, $A \ominus_G B = \emptyset$; see Figure 2.2b.

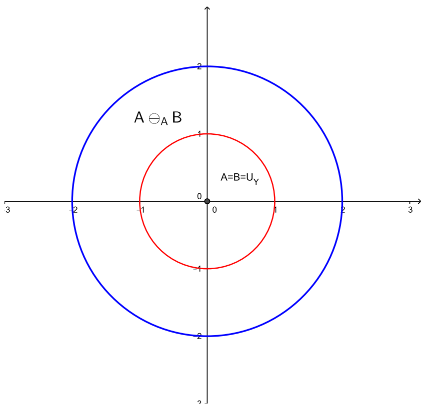
The geometric difference can also be extended to l-difference by Pilecka [60]. In next definition, the l-difference is defined in a finite-dimensional $\mathbb{R}^n$ w.r.t. the relation $\preceq_C^{(iii)}$ given in Definition 2.3.20, where C is a cone in $\mathbb{R}^n$.

Definition 2.4.3. *(Pilecka [60]) Let A, B be arbitrary subsets of $\mathbb{R}^n$, and C be a proper, convex cone in $\mathbb{R}^n$. The l-difference is defined as follows*

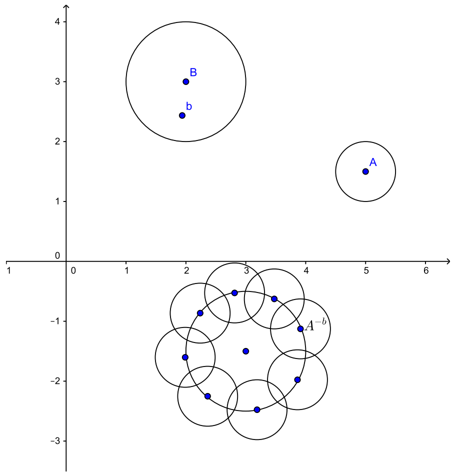
$$A \ominus_l B := \{y \in \mathbb{R}^n : y + B \subseteq A + C\}, \tag{2.4}$$

or the equivalent formulation

$$A \ominus_l B = \{y \in \mathbb{R}^n : A \preceq_C^{(iii)} B + y\},$$



(a) $A \ominus_A B = 2U_Y$



(b) $A \ominus_G B = \emptyset$

Figure 2.2: Illustration for Examples 2.4.2.

where $\preceq_C^{(iii)}$ *is introduced in Definition 2.4.3.*

In [2], Baier and Farkhi defined the algebraic difference and the geometric difference in finite-dimensional spaces. However, it is clear that the nonempty characterization of algebraic difference sets and geometric difference sets does not depend on the dimensionality of the space.

Now we will study Demyanov differences and their modifications. In [2, 15, 64], the Demyanov differences are considered on the class of the convex compact subsets and the class of compact subsets in the n-dimensional Euclidean space $\mathbb{R}^n$, which are essential to make Demyanov difference sets nonempty.

For $Y = \mathbb{R}^n$, we denote by $\mathcal{K}(\mathbb{R}^n)$ the set of nonempty compact subsets of $\mathbb{R}^n$, and by $\mathcal{C}(\mathbb{R}^n)$ the set of nonempty convex compact subsets of $\mathbb{R}^n$. For a given set $A \in \mathcal{K}(\mathbb{R}^n)$, the support function of A is given by

$$\sigma(\ell, A) := \max_{a \in A} \langle \ell, a \rangle \qquad (\ell \in \mathbb{R}^n),$$

and the supporting face of A is given by

$$M(\ell, A) := \{a \in A : \langle \ell, a \rangle = \sigma(\ell, A)\}.$$

Here we denote by $\langle \ell, a \rangle$ the scalar product of ℓ and a, and by $m(\ell, A)$ a point of the supporting face. $\mathcal{S}_A$ denotes the set of $\ell \in \mathbb{R}^n$ such that the supporting face $M(\ell, A)$ consists of only a single point $m(\ell, A)$. In general, one takes $\mathcal{S}_A$ in the unit sphere $S_{n-1} \subset \mathbb{R}^n$. Now we are able to define the Demyanov difference as in [2].

Definition 2.4.4. *(Baier and Farkhi [2]) Let $A, B \in \mathcal{K}(\mathbb{R}^n)$. We define the Demyanov difference as follows*

$$A \ominus_D B := \operatorname{cl}\operatorname{conv}\{m(\ell, A) - m(\ell, B) : \ell \in \mathcal{S}_{\mathcal{A}} \cap \mathcal{S}_{\mathcal{B}}\}. \tag{2.5}$$

There are several modifications of the Demyanov difference in the literature. In the following definitions, we introduce two approaches proposed by Jahn [39], Dempe and Pilecka [14], which restrict the considered directions to vectors contained in the dual and negative dual cone of the ordering cone. Moreover, for these new set differences, new directional derivatives are introduced (see Section 6.7) to formulate optimality conditions in set optimization w.r.t. the set less order relation. To simplify notation, we use the same symbol $\ominus_D$ for all the Demyanov differences under consideration if no confusion arises.

In the next definition, the modified Demyanov difference is proposed in finite-dimensional spaces.

Definition 2.4.5. *(Dempe and Pilecka [14]) Let C be a proper, convex cone in $\mathbb{R}^n$, and A, B be two nonempty sets in $\mathbb{R}^n$. Then, the modified Demyanov difference is given by:*

$$A \ominus_D B := \operatorname{cl}\operatorname{conv}\{m(\ell, A) - m(\ell, B) : \ell \in \mathcal{S}_{\mathcal{A}} \cap \mathcal{S}_{\mathcal{B}} \cap (\mathcal{C}^+ \cup (-\mathcal{C}^+))\}. \tag{2.6}$$

In [39], Jahn derives new Demyanov differences for two arbitrary sets A, B in normed vector space Y which is partially ordered by a convex cone C. However, it is necessary to assume that the solutions of the following minimization and maximization problems are unique for every $\ell \in C_1^+ := C^+ \cap U_{Y^*}$:

$$\min_{a \in A} \langle \ell, a \rangle, \tag{2.7}$$

$$\max_{a \in A} \langle \ell, a \rangle. \tag{2.8}$$

The solutions of the problems (2.7) and (2.8) are denoted by $y_{\min}(\ell, A)$ and $y_{\max}(\ell, A)$, respectively. Note that if the constrained set A is weakly compact, then there exist solutions to these problems.

Definition 2.4.6. *(Jahn [39]) Let Y be a normed vector space, C be a proper, convex cone in Y, and let two sets $A, B \in Y$ be given so that for every $\ell \in C_1^+$ the solutions $y_{\min}(\ell, A), y_{\min}(\ell, B), y_{\max}(\ell, A)$ and $y_{\max}(\ell, B)$ are unique. Then, the Demyanov difference is given by:*

$$A \ominus_D B := \bigcup_{\ell \in C_1^+} \{y_{\min}(\ell, A) - y_{\min}(\ell, B), y_{\max}(\ell, A) - y_{\max}(\ell, B)\}. \tag{2.9}$$

Definitions 2.4.4-2.4.6 use differences of supporting points. However, there are two important differences among these three definitions. The first is how to restrict the continuous linear functionals. Definitions 2.4.5 and 2.4.6 consider them on the closed unit ball, since this restriction fits the vectorization approach in set optimization. The second is that the closure of the convex hull of difference vectors is not needed in Definition 2.4.6. We refer the reader to [14, 39] for more details and comparisons among the aforementioned definitions of Demyanov differences.

We end this part by defining the metric difference of two nonempty compact subsets of $\mathbb{R}^n$.

Definition 2.4.7. *([2]) Let $A, B \in \mathcal{K}(\mathbb{R}^n)$. We define the metric difference as*

$$A \ominus_M B := \{a - b : \|a - b\|_2 = d(a, B) \quad \text{or} \quad \|b - a\|_2 = d(b, A)\}.$$

In the case that $B = \{b\}$ is a singleton, all differences coincide and are equal to the set $A - b$.

We refer the reader to [2, 39] for more details on some properties and the calculus of the aforementioned set differences as well as comparisons between them.

2.5 Scalarizing functionals and separation theorems

In optimization theory, separation theorems play an important role in deriving the necessary and sufficient conditions for solutions of optimization problems; see Chapter 7. In this section, let Y be a topological vector space, and A, B be given nonempty subsets of Y. We introduce separation theorems for convex sets and separation theorems for two arbitrary sets as well as some nonlinear scalarizing functionals.

2.5.1 Separation theorems for convex sets

In mathematics, the Hahn-Banach Theorem is one of three basic principles in functional analysis. The separation theorem for convex sets, an equivalent form of the Hahn-Banach Theorem, asserts that any two nonempty disjoint convex subsets of Y are separated by a hyperplane. We refer the reader to [10, 65] for more details about the Hahn-Banach Theorem. We will present in this part several main results of the Hahn-Banach Theorem without proofs.

Theorem 2.5.1. ***(First separation theorem for convex sets)*** *Let Y be a normed vector space, and let A, B be nonempty convex subsets of Y such that $A \cap B = \emptyset$. If A is open, then there is a continuous linear functional $y^* \in Y^*$, $y^* \neq 0$ separating A and B, i.e.,*

$$y^*(x) \leq y^*(y) \quad \text{for all} \quad x \in A, y \in B,$$

The next corollary follows directly from the theorem above.

Corollary 2.5.2. *Let Y be a normed vector space, and let A, B be nonempty convex subsets of Y. We assume that* $\operatorname{int} A \neq \emptyset$. *If* $\operatorname{int} A \cap B = \emptyset$ *then there is a continuous linear functional $y^* \in Y^*$, $y^* \neq 0$ such that*

$$y^*(x) \leq y^*(y) \quad \textit{for all} \quad x \in A, y \in B,$$

Theorem 2.5.3. ***(Second separation theorem for convex sets)*** *Let Y be a normed vector space, A, B be convex subsets of Y. Moreover, if A is closed, B is compact, and $A \cap B = \emptyset$, then there is a continuous linear functional $y^* \in Y^* \setminus \{0\}$ strictly separating A and B, i.e.,*

$$y^*(x) < y^*(y) \quad \textit{for all} \quad x \in A, y \in B.$$

Proofs of the Theorems 2.5.1 and 2.5.3 can be found in [10].

2.5.2 Separation theorems for not necessarily convex sets

In the past, "the nonlinear scalarizing functional" or "Gerstewitz scalarizing functional" was widely used in vector optimization, set optimization as well as financial mathematics. It was first used in [25] by Tammer (Gerstewitz) and Weidner to prove separation theorems for nonconvex sets, which are important tools for the proof of optimality conditions. In this section, we will discuss this functional and the separation theorems for nonconvex sets.

Now let A be a given proper closed subset of Y, and $e \in Y \setminus \{0\}$ such that

$$A + [0, +\infty) \cdot e \subseteq A. \tag{2.10}$$

We consider the scalarizing functional $\varphi := \varphi_{A,e} : Y \to \overline{\mathbb{R}}$ defined by

$$\varphi_{A,e}(y) := \inf\{\lambda \in \mathbb{R} \mid \lambda \cdot e \in y + A\}, \tag{2.11}$$

where we use the convention $\inf \emptyset := +\infty$, $\sup \emptyset := -\infty$ and $(+\infty) + (-\infty) := +\infty$.

One main purpose of this dissertation is making use of the scalarization technique to study necessary conditions of vector optimization problems stated in Chapter 7 and necessary conditions of set-valued optimization problems stated in Chapter 8. Based on well-studied properties of the functional $\varphi_{A,e}$, we will scalarize objective functions of optimization problems, hence we can characterize solutions of the optimization problems. The nonlinear scalarizing functional is also used to prove the Lipschitzianity of convex set-valued functions in Chapter 5.

We present some important properties of φ in [19, 26, 68] that will be used in sequel.

Theorem 2.5.4. *([26, 68]) Let Y be a topological vector space, and $A \subset Y$ be a proper, closed set. Let e be a given point in $Y \setminus \{0\}$ such that (2.10) holds, then the following properties hold for $\varphi := \varphi_{A,e}$:*

(a) φ is lower semi-continuous, and $\operatorname{dom}\varphi = \mathbb{R}e - A$.

(b) $\forall y \in Y,\ \forall t \in \mathbb{R} : \varphi(y) \leq t$ if and ony if $y \in te - A$.

(c) $\forall y \in Y,\ \forall t \in \mathbb{R} : \varphi(y + te) = \varphi(y) + t$.

(d) φ is convex if and ony if A is convex; $\varphi(\lambda y) = \lambda\varphi(y)$ for all $\lambda > 0$ and $y \in Y$ if and ony if A is a cone.

(e) φ is proper if and ony if A does not contain lines parallel to e, i.e., $\forall y \in Y, \exists t \in \mathbb{R} : y + te \notin A$.

(f) φ takes finite values if and ony if A does not contain lines parallel to e and $\mathbb{R}e - A = Y$.

The following corollary is immediate.

Corollary 2.5.5. *Let Y be a topological vector space, $A, B, C \subset Y$ be proper sets, and C be closed. If $B - C$ is closed, then for every $e \in \operatorname{int} C$ and $t \in \mathbb{R}$, we have*

$$A \subseteq te + B - C \iff \sup_{a \in A} \varphi_{C-B,e}(a) \leq t.$$

Proof. Let $A \subseteq te + B - C$, this is equivalent to

$$a \in te + B - C, \quad \text{for all} \quad a \in A. \tag{2.12}$$

Because of the closedness of $B - C$, all assumptions of Theorem 2.5.4 (b) are fulfilled. Therefore, (2.12) is equivalent to

$$\sup_{a \in A} \varphi_{C-B,e}(a) \leq t.$$

□

Before stating the next result we recall the D-monotonicity of a functional.

Definition 2.5.6. *Let Y be a topological vector space, and D be a nonempty subset of Y. A functional $\varphi : Y \to \overline{\mathbb{R}}$ is called D-monotone, if*

$$\forall y_1, y_2 \in Y : y_1 \in y_2 - D \Rightarrow \varphi(y_1) \leq \varphi(y_2).$$

Moreover, φ is said to be strictly D-monotone, if

$$\forall y_1, y_2 \in Y : y_1 \in y_2 - D \setminus \{0\} \Rightarrow \varphi(y_1) < \varphi(y_2).$$

The following results provide some monotonicity properties of the scalarizing functional φ. These properties are important for characterizing vector and set-valued optimization problems.

Theorem 2.5.7. *([26]) Under the assumptions of Theorem 2.5.4, and take $\emptyset \neq D \subseteq Y$. Then, the following properties hold:*

(a) $\varphi_{A,e}$ *is D-monotone if and only if $A + D \subseteq A$.*

(b) $\varphi_{A,e}$ *is subadditive if and only if $A + A \subseteq A$.*

We present now a separation theorem for not necessarily convex sets.

Theorem 2.5.8. *([26])* ***Nonconvex Separation Theorem.*** *Let Y be a topological vector space, and let $A, B \subseteq Y$ be nonempty sets such that A is closed, $\operatorname{int} A \neq \emptyset$ and $(-\operatorname{int} A) \cap B = \emptyset$. Take $e \in Y$ and assume that one of the following two conditions holds:*

(i) there exists a cone $D \subseteq Y$ such that $e \in \operatorname{int} D$ and $A + \operatorname{int} D \subseteq A$;

(ii) A is convex, $\mathbb{R}e - A = Y$ and (2.10) is satisfied.

Then, $\varphi_{A,e}$ is a finite-valued, continuous function such that

$$\varphi_{A,e}(y) \geq 0 > \varphi_{A,e}(-x) \quad \textit{for all} \quad x \in \operatorname{int} A, y \in B.$$

Moreover, $\varphi_{A,e}(y) > 0$ for every $y \in \operatorname{int} B$.

Now let Y be a Banach space and $f : Y \to \overline{\mathbb{R}}$ be a proper convex function. Recall that the subdifferential or Fenchel subdifferential of f at $\bar{y} \in \operatorname{dom} f$ is given by

$$\partial f(\bar{y}) = \{y^* \in Y^* \mid \forall y \in Y : f(y) - f(\bar{y}) \geq y^*(y - \bar{y})\}, \tag{2.13}$$

for $\bar{y} \notin \operatorname{dom} f$ one puts $\partial f(\bar{y}) = \emptyset$; see Chapter 6 for more details.

Finally, we consider some calculus for the classical (Fenchel) subdifferential of the nonlinear scalarizing functional $\varphi_{C,e}$ given by (2.11).

Theorem 2.5.9. *([19])Let Y be a Banach space, and let C be a closed, convex cone in Y with a nonempty interior. Take $e \in \operatorname{int} C$. Then, we have*

(a) $\partial \varphi_{C,e}(0) = \{y^* \in C^+ \mid y^*(e) = 1\}$.

(b) $\partial \varphi_{C,e}(y) = \{y^* \in C^+ \mid y^*(e) = 1, y^*(y) = \varphi_{C,e}(y)\}$ *for any $y \in Y$.*

(c) $\varphi_{C,e}$ *is $d(e, \operatorname{bd}(C))^{-1}$-Lipschitz and for every $y \in Y$ and $y^* \in \partial \varphi_{C,e}(y)$ one has $\|e\|^{-1} \leq \|y^*\| \leq d(e, \operatorname{bd}(C))^{-1}$.*

For the detailed proofs of Theorems 2.5.7-2.5.9, see [26, Theorem 2.3.1, Theorem 2.3.6] and [19, Lemma 2.4].

2.5.3 The oriented distance function

In [30, 31], Hiriart-Urruty introduced **"the oriented distance function"** to analyse the geometry of nonsmooth optimization problems. This function is an effective tool for scalarizing vector optimization problems; see also Chapter 7.

In this section, Y is a normed vector space, and A is a proper subset of Y (i.e., $A \neq \emptyset, A \neq Y$).

Definition 2.5.10. *The oriented distance function $\Delta_A : Y \to \overline{\mathbb{R}}$ defined for a nonempty set $A \subsetneq Y$, by*

$$\Delta_A(y) := d(y, A) - d(y, Y \setminus A), \tag{2.14}$$

where $d(\cdot, A) : Y \to \mathbb{R}$ is the distance function w.r.t. A.

We will show several important properties of the oriented distance function in the following proposition.

Proposition 2.5.11. *([73, Proposition 3.2])*

(i) Δ_A is Lipschitzian of rank 1.

(ii) $\Delta_A(y) < 0$ for all $y \in \operatorname{int} A$, $\Delta_A(y) = 0$ for all y in the boundary of A, and $\Delta_A(y) > 0$ for all $y \in \operatorname{int}(Y \setminus A)$.

(iii) If A is convex, then Δ_A is convex, and if A is cone, then Δ_A is positively homogeneous.

(iv) If A is a closed, convex cone, then Δ_A is A-monotone (i.e., $y_1 - y_2 \in A$ implies that $\Delta_A(y_1) \leq \Delta_A(y_2)$). Moreover, if A has a nonempty interior, then Δ_A is strictly $\operatorname{int} A$-monotone (i.e., $y_1 - y_2 \in \operatorname{int} A$ implies that $\Delta_A(y_1) < \Delta_A(y_2)$).

One has by the above proposition that Δ_{-C} is convex, positively homogeneous, C-monotone and 1-Lipschitz for every closed, convex cone C. Moreover, if $\operatorname{int} C = \emptyset$, then $\operatorname{cl}(Y \setminus (-C)) = Y$. Therefore, $d(y, Y \setminus (-C)) = 0$ for all $y \in Y$, hence $\Delta_{-C} = d(\cdot, -C)$.

Note that both Δ_A and $d(\cdot, A)$ are convex functions with a convex set A, so we can take their subdifferentials in the sense of Fenchel. For the convenience of the reader we repeat the calculus of subdifferential of the distance function $d(\cdot, A)$ in the following proposition.

Proposition 2.5.12. *([11, Theorem 1]) Let A be a nonempty, closed, and convex subset of Y. Then, $d(\cdot, A)$ is a convex function on Y with a convex subdifferential*

$$\partial d(y, A) = \begin{cases} S_{Y^*} \cap N(y; A_y) & \text{if } y \notin A \\ U_{Y^*} \cap N(y; A) & \text{if } y \in A, \end{cases}$$

where U_{Y^*}, S_{Y^*} are the closed unit ball and the unit sphere in Y^*, $A_y := A + d(y, A)U_Y$, and $N(\bar{a}; A)$ is the normal cone at a point $\bar{a} \in A$ and be given as

$$N(\bar{a}; A) = \left\{ y^* \in Y^* \middle| \forall a \in A : y^*(a - \bar{a}) \leq 0 \right\}.$$

In particular, if $\operatorname{int} C = \emptyset$, then

$$\partial \Delta_{-C}(0) = \partial d(0, -C) = U_{Y^*} \cap N(0; -C) = U_{Y^*} \cap C^+. \tag{2.15}$$

2.6 Solution concepts for vector-valued optimization problems

In order to formulate solution concepts for vector-valued problems and set-valued problems in next sections, we shall lead off with the well-known notions of (weak) Pareto minimal points. In this section, we consider a topological vector space Y, partially ordered by a proper, pointed, convex, closed cone C.

Definition 2.6.1. *Let A be a nonempty subset of Y.*

(i) We define the set of ***Pareto minimal points*** *of A w.r.t. C by*

$$\operatorname{Min}(A; C) := \{\bar{y} \in A \mid A \cap (\bar{y} - C) = \{\bar{y}\}\}.$$

(ii) The set of ***weakly Pareto minimal points*** *of A w.r.t. C (with $\operatorname{int} C \neq \emptyset$) is given by*

$$\operatorname{WMin}(A; C) := \{\bar{y} \in A \mid A \cap (\bar{y} - \operatorname{int} C) = \emptyset\}.$$

The notions of (weak) minimality for vector optimization problem were first introduced by Edgeworth and Pareto. They play an important role in many fields, for example, in engineering and economics. Moreover, many authors defined other concepts of minimality in the literature, such as *strong minimal point, Properly minimal point*, etc. All these concepts and their relationships have been studied systematically in Ha [27, 28], or Khan, Tammer and Zălinescu [44, Section 2.4].

The following lemma indicates that the set of (weak) Pareto minimal points of a set A is exactly the one of the set $A + C$.

Lemma 2.6.2. *([37, Lemma 4.7 and 4.13]) Let A be a nonempty subset of a partially ordered linear space Y, and let C be a proper, pointed, convex, closed ordering cone in Y. The following assertions hold true:*

(i) $\operatorname{Min}(A + C; C) = \operatorname{Min}(A; C)$.

(ii) If $\operatorname{int} C \neq \emptyset$, then $\operatorname{WMin}(A + C; C) = \operatorname{WMin}(A; C)$.

We consider now the ***vector optimization problem***:

$$\text{minimize} \quad f(x) \quad \text{subject to} \quad x \in D, \tag{VP}$$

where X, Y are two topological vector spaces, $D \subseteq X$ is a feasible set, and C is a proper, closed, convex, pointed cone in Y. The objective function $f : X \to Y$ is a single-valued mapping, (VP) is a problem of ***vector optimization***, and "minimization" is to be understood in the sense of the following definition.

Definition 2.6.3. *Let X, Y be two topological vector spaces, D be a nonempty subset of X. Let $f : X \to Y$ be a single-valued mapping.*

*(i) A point $\bar{x} \in D$ is said to be a **Pareto efficient solution** of the problem (VP) for the single-valued mapping f w.r.t. C if $f(\bar{x}) \in \mathrm{Min}(f(D); C)$.*

*(ii) A point $\bar{x} \in D$ is said to be a **weakly Pareto efficient solution** of the problem (VP) for the single-valued mapping f w.r.t. C if $f(\bar{x}) \in \mathrm{WMin}(f(D); C)$*

Of course, there are several other concepts of minimization of the problem (VP) w.r.t. the notions of minimality for a set mentioned right after Definition 2.6.1. However, in this dissertation we restrict ourselves to the concepts in Definition 2.6.3, and study the necessary optimality conditions for (weak) Pareto efficient solutions of vector-valued optimization problems in Chapter 7.

2.7 Solution concepts for set-valued optimization problems

Let X, Y be two topological vector spaces, $D \subseteq X$ be a feasible set, and let C be a proper, closed, convex, pointed cone in Y. This section will be concerned with ***set-valued optimization problems*** given by:

$$\text{minimize} \quad F(x) \quad \text{subject to} \quad x \in D, \tag{SP}$$

where the objective function $F : X \rightrightarrows Y$ is a set-valued mapping, and "minimization" stands for different solution concepts given in definitions below.

We use the notations

$$F(D) = \bigcup_{x \in D} F(x) \text{ and } \mathrm{dom}\, F = \{x \in D \mid F(x) \neq \emptyset\}.$$

There are three different approaches that have been recently studied in the literature for the formulation of optimality notions for the problem (SP): the *vector approach* [3, 4, 5], the *set approach*, and the *lattice approach* [44]. In this dissertation, we restrict ourselves to two first approaches.

Let $\bar{x}$ be a point in X, and $\bar{y}$ be a fix point in $F(\bar{x})$. Next we define solutions of (SP) based on the **vector approach**, that means we consider whether or not $\bar{y}$ is a Pareto minimal point of the image set of F w.r.t. C.

Definition 2.7.1. *Let X, Y be two topological vector spaces, D be a nonempty subset of X. Let $F : X \rightrightarrows Y$ be a set-valued mapping.*

(i) A pair $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with $\bar{x} \in D$ is said to be a ***minimizer*** *of the problem (SP) for the set-valued mapping F w.r.t. C if $\bar{y} \in \operatorname{Min}(F(D); C)$, i.e.*

$$\big(\{\bar{y}\} - C\big) \cap F(D) = \{\bar{y}\}.$$

(ii) A pair $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with $\bar{x} \in D$ is said to be a ***weak minimizer*** *of the problem (SP) for the set-valued mapping F w.r.t. C if $\bar{y} \in \operatorname{WMin}(F(D); C)$, i.e.*

$$\big(\{\bar{y}\} - \operatorname{int} C\big) \cap F(D) = \emptyset.$$

(iii) A pair $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is said to be a ***local minimizer*** *(or* ***local weak minimizer****) of the problem (SP) for the set-valued mapping F w.r.t. C if there is a neighborhood $U \subset X$ of $\bar{x}$ such that $\bar{y} \in \operatorname{Min}(F(U); C)$ (or $\bar{y} \in \operatorname{WMin}(F(U); C)$, respectively).*

The existence of minimizers and weak minimizers is discussed in Chapter 8, in which set-valued functions are epigraphically Lipschitz-like or C-convex with some additional boundedness conditions.

Clearly, the minimizers of the problem (SP) in the sense of Definition 2.7.1 depend only on certain special elements of $F(\bar{x})$, while other elements of $F(\bar{x})$ are not considered, for this reason one derived other approaches and new notions which are more practical. The **set-approach** bases on the set relations introduced in Definitions 2.3.19 and 2.3.20 in order to define the solutions for the problem (SP). Now we consider set-valued optimization problems w.r.t. a set relation $\preceq_C$:

$$\preceq_C -\text{minimize} \quad F(x) \quad \text{subject to} \quad x \in D, \tag{SP$-\preceq_C$}$$

where we denote $\preceq_C$ by one of the set relations introduced in Definitions 2.3.19 and 2.3.20.

Definition 2.7.2. *([44]) Let X, Y be two topological vector spaces, D be a nonempty subset of X, and $F : X \rightrightarrows Y$ be a set-valued mapping. An element $\bar{x} \in D$ is said to be* ***a minimal solution*** *of the problem (SP$-\preceq_C$) w.r.t. the relation $\preceq_C$ if $F(x) \preceq_C F(\bar{x})$ for some $x \in D$ implies that $F(\bar{x}) \preceq_C F(x)$.*

An element $\bar{x} \in D$ is said to be ***a strictly minimal solution*** *of the problem (SP$-\preceq_C$) w.r.t. the relation $\preceq_C$ if there exists no $x \in D \setminus \{\bar{x}\}$ with $F(x) \preceq_C F(\bar{x})$.*

An element $\bar{x} \in D$ is said to be ***a strongly minimal solution*** *of the problem (SP$-\preceq_C$) w.r.t. the relation $\preceq_C$ if $F(x) \preceq_C F(\bar{x})$, for all $x \in D \setminus \{\bar{x}\}$.*

Chapter 3

Lipschitz continuity of vector-valued and set-valued functions

The Lipschitz continuity is an important and useful tool to study many different problems of mathematics. In the theory of differential equations, the Lipschitz continuity is essential for deriving conditions about the existence and uniqueness of the solution to an initial value problem. In variational analysis, the Lipschitz continuity is also used to get some calculus rules in generalized differentiation; see Section 6.4. Furthermore, one can derive the necessary conditions for solutions of optimization problems when objective functions are Lipschitz (see Chapter 7 and Chapter 8). In this chapter, we introduce some concepts of Lipschitzianity not only for scalar- and vector- valued functions but also for set-valued functions. Almost all notions in this chapter are cited in the monographs of Clarke [13] and Mordukhovich [55]. Some of the notions mentioned in Section 3.2 are based on the definitions of set differences in Section 2.4.

3.1 Lipschitz continuity of vector-valued functions

We consider a normed vector space Y endowed with an order structure which is defined in (2.3) by a proper, pointed, convex cone $C \subset Y$. We adjoin a maximal element $+\infty$ to Y, and get $Y^\bullet := Y \cup \{+\infty\}$; see Section 2.3.2. We consider a function $f : X \to Y^\bullet$ between two normed vector spaces, and denote the domain of f by $\operatorname{dom} f := \{x \in X \mid f(x) \in Y\}$.

Definition 3.1.1. *Consider $f : X \to Y^\bullet$;*

(i) f is said to be ***Lipschitz on*** *$U \subseteq X$ if $U \subseteq \operatorname{dom} f$, and there exists $\ell \geq 0$ such*

that

$$\left\|f(x)-f(x')\right\|_Y \le \ell \left\|x-x'\right\|_X, \quad \textit{for all} \quad x, x' \in U.$$

(ii) *f is said to be **Lipschitz around** x if there is a neighborhood U of x such that f is Lipschitz on U (in particular $x \in \operatorname{int}(\operatorname{dom} f)$).*

(iii) *f is said to be **locally Lipschitz** on a nonempty subset D of X, if f is Lipschitz around every point $x \in D$. Hence $D \subseteq \operatorname{int}(\operatorname{dom} f)$.*

It is well known that in finite-dimensional spaces every scalar convex function is locally Lipschitz on the interior of its domain. We will extend this assertion in Section 4.2 for general cases with vector-valued functions in infinite-dimensional spaces.

We recall the notion of *strict Lipschitzianity* in order to show a relationship between the coderivative of a vector-valued function and the subdifferential of its scalarization in Section 6.4. This property is also used to derive necessary conditions for solutions of vector optimization problems in Chapter 7.

Definition 3.1.2. *([55, Definition 3.25]) Consider a function $f : X \to Y$ and a point $\bar{x} \in X$; f is called to be **strictly Lipschitzian** at $\bar{x}$ if f is Lipschitz around $\bar{x}$ and the sequence*

$$y_k := \frac{f(x_k + t_k v) - f(x_k)}{t_k}, \quad k \in \mathbb{N},$$

contains a norm convergent subsequence whenever $v \in X$, $x_k \to \bar{x}$ and $t_k \downarrow 0$.

If Y is a finite-dimensional space, the strictly Lipschitzianity above reduces to the class of locally Lipschitz functions $f : X \to \mathbb{R}^n$. This claim does not hold for the case $\dim Y = +\infty$. For examples and more details on properties of strictly Lipschitzian functions see [55].

We introduce now the concept of equi-Lipschitzianity of a family of functions that are used in Chapter 5 to study the Lipschitzianity of convex set-valued functions.

Definition 3.1.3. *Let $\{f_\alpha\}_{\alpha \in I}$ be a family of functions $f_\alpha : X \to \overline{\mathbb{R}}$, where I is a nonempty index set. We say that the family $\{f_\alpha\}_{\alpha \in I}$ is **equi-Lipschitz** around $x_0 \in X$ if there are a neighborhood U of x_0 and a real number $L > 0$ such that for every $\alpha \in I$, f_α is finite and Lipschitz on U with the same Lipschitz constant L, i.e.,*

$$|f_\alpha(x) - f_\alpha(y)| \le L\|x-y\|_X, \quad \textit{for all} \quad x, y \in U, \alpha \in I.$$

3.2 Lipschitz continuity of set-valued functions

In this section, we will introduce several types of Lipschitz properties for set-valued functions generated by a given proper, convex cone C. Moreover, we derive new Lipschitz continuities w.r.t. set differences given in Section 2.4.

Let us begin with the multi-valued Lipschitz behavior called ***Lipschitz-like*** (also known as the ***Aubin property***, or the ***pseudo-Lipschitzian property***) following the book by Mordukhovich [55, Section 1.2.2]. Let $F : X \rightrightarrows Y$ be a set-valued mapping between normed vector spaces. The ***domain*** of F is $\operatorname{dom} F := \{x \in X \mid F(x) \neq \emptyset\}$. We define the ***graph*** of the set-valued mapping F by

$$\operatorname{gph} F := \big\{(x, y) \in X \times Y \mid y \in F(x)\big\}.$$

Definition 3.2.1. ***(Lipschitz properties of set-valued mappings).*** *Let $F : X \rightrightarrows Y$ with $\operatorname{dom} F \neq \emptyset$.*

*(i) Given nonempty sets $U \subseteq X$ and $V \subseteq Y$, we say that F is **Lipschitz-like** on U relative to V if there is $\ell \geq 0$ such that*

$$\forall x, x' \in U : \qquad F(x) \cap V \subseteq F(x') + \ell \left\|x - x'\right\|_X U_Y. \tag{3.1}$$

Hence, if $F(U) \cap V \neq \emptyset$, then $U \subseteq \operatorname{dom} F$.

*(ii) Given $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, we say that F is **Lipschitz-like** around $(\bar{x}, \bar{y})$ with modulus $\ell \geq 0$ if there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that (3.1) holds, hence necessarily $\bar{x} \in \operatorname{int}(\operatorname{dom} F)$. The infimum of all such moduli ℓ is called the exact Lipschitz bound of F around $(\bar{x}, \bar{y})$ and is denoted by $\operatorname{lip} F(\bar{x}, \bar{y})$.*

*(iii) F is **Lipschitz continuous** on U if (3.1) holds with $V = Y$; the infimum of $\ell \geq 0$ for which (3.1) holds with $V = Y$ is denoted by $\operatorname{lip}_U F(\bar{x})$. Furthermore, F is Lipschitz around $\bar{x}$ if there is a neighborhood U of x such that F is Lipschitz continuous on U.*

Remark 3.2.2. *(i) It follows immediately from Definition 3.2.1 that the Lipschitz-like property of F around $(\bar{x}, \bar{y})$ implies the Lipschitz-like property of F around $(x, y) \in \operatorname{gph} F$ which is close enough to $(\bar{x}, \bar{y})$.*

(ii) If F is Lipschitz-like on U, one has $U \cap \operatorname{dom} F = \emptyset$ or $U \subseteq \operatorname{dom} F$.

For a vector-valued function $f : X \to Y^\bullet$ we associate the set-valued function $F : X \rightrightarrows Y$ given by

$$F(x) := \begin{cases} \{f(x)\} & \text{if } x \in \operatorname{dom} f, \\ \emptyset & \text{otherwise;} \end{cases} \tag{3.2}$$

hence $\operatorname{dom} F = \operatorname{dom} f$; we say that F is at most single-valued. Inversely, for each at most single-valued function $F : X \rightrightarrows Y$, we associate the corresponding vector-valued function $f : X \to Y^\bullet$ given by

$$f(x) := \begin{cases} y & \text{if } x \in \operatorname{dom} F \text{ and } F(x) = \{y\}, \\ +\infty & \text{if } x \notin \operatorname{dom} F. \end{cases} \tag{3.3}$$

Remark 3.2.3. *Of course, if $F : X \rightrightarrows Y$ is at most single-valued, then:*

(i) if $V = Y$, Definition 3.2.1(i) reduces to Definition 3.1.1(i) for the corresponding vector-valued mapping f defined in (3.3),

(ii) if $\bar{x} \in \operatorname{dom} F$, $F(\bar{x}) = \{\bar{y}\}$, and $V = Y$ Definition 3.2.1(ii) reduces to Definition 3.1.1(ii).

We recall that the ***epigraph*** of F w.r.t. the cone C is given by

$$\operatorname{epi} F := \big\{(x, y) \in X \times Y \mid y \in F(x) + C\big\}. \tag{3.4}$$

The ***epigraphical multifunction*** of $F : X \rightrightarrows Y$, $\mathcal{E}_F : X \rightrightarrows Y$, is defined by

$$\mathcal{E}_F(x) := F(x) + C. \tag{3.5}$$

From (3.4) and (3.5), it follows that $\operatorname{gph} \mathcal{E}_F = \operatorname{epi} F$.

Definition 3.2.4. *A set-valued mapping $F : X \rightrightarrows Y$ is **epigraphically Lipschitz-like (ELL)** around a given point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that*

$$\forall x, x' \in U : \qquad \mathcal{E}_F(x) \cap V \subseteq \mathcal{E}_F(x') + \ell \left\|x - x'\right\|_X U_Y. \tag{3.6}$$

In other words, F is ELL at $(\bar{x}, \bar{y})$ if its epigraphical multifunction $\mathcal{E}_F$ is Lipschitz-like around that point.

Now, we introduce concepts of ***upper (lower) C-Lipschitzianity*** w.r.t. the proper, convex cone C. They are used in Section 5.2 to show the C-Lipschitzianity of C-convex set-valued functions.

Definition 3.2.5. *Let $F : X \rightrightarrows Y$ be a set-valued mapping with $\operatorname{dom} F \neq \emptyset$, and $C \subset Y$ be a proper, convex cone.*

*(i) F is said to be **upper (lower) C-Lipschitz around** $x_0 \in X$ if there is a neighborhood U of x_0, and a constant $\ell \geq 0$ such that the following inclusions hold for all $x, x' \in U$*

$$F(x') \subseteq F(x) + \ell \|x' - x\|_X U_Y - C, \tag{3.7}$$

$$\big(F(x') \subseteq F(x) + \ell \|x' - x\|_X U_Y + C, \textit{respectively}\big). \tag{3.8}$$

*(ii) F is said to be **locally upper (lower) C-Lipschitz** on $D \subseteq X$ if it is upper (lower) C-Lipschitz around any point of D.*

Remark 3.2.6. *(i) In Definition 3.2.5(i), if F is upper (lower) C-Lipschitz around x_0, then $x_0 \notin \operatorname{cl}(\operatorname{dom} F)$ or $x_0 \in \operatorname{int}(\operatorname{dom} F)$. Moreover if $U \cap \operatorname{dom} F \neq \emptyset$, then $U \subseteq \operatorname{dom} F$, and thus $x_0 \in \operatorname{int}(\operatorname{dom} F)$.*

(ii) Obviously, if F is lower C-Lipschitz around $\bar{x} \in \operatorname{dom} F$, from (3.8), we have that $\mathcal{E}_F$ is Lipschitz-like continuous around $(\bar{x}, \bar{y})$, for all $\bar{y} \in F(\bar{x})$, so F is (ELL) around $(\bar{x}, \bar{y})$, for all $\bar{y} \in F(\bar{x})$.

(iii) Note that the concepts of C-Lipschitzianity in Definition 3.2.5 are more general than the ones in the sense of Kuwano and Tanaka [50] in Definition 5.3.1, as they fix $x := x_0$ in the right-hand side of the inclusions (3.7) and (3.8).

We now introduce several new Lipschitz continuities of set-valued maps w.r.t. the given set differences presented in Section 2.4. These Lipschitz continuities were investigated for both finite-dimensional spaces in [2, 64], and infinite-dimensional spaces in [39]. In [2], Baier and Farkhi give a good survey on Lipschitz continuities of set-valued maps. Furthermore, they also study the relationships between Lipschitz continuities and existence of selections of set-valued maps. This matter has been attracting the attention of researchers for a long time.

We now define the Lipschitz continuities of set-valued maps w.r.t. the algebraic difference $\ominus_A$ and the geometric difference $\ominus_G$ given in Definition 2.4.1. Since these differences are defined in general vector spaces, we can define Lipschitz continuities in normed vector spaces without any special conditions for set-valued maps.

In the following definition, we use the notion ***Δ-Lipschitz*** standing for ***A-Lipschitz*** (algebraic Lipschitz), and ***G-Lipschitz*** (geometric Lipschitz).

Definition 3.2.7. *([2],[60]) Let X, Y be two normed vector spaces, and $F : X \rightrightarrows Y$ be a set-valued function. F is called* ***Δ-Lipschitz*** *on X w.r.t. the set difference $\ominus_\Delta$ (where $\Delta \in \{A, G\}$) with a constant $L \geq 0$ if*

$$F(x) \ominus_\Delta F(y) \subseteq L\|x - y\|_X U_Y \quad \text{for all} \quad x, y \in X.$$

Note that since the algebraic difference and the geometric difference coincide for singleton, the Lipschitz properties w.r.t. these differences coincide for single-valued map F. Therefore, we get the following proposition.

Proposition 3.2.8. *([2]) Let X, Y be two normed vector spaces, $f : X \to Y$ be a vector-valued function and $F : X \rightrightarrows Y$ be a set-valued function such that $F(x) = \{f(x)\}$. Then, the A-Lipschitzianity and the G-Lipschitzianity for F coincide with classical Lipschitzianity for f.*

For other Lipschitz continuity concepts concerning the partial ordering relation, we derive the upper (lower) G-Lipschitz concepts around $x_0 \in X$. We will use these concepts to study the Lipschitz continuity of extended convex set-valued functions in Section 5.4.

Definition 3.2.9. *Let X, Y be two normed vector spaces, $F : X \rightrightarrows Y$ be a set-valued mapping with $\operatorname{dom} F \neq \emptyset$, and $C \subset Y$ be a proper, convex cone. F is said to be **upper G-Lipschitz (or lower G- Lipschitz) around** $x_0 \in X$ if there is a neighborhood U of x_0, and a constant $\ell \geq 0$ such that*

$$F(x) \ominus_G F(x') \subseteq \ell ||x - x'||_X U_Y + C, \quad \textit{for all} \quad x, x' \in U, \tag{3.9}$$

$$(F(x) \ominus_G F(x') \subseteq \ell ||x - x'||_X U_Y - C, \quad \textit{for all} \quad x, x' \in U, \textit{respectively}). \tag{3.10}$$

Obviously, when C is a normal cone, F is G-Lipschitz if and only if it is upper G-Lipschitz and lower G-Lipschitz.

To this end, we define the Lipschitzianity w.r.t. the metric difference $\ominus_M$, and the D-Lipschitzianity (or Demyanov Lipschitzianity) w.r.t. the Demyanov difference (2.5). We need to restrict the image of F on the set of nonempty compact subsets of $\mathbb{R}^n$ denoted by $\mathcal{K}(\mathbb{R}^n)$ as well as on the set of nonempty convex compact subsets of $\mathbb{R}^n$ denoted by $\mathcal{C}(\mathbb{R}^n)$.

Definition 3.2.10. *([2]) Let X be a vector space, and $F : X \rightrightarrows \mathcal{K}(\mathbb{R}^n)$ (or $F : X \rightrightarrows \mathcal{C}(\mathbb{R}^n)$) be a set-valued function. F is called **Lipschitz** (or D-**Lipschitz**) on X with respect to the metric difference $\ominus_M$ (the Demyanov difference $\ominus_D$, respectively) with a constant $L \geq 0$ if*

$$F(x) \ominus_M F(y) \subseteq L\|x - y\|_X U_{\mathbb{R}^n} \quad \textit{for all} \quad x, y \in X,$$

$$(F(x) \ominus_D F(y) \subseteq L\|x - y\|_X U_{\mathbb{R}^n} \quad \textit{for all} \quad x, y \in X, \textit{respectively}).$$

The following proposition presents the hierarchy of the Lipschitz notions above.

Proposition 3.2.11. *([2]) Let X be a vector space, and $F : X \rightrightarrows \mathbb{R}^n$ be a set-valued function with image in $\mathcal{K}(\mathbb{R}^n)$. Then, we have*

$$F \textit{ is } D\textit{-Lipschitz} \Longrightarrow F \textit{ is Lipschitz} \Longrightarrow F \textit{ is } G\textit{-Lipschitz.}$$

Note that we also can use the formula in Definition 3.2.10 to define the Demyanov Lipschitzianity w.r.t. the Jahn's Demyanov difference (2.9), and $F : X \rightrightarrows Y$ is a set-valued function between two normed vector spaces.

Chapter 4

Lipschitz continuity of cone-convex vector-valued functions

As indicated in Chapter 2, the Lipschitzianity of convex scalar functions lead us to studying the Lipschitzianity for convex vector functions. The present chapter is devoted to study systematically the Lipschitz properties of convex functions in the literature, and also refers to the techniques used to prove them. We not only extend some results in the literature, but also derive some new proofs. This chapter is organized as follows: Section 4.1 is concerned with concepts of C-convex functions that are well known in vector optimization. Section 4.2 is one of the main parts of this dissertation. In Theorem 4.2.7, we derive different proofs for an assertion of Borwein [9] concerning the Lipschitzianity of a convex vector-valued function when the ordered cone C is normal. However, Theorem 4.2.7 is slightly stronger than Borwein's result, because our boundedness condition is weaker. Moreover, in the first proof of Theorem 4.2.7, we can obtain an accurate Lipschitz constant. In order to derive the second proof of Theorem 4.2.7, we start with Luc, Tan and Tinh's result ([52, Theorem 3.1]) for convex-vector functions in finite-dimensional spaces. We use their techniques to extend their result to infinite-dimensional spaces and get the second proof of Theorem 4.2.7. Initially, the proof of Theorem 4.2.7 was for a w-normal cone C, and then we realized that the normal cone and w-normal cone are equivalent in normed vector spaces; see Proposition 2.3.15. This assertion is known from the book by Schaefer [66] but he did not prove it. This chapter is based on the results in the paper [70] by Tuan, Tammer and Zălinescu.

4.1 Cone-convex vector-valued functions

Let X and Y be normed vector spaces, and let $C \subset Y$ be a proper, convex cone. We consider a function f from the normed vector space X to the extended space $Y^\bullet$ ($Y^\bullet = Y \cup \{+\infty_C\}$), and denote the domain of f by $\operatorname{dom} f := \{x \in X \mid f(x) \in Y\}$. In the sequel, we always assume that $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$.

Definition 4.1.1. *Let $f : X \to Y^\bullet$, and $C \subset Y$ be a proper, convex cone. The function f is said to be C-**convex** if for all $x, x' \in X, \lambda \in (0,1)$, one has*

$$\lambda f(x) + (1-\lambda) f(x') \geq_C f(\lambda x + (1-\lambda)x').$$

In the case $Y = \mathbb{R}$ and $C = \mathbb{R}_+ := \{\alpha \in \mathbb{R} \mid \alpha \geq 0\}$, Definition 4.1.1 reduces to the classical definition of convexity for functionals; see [32, 74]. Obviously, the convexity of f implies that $\operatorname{dom} f$ is convex.

In the definition above, the convex function f is defined on the whole of X, and f takes its values in the extended space $Y^\bullet$. However, in several books (see, for example, [32]), a convex function must be defined on a convex subset of X, and its values must be finite. To compare these two definitions for scalar functions, we refer to [32]. In the following definition, we also define a convex vector-valued function f on a convex set of X, and f takes finite values.

Definition 4.1.2. *Let $C \subset Y$ be a proper, convex cone, and A be a nonempty convex set in X. A function $f : A \to Y$ is said to be C-**convex** on A if for all $x, x' \in A, \lambda \in (0,1)$, one has*

$$\lambda f(x) + (1-\lambda) f(x') \geq_C f(\lambda x + (1-\lambda)x').$$

Remark 4.1.3. *If we extend the C-convex function $f : A \to Y$ from Definition 4.1.2 by*

$$\tilde{f}(x) := \begin{cases} f(x) & \text{for} \quad x \in A, \\ +\infty & \text{for} \quad x \notin A, \end{cases}$$

we then obtain a new function $\tilde{f} : X \to Y^\bullet$. It is easy to verify that $\tilde{f}$ is C-convex in the sense of Definition 4.1.1.

Definition 4.1.4. *Consider $f : X \to Y^\bullet$ with $\operatorname{dom} f \neq \emptyset$, and let $C \subset Y$ be a proper, convex cone. The **epigraph** of f w.r.t. the cone C is given by*

$$\operatorname{epi} f := \big\{(x,y) \in X \times Y \mid y \in f(x) + C\big\}. \tag{4.1}$$

The following proposition states that the convexity of epigraphs can be taken as one different definition of convex functions.

Proposition 4.1.5. *Consider $f : X \to Y^\bullet$ with* $\operatorname{dom} f \neq \emptyset$, *and let $C \subset Y$ be a proper, convex cone. The following properties are equivalent:*

(i) f is C-convex,

(ii) its epigraph is a convex set in $X \times Y$.

Proof. **(i)⇒ (ii)** Take $(x_1; y_1), (x_2; y_2) \in \operatorname{epi} f$, we have

$$\begin{cases} y_1 \in f(x_1) + C, \\ y_2 \in f(x_2) + C. \end{cases}$$

Then,

$$\lambda y_1 + (1-\lambda) y_2 \in \lambda f(x_1) + (1-\lambda) f(x_2) + C, \quad \text{for all} \quad \lambda \in (0,1).$$

Since f is C-convex, we get

$$\lambda y_1 + (1-\lambda) y_2 \in f(\lambda x_1 + (1-\lambda) x_2) + C,$$

for all $\lambda \in (0,1)$, so $\operatorname{epi} f$ is convex.

(ii)⇒ (i) As $(x_1, f(x_1)), (x_2, f(x_2)) \in \operatorname{epi} f$, and $\operatorname{epi} f$ is convex,

$$\lambda(x_1, f(x_1)) + (1-\lambda)(x_2, f(x_2)) \in \operatorname{epi} f,$$

for all $\lambda \in (0,1)$. Thus f is C-convex. □

4.2 Lipschitz continuity of cone-convex vector-valued functions

As shown in section 2.2, a proper convex functional is locally Lipschitz. In this section, we prove that this result also holds true for a cone-convex function $f : X \to Y^\bullet$. Luc, Tan and Tinh [52] proved the Lipschitz property of f for the case that X, Y are finite-dimensional spaces. We recall the proof of [52, Theorem 3.1], since it motivates us to investigate cone-convex functions in infinite-dimensional spaces.

Lemma 4.2.1. *([52, Theorem 3.1]) Let $C \subset \mathbb{R}^m$ be a proper, convex cone. Assume that* $\operatorname{cl} C$ *is pointed, D is a proper, open, convex subset of $\mathbb{R}^n$, and $f : D \to \mathbb{R}^m$ is a C-convex vector function. Then, f is locally Lipschitz on D (in the sense of Definition 3.1.1).*

Proof. Taking into account the assumptions of cone C and the properties of cones in Table 2.1, we see that the cone $K := \operatorname{cl} C$ has a convex compact base and $\operatorname{int} K^+ \neq \emptyset$. Therefore, there are m linearly independent vectors $y_1^*, y_2^*, \dots, y_m^* \in K^+$. It is obvious that f is convex w.r.t. the cone K, and $y_i^* \circ f$ is a scalar convex function, for every $i = 1, 2, \dots, m$. Applying Lemma 2.2.9 $y_i^* \circ f$ is locally Lipschitz on D, for every $i = 1, 2, \dots, m$. Take $y^* \in L(\mathbb{R}^m, \mathbb{R})$ arbitrarily, since $(y_i^*,\, i = 1, 2, \dots, m)$ is a base of $L(\mathbb{R}^m, \mathbb{R})$, there are $\alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{R}$ such that $y^* = \sum_{i=1}^{m} \alpha_i y_i^*$. Hence $y^* \circ f$ is also locally Lipschitz on D for every $y^* \in L(\mathbb{R}^m, \mathbb{R})$, and it follows that f is locally Lipschitz on D. □

We will utilize this technique to prove the Lipschitzianity of convex functions between two infinite-dimensional spaces X, Y. Thus, we need to deal with two questions. The first one is whether each vector $y^* \in L(X, Y)$ can be represented through some vectors in $\operatorname{int} C^+$. The second one is whether or not f is locally Lipschitz on D when $y^* \circ f$ is also locally Lipschitz on D for every $y^* \in L(X, Y)$. To answer the first question, we assume that C is a normal cone, and then we can use all properties of normal cones in Section 2.3.2 to present y^* through some vectors in $\operatorname{int} C^+$. We address the second question in the following proposition.

Proposition 4.2.2. *Let X, Y be two normed vector spaces, and let $f : X \to Y^\bullet$ be a mapping. If $y^* \circ f$ is Lipschitz around $x \in \operatorname{dom} f$ for every linear function $y^* \in Y^*$, then f is also Lipschitz around x.*

Proof. Clearly, $x \in \operatorname{int}(\operatorname{dom} f)$. We suppose that f is not Lipschitz around x. Then, there exists $n_0 \in \mathbb{N}^*$ such that f is not Lipschitz on $B(x, \frac{1}{n})$ for every $n \geq n_0$. From Definition 3.1.1, there exist $x_n, x_n' \in B(x, \frac{1}{n})$ such that

$$||f(x_n) - f(x_n')||_Y > n \left\| x_n - x_n' \right\|_X .$$

Because of $x_n, x_n' \in B(x, \frac{1}{n})$, both of the sequences $\{x_n\}, \{x_n'\}$ converge to x. Setting $z_n := \frac{f(x_n) - f(x_n')}{\|x_n - x_n'\|_X} \in Y$, we have $||z_n||_Y \geq n$ for all $n \geq n_0$.

For every $y^* \in Y^*$, $y^* \circ f$ is Lipschitz around x. This means that there exists $\theta = \theta_{y^*} > 0$ such that $y^* \circ f$ is Lipschitz on $B(x, \theta)$. Hence there exists $L_{y^*} > 0$ such that

$$|y^* \circ f(x) - y^* \circ f(x')| \leq L_{y^*} \left\| x - x' \right\|_X , \quad \text{for all} \quad x, x' \in B(x, \theta).$$

Since $x_n, x_n' \to x$, there exists $n_{y^*} \geq n_0$ such that $x_n, x_n' \in B(x, \theta)$ for every $n \geq n_{y^*}$, and so $|y^* \circ f(x_n) - y^* \circ f(x_n')| \leq L_{y^*} \|x_n - x_n'\|_X$, hence $|y^*(z_n)| \leq L_{y^*}$, for every $n \geq n_{y^*}$. It follows that there exists L_{y^*}', such that

$$|y^*(z_n)| \leq L_{y^*}', \quad \text{for all} \quad y^* \in Y^*, n \geq n_0.$$

Therefore, all the assumptions for the normed space Y mentioned in [65, Corollary 3.18] are fulfilled, and then we have that $\{\|z_n\|_Y \mid n \geq n_0\}$ is bounded. This contradicts the fact that $\|z_n\|_Y \geq n$ for all $n \geq n_0$. □

In the following definition we present C-boundedness notions of a mapping $f : X \to Y^\bullet$, where $C \subset Y$ is a proper, convex cone.

Definition 4.2.3. *Consider $f : X \to Y^\bullet$;*

*(i) f is said to be C-**bounded from above** (resp. **below**) on a subset A of X if there exists a constant $\mu > 0$ such that*

$$f(A) \subseteq \mu U_Y - C \quad (\textit{resp. } f(A) \subseteq \mu U_Y + C).$$

*(ii) f is said to be C-**bounded** on a subset A of X if it is C-bounded from above and C-bounded from below on A.*

Remark 4.2.4. *If we assume that $f : X \to Y^\bullet$ is topologically bounded on a neighborhood U of $x_0 \in \operatorname{dom} f$, i.e., there is a positive real μ such that $f(U) \subseteq \mu U_Y$, it is obvious that f is C-bounded on U. Conversely, if X, Y are normed spaces, and C is normal, any C-bounded function around x_0 is topologically bounded around this point. Indeed, from Definition 4.2.3, there exist a neighborhood U of x_0 and a constant $\mu > 0$ such that*

$$f(U) \subseteq \mu U_Y + C \text{ and } f(U) \subseteq \mu U_Y - C,$$

that is

$$f(U) \subseteq [\mu U_Y]_C = \mu [U_Y]_C.$$

Since C is normal, we can take $\mu' > 0$ such that $[U_Y]_C \subseteq \mu' U_Y$, and so

$$f(U) \subseteq [\mu U_Y]_C \subseteq \mu\mu' U_Y;$$

hence f is topologically bounded around x_0.

The following result for vector-valued functions is similar to Lemma 2.2.10 in the case $Y = \mathbb{R}$.

Proposition 4.2.5. *Let X, Y be two normed vector spaces, $C \subset Y$ be a proper, convex cone, and let $f : X \to Y^\bullet$ be C-convex. If f is C-bounded from above on a neighborhood of $x_0 \in \operatorname{int}(\operatorname{dom} f)$ then for every $x \in \operatorname{int}(\operatorname{dom} f)$, f is C-bounded on a neighborhood of x.*

Proof. As $x_0 \in \operatorname{int}(\operatorname{dom} f)$, we take $\theta, \mu_0 > 0$ such that $U := x_0 + \theta U_X \subseteq \operatorname{dom} f$ and $f(U) \subseteq \mu_0 U_Y - C$. Fix some $x \in \operatorname{int}(\operatorname{dom} f)$. Then, there exist $x' \in \operatorname{dom} f$ and $\lambda \in (0, 1)$ such that $x = (1-\lambda)x' + \lambda x_0$. Then for $u \in U_X$, we have that $x + \lambda\theta u = (1-\lambda)x' + \lambda(x_0 + \theta u)$, and so $f(x + \lambda\theta u) \in (1-\lambda)f(x') + \lambda f(x_0 + \theta u) - C \subseteq B_0 - C$, where $B_0 := (1-\lambda)f(x') + \lambda\mu_0 U_Y$. Therefore, f is C-bounded from above on a neighborhood of x. This implies that there exist a constant $\mu > 0$ and a neighborhood $U = B(x, r)$

of x such that $f(U) \subseteq \mu U_Y - C$, so $-f(U) \subseteq \mu U_Y + C$. It is sufficient to prove that f is C-bounded from below on a neighborhood of x.

For every x' in U, we can take $x'' = 2x - x' \in U$, and so that $x = \frac{1}{2}x' + \frac{1}{2}x''$ and $f(x) \in \frac{1}{2}f(x') + \frac{1}{2}f(x'') - C$. Hence $f(x') \in 2f(x) - f(x'') + C \subseteq 2f(x) + \mu U_Y + C$. This completes the proof. □

Now we will show that if the cone C satisfies certain properties related to the topology and order, then all locally C-bounded, C-convex vector functions will be locally Lipschitz in infinite-dimensional spaces. The following result is first proposed and proven by Borwein [9].

Theorem 4.2.6. ([9, Corollary 2.4]) *Let X, Y be normed spaces, C be a normal cone in Y, and $f : X \to Y^\bullet$ be C-convex. If there exist a neighborhood U of $x_0 \in X$ and $y_0 \in Y$ such that $f(x) \leq_C y_0$ for all $x \in U$, then f is Lipschitz around x_0.*

By using a weaker assumption of the boundedness condition, we obtain the following result, which is slightly stronger than Theorem 4.2.6. In addition we obtain a more accurate Lipschitz constant.

Theorem 4.2.7. ([70, Theorem 2]) *Let X, Y be two normed vector spaces, $C \subset Y$ be a normal convex cone, and let $f : X \to Y^\bullet$ be C-convex. Suppose that f is C-bounded from above on a neighborhood of $x_0 \in \operatorname{int}(\operatorname{dom} f)$. Then, f is Lipschitz around x_0. Moreover, f is locally Lipschitz on $\operatorname{int}(\operatorname{dom} f)$.*

Proof.

First proof. Without loss of generality we suppose that $x_0 = 0$ and $f(0) = 0$. there exist $\theta, \mu > 0$ such that $f(U) \subseteq \mu U_Y - C$, where $U := \theta U_X$.

Let x be arbitrary in U; then $f(x) = \mu y - c$ with $\|y\|_Y \leq 1$ (as $y \in U_Y$) and $c \in C$. Take $y^* \in C_1^+ = U_{Y^*} \cap C^+$, that is $\|y^*\|_* \leq 1$ and $y^* \in C^+$. We obtain that

$$y^*(f(x)) = y^*(\mu y - c) = \mu y^*(y) - y^*(c) \leq \mu y^*(y) \leq \mu \|y^*\|_* \|y\|_Y \leq \mu.$$

So $y^*(f(x)) \leq \mu$ for all $y^* \in C_1^+$, $x \in U$ (μ does not depend on y^*).

Since $y^* \circ f$ is proper and convex, for $\theta' \in (0, \theta)$ and $y^* \in C_1^+$, applying Lemma 2.2.12, we get

$$\left|y^*(f(x) - f(x'))\right| \leq L' \left\|x - x'\right\|_X, \quad \text{for all} \quad x, x' \in \theta' U_X, \tag{4.2}$$

where $L' := \mu(\theta + \theta')/[\theta(\theta - \theta')]$. Let us take $\rho > 0$ provided by Lemma 2.3.16. For $y^* \in U_{Y^*}$, we find $y_1^*, y_2^* \in C_1^+$ such that $\rho^{-1} y^* = y_1^* - y_2^*$. From (4.2), we get

$$\left|y_1^*(f(x) - f(x')) - y_2^*(f(x) - f(x'))\right| \leq 2L' \|x - x'\|_X,$$

for all $x, x' \in \theta' U_X$, hence

$$\|f(x) - f(x')\|_Y = \sup_{y^* \in U_{Y^*}} \left| y^*(f(x) - f(x')) \right| \leq 2\rho L' \|x - x'\|_X,$$

for all $x, x' \in \theta' U_X$. This shows that f is Lipschitz on $\theta' U_X$ with the Lipschitz constant $L = 2\rho\mu(\theta + \theta')/[\theta(\theta - \theta')]$. The remaining assertion is deduced from Proposition 4.2.5.

Second proof. Set $f_0 := f|_{\mathrm{int}(\mathrm{dom}\, f)} : \mathrm{int}(\mathrm{dom}\, f) \to Y$, then f_0 also has C-convexity and C-boundedness properties like f.

Since f_0 is C-bounded on a neighborhood U of x_0, $f_0(U) \subseteq \mu U_Y - C$ for some $\mu > 0$; hence for every $x \in U$, there exist $y \in U_Y, c \in C$ such that $f_0(x) = \mu y - c$. For $z^* \in C^+$, we have that $z^*(\mu y') \leq \mu \|z^*\| \cdot \|y'\| \leq \mu \|z^*\| =: \mu'$ for all $y' \in U_Y$. It follows that

$$(z^* \circ f_0)(x) \leq z^*(\mu y) \leq \mu', \quad \text{for all} \quad x \in U.$$

It follows that for every $z^* \in C^+$, $z^* \circ f_0$ is bounded from above on a neighborhood of x_0. By the C-convexity of the function f_0, and according to [51, Proposition 1.6.2], $z^* \circ f_0$ is a scalar convex function. Hence, from [62, Theorem B], $z^* \circ f_0$ is Lipschitz around x_0.

Let $y^* \in Y^*$. Since C is normal, by Proposition 2.3.15(i), we have $Y^* = C^+ - C^+$; hence $y^* = y_1^* - y_2^*$ for some $y_1^*, y_2^* \in C^+$. Since $y_1^*, y_2^* \in C^+$, $y_1^* \circ f_0$ and $y_2^* \circ f_0$ are Lipschitz around x_0, and so $y^* \circ f_0 = y_1^* \circ f_0 - y_2^* \circ f_0$ is also Lipschitz around x_0. Since $y^* \in Y^*$ is arbitrary, applying Proposition 4.2.2, f_0 is Lipschitz around x_0, which completes the proof of the first assertion in Theorem 4.2.7. The second assertion is deduced from Proposition 4.2.5. □

Now we prove that in the case that X, Y are finite-dimensional spaces, we can omit C-boundedness from above in Theorem 4.2.7.

Proposition 4.2.8. *Let C be a cone in R^n, a C-convex function $f : R^m \to R^n$ is locally C-bounded from above; that is, it is C-bounded from above on a neighborhood of each point $x_0 \in R^m$.*

Proof. Let $x_0 \in \mathbb{R}^m$. Take $e_1, e_2, \ldots, e_m$ be m unit vectors in $\mathbb{R}^m$. Set $v_i = x_0 + e_i$ for $i = 1, \ldots m$. We take a cube $U = \mathrm{conv}\, \{v_1, v_2, \ldots, v_m\}$, so for any $x \in U$ we can find scalars λ_i, $i = 1, \ldots, l$ satisfying

$$x = \sum_1^l \lambda_i v_i, \qquad \lambda_i \geq 0, \qquad \sum_1^l \lambda_i = 1.$$

Applying Jensen's inequality to convex functions (see [32, Theorem 1.1.8]), we get

$$\sum_1^l \lambda_i f(v_i) \geq_C f(x), \quad \text{and it follows that} \quad f(x) \in \sum_1^l \lambda_i f(v_i) - C,$$

so f is C-bounded from above on U. □

Remark 4.2.9. *Observe that the convex cone $C \subset \mathbb{R}^n$ is normal if and only if $\operatorname{cl} C$ is pointed; see also [26, Corollary 2.2.11]. Taking into account 4.2.8, from Theorem 4.2.7 one deduces the assertion of Lemma 4.2.1 (compare with [52, Theorem 3.1]).*

From Table 2.1, we know that if a proper, convex cone C is *well-based* ($\Leftrightarrow \operatorname{int} C^+ \neq \emptyset$), then it is also normal. Therefore, we end this chapter by deriving the following Corollary.

Corollary 4.2.10. *Let X, Y be normed vector spaces, and let $f : X \to Y^\bullet$ be C-convex. If the cone C is well-based (or C has a weakly compact base), and f is C-bounded from above on a neighborhood of one point in $\operatorname{int}(\operatorname{dom} f)$, then f is locally Lipschitz on $\operatorname{int}(\operatorname{dom} f)$.*

Chapter 5

Lipschitz continuity of cone-convex set-valued functions

As mentioned in the previous chapters, the convexity and the Lipschitz continuity play an important role in various fields of mathematics. Many authors have investigated the convexity and the Lipschitzianity of vector-valued and set-valued functions, and several new concepts have been introduced. In order to generalize the achieved results for vector-valued functions in Chapter 4, we continue studying the Lipschitzianity for convex set-valued functions. In this chapter we introduce various extended notions of the convexity and the Lipschitzianity for set-valued functions, and then study their relationships.

First, we will define the corresponding convexity notions for set-valued functions in Section 5.1 based on the order relations between two nonempty sets first introduced in [46, 47, 49]. In the sequel we study the relationships between the convexity and the Lipschitzianity concepts introduced in Chapter 3.

The main part of Sections 5.2 is based on the paper [70] by Tuan, Tammer and Zălinescu. We introduce the C-boundedness concepts of set-valued functions, and study their correlation. Consequently, we prove that a C-bounded set-valued function satisfying some additional conditions is C-Lipschitz. It is worth mentioning that the Lipschitzianity of convex set-valued functions were first studied in finite-dimensional spaces by Minh and Tan [53]. They used special functional classes, which scalarize initial set-valued functions to new functional families. Then, the C-Lipschitzianity of initial set-valued functions is equivalent to the equi-Lipschitzianity of the corresponding functional families. We adapt this method to the case of general normed vector spaces to obtain new results in Theorem 5.2.7 and 5.2.8, which are significantly stronger than ones in [53]. Of course, if we restrict ourselves to the special case that the set-valued function is at most single-valued (see (3.2)), then the similar results for vector-valued functions in Chapter 4 will be obtained.

Section 5.3 is based on Kuwano and Tanaka's results; see [50]. In [50], new concepts of C-Lipschitz continuity of set-valued maps are introduced, however, they are weaker than the ones in [53]. The authors applied nonconvex scalarizing functions for set-valued maps to prove the C-Lipschitzianity of convex set-valued functions. In this section we only use the nonlinear scalarizing functional (see Section 2.5.2) to prove the C-Lipschitzianity and to weaken the assumptions of the main theorems in [50].

5.1 Cone-convex set-valued functions

In what follows X and Y are normed spaces, C is a proper, convex cone in Y, and $F : X \rightrightarrows Y$ is a set-valued function. Recall that the *domain* of $F : X \rightrightarrows Y$ is given by $\operatorname{dom} F = \{x \in X \mid F(x) \neq \emptyset\}$. Now we introduce the definition of C-convex of set-valued mappings.

For vector-valued functions between two vector spaces, we have already introduced the convexity concepts based on the order relation between two vectors; see Definition 4.1.1. Based on the six notions of set relations between two sets given in Definition 2.3.20, we define the corresponding cone convexities for set-valued functions.

Definition 5.1.1. *([49]) For each $k = i, \dots, vi$, a set-valued map $F : X \rightrightarrows Y$ is said to be* ***type-(k)-convex*** *if for every $x, y \in \operatorname{dom} F$ and $\lambda \in (0, 1)$,*

$$F(\lambda x + (1-\lambda)y) \preceq_C^{(k)} \lambda F(x) + (1-\lambda)F(y).$$

By Proposition 2.3.21, we also have some implications for the convexities above

Proposition 5.1.2. *([49]) Let $F : X \rightrightarrows Y$ be a set-valued function. Then, the following statements hold:*

$$\begin{array}{ccccc}
\textit{type-(i)-convex} & \Longrightarrow & \textit{type-(ii)-convex} & \Longrightarrow & \textit{type-(iii)-convex} \\
\Downarrow & & & & \Downarrow \\
\textit{type-(iv)-convex} & \Longrightarrow & \textit{type-(v)-convex} & \Longrightarrow & \textit{type-(vi)-convex}
\end{array}$$

There are two ways of generalization to define convexities of functions. The first one is based on the relationships between two sets $\lambda F(x) + (1-\lambda)F(y)$ and $F(\lambda x + (1-\lambda)y)$ as in Definition 5.1.1, while the second one is based on the convexity of the epigraph of F. The following proposition states the correlation between these ways.

Proposition 5.1.3. *Let $F : X \rightrightarrows Y$ be a set-valued function, the following statements hold: If F is type-(k)-convex then*

$$\operatorname{epi}_{(k)} F := \{(x, V) \in X \times \mathcal{V} \mid F(x) \preceq_C^{(k)} V\}$$

is convex; where $k \in \{i, ii, \dots, v\}$. Furthermore, the converse of the above assertion holds if $k \in \{iii, iv, v\}$.

Proof. We will only prove the case $k = iii$, as the other cases can be proved by similar arguments.

If F is type-(iii)-convex, we take $(x_1; V_1), (x_2; V_2) \in \operatorname{epi}_{(iii)} F$, we have

$$\begin{cases} V_1 \subseteq F(x_1) + C; \\ V_2 \subseteq F(x_2) + C. \end{cases}$$

These inclusions imply that $\lambda V_1 + (1-\lambda)V_2 \subseteq \lambda F(x_1) + (1-\lambda)F(x_2) + C \subseteq F(\lambda x_1 + (1-\lambda)x_2) + C$ for all $\lambda \in (0,1)$, which shows that $\operatorname{epi}_{(iii)} F$ is convex.
Conversely, since $\operatorname{epi}_{(iii)} F$ is convex, and $(x_1, F(x_1)), (x_2, F(x_2)) \in \operatorname{epi}_{(iii)} F$, we have

$$\big(\lambda x_1 + (1-\lambda)x_2, \lambda F(x_1) + (1-\lambda)F(x_2)\big) \in \operatorname{epi}_{(iii)} F.$$

Thus F is type-(iii)-convex. $\square$

Type-(iii)- and type-(iv)-convexity above are also known as upper C-convexity and lower C-convexity, which will be shown again in the following definition.

Definition 5.1.4. *Let $F : X \rightrightarrows Y$ with $\operatorname{dom} F \neq \emptyset$, and C be a proper, convex cone; F is said to be **upper C-convex** (or **lower C-convex**) if*

$$F(\alpha x + (1-\alpha)y) \subseteq \alpha F(x) + (1-\alpha)F(y) - C,$$

$$(\alpha F(x) + (1-\alpha)F(y) \subseteq F(\alpha x + (1-\alpha)y) + C, \textit{respectively}),$$

holds for all $x, y \in \operatorname{dom} F$ and $\alpha \in (0,1)$.

Remark 5.1.5. *(i) If $F : X \rightrightarrows Y$ is at most single-valued, then F is upper (lower) C-convex in the sense of Definition 5.1.4 if and only if the corresponding vector-valued function $f : X \to Y^\bullet$ given by*

$$f(x) := \begin{cases} y & \text{if } x \in \operatorname{dom} F \text{ and } F(x) = \{y\}, \\ +\infty & \text{if } x \notin \operatorname{dom} F. \end{cases}$$

is C-convex in the sense of Definition 4.1.1.

(ii) Obviously, if F is lower C-convex, then $\operatorname{dom} F$ is convex, and $F(x) + C$ is a convex set for all $x \in \operatorname{dom} F$.

In order to study properties of set-valued mappings, Minh and Tan [53] used a scalarization method for set-valued mappings. For a given set-valued function $F : X \rightrightarrows Y$ between two normed vector spaces X, Y and a proper, convex cone C in Y. The functions $G_{y^*}, g_{y^*} : X \to \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ are defined for each $y^* \in C^+$ as follows:

$$G_{y^*}(x) := \sup_{y \in F(x)} y^*(y), \quad x \in X, \tag{5.1}$$

$$g_{y^*}(x) := \inf_{y \in F(x)} y^*(y), \quad x \in X, \tag{5.2}$$

with the convention $\inf \emptyset := +\infty$, $\sup \emptyset := -\infty$.

Obviously, $\operatorname{dom} g_{y^*} = \operatorname{dom} F$ and (for $y^* = 0$) $g_0 = \delta_{\operatorname{dom} F}$, where δ_A is the indicator function of A defined by $\delta_A(x) = 0$ if $x \in A$, and $\delta_A(x) = +\infty$ otherwise.

We will recall some properties of the scalar functions G_{y^*}, g_{y^*} corresponding to properties of F; see [53, 54]. The following propositions are stated in [54, Proposition 2.2] without proof. For convenience of the reader, we prove these propositions in detail.

Proposition 5.1.6. *Let $F : X \rightrightarrows Y$ be a set-valued function, and $\operatorname{dom} F$ be convex and nonempty. Let C be a proper, convex cone. Then, the following implications hold:*

(i) If F is an upper C-convex mapping, then G_{y^*} is convex on $\operatorname{dom} F$ for all $y^* \in C^+$.

(ii) Conversely, if $F(x) - C$ is closed and convex for all $x \in \operatorname{dom} F \neq \emptyset$, and G_{y^*} is convex for all $y^* \in C^+$, then F is upper C-convex.

Proof. (i) Let F be upper C-convex, and $y^* \in C^+$ be chosen arbitrarily. For every $\lambda \in (0,1), x_1, x_2 \in \operatorname{dom} F$, we have

$$\begin{aligned}
G_{y^*}(\lambda x_1 + (1-\lambda)x_2) &= \sup_{y \in F(\lambda x_1 + (1-\lambda)x_2)} y^*(y) \\
&\leq \sup_{y \in \lambda F(x_1) + (1-\lambda)F(x_2) - C} y^*(y) \\
&\leq \sup_{y \in \lambda F(x_1) + (1-\lambda)F(x_2)} y^*(y) \\
&= \sup_{y \in \lambda F(x_1)} y^*(y) + \sup_{y \in (1-\lambda)F(x_2)} y^*(y) \\
&= \lambda \sup_{y \in F(x_1)} y^*(y) + (1-\lambda) \sup_{y \in F(x_2)} y^*(y) \\
&= \lambda G_{y^*}(x_1) + (1-\lambda) G_{y^*}(x_2).
\end{aligned}$$

Therefore, G_{y^*} is convex on $\operatorname{dom} F$ for all $y^* \in C^+$.

(ii) Suppose by contradiction that F is not upper C-convex, so there exist $x_1, x_2 \in \operatorname{dom} F$ and $\lambda \in (0,1)$ such that

$$F(\lambda x_1 + (1-\lambda)x_2) \nsubseteq \lambda F(x_1) + (1-\lambda)F(x_2) - C.$$

One can take $\bar{y} \in F(\lambda x_1 + (1-\lambda)x_2) \neq \emptyset$ such that $\bar{y} \notin \lambda F(x_1) + (1-\lambda)F(x_2) - C$. Since $\lambda F(x_1) + (1-\lambda)F(x_2) - C$ is closed and convex, there exists $y^* \in Y^*$ such that

$$y^*(\bar{y}) > \sup\{y^*(y) \big| y \in \lambda F(x_1) + (1-\lambda)F(x_2) - C\}.$$

It follows that $y* \in C^+ \setminus \{0\}$ and

$$\begin{aligned} G_{y^*}(\lambda x_1 + (1-\lambda)x_2) &= \sup_{y \in F(\lambda x_1 + (1-\lambda)x_2)} y^*(y) \geq y^*(\bar{y}) \\ &> \sup_{y \in \lambda F(x_1) + (1-\lambda)F(x_2) - C} y^*(y) \\ &= \lambda G_{y^*}(x_1) + (1-\lambda) G_{y^*}(x_2). \end{aligned}$$

This contradicts our assumption on the convexity of G_{y^*}. □

Proposition 5.1.7. *Let $F : X \rightrightarrows Y$ with* $\operatorname{dom} F \neq \emptyset$*, and C be a proper, convex cone; the following implications hold:*

(i) If F is a lower C-convex mapping, then g_{y^*} is convex for all $y^* \in C^+$.

(ii) Conversely, if $F(x) + C$ is closed and convex for all $x \in \operatorname{dom} F \neq \emptyset$, and g_{y^*} is convex for all $y^* \in C^+$, then F is lower C-convex.

Proof. (i) Let F be lower C-convex, and $y^* \in C^+$ be chosen arbitrarily; for every $\lambda \in (0,1), x_1, x_2 \in \operatorname{dom} g_{y^*} = \operatorname{dom} F$, we have

$$\begin{aligned} g_{y^*}(\lambda x_1 + (1-\lambda)x_2) &= \inf_{y \in F(\lambda x_1 + (1-\lambda)x_2)} y^*(y) = \inf_{y \in F(\lambda x_1 + (1-\lambda)x_2) + C} y^*(y) \\ &\leq \inf_{y \in \lambda F(x_1) + (1-\lambda)F(x_2)} y^*(y) \\ &= \inf_{y \in \lambda F(x_1)} y^*(y) + \inf_{y \in (1-\lambda)F(x_2)} y^*(y) \\ &= \lambda \inf_{y \in F(x_1)} y^*(y) + (1-\lambda) \inf_{y \in F(x_2)} y^*(y) \\ &= \lambda g_{y^*}(x_1) + (1-\lambda) g_{y^*}(x_2). \end{aligned}$$

Therefore, g_{y^*} is convex for all $y^* \in C^+$.

(ii) Since $g_0 = \delta_{\operatorname{dom} F}$ is convex, so is $\operatorname{dom} F$. Suppose by contradiction that F is not lower C-convex, there exist $x_1, x_2 \in \operatorname{dom} F$ and $\lambda \in (0,1)$ such that

$$\lambda F(x_1) + (1-\lambda)F(x_2) \nsubseteq F(\lambda x_1 + (1-\lambda)x_2) + C.$$

One can take $\bar{y} \in \lambda F(x_1) + (1-\lambda)F(x_2)$ such that $\bar{y} \notin F(\lambda x_1 + (1-\lambda)x_2) + C \neq \emptyset$. Since $F(\lambda x_1 + (1-\lambda)x_2) + C$ is closed and convex, there exists $y^* \in Y^*$ such that

$$y^*(\bar{y}) < \inf\{y^*(y) \big| y \in F(\lambda x_1 + (1-\lambda)x_2) + C\}.$$

It follows that $y^* \in C^+ \setminus \{0\}$ and

$$\begin{aligned} g_{y^*}(\lambda x_1 + (1-\lambda)x_2) &= \inf_{y \in F(\lambda x_1 + (1-\lambda)x_2) + C} y^*(y) > y^*(\bar{y}) \\ &\geq \inf_{y \in \lambda F(x_1) + (1-\lambda)F(x_2)} y^*(y) \quad (\text{as } \bar{y} \in \lambda F(x_1) + (1-\lambda)F(x_2)) \\ &= \lambda g_{y^*}(x_1) + (1-\lambda) g_{y^*}(x_2). \end{aligned}$$

This contradicts our assumption on the convexity of g_{y^*}. $\square$

Moreover, we shall recall the cone convexity for set-valued functions w.r.t the set less order relation introduced in Definition 2.3.19. We will use this concept to prove the G-Lipschitzianity of set-valued functions in the next section.

Definition 5.1.8. *Let $F : X \rightrightarrows Y$ with $\operatorname{dom} F \neq \emptyset$, and C be a proper, convex cone; F is said to be* $\mathfrak{Cs}$***-convex*** *if*

$$F(\alpha x + (1-\alpha)y) \preceq_C^s \alpha F(x) + (1-\alpha)F(y),$$

holds for all $x, y \in \operatorname{dom} F$ and $\alpha \in (0,1)$ (hence $\operatorname{dom} F$ is convex).

Remark 5.1.9. *Obviously, F is $\mathfrak{Cs}$-convex if and only if F is type-(iii)- and type-(v)-convex (or lower C-convex and upper C-convex in Definition 5.1.4).*

5.2 The C-Lipschitzianity of convex set-valued functions

In [53], Minh and Tan already studied the C-Lipschitzianity of C-convex set-valued mappings $F : X \rightrightarrows Y$, where X is a finite dimensional space, and Y is a Banach space. In this section, we derive the corresponding results in general normed spaces.

We shall present C-boundedness notions of a set-valued mapping $F : X \rightrightarrows Y$, where $C \subset Y$ is a proper, convex cone.

Definition 5.2.1. *Let $F : X \rightrightarrows Y$ be a set-valued function, and C be a proper, convex cone in Y.*

*(i) F is said to be C**-bounded from above** (resp. **below**) on a subset A of X if there exists a constant $\mu > 0$ such that*

$$F(A) \subseteq \mu U_Y - C \quad (\textit{resp. } F(A) \subseteq \mu U_Y + C).$$

*(ii) F is said to be C**-bounded** on a subset A of X if it is C-bounded from above and C-bounded from below on A.*

Definition 5.2.2. *([44, Definition 3.1.26]) We say that $F : X \rightrightarrows Y$ is **weakly C-upper (lower) bounded** on a set $A \subseteq X$ if there exists $\mu' > 0$ such that $F(x) \cap (\mu' U_Y - C) \neq \emptyset$ ($F(x) \cap (\mu' U_Y + C) \neq \emptyset$, respectively) for all $x \in A$.*

In the next proposition, we study the equivalence between the upper C-Lipschitzianity of a given set-valued mapping F (see Definition 3.2.5) and the equi-Lipschitzianity (see Definition 3.1.3) of the scalar functional family $\{G_{y^*} : X \to \overline{\mathbb{R}} \mid y^* \in C^+, \|y^*\|_* = 1\}$ corresponding to F.

Proposition 5.2.3. *Let X, Y be two normed spaces, $F : X \rightrightarrows Y$, and $F(x) - C$ be convex for all $x \in X$. Let x_0 be a given point in* $\operatorname{int}(\operatorname{dom} F)$ *such that $F(x_0)$ is C-bounded from above. Then, F is upper C-Lipschitz around x_0 if and only if the family $\{G_{y^*} | y^* \in C^+, \|y^*\|_* = 1\}$ is equi-Lipschitz around x_0.*

Proof. As F is upper C-Lipschitz around x_0, there exist a neighborhood $U \subseteq \operatorname{dom} F$ of x_0 and a real number $\ell > 0$ such that

$$F(x) \subseteq F(x') + \ell \|x - x'\|_X U_Y - C, \quad \text{for all} \quad x, x' \in U. \tag{5.3}$$

As $F(x_0)$ is C-bounded from above, we assume that $F(x_0) \subseteq \mu U_Y - C$ for some $\mu > 0$; see Definition 4.2.3. From (5.3) we get

$$F(x) \subseteq (\mu + \ell \|x - x'\|_X) U_Y - C, \quad \text{for all} \quad x \in U. \tag{5.4}$$

Hence, for all $x \in U, y^* \in C^+$, $\|y^*\|_* = 1$, we get

$$G_{y^*}(x) = \sup_{y \in F(x)} y^*(y) \leq \mu + \ell \|x - x'\|_X < +\infty.$$

Therefore, G_{y^*} is finite on U, for all $y^* \in C^+$ that satisfies $\|y^*\|_* = 1$.
Taking into account (5.3), we have the following estimation for all $x, x' \in U \subseteq \operatorname{dom} F$

$$G_{y^*}(x) = \sup_{y \in F(x)} y^*(y) \leq \sup_{y \in F(x')} y^*(y) + \ell \|x - x'\|_X = G_{y^*}(x') + \ell \|x - x'\|_X.$$

Hence

$$G_{y^*}(x) - G_{y^*}(x') \leq \ell \|x - x'\|_X, \quad \text{for all} \quad x, x' \in U, y^* \in C^+, \|y^*\|_* = 1.$$

By interchanging x and x', we get

$$|G_{y^*}(x') - G_{y^*}(x)| \leq \ell \|x - x'\|_X, \quad \text{for all} \quad x, x' \in U, y^* \in C^+, \|y^*\|_* = 1.$$

This shows that the family $\{G_{y^*} | y^* \in C^+, \|y^*\|_* = 1\}$ is equi-Lipschitz around x_0.

Now we prove the converse implication by contradiction: if the family $\{G_{y^*} | y^* \in C^+, \|y^*\|_* = 1\}$ is equi-Lipschitz around x_0, then F is upper C-Lipschitz around x_0. Assume that F is not upper C-Lipschitz around x_0, i.e., for any $n \in \mathbb{N}^*$, there are $x_n, x'_n \in B(x_0, \frac{1}{n})$ such that

$$F(x_n) \not\subseteq F(x'_n) + n \|x_n - x'_n\|_X U_Y - C;$$

and hence $x_n \neq x'_n$ for all $n \in \mathbb{N}^*$.
Since $x_0 \in \operatorname{int}(\operatorname{dom} F)$, for n large enough, $B(x_0, \frac{1}{n}) \subseteq \operatorname{dom} F$, and then we can take $y_n \in F(x_n)$ such that

$$y_n \notin B_n := F(x'_n) + n \|x_n - x'_n\|_X U_Y - C.$$

Since the set B_n is convex and $\operatorname{int} B_n \neq \emptyset$, one can find $y_n^* \in Y^*$ with $\|y_n^*\|_* = 1$ and

$$y_n^*(y_n) \geq y_n^*(v) \quad \text{for all} \quad v \in B_n.$$

Hence,

$$y_n^*(y_n) \geq \sup y_n^*(B_n) = \sup y_n^*(F(x_n')) + n\|x_n - x_n'\|_X + \sup y_n^*(-C).$$

It follows that $y_n^* \in C^+$ for large $n \in \mathbb{N}$ and

$$G_{y_n^*}(x_n) \geq G_{y_n^*}(x_n') + n\|x_n - x_n'\|_X,$$

which yields

$$n\|x_n - x_n'\|_X \leq G_{y_n^*}(x_n) - G_{y_n^*}(x_n') \leq \ell\|x_n - x_n'\|_X,$$

and hence $n \leq \ell$, which could not hold true for n sufficiently large. □

Similarly, in the following proposition, we study the equivalence between the lower C-Lipschitzianity of F (see Definition 3.2.5) and the equi-Lipschitzianity (see Definition 3.1.3) of the scalar functional family $\{g_{y^*} | y^* \in C^+, \|y^*\|_* = 1\}$ corresponding to F.

Proposition 5.2.4. *Let X, Y be two normed vector spaces, $F : X \rightrightarrows Y$, and $F(x) + C$ be convex for all $x \in X$. Let x_0 be a given point in $\operatorname{int}(\operatorname{dom} F)$ such that $F(x_0)$ is C-bounded from below. Then, F is lower C-Lipschitz around x_0 if and only if the family $\{g_{y^*} | y^* \in C^+, \|y^*\|_* = 1\}$ is equi-Lipschitz around x_0.*

Proof. As F is lower C-Lipschitz around x_0, there exist a neighborhood $U \subseteq \operatorname{dom} F$ of x_0 and a real number $\ell > 0$ such that

$$F(x) \subseteq F(x') + \ell\|x - x'\|_X U_Y + C, \quad \text{for all} \quad x, x' \in U. \tag{5.5}$$

As $F(x_0)$ is C-bounded from below, we assume that $F(x_0) \subseteq \mu U_Y + C$ for some $\mu > 0$; see Definition 4.2.3. Due to (5.5), we get

$$F(x) \subseteq (\mu + \ell\|x - x'\|_X) U_Y + C, \quad \text{for all} \quad x \in U. \tag{5.6}$$

Hence, for all $x \in U, y^* \in C^+, \|y^*\|_* = 1$, we have

$$g_{y^*}(x) = \inf_{y \in F(x)} y^*(y) \geq -(\mu + \ell\|x - x'\|_X) > -\infty.$$

Thus g_{y^*} is finite on U, for all $y^* \in C^+$ that satisfies $\|y^*\|_* = 1$.
Taking into account (5.5), we have the following estimation for all $x, x' \in U \subseteq \operatorname{dom} F$

$$g_{y^*}(x) = \inf_{y \in F(x)} y^*(y) \geq \inf_{y \in F(x')} y^*(y) - \ell\|x - x'\|_X = g_{y^*}(x') - \ell\|x - x'\|_X.$$

Hence,

$$g_{y^*}(x') - g_{y^*}(x) \leq \ell\|x - x'\|_X, \quad \text{for all} \quad x, x' \in U, y^* \in C^+, \|y^*\|_* = 1.$$

By interchanging x and x', we get

$$|g_{y^*}(x) - g_{y^*}(x')| \leq \ell\|x - x'\|_X, \quad \text{for all} \quad x, x' \in U, y^* \in C^+, \|y^*\|_* = 1.$$

This shows that the family $\{g_{y^*}|y^* \in C^+, \|y^*\|_* = 1\}$ is equi-Lipschitz around x_0.

We prove the converse implication by contradiction: if the family $\{g_{y^*}|y^* \in C^+, \|y^*\|_* = 1\}$ is equi-Lipschitz around x_0 then F is lower C-Lipschitz around x_0. Suppose that F is not lower C-Lipschitz around x_0. Then, there exist $x_n, x'_n \in B(x_0, \frac{1}{n})$ such that

$$F(x_n) \nsubseteq F(x'_n) + n\|x_n - x'_n\|_X U_Y + C \quad \text{for all} \quad n \in \mathbb{N}^*;$$

and hence $x_n \neq x'_n$ for all $n \in \mathbb{N}^*$.
Since $x_0 \in \operatorname{int}(\operatorname{dom} F)$, for n large enough, $B(x_0, \frac{1}{n}) \subseteq \operatorname{dom} F$, we can take $y_n \in F(x_n)$ such that

$$y_n \notin B_n := F(x'_n) + n\|x_n - x'_n\|_X U_Y + C.$$

Since the set B_n is convex and $\operatorname{int} B_n \neq \emptyset$, one can find $y_n^* \in Y^*$ with $\|y_n^*\|_* = 1$ such that

$$y_n^*(y_n) \leq y_n^*(v) \quad \text{for all} \quad v \in B_n.$$

Hence,

$$y_n^*(y_n) \leq \inf y_n^*(B_n) = \inf y_n^*(F(x'_n)) - n\|x_n - x'_n\|_X + \inf y_n^*(C).$$

It follows that $y_n^* \in C^+$ for large $n \in \mathbb{N}$ and

$$g_{y_n^*}(x_n) \leq g_{y_n^*}(x'_n) - n\|x_n - x'_n\|_X,$$

which yields that

$$n\|x_n - x'_n\|_X \leq g_{y_n^*}(x'_n) - g_{y_n^*}(x_n) \leq \ell\|x_n - x'_n\|_X,$$

and hence $n \leq \ell$, which could not hold true for n sufficiently large. □

Remark 5.2.5. *Proposition 5.2.4 is stated in [53, Theorem 2.5] without the assumption that $F(x_0)$ is C-bounded from below. Taking $F(x) = Y$ for all $x \in X$, it is clear that F is lower C-Lipschitz, but $\{g_{y^*}\big|y^* \in C^*, \|y^*\| = 1\}$ is not equi-Lipschitz.*

Theorem 5.2.6. *Let X, Y be two normed spaces, C be a proper, convex cone. Let $F : X \rightrightarrows Y$ be upper C-convex, and $F(x) - C$ be convex for all $x \in X$. If F is both C-bounded from above and weakly C-lower bounded on a neighborhood of $x_0 \in \operatorname{int}(\operatorname{dom} F)$, then F is upper C-Lipschitz around x_0.*

Proof. Without loss of generality we suppose that $x_0 = 0$ and $0 \in F(0)$. As F is C-bounded from above and weakly C-lower bounded on a neighborhood $U = \theta U_X \subseteq \operatorname{dom} F$ of 0 ($\theta > 0$), Definition 5.2.1 and Definition 5.2.2 imply that there exists a real number $\mu > 0$ such that $F(U) \subseteq \mu U_Y - C$ and $F(x) \cap (\mu U_Y + C) \neq \emptyset$ for all $x \in U$.
Take $y^* \in C^+$ with $\|y^*\|_* = 1$. Let $\bar{x} \in U$ be arbitrary, $\bar{y} \in F(\bar{x})$, $c \in C$, and $y' \in \mu U_Y$ such that $\bar{y} = y' + c$. We then have

$$\begin{aligned} G_{y^*}(\bar{x}) = \sup_{y \in F(\bar{x})} y^*(y) &\geq y^*(\bar{y}) = y^*(y' + c) = y^*(y') + y^*(c) \\ &\geq y^*(y') \geq -\|y^*\|_* \|y'\|_Y = -\|y'\|_Y \geq -\mu, \quad \text{for all} \quad \bar{x} \in U. \end{aligned}$$

Analogously, from $F(U) \subseteq \mu U_Y - C$, we get $G_{y^*}(x) = \sup_{y \in F(x)} y^*(y) \leq \mu$ for every $x \in U$. This follows that G_{y^*} is finite on U and

$$G_{y^*}(x) \leq G_{y^*}(0) + 2\mu, \quad \text{for all} \quad x \in U = \theta U_X.$$

Taking into account Proposition 5.1.6, G_{y^*} is convex on U. Applying Lemma 2.2.12 to the convex function G_{y^*} and $\theta' \in (0, \theta)$, we get

$$|G_{y^*}(x) - G_{y^*}(x')| \leq L\|x - x'\|_X, \quad \text{for all} \quad x, x' \in \theta' U_X,$$

where $L := 2\mu(\theta + \theta')/[\theta(\theta - \theta')]$, which does not depend on y^*.
So $\{G_{y^*} \mid y^* \in C^+, \|y^*\|_* = 1\}$ is equi-Lipschitz around x_0 with the Lipschitz constant L. Applying Proposition 5.2.3, we have that F is upper C-Lipschitz around x_0. $\square$

Theorem 5.2.7. ([70, Theorem 3]) *Let X, Y be two normed spaces, C be a proper, convex cone. Let $F : X \rightrightarrows Y$ be lower C-convex, C-bounded from below and weakly C-upper bounded on a neighborhood of $x_0 \in \operatorname{int}(\operatorname{dom} F)$, then F is lower C-Lipschitz around x_0.*

Proof. Without loss of generality, we suppose that $x_0 = 0$ and $0 \in F(0)$. As F is C-bounded from below and weakly C-upper bounded on a neighborhood $U = \theta U_X \subseteq \operatorname{dom} F$ of 0 ($\theta > 0$), taking into account Definition 5.2.1 and Definition 5.2.2 there exists a real number $\mu > 0$ such that $F(U) \subseteq \mu U_Y + C$ and $F(x) \cap (\mu U_Y - C) \neq \emptyset$ for all $x \in U$.
Take $y^* \in C^+$ with $\|y^*\|_* = 1$. Let $\bar{x} \in U$ be arbitrary, and take $\bar{y} \in F(\bar{x})$, $c \in C$, and $y' \in \mu U_Y$ such that $\bar{y} = y' - c$. We then have

$$\begin{aligned} g_{y^*}(\bar{x}) = \inf_{y \in F(\bar{x})} y^*(y) &\leq y^*(\bar{y}) = y^*(y' - c) = y^*(y') - y^*(c) \\ &\leq y^*(y') \leq \|y^*\|_* \|y'\|_Y = \|y'\|_Y \leq \mu, \quad \text{for all} \quad \bar{x} \in U. \end{aligned}$$

Analogously, from $F(U) \subseteq \mu U_Y + C$, we get $g_{y^*}(x) = \inf_{y\in F(x)} y^*(y) \geq -\mu$ for every $x \in U$. It follows that g_{y^*} is finite on U and

$$g_{y^*}(x) \leq g_{y^*}(0) + 2\mu, \quad \text{for all} \quad x \in U = \theta U_X.$$

By Proposition 5.1.7, g_{y^*} is convex on U. Applying Lemma 2.2.12 to the convex function g_{y^*} and $\theta' \in (0, \theta)$, we get

$$|g_{y^*}(x) - g_{y^*}(x')| \leq L\|x - x'\|_X, \quad \text{for all} \quad x, x' \in \theta' U_X,$$

where $L := 2\mu(\theta + \theta')/[\theta(\theta - \theta')]$, which does not depend on y^*.
So $\{g_{y^*} \mid y^* \in C^+, \|y^*\|_* = 1\}$ is equi-Lipschitz around x_0 with the Lipschitz constant L. Because of the convexity of F, $F(x) + C$ is convex for all $x \in X$. Applying Proposition 5.2.4, we have that F is lower C-Lipschitz around x_0. □

The next result follows directly from Theorem 5.2.7.

Theorem 5.2.8. *Let X, Y be two normed vector spaces, $C \subset Y$ be a proper, convex cone, and let $F : X \rightrightarrows Y$ be lower C-convex. If F is C-bounded from below and weakly C-upper bounded on a neighborhood of some point $x \in \operatorname{int}(\operatorname{dom} F)$, then F is locally lower C-Lipschitz on $\operatorname{int}(\operatorname{dom} F)$.*

Remark 5.2.9. *(i) It is clear that the assumptions in Theorem 5.2.8 are much weaker than those in [53, Theorem 2.9]; while the assumption that X is finite-dimensional is unnecessary. We do not even need any additional conditions for the cone C such as $C^+ = \operatorname{cone}(\operatorname{conv}\{y_1^*, \dots, y_n^*\})$ for some $y_1^*, \dots, y_n^* \in Y^*$ and $0 \notin \operatorname{conv}\{y_1^*, \dots, y_n^*\}$ as in [53].*

(ii) By [26, Proposition 2.6.2], if F is C-bounded from below on a neighborhood of $x_0 \in \operatorname{int}(\operatorname{dom} F)$ then F is C-bounded from below on a neighborhood of x for every $x \in \operatorname{dom} F$.

(iii) If F is weakly C-upper bounded on a neighborhood of $x_0 \in \operatorname{int}(\operatorname{dom} F)$ then F is weakly C-upper bounded on a neighborhood of x for every $x \in \operatorname{int}(\operatorname{dom} F)$. Indeed, assume that $F(x) \cap (\mu U_Y - C) \neq \emptyset\ \forall x \in B(x_0, r)$ and fix $\bar{x} \in \operatorname{int}(\operatorname{dom} F)$, then there exist $x_1 \in \operatorname{dom} F$, $\lambda \in (0, 1)$ such that $\bar{x} = \lambda x_0 + (1 - \lambda)x_1$. Fix $y_1 \in F(x_1)$ and take $u \in rU_X$, then $\bar{x} + \lambda u = \lambda(x_0 + u) + (1 - \lambda)x_1$, and there exists $y_u \in F(x_0 + u) \cap (\mu U_Y - C)$. Hence, $y_u = \mu v_u - c_u$ with $\|v_u\| \leq 1$, $c_u \in C$. Then, $\lambda(\mu v_u - c_u) + (1-\lambda)y_1 \in F(\bar{x} + \lambda u) + C$, hence $\exists c'_u \in C$ and $\bar{y} \in F(\bar{x} + \lambda u)$ such that $\bar{y} = \lambda(\mu v_u - c_u) + (1 - \lambda)y_1 - c'_u \in \bar{\mu} U_Y - C$ with $\bar{\mu} = \lambda\mu + (1 - \lambda)\|y_1\|$.

Therefore, Theorem 5.2.8 is a consequence of Theorem 5.2.7 and Remark (ii) above.

In particular, when $F : X \rightrightarrows Y$ is an at most single-valued mapping, we have the following result, which is similar to Theorem 5.2.7 for the corresponding vector-valued mapping $f : X \to Y^\bullet$; see Remark 3.2.2.

Theorem 5.2.10. *Under the hypotheses of Theorem 5.2.7, if $F : X \rightrightarrows Y$ is an at most single-valued mapping and C is a normal cone, then the function $f : X \to Y^\bullet$ defined in (3.3) is Lipschitz around x_0.*

Proof. Theorem 5.2.7 implies that F is lower C-Lipschitz around $x_0 \in \operatorname{int}(\operatorname{dom} F)$. Thus, there is a neighborhood U of x_0 in $\operatorname{dom} F$ and a constant $\ell > 0$ such that

$$F(x) \subseteq F(x') + \ell\|x - x'\|_X U_Y + C, \quad \text{for all} \quad x, x' \in U,$$

or equivalently

$$f(x) \in f(x') + \ell\|x - x'\|_X U_Y + C, \quad \text{for all} \quad x, x' \in U. \tag{5.7}$$

Since C is normal, there is $\rho > 0$ such that

$$(\rho U_Y + C) \cap (\rho U_Y - C) \subseteq U_Y.$$

From (5.7), we have

$$\frac{\rho(f(x) - f(x'))}{\ell\|x - x'\|_X} \subseteq \rho U_Y + C, \quad \text{for all} \quad x, x' \in U, x \neq x'.$$

By interchanging x and x', we also have

$$\frac{\rho(f(x) - f(x'))}{\ell\|x - x'\|_X} \subseteq \rho U_Y - C, \quad \text{for all} \quad x, x' \in U, x \neq x'.$$

Therefore,

$$\frac{\rho(f(x) - f(x'))}{\ell\|x - x'\|_X} \subseteq (\rho U_Y + C) \cap (\rho U_Y - C) \subseteq U_Y,$$

for all $x, x' \in U, x \neq x'$. This shows that f is Lipschitzian around x_0. □

When C is normal, by Proposition 4.2.5, the C-boundedness from below and weakly C-upper boundedness of f in Theorem 5.2.10 can be replaced by the C-boundedness from above, and then we obtain again the assertions of Theorem 4.2.7.

5.3 Kuwano and Tanaka's C-Lipschitzianity

In [50], Kuwano and Tanaka introduced a new concept of the locally Lipschitz continuity of set-valued maps, and then they used the nonlinear scalarizing functional to prove the locally Lipschitz continuity of convex set-valued functions. In this section, we also use the nonlinear scalarizing functionals but we propose another scalarization approach to lighten the assumptions of the main Theorem 3.2 in [50]. Throughout this section X

is a normed space, Y is a linear topological space, $C \subset Y$ is a proper, closed, pointed, convex cone with a nonempty interior, and $e \in \operatorname{int} C$. We begin this section by recalling the following concepts of Lipschitz continuity in [50].

Definition 5.3.1. *Let X be a normed space, Y be a linear topological space. Let $C \subset Y$ be a proper, closed, pointed, convex cone with a nonempty interior, $e \in \operatorname{int} C$, and $F : X \rightrightarrows Y$. Then, F is said to be **locally upper (lower) C-Lipschitz continuous** at $x \in X$ if there exist a positive constant L and a neighborhood U_x of x such that for any $x' \in U_x$,*

$$F(x') \subseteq F(x) + L\|x - x'\|e - C,$$

$$\big(F(x') \subseteq F(x) - L\|x - x'\|e + C, \textit{respectively}\big).$$

*F is said to be **locally C-Lipschitz continuous** at $x \in X$ if F is locally upper C-Lipschitz continuous and locally lower C-Lipschitz continuous at this point.*

It is easy to see that these concepts are weaker than the ones in Definition 3.2.5, because the element x is taken as a fixed point in Definition 5.3.1. It means that the locally upper (lower) C-Lipschitz continuity (in the sense of Definition 5.3.1) at a given point $x \in X$ implies the upper (lower) C-Lipschitzianity (in the sense of Definition 3.2.5) around this point.

In this section, we only study the C-Lipschitz continuity in the sense of Definition 5.3.1. Now we recall the boundedness concepts in [50].

Definition 5.3.2. *Let X be a normed space, Y be a linear topological space, and $F : X \rightrightarrows Y$ be a set-valued function. Let $C \subset Y$ be a proper, closed, pointed, convex cone with a nonempty interior, and e be a given point in $\operatorname{int} C$. Then, F is said to be **C-bounded from above (resp. below)** around a point $x_0 \in X$ if there exist a positive t and a neighborhood U of x_0 such that*

$$F(U) \subseteq te - C, \quad (\textit{resp. } F(U) \subseteq -te + C).$$

*Furthermore, F is called **C-bounded** around a point $x_0 \in X$ if it is C-bounded from above and C-bounded from below around that point.*

Remark 5.3.3. *We will prove that these concepts are equivalent to other boundedness concepts in Definition 5.2.1. Indeed, since $e \in \operatorname{int} C$, there is $\mu' > 0$ such that $-e + \mu' U_Y \subseteq -C$. If $F(U) \subseteq \mu U_Y - C$, then $F(U) \subseteq \frac{1}{\mu'} e - C := te - C$. It is obvious that if $F(U) \subseteq te - C$, then there is μ such that $F(U) \subseteq \mu U_Y - C$, and, therefore, the converse assertion is clear. Similar arguments can be applied to the case C-bounded from below, and then the proof is complete.*

Now we will prove that an upper C-convex set-valued function (see Definition 5.1.4) is locally upper C-Lipschitz continuous in the sense of Definition 5.3.1.

Theorem 5.3.4. *Let X be a normed space, Y be a linear topological space. Let $C \subset Y$ be a proper, closed, pointed, convex cone with a nonempty interior, and e be a given point in $\operatorname{int} C$. Let $F : X \rightrightarrows Y$ be an upper C-convex set-valued function, and $C - F(x)$ is closed and convex for every $x \in X$. If F is C-bounded around a point $x_0 \in \operatorname{int}(\operatorname{dom} F)$, then F is locally upper C-Lipschitz continuous at x_0.*

Proof. Let $e \in \operatorname{int} C$. Because of the C-boundedness of F around x_0 (see Definition 5.3.2), there exist a positive t and a neighborhood $U := x_0 + \mu U_X \subseteq \operatorname{int}(\operatorname{dom} F)$ of x_0 such that $F(U) \subseteq te - C$ and $F(U) \subseteq -te + C$, which is equivalent to $-F(U) \subseteq te - C$. Thus, for every $x \in U$, we have $F(x) - F(x_0) \subseteq 2te - C$. This implies that

$$F(x) \subseteq 2te + F(x_0) - C. \tag{5.8}$$

From the assumptions of $C - F(x_0)$, it follows directly that $A := C - F(x_0)$ is closed and (2.10) is fullfilled. Thus, using the functional $\varphi_{A,e}$ introduced in (2.11) with $A = C - F(x_0)$ and $e \in \operatorname{int} C$, we can consider a function $H : X \to \overline{\mathbb{R}}$ given by

$$H(x) := \sup_{a \in F(x)} \varphi_{A,e}(a),$$

where C is a proper, closed, pointed, convex cone and $e \in \operatorname{int} C$.

Applying Theorem 2.5.7(a), we get that $\varphi_{A,e}$ is C-monotone. Moreover, taking into account the inclusion (5.8), for all $a \in F(x)$, we can choose $b \in F(x_0)$ such that $a \in 2te + b - C$, hence

$$\varphi_{A,e}(a) \leq \varphi_{A,e}(2te + b) = 2t + \varphi_{A,e}(b) \leq 2t + H(x_0).$$

It follows directly from the definition of the functional H that

$$H(x) \leq H(x_0) + 2t. \tag{5.9}$$

This implies that H is bounded from above around x_0.

Now we prove that H is convex. Indeed, it follows from the upper C-convexity of F (see Definition 5.1.4) and the convexity of $\varphi_{A,e}$ (since $A = C - F(x_0)$ is convex) that for every $x, y \in X$ and $\alpha \in (0,1)$ we have

$$\begin{aligned}
H(\alpha x + (1-\alpha)y) &= \sup_{a \in F(\alpha x + (1-\alpha)y)} \varphi_{A,e}(a) \\
&\leq \sup_{a \in \alpha F(x) + (1-\alpha)F(y) - C} \varphi_{A,e}(a) \\
&= \sup_{a_1 \in F(x), a_2 \in F(y), c \in C} \varphi_{A,e}\big(\alpha a_1 + (1-\alpha)a_2 - C\big) \\
&= \sup_{a_1 \in F(x), a_2 \in F(y)} \varphi_{A,e}\big(\alpha a_1 + (1-\alpha)a_2\big) \\
&\leq \alpha \sup_{a_1 \in F(x)} \varphi_{A,e}(a_1) + (1-\alpha) \sup_{a_2 \in F(y)} \varphi_{A,e}(a_2) \\
&= \alpha H(x) + (1-\alpha)H(y).
\end{aligned}$$

Applying Lemma 2.2.12 to the scalar proper convex function H, we get that H is Lipschitz on a neighborhood $U' := x_0 + \mu' U_X$ of x_0 ($0 < \mu' < \mu$) with the Lipschitz constant $L > 0$, which means that

$$\|H(x) - H(x_0)\| \leq L\|x - x_0\| \quad \text{for all} \quad x \in U'.$$

Since $H(x_0) \leq 0$, we get

$$H(x) \leq L\|x - x_0\| \quad \text{for all} \quad x \in U',$$

which induces the conclusion due to Corollary 2.5.5. □

Similarly we can prove that an upper $(-C)$-convex set-valued function (see Definition 5.1.4) is locally lower C-Lipschitz continuous in the sense of Definition 5.3.1.

Theorem 5.3.5. *Let X be a normed space, and Y be a linear topological space. Let $C \subset Y$ be a proper, closed, pointed, convex cone with a nonempty interior, and e be a given point in $\operatorname{int} C$. Let $F : X \rightrightarrows Y$ be a upper $(-C)$-convex set-valued function, and $F(x) + C$ is closed and convex for every $x \in X$. If F is C-bounded around a point $x_0 \in \operatorname{int}(\operatorname{dom} F)$, then F is locally lower C-Lipschitz continuous at x_0.*

Proof. Let $e \in \operatorname{int} C$. Because of the C-boundedness of F around x_0 (see Definition 5.3.2), there exist $t > 0$ and a neighborhood $U := x_0 + \mu U_X \subseteq \operatorname{int}(\operatorname{dom} F)$ of x_0 such that $F(U) \subseteq te - C$ and $F(U) \subseteq -te + C$. It follows that $-F(U) \subseteq -te + C$. Thus, for every $x \in U$, we have that $F(x) - F(x_0) \subseteq -2te + C$. This implies that

$$F(x) \subseteq -2te + F(x_0) + C. \tag{5.10}$$

From the assumptions of $F(x_0) + C$, it is obvious that $A := -F(x_0) - C$ is closed and (2.10) is fullfilled. Using the functional $\varphi_{A,-e}$ introduced in (2.11) with $A = -F(x_0) - C$ and $e \in \operatorname{int} C$, we can consider a function $G : X \to \overline{\mathbb{R}}$ given by

$$G(x) := \sup_{a \in F(x)} \varphi_{A,-e}(a),$$

where C is a proper, closed, pointed, convex cone and $e \in \operatorname{int} C$.

Applying Theorem 2.5.7(a), we get that $\varphi_{A,e}$ is C-monotone. Moreover, taking into account the inclusion (5.10), for all $a \in F(x)$, we can choose $b \in F(x_0)$ such that $a \in -2te + b + C$, and hence

$$\varphi_{A,-e}(a) \leq \varphi_{A,-e}(-2te + b) = 2t + \varphi_{A,-e}(b) \leq 2t + G(x_0).$$

It follows from the definition of the functional G that

$$G(x) \leq G(x_0) + 2t. \tag{5.11}$$

Thus, G is bounded from above around x_0.

Now we prove that G is convex. Indeed, it follows from the upper $(-C)$-convexity of F (see Definition 5.1.4) and the convexity of $\varphi_{A,-e}$ (since $A = -F(x_0) - C$ is convex) that for every $x, y \in X$ and $\alpha \in (0,1)$ we have

$$\begin{aligned} G(\alpha x + (1-\alpha)y) &= \sup_{a \in F(\alpha x + (1-\alpha)y)} \varphi_{A,-e}(a) \\ &\leq \sup_{a \in \alpha F(x) + (1-\alpha)F(y) + C} \varphi_{A,-e}(a) \\ &= \sup_{a_1 \in F(x), a_2 \in F(y), c \in C} \varphi_{A,-e}\big(\alpha a_1 + (1-\alpha)a_2 + C\big) \\ &= \sup_{a_1 \in F(x), a_2 \in F(y)} \varphi_{A,-e}\big(\alpha a_1 + (1-\alpha)a_2\big) \\ &\leq \alpha \sup_{a_1 \in F(x)} \varphi_{A,-e}(a_1) + (1-\alpha) \sup_{a_2 \in F(y)} \varphi_{A,-e}(a_2) \\ &= \alpha G(x) + (1-\alpha)G(y). \end{aligned}$$

Applying Lemma 2.2.12 to the scalar proper convex function G, we get that G is Lipschitz on a neighborhood $U' := x_0 + \mu' U_X$ of x_0 $(0 < \mu' < \mu)$ with the Lipschitz constant $L > 0$, which means that

$$\|G(x) - G(x_0)\| \leq L\|x - x_0\| \quad \text{for all} \quad x \in U.$$

Since $G(x_0) \leq 0$, we get

$$G(x) \leq L\|x - x_0\| \quad \text{for all} \quad x \in U,$$

which induces the conclusion due to Corollary 2.5.5. $\square$

Let us compare the results obtained in Theorem 5.3.4 and Theorem 5.3.5 with those previously known in the literature derived by Kuwano and Tanaka [50]. The assumptions in Theorem 5.3.4 and Theorem 5.3.5 are weaker than the ones in [50], since we do not need the conditions that C is a normal cone and the space X is a finite-dimensional space.

5.4 Lipschitz continuity of $\mathfrak{C}\mathfrak{s}$-convex set-valued functions

This section is devoted to the relationships of the $\mathfrak{C}\mathfrak{s}$-convexity of set-valued functions introduced in Definition 5.1.8 and the upper (lower) G-Lipschitzianity given in Definition 3.2.9. We also prove that a C-bounded, $\mathfrak{C}\mathfrak{s}$-convex function $F : X \rightrightarrows Y$ on a neighborhood of $x_0 \in \operatorname{int}(\operatorname{dom} F)$, then F is G-Lipschitz (see Definition 3.2.7) around x_0. In order to prove these assertions, we will use the equi-Lipschitzianity of a functional family in the sense of Definition 3.1.3

From now on (in this section), X, Y are two normed spaces, C is a proper, convex cone and $F : X \rightrightarrows Y$ is a set-valued mapping. We consider the functions G_{y^*}, g_{y^*} given by (5.1), (5.2) w.r.t. F.

At first, we need the following propositions.

Proposition 5.4.1. *Let X, Y be two normed spaces, $F : X \rightrightarrows Y$, and $x_0 \in \operatorname{int}(\operatorname{dom} F)$. If the family $\{G_{y^*} | y^* \in C^+, ||y^*||_* = 1\}$ is equi-Lipschitz around x_0, then F is lower G-Lipschitz around x_0.*

Proof. Suppose by contradiction that F is not lower G-Lipschitz around x_0, it follows that for any $n \in \mathbb{N}^*$, there are $x_n, x'_n \in B(x_0, \frac{1}{n})$ such that

$$F(x_n) \ominus_G F(x'_n) \nsubseteq n||x_n - x'_n||_X U_Y - C,$$

hence $x_n \neq x'_n$ for all $n \in \mathbb{N}^*$.
Since $x_0 \in \operatorname{int}(\operatorname{dom} F)$, for n large enough, $B(x_0, \frac{1}{n}) \subseteq \operatorname{dom} F$, and we can take $y_n \in F(x_n) \ominus_G F(x'_n)$ such that $y_n + F(x'_n) \subseteq F(x_n)$ and

$$y_n \notin B_n := n||x_n - x'_n||_X U_Y - C.$$

Since B_n is convex and $\operatorname{int} B_n \neq \emptyset$, one can find $y_n^* \in Y^*$ such that $||y_n^*||_* = 1$ and

$$y_n^*(y_n) \geq y_n^*(v) \quad \text{for all} \quad v \in B_n.$$

This implies that

$$y_n^*(y_n) \geq \sup y_n^*(B_n) = n||x_n - x'_n||_X - \sup y_n^*(C).$$

It follows that $y_n^* \in C^+$ for all $n \in \mathbb{N}$ and

$$G_{y_n^*}(x_n) = \sup y_n^*(F(x_n)) \geq \sup y_n^*(F(x'_n)) + y_n^*(y_n) \geq G_{y_n^*}(x'_n) + n||x_n - x'_n||_X.$$

Therefore,

$$n||x_n - x'_n||_X \leq G_{y_n^*}(x_n) - G_{y_n^*}(x'_n) \leq \ell||x_n - x'_n||_X.$$

This yields that $n \leq \ell$, which could not hold true for arbitrarily large n. □

Proposition 5.4.2. *Let X, Y be two normed spaces, $F : X \rightrightarrows Y$, and $x_0 \in \operatorname{int}(\operatorname{dom} F)$. If the family $\{g_{y^*} | y^* \in C^+, ||y^*||_* = 1\}$ is equi-Lipschitz around x_0, then F is upper G-Lipschitz around x_0.*

Proof. We prove by contradiction: Assuming that F is not upper G-Lipschitz around x_0, it follows that for any $n \in \mathbb{N}^*$, there are $x_n, x'_n \in B(x_0, \frac{1}{n})$ with

$$F(x_n) \ominus_G F(x'_n) \nsubseteq n||x_n - x'_n||_X U_Y + C,$$

hence $x_n \neq x'_n$ for all $n \in \mathbb{N}^*$.
Since $x_0 \in \operatorname{int}(\operatorname{dom} F)$, for n large enough, $B(x_0, \frac{1}{n}) \subseteq \operatorname{dom} F$, and we can take $y_n \in F(x_n) \ominus_G F(x'_n)$ such that $y_n + F(x'_n) \subseteq F(x_n)$ and

$$y_n \notin B_n := n||x_n - x'_n||_X U_Y + C.$$

Since the set B_n is convex and $\operatorname{int} B_n \neq \emptyset$, one can find $y^*_n \in Y^*$ such that $||y^*_n||_* = 1$ and

$$y^*_n(y_n) \leq y^*_n(v) \quad \text{for all} \quad v \in B_n.$$

Hence,

$$y^*_n(y_n) \leq \inf y^*_n(B_n) = -n||x_n - x'_n||_X + \inf y^*_n(C).$$

It follows that $y^*_n \in C^+$ for all $n \in \mathbb{N}$ and

$$g_{y^*_n}(x_n) = \inf y^*_n(F(x_n)) \leq \inf y^*_n(F(x'_n)) + y^*_n(y_n) \leq g_{y^*_n}(x'_n) - n||x_n - x'_n||_X.$$

Therefore

$$n||x_n - x'_n||_X \leq g_{y^*_n}(x'_n) - g_{y^*_n}(x_n) \leq \ell||x_n - x'_n||_X.$$

This yields that $n \leq \ell$, which could not hold true for arbitrarily large n. $\square$

Theorem 5.4.3. *Let X, Y be two normed spaces, C be a proper normal cone, and $F : X \rightrightarrows Y$ be $\mathfrak{C}\mathfrak{s}$-convex. If F is C-bounded on a neighborhood of $x_0 \in \operatorname{int}(\operatorname{dom} F)$, then F is G-Lipschitz around x_0.*

Proof. Without loss of generality we suppose that $x_0 = 0$ and $0 \in F(0)$. As F is C-bounded on a neighborhood $U = \theta U_X \subseteq \operatorname{dom} F$ of 0 ($\theta > 0$), and taking into account Definition 4.2.3, there exist real numbers $\mu, \mu' > 0$ such that $F(U) \subseteq \mu U_Y + C$ and $F(x) \subseteq \mu U_Y - C$ for all $x \in U$.
Take $y^* \in C^+$ with $||y^*||_* = 1$. Let $\bar{x} \in U$ be arbitrary, $\bar{y} \in F(\bar{x})$, $c \in C$, and $y' \in \mu' U_Y$ such that $\bar{y} = y' - c$. It follows from the definition g_{y^*} given by (5.2) that

$$\begin{aligned} g_{y^*}(\bar{x}) = \inf_{y \in F(\bar{x})} y^*(y) &\leq y^*(\bar{y}) = y^*(y' - c) \leq y^*(y') - y^*(c) \\ &\leq y^*(y') \leq \|y^*\|_* \|y'\|_Y = \|y'\|_Y \leq \mu', \quad \text{for all} \quad \bar{x} \in U. \end{aligned}$$

Analogously, from $F(U) \subseteq \mu U_Y + C$, we get $g_{y^*}(x) = \inf\limits_{y \in F(x)} y^*(y) \geq -\mu$ for every $x \in U$.
It follows that g_{y^*} is finite on U and

$$g_{y^*}(x) \leq g_{y^*}(0) + \mu + \mu', \quad \text{for all} \quad x \in U = \theta U_X.$$

By Proposition 5.1.7, g_{y^*} is convex. Applying Lemma 2.2.12 to the convex function g_{y^*} and $\theta' \in (0, \theta)$, we get

$$|g_{y^*}(x) - g_{y^*}(x')| \leq L\|x - x'\|_X, \quad \text{for all} \quad x, x' \in \theta' U_X,$$

where $L := (\mu + \mu')(\theta + \theta')/[\theta(\theta - \theta')]$, which clearly does not depend on y^*.
So $\{g_{y^*} \mid y^* \in C^+, ||y^*||_* = 1\}$ is equi-Lipschitz around x_0 with the Lipschitz constant L. Applying Proposition 5.4.2, we have that F is upper C-Lipschitz around x_0.
Analogously, $\{G_{y^*} \mid y^* \in C^+, ||y^*||_* = 1\}$ is equi-Lipschitz around x_0. Applying Proposition 5.4.1, F is lower C-Lipschitz around x_0. Hence, there exists $\ell > 0$ such that:

$$F(x) \ominus_G F(x') \subseteq \ell ||x - x'||_X U_Y + C, \quad \text{for all} \quad x, x' \in U, \tag{5.12}$$

and

$$F(x) \ominus_G F(x') \subseteq \ell ||x - x'||_X U_Y - C, \quad \text{for all} \quad x, x' \in U. \tag{5.13}$$

Since C is normal, there is $\rho > 0$ such that

$$(\rho U_Y + C) \cap (\rho U_Y - C) \subseteq U_Y. \tag{5.14}$$

From (5.12),(5.13),(5.14), we have

$$\frac{\rho(F(x) \ominus_G F(x'))}{\ell ||x - x'||_X} \subseteq (\rho U_Y + C) \cap (\rho U_Y - C) \subseteq U_Y,$$

for all $x, x' \in U, x \neq x'$. This shows that F is G-Lipschitzian around x_0. □

The following example, which is first given in [2, Example 6.2], illustrates the G-Lipschitz continuity of a convex set-valued mappings in Theorem 5.4.3.

Example 5.4.4. *Set $F : [0, +\infty) \times \mathbb{R} \rightrightarrows \mathbb{R}^2$ assuming*

$$F(x) = \operatorname{conv} \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ \sqrt{x_1} \end{pmatrix} \right\} \quad (x \in \mathbb{R}^2).$$

Obviously, F is C-bounded around every point $x \in \mathbb{R}^2$, and $\mathfrak{Cs}$-convex with $C := \mathbb{R}^2_+$. Hence F is locally G-Lipschitz. Like in [2, Example 6.2], F is not Lipschitz.

Chapter 6

Differentiability properties

For convenience of the reader, in this chapter, we recall some preliminary materials on basic normal cones, subdifferentials, derivatives, coderivatives and generalized differentiation, which will be used in the following chapters. These concepts will be considered not only for vector-valued functions, but also for set-valued functions. We refer the reader to [13, 35, 36, 59, 63, 67, 74] for more references and discussions.

6.1 Basic definitions

We begin this part by recalling several basic derivatives for vector-valued functions. Let X, Y be Banach spaces, and C be a proper, pointed, convex cone in Y. Consider a vector-valued function $f : X \to Y$. For each $\bar{x} \in X$, the "one-sided" ***directional derivative*** of f at $\bar{x}$ in the direction $v \in X$ is defined by

$$f'(\bar{x}, v) := \lim_{t \to 0^+} \frac{1}{t}(f(\bar{x} + tv) - f(\bar{x})), \tag{6.1}$$

when the limit exists in $\overline{\mathbb{R}}$.

The function f is said to be ***Gâteaux differentiable*** at $\bar{x} \in X$, if there exists a continuous linear functional denoted by $f'(\bar{x}) : X \to Y$, such that for every $v \in X$, $f'(\bar{x}, v)$ exists and $f'(\bar{x}, v) = f'(\bar{x})(v)$. The function $f'(x)$ is called the ***Gâteaux derivative*** (or ***Gâteaux differential***) of f at $x \in X$. This means that the following difference quotient conveges for each $v \in X$:

$$f'(\bar{x})(v) = \lim_{t \to 0^+} \frac{1}{t}(f(\bar{x} + tv) - f(\bar{x})), \tag{6.2}$$

and the convergence is uniform w.r.t. v in finite sets. If the convergence (6.2) is uniform w.r.t. v in bounded sets, f is said to be ***Fréchet differentiable*** at $\bar{x}$. This is equivalent to:

$$\lim_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - f'(\bar{x})(x - \bar{x})}{\|x - \bar{x}\|} = 0. \tag{6.3}$$

Moreover, the function f is said to be ***strictly differentiable*** at $\bar{x}$ if

$$\lim_{x\to\bar{x},u\to\bar{x}} \frac{f(x)-f(u)-f'(\bar{x})(x-u)}{\|x-u\|} = 0. \tag{6.4}$$

Note that if f is strictly differentiable at $\bar{x}$, then f is Lipschitz around this point; see Clarke [13, Proposition 2.2.1].

Example 6.1.1. *Consider the convex function $f(x) = \|x\|$, it is easy to verify that f has directional derivative at every point $x \in X$ and f is Gâteaux differentiable at every point $x \neq 0$, but there does not exist a Gâteaux derivative at $x = 0$.*

6.2 Subdifferentials of convex functions

This section contains a brief summary of the subdifferentials of convex functionals as well as the subdifferentials of convex vector-valued functions.

6.2.1 The Fenchel subdifferential of convex analysis

In this part, we consider a convex function $f : X \to \overline{\mathbb{R}}$. We have the following proposition which is fundamental to the study of differentiability of convex functions.

Proposition 6.2.1. *([59, Lemma 1.2]) Let X be a Banach space. If $f : X \to \overline{\mathbb{R}}$ is a proper convex function, then the directional derivative of f exists at every point $x \in \operatorname{dom} f$ and*

$$f'(x,v) = \inf_{t>0} \frac{1}{t}(f(x+tv) - f(x)), \tag{6.5}$$

In the following definitions, we consider the normal cone and the subdifferential in the sense of convex analysis (or Fenchel subdifferential) of convex functions defined as follows.

Definition 6.2.2. *([74, Section 2.4]) Let X be a Banach space and $f : X \to \overline{\mathbb{R}}$ be a proper convex function, the subdifferential or Fenchel subdifferential of f at $\bar{x} \in \operatorname{dom} f$ is defined by*

$$\partial f(\bar{x}) := \{x^* \in X^* \mid \forall x \in X : f(x) - f(\bar{x}) \geq x^*(x-\bar{x})\}, \tag{6.6}$$

for $\bar{x} \notin \operatorname{dom} f$ one puts $\partial f(\bar{x}) = \emptyset$. If $\partial f(\bar{x})$ is nonempty, f is said to be subdifferentiable at $\bar{x}$.

The formulation (6.6) can also be written as

$$\partial f(\bar{x}) = \{x^* \in X^* \mid \forall x \in X : x^*(x) \leq f'(\bar{x}, x)\}, \quad \bar{x} \in \operatorname{dom} f. \tag{6.7}$$

Taking into account (6.7), it is easy to see that $f'(\bar{x}, \cdot) \in \partial f(\bar{x})$. Moreover, if f is a proper convex function, then $\partial f(\bar{x}) \neq \emptyset$.

Let us note that a continuous convex f is Gâteaux differentiable at $\bar{x} \in \operatorname{dom} f$ if and only if $\partial f(\bar{x})$ is a singleton; see [74, Corollary 2.4.10].

Definition 6.2.3. *Let A be a nonempty convex subset of a Banach space X. The normal cone to A at $\bar{x} \in A$ is defined by*

$$N(\bar{x}, A) := \{x^* \in X^* \mid x \in A : x^*(x - \bar{x}) \leq 0\}.$$

It follows directly from Definition 6.2.2 and 6.2.3 that the normal cone to a set A at a given point can also be equivalently defined by the subdifferential of the indicator function associated with this set at that point,

$$N(\bar{x}, A) = \partial \delta_A(\bar{x}), \tag{6.8}$$

where δ_A is the indicator function of A.

The following proposition presents some calculus rules for subdifferentials of convex functions.

Proposition 6.2.4. *([74, Theorem 2.4.2]) Let $f, g : X \to \overline{\mathbb{R}}$ be proper convex functions on X, $x \in X$. We have some basic formulae:*

(i) For any scalar t, we have
$$\partial(tf)(x) = t\partial f(x).$$

(ii) We have the following sum rule

$$\partial f(x) + \partial g(x) \subseteq \partial(f + g)(x).$$

The equality holds if $x \in \operatorname{dom} f \cap \operatorname{dom} g$ and one of the functions is continuous.

6.2.2 Subdifferential of convex vector-valued functions

In the sequel X, Y are are Banach spaces and C is a proper, pointed, convex cone in Y. Now we consider a proper vector-valued function $f : X \to Y^\bullet$.

Definition 6.2.5. *The* ***subdifferential*** *of f at $\bar{x} \in \operatorname{dom} f$ is given by*

$$\partial^{\leq} f(\bar{x}) := \{T \in L(X, Y) \mid \forall x \in X : f(x) - f(\bar{x}) \geq_C T(x - \bar{x})\}. \tag{6.9}$$

If $\bar{x} \in X \setminus \operatorname{dom} f$ we set $\partial^{\leq} f(\bar{x}) = \emptyset$. An element $T \in \partial^{\leq} f(\bar{x})$ is called a ***subgradient*** *of f at $\bar{x}$.*

In the case that $Y = \mathbb{R}$, $C = \mathbb{R}_+ := \{\alpha \in \mathbb{R} \mid \alpha \geq 0\}$ and f is convex, (6.9) reduces to the classical definition of the subdifferential in the sense of Definition 6.2.2.

We recall some properties of (strong) subdifferentials of convex functions in the following proposition.

Proposition 6.2.6. *(i) ([44, Corollary 6.1.10]) Let $f : X \to Y^\bullet$ be a proper C- convex mapping, and $\bar{x}$ be a given point in* $\operatorname{int}(\operatorname{dom} f)$*. Then, $\partial^{\leq} f(\bar{x})$ is nonempty.*

(ii) ([67, Lemma 2.2]) Let $f : X \to Y^\bullet$ be a proper C-convex vector-valued function and let $g : Y \to \mathbb{R} \cup \{+\infty\}$ be convex and C-monotone on Y. If there exists $(x_0; y_0) \in \operatorname{epi} f$ such that g is continuous at y_0, then for $\bar{y} = f(\bar{x}) \in \operatorname{dom} g$ one has

$$\partial(g \circ f)(\bar{x}) = \bigcup_{y^* \in \partial g(\bar{y})} \partial(y^* \circ f)(\bar{x}).$$

Furthermore, if we assume additionally that C has a weakly compact base, we get the well-known result of Valadier [71], which is useful in the sequel.

Theorem 6.2.7. (Valadier [71]) *Let X, Y be real reflexive Banach spaces, and $C \subset Y$ be a proper, convex cone with a weakly compact base. If $f : X \to Y$ is a C-convex mapping, continuous at some point of its domain, then for every $x \in X$ and $y^* \in C^+$ one has*

$$y^* \circ \partial^{\leq} f(x) = \partial(y^* \circ f)(x).$$

6.3 Clarke's normal cone and subdifferential

Now we extend the notions of the directional derivative and the subdifferential from convex functions to locally Lipschitz functions by defining generalized directional derivatives and generalized gradients (or Clarke's subdifferentials) that were first introduced by Clarke [13].

Definition 6.3.1. *([13]) Let X be a Banach space and $f : X \to \mathbb{R}$ be Lipschitz around a given point $x \in \operatorname{dom} f$. For each $v \in X$, the **generalized directional derivative** of f at x in the direction v is defined by*

$$f^\circ(x, v) := \limsup_{y \to x, t \to 0^+} \frac{1}{t}(f(y + tv) - f(y)). \tag{6.10}$$

Definition 6.3.2. *([13]) Let X be a Banach space and $f : X \to \mathbb{R}$ be locally Lipschitz, the **generalized gradient** of f at $\bar{x} \in \operatorname{dom} f$ is defined by*

$$\partial_C f(\bar{x}) := \{x^* \in X^* \mid \forall v \in X : f^\circ(x, v) \geq x^*(v)\}. \tag{6.11}$$

Note that if f is Lipschitz around $\bar{x} \in \operatorname{dom} f$ and admits a Gâteaux derivative $f'(\bar{x})$, then $f'(\bar{x}) \in \partial_C f(\bar{x})$; see Clarke [13].

Proposition 6.3.3. *(Clarke [13, Proposition 2.2.7 and Section 2.3]) Let X be a Banach space. We assume that functions $f, g : X \to \mathbb{R}$ are Lipschitz around a given point $x \in X$. We have some basic calculus:*

(i) For any scalar t, we have

$$\partial_C(tf)(x) = t\partial_C f(x).$$

(ii) The sum rule

$$\partial_C(f+g)(x) \subseteq \partial_C f(x) + \partial_C g(x).$$

The equality holds if one of the functions is strictly differentiable at x.

(iii) If f is convex on an open convex subset U of X and f is Lipschitz around a given point x, then the generalized gradient of f coincides with the Fenchel's subdifferential of f at x, and the generalized directional derivative of f coincides with its directional derivative.

Let A be a nonempty subset of a Banach space X; it is not necessary to suppose A to be convex. Taking a point $x \in A$, we define the set of all tangents to A at x by

$$T(A,x) := \{v \in X \mid d_A^\circ(x,v) = 0\},$$

where d_A° is the generalized directional derivative of the distance function $d(\cdot, A)$ at x in the direction v; see Clarke [13, Section 2.4]. A vector $v \in T(A,x)$ is called a tangent to A at x.

We denote the **Clarke normal cone** to A at $\bar{x} \in A$ by $N_C(\bar{x}, A)$ and

$$N_C(\bar{x}, A) := \{x^* \in X^* \mid \forall v \in T(A,x) : x^*(x) \leq 0\}.$$

We have the following property which is considered as another definition of normal cone using Clarke' subdifferential of the distance function.

$$N_C(\bar{x}, A) = \operatorname{cl}^*\big(\cup_{\lambda>0} \lambda \partial_C d(x.A)\big), \tag{6.12}$$

where cl^* denotes the weak* closure. This cone is also equivalent to

$$N_C(\bar{x}, A) = \partial_C \delta_A(\bar{x}). \tag{6.13}$$

The following proposition presents the correlation between the generalized gradient of a function and the normal cone to its epigraph.

Proposition 6.3.4. *([13, Corollary 2.4.9]) Let X be a Banach space, $f : X \to \mathbb{R}$ be Lipschitz around a given point x. An element x^* of X^* belongs to $\partial_C f(x)$ if and only if $(x^*, -1)$ belongs to $N_C\big((x, f(x)), \operatorname{epi} f\big)$.*

In [13], Clarke first defined the generalized gradient for a locally Lipschitz function, and then defined the corresponding normal cones via the generalized gradient of the distance function as in (6.12), and finally, the author extended Clarke subdifferentials to functions which are not necessarily locally Lipschitz.

Definition 6.3.5. *([13]) Let X be a Banach space and $f : X \to \overline{\mathbb{R}}$. The* ***Clarke subdifferential*** *of f at $\bar{x}$ is the set*

$$\partial_C f(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in N_C\big((x, f(x)), \operatorname{epi} f\big)\} \tag{6.14}$$

if $\bar{x} \in \operatorname{dom} f$ and $\partial_C f(\bar{x}) := \emptyset$ if $\bar{x} \notin \operatorname{dom} f$.

We also have the sum rule for the Clarke subdifferential, when f is Lipschitz around x and g is lower semicontinuous around x,

$$\partial_C(f+g)(x) \subseteq \partial_C f(x) + \partial_C g(x).$$

Proposition 6.3.4 guarantees that the definition of Clarke subdifferential given by (6.14) is consistent with the generalized gradient for the locally Lipschitz case. In Section 6.4, we will recall this approach for Mordukhovich's subdifferential, by which the normal cones to a set will be defined first, and then the corresponding subdifferentials of a function are defined thanks to the normal cones to its epigraph.

6.4 Mordukhovich's limiting subdifferential

This section is devoted to presenting definitions and properties of basic generalized differential constructions held in ***Asplund spaces***. In [55], Mordukhovich studied the hierarchy of generalized normal cones, coderivatives, and limiting subdifferentials. We recall in this part some main calculus for normal cones and coderivatives, for more details, the reader can find them in Mordukhovich's books [55, 56].

Definition 6.4.1. *([59]) A Banach space X is called an Asplund space if every convex continuous function on a nonempty open convex subset D of X is Fréchet differentiable at each point of some nonempty dense G_δ subset of D.*

The class of Asplund spaces is quite broad, and contains every reflexive Banach space, as well as every Banach space with the separable dual. In particular, c_0 and l_p, $L^p[0,1]$ for $1 < p < +\infty$ are Asplund spaces, but l_1 and l_∞ are not Asplund spaces.

Consider a set-valued mapping $F : X \rightrightarrows X^*$ between an Asplund space and its dual, and a subset Ω of X. We define the Painlevé-Kuratowski outer limit of F at $\bar{x}$ w.r.t. the norm topology of X and the weak* topology of X^* by

$$\limsup_{x \to \bar{x}} F(x) := \{x^* \in X^* \mid \forall k \in \mathbb{N},\ \exists (x_k, x_k^*) \in \operatorname{gph} F : x_k \to \bar{x},\ x_k^* \xrightarrow{w^*} x^*\}.$$

In this section, we use the notation $x' \xrightarrow{\Omega} x$ for $x' \to x$ with $x' \in \Omega$. We define the generalized normal cone to Ω at $x \in \Omega$ in a Banach space as follows.

Definition 6.4.2. *([55, Definition 1.1]) Let Ω be a nonempty subset of a Banach space X.*

1. *Given $x \in \Omega$ and $\epsilon \geq 0$, define the **set of ϵ-normals** to Ω at x by*
$$\hat{N}_\epsilon(x,\Omega) := \left\{ x^* \in X^* \mid \limsup_{u \xrightarrow{\Omega} x} \frac{x^*(u-x)}{\|u-x\|} \leq \epsilon \right\}. \tag{6.15}$$
*When $\epsilon = 0$, the set (6.15) is called **Fréchet normal cone** to Ω at x, denoted by $\hat{N}(x,\Omega)$. If $x \notin \Omega$, we put $\hat{N}_\epsilon(x,\Omega) := \emptyset$ for all $\epsilon \geq 0$.*

2. *Let $\bar{x} \in \Omega$, the **(basic, limiting, or Mordukhovich) normal cone** to Ω at $\bar{x}$ is defined by*
$$N_L(\bar{x};\Omega) := \limsup_{\substack{x \to \bar{x} \\ \epsilon \downarrow 0}} \hat{N}_\epsilon(x;\Omega). \tag{6.16}$$
Put $N_L(\bar{x};\Omega) := \emptyset$ for $\bar{x} \notin \Omega$.

Now, if X is an Asplund space and Ω is closed around a given point $\bar{x} \in \Omega$, i.e., there is a neighborhood U of $\bar{x}$ such that $\Omega \cap U$ is a closed set. Then, the limiting normal cone to Ω at $\bar{x}$ is also presented by (compare [55, Theorem 1.6])

$$\begin{aligned} N_L(\bar{x};\Omega) &= \limsup_{x \to \bar{x}} \quad \hat{N}(x;\Omega) \qquad (6.17) \\ &= \{x^* \in X^* \mid \exists x_k \xrightarrow{\Omega} \bar{x}, x_k^* \xrightarrow{w^*} x^*, \forall k \in \mathbb{N} : x_k^* \in \hat{N}(x_k;\Omega)\}. \end{aligned}$$

If Ω is a convex set, then both the Fréchet normal cone and the limiting normal cone reduce to the normal cone of convex analysis; see [55, Proposition 1.5]:

$$\hat{N}(\bar{x};\Omega) = N_L(\bar{x};\Omega) = \{x^* \in X^* \mid \forall x \in \Omega : x^*(x-\bar{x}) \leq 0\}. \tag{6.18}$$

Let us now recall the ***sequential normal compactness*** property of sets, which shows the equivalence between the weak* and norm convergences to zero of ϵ-normals (6.15) in dual spaces.

Definition 6.4.3. *([55, Definition 1.20]) Let Ω be a nonempty subset of a Banach space X and $\bar{x}$ be a given point of Ω. We say that Ω is **sequentially normally compact** (SNC) at $\bar{x}$ if for any sequence $(\epsilon_k, x_k, x_k^*) \in [0,\infty] \times \Omega \times X^*$ satisfying*

$$\epsilon_k \downarrow 0, x_k \to \bar{x}, x_k^* \in \hat{N}_{\epsilon_k}(x_k;\Omega), \text{ and } x_k^* \xrightarrow{w^*} 0,$$

one has $\|x_k^*\| \to 0$ as $k \to +\infty$.

Remark 6.4.4. *As a consequence of [55, Theorem 1.21], every nonempty set in a finite dimensional space is SNC at each of its points. In addition, this property is also fulfilled for a convex set with a nonempty interior; see [55, Proposition 1.25, and Theorem 1.26].*

As pointed out in Section 6.3, it is possible to define subdifferentials of an extended-real-valued function through the normal cones to its epigraph. Next, we define the (basic, limiting, Mordukhovich) subdifferential.

Definition 6.4.5. *([55, Definition 1.77]) The* ***(basic, limiting, Mordukhovich) subdifferential*** *for a given function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ at $\bar{x} \in X$ with $|f(\bar{x})| < +\infty$ is defined by*

$$\partial_L f(\bar{x}) := \big\{x^* \in X^* \mid (x^*, -1) \in N_L\big((\bar{x}, f(\bar{x})); \operatorname{epi} f\big)\big\}.$$

We put $\partial_L f(\bar{x}) := \emptyset$ if $|f(\bar{x})| = +\infty$.

Note that the limiting subdifferential agrees with the classical gradient for strictly differentiable functions, and becomes the subdifferential of convex analysis when f is convex; see [55, Theorem 1.93].

Now we recall the sum rule (see [55, Theorem 3.36]) and the chain rule (see [55, Theorem 3.41 and Corollary 3.43]) for the limiting subdifferential of locally Lipschitzian functions.

Proposition 6.4.6. *We consider Asplund spaces X and Y.*

(i) ***(Sum rule)*** *Let $\varphi_i : X \to \mathbb{R} \cup \{\pm\infty\}, i = 1, 2, \ldots, n, n \geq 2$, be lower semicontinuous around $\bar{x}$, and let all but one of these functions be locally Lipschitz around $\bar{x}$. Then, one has the following inclusion*

$$\partial_L(\varphi_1 + \varphi_2 + \ldots + \varphi_n)(\bar{x}) \subseteq \partial_L\varphi_1(\bar{x}) + \partial_L\varphi_2(\bar{x}) + \ldots + \partial_L\varphi_n(\bar{x}). \tag{6.19}$$

In addition, if each φ_i is convex (or strictly differentiable), then (6.19) holds as equality.

(ii) ***(Chain rule)*** *Let $g : X \to Y$ be strictly Lipschitz at $\bar{x}$, and $\varphi : Y \to \mathbb{R}$ be locally Lipschitzian around $g(\bar{x})$. Then, one has*

$$\partial_L(\varphi \circ g)(\bar{x}) \subseteq \bigcup_{y^* \in \partial_L\varphi(g(\bar{x}))} \partial_L(y^* \circ g)(\bar{x}). \tag{6.20}$$

Now considering a set-valued function $F : X \rightrightarrows Y$ between two Banach spaces, and a proper, convex cone C in Y. The graph and the epigraph of F w.r.t. the cone C are defined by

$$\begin{aligned} \operatorname{gph} F &:= \big\{(x, y) \in X \times Y \mid y \in F(x)\big\}, \\ \operatorname{epi} F &:= \big\{(x, y) \in X \times Y \mid y \in F(x) + C\big\}. \end{aligned}$$

The epigraphical multifunction of $F : X \rightrightarrows Y$, $\mathcal{E}_F : X \rightrightarrows Y$, is defined by

$$\mathcal{E}_F(x) := F(x) + C;$$

and hence, $\operatorname{gph} \mathcal{E}_F = \operatorname{epi} F$.

We continue defining the ***(basic, normal, Mordukhovich)*** **coderivative** at the reference point.

Definition 6.4.7. *([55, Definition 1.32]) Let $F : X \rightrightarrows Y$ be a set-valued function between two Banach spaces with* $\operatorname{dom} F \neq \emptyset$.

1. *Given $(x, y) \in X \times Y$ and $\epsilon \geq 0$, we define the **ϵ-coderivative** of F at (x, y) as a multifunction $\hat{D}^*_\epsilon F(x, y) : Y^* \rightrightarrows X^*$ with the values*

$$\hat{D}^*_\epsilon F(x, y)(y^*) := \left\{ x^* \in X^* \mid \quad (x^*, -y^*) \in \hat{N}_\epsilon((x, y); \operatorname{gph} F) \right\}. \tag{6.21}$$

   *When $\epsilon = 0$, it is called the **precoderivative or Fréchet coderivative** of F at (x, y) and is denoted by $\hat{D}^* F(x, y)$. It follows directly from the definition that $\hat{D}^*_\epsilon F(x, y)(y^*) = \emptyset$ for all $\epsilon \geq 0$ and $y^* \in Y^*$ if $(x, y) \notin \operatorname{gph} F$.*

2. *The **(basic, normal, Mordukhovich) coderivative** of F at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is a multifunction $D^*_N F(x, y) : Y^* \rightrightarrows X^*$ defined by*

$$D^* F(\bar{x}, \bar{y})(y^*) := \limsup_{\substack{(x,y) \to (\bar{x}, \bar{y}) \\ y^* \xrightarrow{\omega^*} \bar{y}^* \\ \epsilon \downarrow 0}} \hat{D}^*_\epsilon F(x, y)(y^*). \tag{6.22}$$

   We put $D^* F(\bar{x}, \bar{y})(y^*) := \emptyset$ for all $y^* \in Y^*$ if $(\bar{x}, \bar{y}) \notin \operatorname{gph} F$.

Because of the definition of the limiting normal cone, the (basic, normal, Mordukhovich) *coderivative* of F can be defined through the corresponding normal cone as follows

$$D^* F(\bar{x}, \bar{y})(y^*) = \{x^* \in X^* \mid (x^*, -y^*) \in N_L\big((\bar{x}, \bar{y}); \operatorname{gph} F\big)\}. \tag{6.23}$$

We can omit $\bar{y}$ in the coderivative notation above if $F = f : X \to Y$ is a vector-valued function. If f is strictly differentiable at $\bar{x}$, then

$$D^* f(\bar{x})(y^*) = \{\nabla f(\bar{x})^* y^*)\} \quad \text{for all} \quad y^* \in Y^*.$$

Furthermore, if f is strictly Lipschitzian at $\bar{x}$, the relationship between the coderivative of a vector function and the subdifferential of its scalarization is given by [55, Theorem 3.28]:

$$D^* f(\bar{x})(y^*) = \partial_L (y^* \circ f)(\bar{x}).$$

In [5], Bao and Mordukhovich introduced a subdifferential notion for vector-valued and set-valued mappings with values in partially ordered spaces by using coderivatives of the epigraphical multifunction.

Definition 6.4.8. *([5]) Let $F : X \rightrightarrows Y$ be a set-valued function between two Banach spaces with $\operatorname{dom} F \neq \emptyset$. Given $(\bar{x}, \bar{y}) \in X \times Y$, we define the **basic/normal subdifferential** of F at $(\bar{x}, \bar{y})$ in direction $y^* \in Y^*$ by*

$$\partial F(\bar{x}, \bar{y})(y^*) := D^* \mathcal{E}_F(\bar{x}, \bar{y})(y^*). \tag{6.24}$$

In the case of a single-valued function $F := f : X \to Y$, the subdifferential of f at $\bar{x}$ in (6.24) is given by

$$\partial f(\bar{x})(y^*) := D^*\mathcal{E}_f(\bar{x})(y^*). \tag{6.25}$$

Moreover, when $F = f : X \to (-\infty, \infty]$ is a lower semicontinuous function, the subdifferential (6.24) with $\|y^*\| = 1$ agrees with the limiting subdifferential.

6.5 Ioffe's approximate subdifferential

In this section, we study Ioffe's approximate subdifferential and approximate normal cone, which are considered in arbitrary Banach spaces. These structures were first introduced in the series of works by Ioffe starting from 1981.

We suppose that X is a Banach space and $f : X \to \overline{\mathbb{R}}$ is lower semicontinuous on X, and $x \in \operatorname{dom} f$. Let $\mathcal{F}$ be the collection of all finite-dimensional subspaces of X. In [35], the ***approximate subdifferential*** of f at x is given by

$$\partial_A f(x) := \bigcap_{L \in \mathcal{F}} \limsup_{(\epsilon, y) \to (+0, x)} \partial_\epsilon^- f_{y+L}(y),$$

where

$$f_{y+L}(u) := \begin{cases} f(u) & \text{if} \quad u \in y + L, \\ \infty & \text{otherwise,} \end{cases}$$

and for $\epsilon > 0$,

$$\partial_\epsilon^- f_{y+L}(y) := \Big\{ x^* \in X^* \mid \forall v \in X : x^*(v) \le \epsilon \|v\| + \liminf_{t \to +0} t^{-1}[f_{y+L}(y + tv) - f_{y+L}(y)] \Big\}.$$

Note that the construction of approximate normal cones is similar to that of Clarke normal cones. Therefore, we can define the ***approximate normal cone*** to $\Omega \subseteq X$ at $x \in \Omega$ via the approximate subdifferential of either the distance function or the indicator function associated with this set as

$$N_A(x; \Omega) := \bigcup_{\lambda > 0} \lambda \partial_A d(x, \Omega) = \partial_A \delta_\Omega(x).$$

Moreover, one can present the approximate subdifferential via the approximate normal cone above by the following equality

$$\partial_A f(\bar{x}) = \big\{ x^* \in X^* \mid (x^*, -1) \in N_A\big((\bar{x}, f(\bar{x})); \operatorname{epi} f\big) \big\}.$$

Using the approximate normal cone, we can define the approximate coderivative of a set-valued function $F : X \rightrightarrows Y$ as in the next definition.

Definition 6.5.1. *Let $F : X \rightrightarrows Y$ be a set-valued function between two Banach spaces with $\operatorname{dom} F \neq \emptyset$, and $(x, y) \in \operatorname{gph} F$. Assuming that F is closed, and the **approximate coderivative** of F at (x, y) is a multifunction $D_A^* F(x, y) : Y^* \rightrightarrows X^*$ defined by*

$$D_A^* F(x, y)(y^*) := \big\{ x^* \in X^* \mid (x^*, -y^*) \in N_A\big((\bar{x}, \bar{y}); \operatorname{gph} F\big) \big\}. \tag{6.26}$$

Now we show some properties of the approximate subdifferential for Banach spaces X and Y.

Proposition 6.5.2. *(Ioffe [35, Section 3]) We assume that given functions $f, g : X \to \overline{\mathbb{R}}$ are lower semicontinuous on their domains. Assuming that $x \in \operatorname{dom} f$ or $x \in \operatorname{dom} g$, we have the following properties:*

(i) ***(Sum rule)*** *If f is Lipschitz around x and $x \in \operatorname{dom} f \cap \operatorname{dom} g$, then*

$$\partial_A (f + g)(x) \subseteq \partial_A f(x) + \partial_A g(x).$$

(ii) If f attains a local minimum at x, then $0 \in \partial_A f(x)$.

(iii) If f is strictly differentiable at x, then $\partial_A f(x) = \{f'(x)\}$.

In the previous sections, we described the constructions of Fenchel subdifferentials, Clarke subdifferentials as well as Mordukhovich subdifferentials. To finish this section, we establish relationships between them and Ioffe subdifferentials in the framework of Banach spaces and Asplund spaces.

Proposition 6.5.3. *(Mordukhovich [55, Section 3.2.3], Ioffe [35, Section 3]) Let X be an Asplund space, and $f : X \to \overline{\mathbb{R}}$ be lower semicontinuous and Lipschitz around $x \in \operatorname{dom} f$. We have the following properties:*

$$\partial_C f(x) = \operatorname{cl} \operatorname{conv} \partial_A f(x) = \operatorname{cl}^* \operatorname{conv} \partial_L f(x),$$

$$\partial_A f(x) = \operatorname{cl}^* \partial_L f(x),$$

and hence

$$\partial_L f(x) \subseteq \partial_A f(x) \subseteq \partial_C f(x).$$

Moreover, if X is a weakly compactly generated space (i.e., $X = \operatorname{cl}(\operatorname{span} K)$ for some weakly compact set $K \subset X$), then

$$\partial_L f(x) = \partial_A f(x).$$

In the latter case if f is convex and f continuous at least at one point (i.e., f is Lipschitz around that point, see Chapter 4), then these subdifferentials coincide with the subdifferential of convex analysis.

Note that the relationships between the Clarke subdifferential and Ioffe subdifferential above also hold in Banach spaces.

6.6 Derivatives of set-valued functions

Derivatives of set-valued functions are essential in optimization theory. They are used in primal-space approaches to derive optimality conditions for solutions of set-valued optimization problems. There are many approaches to define derivatives of set-valued functions, for instance: contingent (Bouligand), Ursescu, Dubovitskij - Miljutin and Dini derivatives, etc. In this section, we focus on contigent derivatives and contingent epiderivatives, which are motivated by the geometric interpretation of the classical notion of derivative for single-valued functions as a local approximation of their graphs and epigraphs.

We begin with a brief introduction of the contingent cone (or the Bouligand tangent cone) of a set $S \subseteq X$ at a given point $x \in X$.

Definition 6.6.1. *Let X be a normed vector space, S be a subset of X, and $x \in X$ be given. The* ***contingent cone*** *of S at x is a set*

$$T(S,x) := \{u \in X \mid \exists (t_n) \downarrow 0, \exists (u_n) \to u, x + t_n u_n \in S\},$$

where $(t_n) \downarrow 0$ means $(t_n) \subset (0,+\infty)$ and $(t_n) \to 0$.

In the literature, the contingent cone has been widely used in optimization theory and variational analysis, and it has been known under many different names such as the **Bouligand tangent cone**, the **tangent cone**, the **cone of adherent dispalcements**, the **outer tangent cone**, etc. We refer the reader to [44] for more notions and more discussions of contingent cones. In [44, Theorem 4.1.12], the authors introduce an equivalence of sixteen different characterizations of the contingent cone including detailed proofs. We present here two characterizations which are more popular than the others.

Proposition 6.6.2. *([44, Theorem 4.1.12]) Let X be a normed vector space, S be a subset of X, and let $x \in \operatorname{cl} S$. Then, for $i \in \{1,2\}$, we have $T(S,x) = T_i(S,x)$, where $T_i(S,x)$ are given as follows:*

$$T_1(S,x) := \{u \in X \mid \exists (t_n) \subset \mathbb{R}, \exists (u_n) \subset S \text{ such that } u_n \to x, t_n(u_n - x) \to u\}$$
$$T_2(S,x) := \{u \in X \mid \exists (t_n) \subset \mathbb{R}, \exists (u_n) \subset S \text{ such that } t_n \downarrow 0, \frac{u_n - x}{t_n} \to u\}.$$

Note that, in the special case that S is convex, we have $T(S,x) = \operatorname{cl} \operatorname{cone}(S - x)$.

Now we introduce the notion of the contingent derivative via the contigent cone above.

Definition 6.6.3. *Let X and Y be normed spaces, $F : X \rightrightarrows Y$ be a set-valued function, and $(x,y) \in \operatorname{gph} F$ be given. A* ***contingent derivative*** *of F at (x,y) is a set-valued map $D_c F(x,y) : X \rightrightarrows Y$ such that*

$$\operatorname{gph}(D_c F(x,y)) := T(\operatorname{gph} F, (x,y)).$$

Because of the equivalence of the characterizations of the contingent cone, and due to Definition 6.6.1 we get the following properties of the contingent derivative.

Proposition 6.6.4. *([44, Theorem 11.1.8]) Let X, Y be two normed vector spaces, S be a subset of X. Let $F : X \rightrightarrows Y$ be a set-valued function, and $(x, y) \in \operatorname{gph} F$. The following assertions hold:*

(i) A pair (u, v) belongs to $\operatorname{gph}(D_c F(x, y))$ if and only if there are sequences $\{t_n\} \downarrow 0$ and $\{(u_n, v_n)\} \subset X \times Y$ with $(u_n, v_n) \to (u, v)$ such that $y + t_n v_n \in F(x + t_n u_n)$, for every $n \in N$.

(ii) A pair (u, v) belongs to $\operatorname{gph}(D_c F(x, y))$ if and only if

$$\liminf_{(\bar{u},t)\to(u,0^+)} d\Big(\frac{F(x + t\bar{u}) - y}{t}, v\Big) = 0.$$

It follows from the definitions of the contigent derivative and the properties of the contingent cone that the contigent derivative is a natural extension of the Fréchet differentiability concept to the set-valued case; see [37, Remark 15.2].

Now we consider an ordering relation $\leq_C$ on the normed space Y, which is generated by a proper convex cone $C \subset Y$, and we recall that the *epigraph* of $F : X \rightrightarrows Y$ with respect to C is given by

$$\operatorname{epi} F := \big\{(x, y) \in X \times Y \mid y \in F(x) + C\big\}.$$

In case $F : S \rightrightarrows Y$, where S is a nonempty subset of X, the *epigraph* of F, also denoted by $\operatorname{epi} F$ if there is no confusion, is given by

$$\operatorname{epi} F := \big\{(x, y) \in X \times Y \mid x \in S, y \in F(x) + C\big\}.$$

We introduce another notion of derivatives of set-valued maps in the following definition. This is a useful tool for the formulation of optimality conditions in set optimization.

Definition 6.6.5. *Let X and Y be normed spaces, let S be a nonempty subset of X, and let C be a proper, convex cone. Let $F : S \rightrightarrows Y$ be a set-valued function, and let a pair $(x, y) \in \operatorname{gph} F$ with $x \in S$ be given. A **contingent epiderivative** of F at (x, y) is a single-valued map $D_e F(x, y) : X \to Y$ such that*

$$\operatorname{epi}(D_e F(x, y)) = T(\operatorname{epi} F, (x, y)).$$

The contingent epiderivative was originally proposed by Jahn and Rauh [41]. Following the idea of the definition above but taking a different tangent cone for the local approximation, one can derive other epiderivatives such as the *adjacent epiderivative* and the *Clarke epiderivative*; see [44, Chapter 11]. Note that the contingent epiderivative is a single-valued function while the contingent derivative is a set-valued function. Moreover, in the case that F is lower C-convex, we get the following relationship between the contingent derivative and the contingent epiderivative.

Proposition 6.6.6. *(see [37, Theorem 15.9]) Let X and Y be normed spaces, let S be a nonempty convex subset of X, and let C be a proper, pointed, convex cone. Let $F : X \rightrightarrows Y$ be a lower C-convex set-valued function, and let a pair $(x, y) \in \operatorname{gph} F$ with $x \in S$ be given. If both the contingent derivative $D_c F(x, y)$ and the contingent epiderivative $D_e F(x, y)$ exist, then*

$$\operatorname{epi}(D_c F(x, y)) \subseteq \operatorname{epi}(D_e F(x, y)).$$

When F is lower C-convex, we also obtain the following property of the contigent epiderivative.

Proposition 6.6.7. *(see [37, Theorem 15.11]) Let X and Y be normed spaces, let S be a nonempty convex subset of X, and let C be a proper, pointed, convex cone. Let $F : S \rightrightarrows Y$ be a lower C-convex set-valued function, and let a pair $(x, y) \in \operatorname{gph} F$ with $x \in S$ be given. If the contingent epiderivative $D_e F(x, y)$ exists, then it is sublinear, i.e.,*

(i) $D_e F(x, y)(\alpha z) = \alpha D_e F(x, y)(z)$ for all $\alpha \geq 0$ and for all $z \in X$ (positive homogenity),

(ii) $D_e F(x, y)(z_1 + z_2) \in \{D_e F(x, y)(z_1) + D_e F(x, y)(z_2)\} - C$ for all $z_1, z_2 \in X$ (subadditivity).

For deeper discussions of the contingent derivative as well as the contingent epiderivative, we refer the reader to [1, 37, 44].

6.7 Directional derivatives of set-valued functions

The aim of this section is to present an overview of the directional derivatives of set-valued functions studied by Jahn [39], Pilecka [60], Dempe and Pilecka [14]. They proposed at least two approaches: the first one is to construct the difference quotients of the minimal and maximal solution functions, the second one is based on the aid of the Painlevé-Kuratowski outer limit. In the literature, Kuroiwa is the first author investigating directional derivatives of set-valued maps. He used a special embedding technique to derive directional derivatives in [48]. Among the other approaches, we mention results by Hoheisel, Kanzow, Mordukhovich and Phan [33, 34], as well as by Hamel, Heyde, Löhne, Rudloff and Schrage [29].

In this section, we also use the same notation $\ominus_D$ for all Demyanov differences if there is no confusion; see Section 2.4.

First, we consider a set-valued function $F : S \rightrightarrows Y$ from a subset S of the real linear space X with a nonempty interior to the real normed space Y partially ordered by a convex cone C. We assume that F takes strictly convex and weakly compact set-values. In order to define the differentials of set-valued functions, we will use the Demyanov differences in the sense of (2.9).

In [39], the Demyanov difference (2.9) was chosen to define the directional derivatives of set-valued functions because the author considered a difference quotient $\frac{1}{\lambda}(F(x+\lambda d) \ominus_D F(x))$, which is nearly of the form $\frac{0}{0}$ as $\lambda \to 0^+$. Hence the chosen set difference ensures that $F(x+\lambda d) \ominus_D F(x)$ becomes small for nearly the same sets. Before giving directional derivatives, we use the following convention:

$$\lim_{\lambda\to 0^+} \bigcup_{l\in C_1^+} \{G(\lambda,l)\} = \bigcup_{l\in C_1^+} \{\lim_{\lambda\to 0^+} G(\lambda,l)\}.$$

Hence,

$$\begin{aligned}\lim_{\lambda\to 0^+}\frac{1}{\lambda}\big(F(x+\lambda d)\ominus_D F(x)\big) &= \lim_{\lambda\to 0^+}\frac{1}{\lambda}\bigcup_{l\in C_1^+}\{y_{\min}(l,F(x+\lambda d)) - y_{\min}(l,F(x)),\\ &\qquad y_{\max}(l,F(x+\lambda d)) - y_{\max}(l,F(x))\big\}\\ &= \bigcup_{l\in C_1^+}\Big\{\lim_{\lambda\to 0^+}\frac{1}{\lambda}\big(y_{\min}(l,F(x+\lambda d)) - y_{\min}(l,F(x))\big),\\ &\qquad \lim_{\lambda\to 0^+}\frac{1}{\lambda}\big(y_{\max}(l,F(x+\lambda d)) - y_{\max}(l,F(x))\big)\Big\}\end{aligned}$$

We define the directional derivatives $D_{\min}F(x,d,l)$ and $D_{\max}F(x,d,l)$ by

$$D_{\min}F(x,d,l) := \lim_{\lambda\to 0^+}\frac{1}{\lambda}\big(y_{\min}(l,F(x+\lambda d)) - y_{\min}(l,F(x))\big) \text{ for all } l\in C_1^+, \tag{6.27}$$

and

$$D_{\max}F(x,d,l) := \lim_{\lambda\to 0^+}\frac{1}{\lambda}\big(y_{\max}(l,F(x+\lambda d)) - y_{\max}(l,F(x))\big) \text{ for all } l\in C_1^+. \tag{6.28}$$

Definition 6.7.1. *([39]) Consider a set-valued function $F : S \rightrightarrows Y$ taking strictly convex and weakly compact set-values, where X is a real linear space, $S \subseteq X$ with a nonempty interior, and Y is a real normed space ordered by a convex cone C. Take $x \in \operatorname{int} S$, and some $d \in X$. Let the directional derivatives $D_{\min}F(x,d,l)$ and $D_{\max}F(x,d,l)$ exist for all $l \in C_1^+$. The set*

$$D_J F(x,d) := \bigcup_{l\in C_1^+}\big\{D_{\min}F(x,d,l), D_{\max}F(x,d,l)\big\} \tag{6.29}$$

*is called the **directional derivative** of F at x in the direction d.*

The following property obviously holds for all $x \in \operatorname{int} S$ and $\lambda \geq 0$:

$$D_J F(x, \lambda d) = \lambda D_J F(x, d).$$

Next we consider the second approach by Dempe and Pilecka [14], where they used the modified Demyanov difference (2.6) to derive the differentials of set-valued functions. Now let C be a convex cone in $\mathbb{R}^n$ with a nonempty interior, and let $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be a set-valued function, which takes convex and weakly compact set-values and $\operatorname{dom} F \neq \emptyset$.

Definition 6.7.2. *([14]) Consider a set-valued function $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$. Take $x \in \operatorname{int}(\operatorname{dom} F)$, the directional derivatives $D_P F(x, d)$ at x in the direction $d \in \mathbb{R}^m$ is defined by*

$$D_P F(x, d) := \limsup_{t \to 0^+} \frac{F(x + td) \ominus_D F(x)}{t} \tag{6.30}$$

*is called the **directional derivative** of F at x in the direction d, where $\ominus_D$ is the modified Demyanov difference in the sense of (2.6).*

Proposition 6.7.3. *([14, Lemma 3.2]) Let $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be Lipschitz around $x \in \operatorname{int}(\operatorname{dom} F)$ in the sense of Definition 3.2.10 w.r.t. the difference (2.6) with Lipschitz modulus L. Then, for each direction $d \in \mathbb{R}^m$, the directional derivative of F at x is non-empty, bounded and satisfies*

$$D_P F(x, d) \subseteq L\|d\| U_{\mathbb{R}^n}. \tag{6.31}$$

We note that in [60], Pilecka used formula (6.30) to define the directional derivative with respect to the l-difference (2.4),

$$D_l F(x, d) := \limsup_{t \to 0^+} \frac{F(x + td) \ominus_l F(x)}{t}, \tag{6.32}$$

which leads to results similar to Proposition 6.7.3.

Chapter 7

Optimality conditions for vector optimization problems

Many problems in our daily life lead us to make decisions satisfying various objectives and conflicting goals, which can be mathematically modelled by ***vector optimization problems***. They are also well known by other names, such as multiobjective optimization problems or multi-objective optimization problems. Each application to real problems, for example in industrial systems, politics, business, industrial systems, control theory, management science, and networks, makes new models or new research branches for vector optimization problems; see more examples and models in the introductory chapter of [26]. The main goal of this chapter is to study necessary optimal conditions for solutions of the vector optimization problem (VP):

$$\text{minimize} \quad f(x) \quad \text{subject to} \quad x \in D, \tag{VP}$$

where X, Y are normed vector spaces, $f : X \to Y$ is a single-valued mapping, $D \subseteq X$ is non-convex, and C is a proper, closed, convex, pointed cone in Y. Recall that some solution concepts of the problem (VP) have been given in Section 2.6.

In the first section of this chapter, we will collect some recent and interesting techniques to scalarize the vector optimization problem (VP). These techniques are based on the scalaring functions introduced in Section 2.5. In the second section we give some necessary optimality conditions for (weakly) Pareto efficient solutions for the problem (VP) where the objective function f is either Lipschitz or C-convex, no matter whether $\operatorname{int} C$ is empty or not. The last section is devoted to the vector control approximation problem, which is a special form of the problem (VP), and is applied in many practical problems. We will derive necessary conditions for approximate solutions of this problem in infinite-dimensional reflexive Banach spaces.

7.1 Characterization of solutions of vector optimization problems by scalarization.

Our objective now is to present some methods to scalarize the vector optimization problem (VP). We prove that the vector optimization problem (VP) and its suitable scalar optimization problem have the same solution sets. Of course, solving the new problem is more advantageous than solving (VP), since we can use the optimality conditions for scalar optimization problems introduced in Appendix A.

First of all, we will scalarize the convex optimization problems by using the separation theorem for two convex sets in Section 2.5.1.

Proposition 7.1.1. *([37, Theorem 5.4]) Let $C \subset Y$ be a closed, convex cone.*

(i) Given a nonempty subset A of Y such that $A + C$ is convex and has a nonempty interior, one has that a point $\bar{y} \in A$ is a Pareto minimal point of A w.r.t. C if there exists $y^* \in C^+ \setminus \{0\}$ such that $\bar{y}$ is a solution of the problem $\min_{y \in A} y^*(y)$, i.e., $y^*(y) \geq y^*(\bar{y})$, for all $y \in A$.

(ii) Consider the problem (VP), assume that D is closed convex and $f : X \to Y$ is a C-convex function such that the set $f(D) + C$ has a nonempty interior. A point $\bar{x} \in D$ is a Pareto efficient solution of (VP) if there exists $y^* \in C^+ \setminus \{0\}$ such that

$$y^*(f(x) - f(\bar{x})) \geq 0 \quad \forall x \in D. \tag{7.1}$$

The following results will handle weakly Pareto minima with solid ordering cone.

Proposition 7.1.2. *([37, Theorem 5.13]) Let $C \subset Y$ be a closed, convex cone with a nonempty interior.*

(i) Given a nonempty subset A of Y such that $A + C$ is convex, one has that a point $\bar{y} \in A$ is a weakly Pareto minimal point of A w.r.t. C if there exists $y^* \in C^+ \setminus \{0\}$ such that $\bar{y}$ is a solution of the problem $\min_{y \in A} y^*(y)$, i.e., $y^*(y) \geq y^*(\bar{y})$, for all $y \in A$.

(ii) Consider the problem (VP), assume that D is closed convex, $f : X \to Y$ is a C-convex function. A point $\bar{x} \in D$ is a weakly Pareto efficient solution of (VP) if there exists $y^* \in C^+ \setminus \{0\}$ such that

$$y^*(f(x) - f(\bar{x})) \geq 0 \quad \forall x \in D. \tag{7.2}$$

Now to deal with scalarization of general vector optimization problems (some convex assumptions are not necessary), there are at least three successful approaches in the literature. The first one is to change the scalarization procedure by using the oriented distance function; see, for example, [27, 73]. The second one is to use the nonlinear scalarizing functional $\varphi_{C,e}$, which has also become popular in the last few years; see [8, 16, 17, 19]. The last one is to consider new approximate solution concepts, and then derive optimal conditions for the new solutions (see [16, 17, 19]). However, the third approach is beyond the context of this work, and will not be discussed further.

The following proposition presents the scalarization procedure of the problem (VP) using the oriented distance function introduced in Section 2.5.3. Since this proposition is a direct consequence of Proposition 2.5.11(ii), for brevity we will omit the proof.

Proposition 7.1.3. *([73, Theorem 4.3]) Let $C \subset Y$ be a closed, convex cone, and Δ_{-C} be given by* (2.14).

(i) Given a nonempty subset A of Y, one has that $\bar{y} \in A$ is a Pareto minimal point of A w.r.t. C if and only if $\bar{y}$ is a unique solution of the problem $\min_{y \in A} \Delta_{-C}(y - \bar{y})$, i.e., $\Delta_{-C}(y - \bar{y}) > 0$, for all $y \in A, y \neq \bar{y}$.

(ii) Moreover, consider the problem (VP)*, $\bar{x} \in D$ is a Pareto efficient solution of* (VP) *if and only if*

$$\Delta_{-C}(f(x) - f(\bar{x})) \geq 0 \quad \forall x \in D. \tag{7.3}$$

Now using the nonlinear scalarizing functional, the following proposition is a direct consequence of Theorem 2.5.8.

Proposition 7.1.4. *([19]) Let C be a closed, convex cone with a nonempty interior, $e \in \operatorname{int} C$, and $\varphi_{C,e}$ be given by* (2.11).

(i) Given a nonempty subset A of Y, one has that $\bar{y} \in A$ is a weakly Pareto minimal point of A w.r.t. C, then

$$\varphi_{C,e}(y - \bar{y}) \geq 0 \quad \textit{for all} \quad y \in A.$$

(ii) Moreover, consider the problem (VP)*, if $\bar{x} \in D$ is a weakly Pareto efficient solution of* (VP)*, then $\bar{x}$ is minimum of the following problem*

$$\textit{minimize} \quad \varphi_{C,e}(f(x) - f(\bar{x})) \quad \textit{subject to} \quad x \in D. \tag{7.4}$$

The most important condition to make use of the nonlinear scalaring functional $\varphi_{C,e}$ is that the ordering cone C has a nonempty interior. However, the class of ordering cones with nonempty interiors in infinite-dimensional spaces is not very broad. In the case that $\operatorname{int} C = \emptyset$, Bao and Tammer [8] constructed a new appropriate solid cone such that the Pareto minimal points w.r.t. the original cone C are also the Pareto minimal points w.r.t. the new cone.

From now on in this section, we consider a normed vector space Y with a proper pointed convex closed ordering cone C. For each point $e \in C \setminus \{0\}$ and for each $\epsilon \in (0, \|e\|)$ we consider the following cone

$$\Theta_{e,\epsilon} := \operatorname{cone}\big(B(e,\epsilon)\big) = \{t \cdot y, \quad y \in B(e,\epsilon)\}. \tag{7.5}$$

It is easy to see that the new cone $\Theta_{e,\epsilon}$ might not contain the given cone C or be contained in it. Obviously, $\Theta_{e,\epsilon}$ is a proper pointed convex closed cone with a nonempty interior, since $e \in \operatorname{int}\Theta_{e,\epsilon}$.

Proposition 7.1.5. *([8, Theorem 3.1]) Let Y be a normed vector space, C be a proper, closed convex cone. Let A be a nonempty subset of Y and $\bar{y} \in \operatorname{Min}(A, C)$. Then, for each $e \in C \setminus \{0\}$ satisfying*

$$-e \notin \operatorname{cl}\operatorname{cone}(A + C - \bar{y}), \tag{7.6}$$

there exists a positive real number $\epsilon > 0$ such that $\bar{y} \in \operatorname{Min}(A + C, \Theta_{e,\epsilon})$, where $\Theta_{e,\epsilon}$ is given in (7.5). Moreover, $\bar{y}$ is a minimum of the scalarization function $\varphi := \varphi_{\Theta_{e,\epsilon},e}$ over $A + C$:

$$\textit{minimize} \quad \varphi(y - \bar{y}) \quad \textit{subject to} \quad y \in A + C, \tag{7.7}$$

where φ w.r.t. $\Theta_{e,\epsilon}$ is given by (2.11).

In some other works, the assumption (7.6) could be replaced by a stronger condition that $\operatorname{cone}(A + C - \bar{y})$ is closed; see, for instance, [17, Theorem 2.3]. Furthermore, several new results about the asymptotic cone and the Bouligand tangent cone are given to derive necessary optimality conditions for Pareto minimal points without the assumption (7.6); see [17, Theorem 2.5 and Corollary 2.1]. In the following proposition, A is locally closed at Pareto minimal point $\bar{y}$, and clearly this condition is weaker than (7.6).

Proposition 7.1.6. *([17, Proposition 2.1]) Let Y be a normed vector space, C be a proper, closed, convex cone. Let A be a nonempty subset of Y, $\bar{y} \in \operatorname{Min}(A, C)$ such that A is locally closed at $\bar{y}$. Then, for each $e \in C \setminus \{0\}$, there exists a positive real number $\epsilon > 0$ such that $0 \in \operatorname{Min}(\operatorname{cone}(A \cap B(\bar{y}, \epsilon) - \bar{y} + e), \operatorname{cone} B(e, \epsilon))$. Moreover, 0 is a minimum of the scalarization function φ over $\operatorname{cone}(A \cap B(\bar{y}, \epsilon) - \bar{y} + e)$:*

$$\textit{minimize} \quad \varphi(y) \quad \textit{subject to} \quad y \in \operatorname{cone}(A \cap B(\bar{y}, \epsilon) - \bar{y} + e), \tag{7.8}$$

where $\varphi := \varphi_{A,e}$ with $A = \operatorname{cone} B(e, \epsilon)$ is given by (2.11).

However, this approach has the disadvantage that the minimum of the new scalar problem is not attained at the original minimal point $\bar{y}$, but at 0.

7.2 Necessary optimality conditions

In this section, we consider the problem (VP) with certain assumptions concerning the Lipschitzianity and the C-convexity of the objective function f, and we study both whether the interior of cone C is empty or not. In the case that $\operatorname{int} C = \emptyset$, we derive necessary conditions for Pareto efficient solutions in Theorem 7.2.4 and Theorem 7.2.5, and in the latter case, we derive the necessary conditions for weakly Pareto efficient solutions in Theorem 7.2.2 and Theorem 7.2.3. We are interested in deriving necessary optimality conditions in terms of the (basic, normal, Mordukhovich) coderivative mapping, and hence, the problem (VP) will be investigated in Asplund spaces.

Throughout the section, we use the following assumption:

Assumption 7.2.1. *Let X, Y be Asplund spaces, $D \subseteq X$ be a nonempty subset of X (D is not necessarily convex), let C be a proper, closed, convex, pointed cone in Y, and let $f : X \to Y$ be a vector-valued function such that $D \subseteq \operatorname{dom} f$. We consider a pair $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in D$, D is closed around $\bar{x}$, and $\bar{y} = f(\bar{x})$ (i.e., $(\bar{x}, \bar{y}) \in \operatorname{gph} f$).*

We recall the definition of the epigraphical multifunction of $f : X \to Y$, $\mathcal{E}_f : X \rightrightarrows Y$ given by (compare with (3.5) for set-valued mappings F)

$$\mathcal{E}_f(x) := f(x) + C.$$

We begin with the case $\operatorname{int} C \neq \emptyset$. Note that a convex ordering cone with a nonempty interior has the SNC property; see Definition 6.4.3. In the case that the ordering cone C has the SNC property, necessary conditions for minimizers of set-valued optimization problems are derived in [3, Theorem 4.1], [5, Theorem 5.3]. As a direct consequence of [3, Theorem 4.1], we get the following necessary conditions for weakly Pareto efficient solutions of vector optimization problems without convexity assumptions concerning the constraint set D and the objective function f. In the next theorem, we consider the vector optimization problem in terms of the Mordukhovich coderivative, and limiting subdifferential in infinite-dimensional spaces.

Theorem 7.2.2. *Consider the vector optimization problem* (VP) *under Assumption 7.2.1, and, in addition, assume that $\operatorname{int} C \neq \emptyset$, and f is Lipschitz around $\bar{x}$. If $\bar{x} \in D$ is a weakly Pareto efficient solution of* (VP)*, then there exists $y^* \in C^+ \setminus \{0\}$ such that*

$$0 \in D^*\mathcal{E}_f(\bar{x}, f(\bar{x}))(y^*) + N_L(\bar{x}; D). \tag{7.9}$$

Furthermore, if f is strictly Lipschitzian at $\bar{x}$, then (7.9) implies

$$0 \in \partial_L(y^* \circ f)(\bar{x}) + N_L(\bar{x}; D). \tag{7.10}$$

Proof. Since $\operatorname{int} C \neq \emptyset$, C has the SNC property. Because of the Lipschitzianity of f, the qualification condition (5.28) in [3, Theorem 4.1] is fulfilled. Hence, the necessary condition (7.9) follows immediately from [3, Theorem 4.1], and [4, Theorem 5].

For the remaining assertion, if we assume additionally that f is strictly Lipschitzian at $\bar{x}$, using [55, Theorem 3.28] we get that $D^*f(\bar{x})(y^*) = \partial_L(y^* \circ f)(\bar{x})$. By the definition of the coderivative, it is easy to see that $D^*\mathcal{E}_f(\bar{x}, f(\bar{x}))(y^*) \subseteq D^*f(\bar{x})(y^*)$. Thus $D^*\mathcal{E}_f(\bar{x}, f(\bar{x}))(y^*) \subseteq \partial_L(y^* \circ f)(\bar{x})$, which completes the proof. □

We now consider the problem (VP), in which the feasible set D is not necessarily convex, and the objective function $f : X \to Y$ is C-convex (see Definition 4.1.1). We observe from Chapter 4 that a C-convex, locally C-bounded function is locally Lipschitz under the assumption that the cone C is normal. Hence, all the calculus rules for coderivatives and generalized differentiations for locally Lipschitz mappings in [55, 56] are fulfilled for the class of C-convex mappings.

In our next theorem, under the assumption that f is C-convex, we will establish the Lagrangian necessary condition in the form of (7.10) using the subdifferentials of convex analysis.

Theorem 7.2.3. *Consider the vector optimization problem* (VP) *under Assumption 7.2.1, and, in addition, assume that C is a normal cone in Y with a nonempty interior, f is C-convex and C-bounded from above on a neighborhood U of $\bar{x}$. If $\bar{x} \in D$ is a weakly Pareto efficient solution of* (VP)*, then there exists $y^* \in C^+ \setminus \{0\}$ such that*

$$0 \in \partial(y^* \circ f)(\bar{x}) + N_L(\bar{x}; D), \tag{7.11}$$

Proof. Since C is normal, the cone C satisfies the assumptions of Theorem 4.2.7, hence f is C-convex and C-bounded from above around $\bar{x}$. Thus, f is Lipschitz around $\bar{x}$. Hence, all the calculus rules of coderivatives and generalized differentiations for locally Lipschitz mappings in [55, Chapter 1] are valid for the class of C-convex mappings. Since all assumptions of Theorem 7.2.2 are fulfilled, there exists $y^* \in C^+ \setminus \{0\}$ such that (7.9) holds. For $\bar{y} = f(\bar{x})$, we have that

$$x^* \in D^*\mathcal{E}_f(\bar{x}, \bar{y})(y^*) \iff (x^*, -y^*) \in N_L((\bar{x}, \bar{y}); \operatorname{epi} f), \tag{7.12}$$

where the *epigraph* of f is given by

$$\operatorname{epi} f := \big\{(x, y) \in X \times Y \mid y \in f(x) + C\big\}.$$

As f is C-convex, it follows that $\operatorname{epi} f$ is convex. Because of the convexity of $\operatorname{epi} f$, [55, Proposition 1.5] can be applied such that we get $N_L((\bar{x}, \bar{y}); \operatorname{epi} f) = \hat{N}((\bar{x}, \bar{y}); \operatorname{epi} f)$, i.e., representation (6.18) holds. We can rewrite (7.12) as

$$\begin{aligned} x^* \in D^*\mathcal{E}_f(\bar{x}, \bar{y})(y^*) &\iff (x^*, -y^*) \in \hat{N}((\bar{x}, \bar{y}); \operatorname{epi} f) \\ &\iff x^*(x - \bar{x}) - y^*(y - \bar{y}) \leq 0 \quad \text{for all} \quad (x, y) \in \operatorname{epi} f. \end{aligned}$$

Note that $y^* \in C^+$; and hence we have

$$x^* \in D^*\mathcal{E}_f(\bar{x},\bar{y})(y^*) \Longleftrightarrow x^*(x-\bar{x}) - y^*(f(x)-f(\bar{x})) \leq 0 \quad \text{for all} \quad x \in X.$$
$$\Longleftrightarrow x^* \in \partial(y^* \circ f)(\bar{x}).$$

It follows that $D^*\mathcal{E}_f(\bar{x},\bar{y})(y^*) = \partial(y^* \circ f)(\bar{x})$. This gives (7.11) when substituted in (7.9), and the proof is completed. □

Our next goal is to find Lagrangian necessary conditions for Pareto efficient solutions of the problem (VP) in the case that $\operatorname{int} C = \emptyset$, which is much harder than the previous one. In order to overcome the difficulties of this case, we refer the reader to [8, 17] for more references and discussions. Durea et al. [17] mentioned three possibilities to deal with this case, however in this section we only consider the following result of Bao and Tammer [8].

Theorem 7.2.4. *([8, Theorem 3.8]) Consider the vector optimization problem (VP) under Assumption 7.2.1, and, in addition, assume that* $\operatorname{int} C = \emptyset$*,* f *is Lipschitz around* $\bar{x}$*. Moreover, suppose that* $\operatorname{cone}(f(D)+C-\bar{y})$ *is closed. If* $\bar{x} \in D$ *is a Pareto efficient solution of (VP), then for every* $e \in C \setminus \{0\}$*, there exists* $y^* \in C^+$ *with* $y^*(e) = 1$ *such that*

$$0 \in D^*\mathcal{E}_f(\bar{x}, f(\bar{x}))(y^*) + N_L(\bar{x}; D). \tag{7.13}$$

Furthermore, if f *is strictly Lipschitzian at* $\bar{x}$*, then (7.9) implies*

$$0 \in \partial_L(y^* \circ f)(\bar{x}) + N_L(\bar{x}; D). \tag{7.14}$$

In comparison with the necessary condition in Theorem 7.2.2, we do not assume the closedness of $\operatorname{cone}(f(D)+C-\bar{y})$ from the hypotheses of Theorem 7.2.4, but we suppose additionally that $\operatorname{int} C \neq \emptyset$ in Theorem 7.2.2.

In the following theorem, we suppose the C-convexity and C-boundedness of the objective function f without the Lipschitzianity assumptions. We will get the following result similar to Theorem 7.2.3.

Theorem 7.2.5. *Consider the vector optimization problem (VP) under Assumption 7.2.1, and, in addition, assume that* C *is a normal cone in* Y *with an empty interior,* f *is* C*-convex and* C*-bounded from above on a neighborhood* U *of* $\bar{x}$*. Furthermore, suppose that* $\operatorname{cone}(f(D)+C-\bar{y})$ *is closed. If* $\bar{x} \in D$ *is a Pareto efficient solution of (VP), then for every* $e \in C \setminus \{0\}$*, there exists* $y^* \in C^+$ *with* $y^*(e) = 1$ *such that*

$$0 \in \partial(y^* \circ f)(\bar{x}) + N_L(\bar{x}; D). \tag{7.15}$$

Proof. Taking into account Theorem 4.2.7, we see that any C-convex, C-bounded function is locally Lipschitz. Following the same lines used in the proof of Theorem

7.2.3 and applying Theorem 7.2.4 and [8, Corollary 3.2], we obtain the assertion (7.15). □

As shown in [8, 17], the assumption that $\text{cone}\,(f(D) + C - \bar{y})$ is closed, is quite strict and may not hold true even in simple examples; see [17]. It can be replaced by weaker assumptions that $(f(D) + C)$ is locally closed at $\bar{y}$ and $e \in C \setminus \{0\}$ with

$$-e \notin \text{cl}\,\text{cone}\,(f(D) + C - \bar{y}). \tag{7.16}$$

Moreover, the existence of a vector e in the condition (7.16) is ensured provided that the following condition holds:

$$(-C \setminus \{0\}) \cap \text{bd}\,\text{cone}\,(f(D) + C - \bar{y}) = \emptyset.$$

In [17], the authors derived necessary conditions without the assumption that the generated cone is closed. However, the Lagrange multiplier $y^* \in Y^*$ is nontrivial, but it is positive only in a certain direction $c \in C \setminus \{0\}$; see [17, Proposition 2.1 and 2.2]. In [8, Theorem 3.8] as well as Theorem 7.2.3 and Theorem 7.2.5 one gets the stronger condition that is $y^* \in C^+$.

Note that when D is closed around $\bar{x}$ and $\text{epi}\, f$ is closed around $(\bar{x}, f(\bar{x}))$, then the indicator functions of the sets D and $\text{epi}\, f$ are lower-semicontinuous around $\bar{x}$ and $(\bar{x}, f(\bar{x}))$, respectively. Thus, the local closedness assumptions are essential in all theorems of this section in order to guarantee the necessary conditions of [3, Theorem 4.1], in comparison with the ones in [8, 4, 5, 27, 28].

Remark 7.2.6. ***(Comparison with necessary conditions presented in the literature).*** *To obtain necessary optimality conditions for vector optimization problems, Dutta and Tammer [20] used Mordukhovich's subdifferential when X is an Asplund space, Y is finite dimensional (see [20, Theorem 3.2]), and Ioffe's approximate subdifferential in general Banach spaces; see [20, Theorem 3.1]. Obviously, the assertion of [20, Theorem 3.2] can be deduced from Theorem 7.2.2. In Durea and Tammer [19], the authors enlarged the framework of the paper [20] to the concepts of abstract subdifferentials satisfying certain axioms, and considered not only "exact calculus rules" (see [19, Theorem 3.1]) but also "fuzzy calculus rules"; see [19, Theorem 4.1]. Theorem 7.2.3 and Theorem 7.2.5 show necessary conditions for vector optimization problems where the subdifferential of convex analysis for scalar functions is involved, since the objective function is supposed to be C-convex. Moreover, in order to get the corresponding necessary condition (7.10) for Mordukhovich subdifferentials ∂_L, we need the strong assumption that f is strictly Lipschitz.*

7.3 Applications in approximation theory

We consider in this part some applications of Theorems 7.2.3, and 7.2.5 for some vector control approximation problems. Since many practical problems can be described as vector approximation problems, they are of interest from both theoretical and practical points of view. For the convenience of the reader, we will recall relevant materials from Jahn [37], and Göpfert, Riahi, Tammer and Zălinescu [26].

Let X, Y, Z be real reflexive Banach spaces, and $C \subset Y$ be a proper, closed, pointed, convex cone. We denote the set of linear continuous mappings from X to Y by $L(X, Y)$.

Consider a **vector-valued norm** $|\!|\!| \cdot |\!|\!| : Z \to C$, for all $z, z_1, z_2 \in Z$ and $\lambda \in \mathbb{R}$, we have

1. $|\!|\!| z |\!|\!| = 0 \Longleftrightarrow z = 0$;
2. $|\!|\!| \lambda z |\!|\!| = |\lambda| \, |\!|\!| z |\!|\!|$;
3. $|\!|\!| z_1 + z_2 |\!|\!| \in |\!|\!| z_1 |\!|\!| + |\!|\!| z_2 |\!|\!| - C$.

We recall the subdifferential for vector-valued functions (denoted by $\partial^{\leq}$) defined in Chapter 6.

$$\partial^{\leq} f(z_0) = \{T \in L(Z, Y) \mid \forall z \in Z : f(z) - f(z_0) \in T(z) - T(z_0) + C\}.$$

It follows that

$$\partial^{\leq} |\!|\!| \cdot |\!|\!| (0) = \{T \in L(Z, Y) \mid \forall z \in Z : |\!|\!| z |\!|\!| - T(z) \in C\},$$

and

$$\partial^{\leq} |\!|\!| \cdot |\!|\!| (z) = \{T \in \partial^{\leq} |\!|\!| \cdot |\!|\!| (0) \mid T(z) = \ |\!|\!| z |\!|\!| \} \quad \text{for all} \quad z \in Z. \tag{7.17}$$

Moreover, if $|\!|\!| \cdot |\!|\!|$ is continuous, and C is Daniell , then $\partial^{\leq} |\!|\!| \cdot |\!|\!| \neq \emptyset$; see Jahn [37, Lemma 2.24].

Let $C \subset Y$ now be a proper, closed, pointed, convex cone, $D \subseteq X$ be closed and not supposed to be convex, and $f : X \to Y$ be given by

$$f(x) := f_1(x) + \sum_{i=1}^{n} \alpha_i |\!|\!| A_i(x) - a^i |\!|\!|,$$

with $f_1 : X \to Y, A_i \in L(X, Z)$, $a_i \in Z$. We will consider the following **vector control approximation problem** w.r.t. the concept of weakly Pareto efficient solution introduced in Definition 2.7.1

$$\mathrm{WMin}(f(D); C). \tag{7.18}$$

Next, we study the convexity and the Lipschitz continuity of the functions f and f_1 above via the following lemma.

Lemma 7.3.1. *Assume that C is a normal cone. If the vector-valued norm $|||\cdot||| : Z \to C$ is continuous around a given point $z \in Z$, then $|||\cdot|||$ is Lipschitz.*

Proof. It follows from the continuity assumption that $|||\cdot|||$ is bounded around z. Therefore $|||\cdot|||$ is C-bounded from above around z. Obviously, $|||\cdot|||$ is C-convex due to the definition of the vector-valued norm. Applying Theorem 4.2.7, then $|||\cdot|||$ is locally Lipschitz. Hence there exist $r, \ell > 0$ such that

$$\big|\big|\, |||z_1||| - |||z_2|||\, \big|\big|_Y \le \ell \big|\big|z_1 - z_2\big|\big|_Z \quad \text{for all} \quad z_1, z_2 \in rU_Z.$$

Consider arbitrary vectors $z_1, z_2 \in Z$, there exists $\alpha > 0$ such that $\alpha z_1, \alpha z_2 \in rU_Z$. Then

$$\big|\big|\, |||\alpha z_1||| - |||\alpha z_2|||\, \big|\big|_Y \le \ell \big|\big|\alpha z_1 - \alpha z_2\big|\big|_Z,$$

and hence $\big|\big|\, |||z_1||| - |||z_2|||\, \big|\big|_Y \le \ell \big|\big|z_1 - z_2\big|\big|_Z$. □

Remark 7.3.2. *Let $a \in Z$ and $A \in L(X, Z)$ be given. It holds that if $|||\cdot|||$ is continuous, then $|||A_i(\cdot) - a^i|||$ is also continuous. In addition, it is clear that $|||A_i(\cdot) - a^i|||$ is also C-convex. Taking into account Lemma 7.3.1, we get that $|||A_i(\cdot) - a^i|||$ is Lipschitz. Therefore, it is bounded by $\|\cdot\|_Y$ around a given point $\bar{x} \in X$.*

The following theorem presents a necessary condition for weakly Pareto efficient solutions of the problem (7.18) under the assumption that C has a nonempty interior, and f_1 is C-bounded and C-convex.

Theorem 7.3.3. *Suppose that X, Y, Z are reflexive Banach spaces, D is a closed subset of X, $C \subset Y$ is a proper pointed closed convex Daniell cone with a weakly compact base and a nonempty interior, and f_1 is C-convex. Let $\bar{y} = f(\bar{x})$ with $\bar{x} \in D$ be a weakly Pareto efficient solution of (7.18). If f_1 is C-bounded from above around $\bar{x}$, and $|||\cdot|||$ is continuous, then there exists $y^* \in C^+ \setminus \{0\}$ such that*

$$0 \in y^* \circ \partial^{\le} f_1(\bar{x}) + \sum_{i=1}^{n} \alpha_i A_i^*(y^* T_i) + N_L(\bar{x}; D), \tag{7.19}$$

where $T_i \in L(Z, Y)$ and

$$T_i \in \partial^{\le} |||\cdot|||(A_i(\bar{x}) - a^i), \quad i = 1, \dots, n.$$

Proof. By the assumptions on C, it is easy to see that C is normal; see [26], Section 2.2. Remark 7.3.2 shows that $|||A_i(\cdot) - a^i|||$ is C-convex and C-bounded from above around $\bar{x}$; so f has the same properties, hence the assumptions of Theorem 7.2.3 are fulfilled. Consequently, for every $e \in \operatorname{int} C$, we get the existence of $y^* \in C^+$ with $y^*(e) = 1$ such that

$$0 \in \partial(y^* \circ f)(\bar{x}) + N_L(\bar{x}; D).$$

The sum rule for subdifferentials of convex continuous functions (see [59, Theorem 3.16]) and Theorem 6.2.7 yield the relation

$$\begin{aligned}\partial(y^*\circ f)(\bar{x}) &= \partial\Big(y^*(f_1(\cdot)) + \sum_{i=1}^{n}\alpha_i y^*\|A_i(\cdot)-a^i\|\Big)(\bar{x})\\ &= y^*\partial^{\leq} f_1(\bar{x}) + \sum_{i=1}^{n}\alpha_i A_i^*\left(\partial y^*\circ\|\cdot\|(A_i(\bar{x})-a^i)\right)\\ &= y^*\partial^{\leq} f_1(\bar{x}) + \sum_{i=1}^{n}\alpha_i A_i^*\left(y^*\circ\partial^{\leq}(\|A_i(\bar{x})-a^i\|)\right).\end{aligned}$$

It follows that there exist $T_i \in \partial^{\leq}\|\cdot\|(A_i(\bar{x})-a^i)$ for $i=1,\dots,n$ such that (7.19) is satisfied. □

In comparison with the corresponding results in Dutta and Tammer [20, Theorem 4.1], the function f_1 is assumed to be Lipschitz from an Asplund space X to a finite-dimensional space Y. It is worth noting that the Lipschitz continuity, the strictly Lipschitz continuity and the strongly compactly Lipschitz continuity are equivalent in finite dimensional spaces. In [19, Theorem 5.2] a similar result to that of Theorem 7.3.3 is mentioned in terms of an abstract subdifferential (a subdifferential satisfying certain axioms).

In the case that $\operatorname{int} C \neq \emptyset$, and f_1 is strictly Lipschitz, Bao and Tammer [8, Theorem 4.4] derived Lagrange multiplier rules for the vector control approximation problems for Pareto efficient solutions. Now if we suppose the C-convexity and C-boundedness replace to the strictly Lipschitz continuity of f_1, then we will also get a similar result to [8, Theorem 4.4] for Pareto efficient solutions.

Theorem 7.3.4. *Under the hypotheses of Theorem 7.3.3 with the condition that* $\operatorname{int} C = \emptyset$, *we furthermore suppose that the* $\operatorname{cone}(f(D)+C-\bar{y})$ *is closed. If* $\bar{x}\in D$ *is a Pareto efficient solution of (7.18), then for every* $e \in C\setminus\{0\}$, *there exists* $y^*\in C^+$ *with* $y^*(e)=1$ *such that* (7.19) *holds.*

The proof is based on the same technique that was used in the proof of Theorem 7.3.3.

Chapter 8

Optimality conditions for set-valued optimization problems

Let X, Y be normed vector spaces, let $D \subseteq X$ be nonempty and not necessarily convex, let C be a proper, closed, convex, pointed cone in Y, and let $F : X \rightrightarrows Y$ be a set-valued map. In this chapter, we investigate the set-valued problem:

$$\text{minimize} \quad F(x) \quad \text{subject to} \quad x \in D. \tag{SP}$$

We will establish necessary conditions for solutions of the problem (SP) based on the primal-space approach and the dual-space approach. The principal difference between these two approaches is that the primal-space approach provides optimality conditions in primal spaces (using contingent derivatives, contingent epiderivatives and directional derivatives, etc), while the dual-space approach derives optimality conditions in dual spaces (using coderivatives, subdifferentials, etc). In this chapter, the interior of the cone C may be chosen empty or nonempty, depending on the solution types of the problem (SP).

8.1 The primal-space approach

This section is devoted to necessary and sufficient optimality conditions for solutions of the problem (SP) using some suitable derivatives and epiderivatives of the involved objective function. There are three different approaches for defining solutions of the problem (SP); see Section 2.7. The focus of this section is obtaining optimality conditions for solutions of the problem (SP) based on the vector approach as well as the set approach.

We state the following assumption that will be used within this section.

Assumption 8.1.1. *Let X, Y be normed vector spaces, $D \subseteq X$ be nonempty subset of X (D is not necessarily convex). Let C be a proper, closed, convex, pointed cone in Y,* *and let $F : X \rightrightarrows Y$ be a set-valued function such that $D \subseteq \operatorname{dom} F$.*

First, we present a necessary optimality condition for weak minimizers (see Definition 2.7.1) for the unconstrained optimization problem (SP), when $D = X$. Recall that $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is a weak minimizer of set-valued optimization problem (SP), if and only if, $\big(\{\bar{y}\} - \operatorname{int} C\big) \cap F(D) = \emptyset$.

Theorem 8.1.2. *Let Assumption 8.1.1 be satisfied, and, in addition, let $D = X$ and $\operatorname{int} C \neq \emptyset$. Assume that $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is a weak minimizer of the problem (SP) and the contingent derivative $D_c F(\bar{x}, \bar{y})$ exists, then*

$$D_c F(\bar{x}, \bar{y})(X) \cap (-\operatorname{int} C) = \emptyset.$$

Proof. We argue by contradiction, and assume that $u \in X$ and $v \in D_c F(\bar{x}, \bar{y})(u) \cap (-\operatorname{int} C)$. By the definition of contingent derivative, we have

$$(u, v) \in T\big(\operatorname{gph} F, (\bar{x}, \bar{y})\big).$$

Then, there exist $t_n \to 0$, $t_n \in (0, +\infty)$ and $(u_n, v_n) \in X \times Y$ such that $(u_n, v_n) \to (u, v)$ and $(\bar{x} + t_n u_n, \bar{y} + t_n v_n) \in \operatorname{gph} F$, i.e.,

$$\bar{y} + t_n v_n \in F(\bar{x} + t_n u_n) \quad \text{for all} \quad n \in N,$$

which implies that

$$t_n v_n \in F(\bar{x} + t_n u_n) - \bar{y} \quad \text{for all} \quad n \in N.$$

For n large enough, $t_n v_n \in -\operatorname{int} C$, hence

$$\big(F(\bar{x} + t_n u_n) - \bar{y}\big) \cap -\operatorname{int} C \neq \emptyset,$$

contradicts to the weak minimalty of $(\bar{x}, \bar{y})$, and completes the proof. $\square$

For $D \subsetneq X$ and a set-valued function $F : D \rightrightarrows Y$, we define a new function $F_D : X \rightrightarrows Y$ by

$$F_D(x) = \begin{cases} \{0\} & \text{if } x \in X \setminus D, \\ F(x) & \text{if } x \in D. \end{cases}$$

Then applying the result in Theorem 8.1.2, one can obtain a necessary condition similar to [12, Theorem 2.48]. It is easy to check that Theorem 8.1.2 is a consequence of [12, Theorem 2.64] and [18, Proposition 3.1], in which one considers the problem (SP) with a variable ordering structure.

The necessary condition for weak minimizers for the optimization problem (SP), using the contingent epiderivative, was derived by Jahn [37]. For the convenience of the reader, we recall the following theorem without proof.

Theorem 8.1.3. *([37, Theorem 17.3]) Let Assumption 8.1.1 be satisfied, and, in addition,* $\operatorname{int} C \neq \emptyset$*. Assume that* $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ *is a weak minimizer of the problem* (SP) *and the contingent epiderivative* $D_e F(\bar{x}, \bar{y})$ *exists, then*

$$D_e F(\bar{x}, \bar{y})(x - \bar{x}) \notin -\operatorname{int} C, \text{ for all } x \in D.$$

Moreover, [37] also presented the sufficient condition for solutions of the problem (SP) under an appropriate convexity assumption.

Theorem 8.1.4. *([37, Theorem 17.4]) Let Assumption 8.1.1 be satisfied. Assume that* $\operatorname{int} C \neq \emptyset$*,* D *is convex, and* $F : D \rightrightarrows Y$ *is lower* C*-convex in the sense of Definition 5.1.4. Assume that the contingent epiderivative* $D_e F(\bar{x}, \bar{y})$ *exists at a pair* $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ *and satisfies*

$$D_e F(\bar{x}, \bar{y})(x - \bar{x}) \notin -\operatorname{int} C, \text{ for all } x \in D.$$

Then, $(\bar{x}, \bar{y})$ *is a weak minimizer of the problem* (SP)*.*

In the following, we use the set approach to study solutions of problem (SP), which are defined by the relations $\preceq_C^{(t)}$, where $t \in \{i, ii, iii, iv, v, vi\}$, and $\preceq_C^{(t)}$ is given in Definition 2.3.20. Under Assumption 8.1.1, we consider the following set-valued optimization problem w.r.t. the relation $\preceq_C^{(t)}$:

$$\preceq_C^{(t)} -\text{minimize} \quad F(x) \quad \text{subject to} \quad x \in D, \tag{SP$-\preceq_C$}$$

where the minimum is taken in the sense of Definition 2.7.2

Next we establish a relationship between weak minimizers of the problem (SP) introduced in Definition 2.7.1, and strictly minimal solutions of the problem (SP$- \preceq_C$) w.r.t. the relation $\preceq_C^{(vi)}$ introduced in Definition 2.7.2. We also get a necessary condition for solutions of the problem (SP$- \preceq_C$) w.r.t. $\preceq_C^{(vi)}$.

Theorem 8.1.5. *Let Assumption 8.1.1 be satisfied,* $\operatorname{int} C \neq \emptyset$*, and let* $D = X$*. Assume that* $\bar{x}$ *is a strictly minimal solution of* (SP$- \preceq_C$) *w.r.t. the relation* $\preceq_C^{(vi)}$ *and there exists* $\bar{y} \in F(\bar{x})$ *such that*

$$\bar{y} \notin F(\bar{x}) + C \setminus \{0\}, \tag{8.1}$$

then $(\bar{x}, \bar{y})$ *is a minimizer of the problem* (SP)*. Moreover,*

$$D_c F(\bar{x}, \bar{y})(X) \cap (-\operatorname{int} C) = \emptyset. \tag{8.2}$$

Proof. Since $\bar{x}$ is a strictly minimal solution of (SP$- \preceq_C$) w.r.t. the relation $\preceq_C^{(vi)}$, we have

$$F(x) \not\preceq_C^{(vi)} F(\bar{x}), \ \forall x \in X \setminus \{\bar{x}\}.$$

This yields that

$$F(\bar{x}) \cap (F(x) + C) = \emptyset, \ \forall x \in X \setminus \{\bar{x}\}. \tag{8.3}$$

As there exists $\bar{y} \in F(\bar{x})$ satisfying (8.1), and taking into account (8.3), we get

$$\bar{y} \notin F(x) + C \setminus \{0\}, \ \forall x \in X.$$

Therefore,

$$(\bar{y} - C) \cap F(X) = \{\bar{y}\},$$

which completes the first assertion of this theorem.

Finally, the necessary condition (8.2) for strictly minimal solutions of the problem (SP$-\preceq_C$) is a direct consequence of Theorem 8.1.2. $\square$

As shown in Proposition 2.3.21, $\preceq_C^{(vi)}$ is the weakest relation. Hence, Theorem 8.1.5 also holds true for strictly minimal solutions of (SP$-\preceq_C$) w.r.t. any other relation definied in Definition 2.3.20.

In the following theorem we show a corresponding result for weak minimizers of the problem (SP) introduced in Definition 2.7.1, and strongly minimal solutions of the problem (SP$-\preceq_C$) w.r.t. the relation $\preceq_C^{(iii)}$ introduced in Definition 2.7.2. We also get a necessary condition for solutions of the problem (SP$-\preceq_C$) w.r.t. $\preceq_C^{(iii)}$.

Theorem 8.1.6. *Let Assumption 8.1.1 be satisfied,* $\operatorname{int} C \neq \emptyset$*, and let* $D = X$*. Assume that* $\bar{x}$ *is a strongly minimal solution of* (SP$-\preceq_C$) *w.r.t. the relation* $\preceq_C^{(iii)}$ *and there exists* $\bar{y} \in F(\bar{x})$ *such that* (8.1) *is satisfied, then* $(\bar{x}, \bar{y})$ *is a minimizer of the problem* (SP)*. Moreover, the necessary condition* (8.2) *for strongly minimal solutions of the problem* (SP$-\preceq_C$) *holds true.*

Proof. Since $\bar{x}$ is a strongly minimal solution of (SP$-\preceq_C$) w.r.t. the relation $\preceq_C^{(iii)}$, we have

$$F(x) \preceq_C^{(iii)} F(\bar{x}), \ \forall x \in X \setminus \{\bar{x}\}.$$

This yields that

$$F(x) \subset (F(\bar{x}) + C), \ \forall x \in X \setminus \{\bar{x}\}. \tag{8.4}$$

As there exists $\bar{y} \in F(\bar{x})$ satisfying (8.1), and taking into account (8.4), we get

$$\bar{y} \notin F(x) + C \setminus \{0\}, \ \forall x \in X.$$

Therefore,

$$(\bar{y} - C) \cap F(X) = \{\bar{y}\},$$

which completes the first assertion of this theorem. The second one is proved similarly to Theorem 8.1.5. $\square$

Taking into account Proposition 2.3.21, Theorem 8.1.6 also holds true for strongly minimal solutions of (SP$-\preceq_C$) w.r.t. the relation $\preceq_C^{(i)}$ or $\preceq_C^{(ii)}$.

In [24, 43], the problem (SP$-\preceq_C$) was considered w.r.t. variable domination structures. Some relationships between strictly (strongly) minimal solutions of the problem

(SP$- \preceq_C$) and weak minimizers of the problem (SP) were also studied. All the relationships in Theorems 8.1.5, 8.1.6 can be realized as the consequences of [24, Lemma 11] and [43, Theorem 2].

Recently, some results concerning optimality conditions of set-valued optimization problems w.r.t. the set less order relation $\preceq^s_C$ have been derived by Dempe and Pilecka [14], and Jahn [39]. They used the modified Demyanov differences (see Section 2.4) in order to define the corresponding directional derivatives for set-valued functions (see Section 6.7), which are appropriate tools for deriving optimality conditions for solutions of set-valued optimization problems with respect to $\preceq^s_C$.

8.2 The dual-space approach

In this section, X, Y are Asplund spaces, and $D \subseteq X$ is a nonempty subset of X (D is not necessarily convex). Let C be a proper, closed, convex, pointed cone in Y, and $F : X \rightrightarrows Y$ be a set-valued function such that $D \subseteq \operatorname{dom} F$. We take a pair $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and suppose that $\operatorname{epi} F$ is closed around $(\bar{x}, \bar{y})$, and the constraint set D is closed around $\bar{x}$.

This section considers again the problem (SP). We will derive necessary conditions for solutions of the optimization problem (SP) using coderivatives and subdifferentials in the sense of Mordukhovich in Asplund spaces. For the next results, we need the two following assumptions about the objective function F.

(A1) F is (ELL) around $(\bar{x}, \bar{y})$.

(A2) F is lower C-convex. In addition, F is C-bounded from below and weakly C-upper bounded on a neighborhood of $\bar{x}$.

It follows from Theorem 5.2.7 and Remark 3.2.6(ii) that if F is lower C-convex, C-bounded from below and weakly C-upper bounded on a neighborhood of $\bar{x}$, then F is (ELL) at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with $\bar{y} \in F(\bar{x})$. Hence, the assumption $(A2)$ is stronger than $(A1)$.

Recall again (see Definition 3.2.1(iv)) that a set-valued function $F : X \rightrightarrows Y$ is epigraphically Lipschitz-like (ELL) around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with modulus $l \geq 0$ if there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$\forall x, u \in U : \quad \mathcal{E}_F(x) \cap V \subseteq \mathcal{E}_F(u) + l \, \|x - u\| \, U_Y.$$

In the following, we present an optimality condition for weak minimizers of (SP) where $\operatorname{int} C \neq \emptyset$.

Theorem 8.2.1. *Consider the set-valued optimization problem* (SP) *with Assumption* $(A1)$*, and, in adddition, assume that* $\operatorname{int} C \neq \emptyset$*. If* $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ *is a weak minimizer* *of* (SP)*, then for every* $e \in \operatorname{int} C$*, there exists a dual element* $y^* \in C^+$ *with* $y^*(e) = 1$ *such that*

$$0 \in D^*\mathcal{E}_F(\bar{x}, \bar{y})(y^*) + N_L(\bar{x}; D). \tag{8.5}$$

Proof. From Remark 6.4.4, a convex cone C with a nonempty interior has the SNC property; see Definition 6.4.3. Because of the (ELL) property of F, the qualification condition (4.1) in [3, Theorem 4.1] is fulfilled. Hence, the necessary condition (8.5) follows immediately from [3, Theorem 4.1]. □

Observe that, since the assumption (A2) is stronger than (A1), the next result can be considered as a consequence of the previous theorem.

Corollary 8.2.2. *Consider the set-valued optimization problem* (SP) *with Assumption (A2), and, in adddition, assume that* $\operatorname{int} C \neq \emptyset$*. If* $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ *is a weak minimizer of* (SP)*, then for every* $e \in \operatorname{int} C$*, there exists a dual element* $y^* \in C^+$ *with* $y^*(e) = 1$ *such that* (8.5) *holds.*

Proof. The result is a direct consequence of Theorem 8.2.1. □

In oder to deal with the problem (SP), Ha [27] also derived optimality conditions in terms of the Clarke coderivatives and the Ioffe coderivatives for several types of efficient solutions by transferring them to be solutions of a set-valued optimization problem equipped with an open cone. In addition, [3, 8] studied the problem (SP) in terms of the Mordukhovich coderivatives. To scalarizing the problem (SP) Ha [27] used the oriented functional, and Bao and Tammer [8] used the nonlinear scalarizing functional. However, Bao and Mordukhovich used the technique related to extremal principle of variational analysis.

Now, we recall necessary optimality conditions given by Bao and Tammer [8, Theorem 3.10] for the problem (SP), in which the set-valued function F is (ELL), without assuming that the ordering cone has a nonempty interior and the constraint set D is convex.

Theorem 8.2.3. *([8, Theorem 3.10]) Consider the set-valued optimization problem* (SP) *with Assumption (A1), and, in adddition, assume that* $\operatorname{cone}\big(F(D) + C - \bar{y}\big)$ *is closed. If* $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ *is a minimizer of* (SP)*, then for every* $e \in C \setminus \{0\}$*, there exists a dual element* $y^* \in C^+$ *with* $y^*(e) = 1$ *such that* (8.5) *holds.*

The following result presents a necessary optimality condition for solutions of the problem (SP), where F is a lower C-convex set-valued function. We omit the proof, since it is a direct consequence of Theorem 8.2.3

Corollary 8.2.4. *Consider the set-valued optimization problem* (SP) *with Assumption (A2), and, in adddition, assume that* $\operatorname{cone}\big(F(D) + C - \bar{y}\big)$ *is closed. If* $(\bar{x}, \bar{y}) \in \operatorname{gph} F$

is a minimizer of (SP)*, then for every* $e \in C \setminus \{0\}$*, there exists a dual element* $y^* \in C^+$ *with* $y^*(e) = 1$ *such that* (8.5) *holds.*

Remark 8.2.5. *If* $F = f : X \to Y$ *is at most single-valued, it is clear that the necessary condition in Corollary 8.2.4 reduces to that in Theorem 7.2.5.*

Now we consider the problem (SP$-\preceq_C$). In Section 8.1, Theorem 8.1.5 and 8.1.6 illustrate the relationships between strictly (strongly) minimal solutions of the problem (SP$-\preceq_C$) and minimizers of the problem (SP). Therefore, taking into account Theorems 8.2.1, 8.2.3 (Corollaries 8.2.2, 8.2.4) we could derive necessary optimality conditions for strictly (strongly) solutions of the problem (SP$-\preceq_C$) where the objective function F satisfies Assumption (A1) (Assumption (A2), respectively). For the sake of shortness, we skip presenting these results in this work.

Among abundant developments in set-valued optimization, we refer the reader to the monographs by Khan, Tammer and Zălinescu [44], Jahn [37] and Mordukhovich [55, 56]. We especially emphasize the vector optimization problems with variable ordering structure based on general domination set mappings, in which many solution concepts, optimality conditions and numerical procedures are derived. They have been among the primary motivations for developing new issues and applications of optimization theory. For more details and discussions on the set-valued problems with variable ordering structure, we mention the recent research by Chen, Huang, and Yang [12], Eichfelder [21, 22], Eichfelder and Pilecka [23, 24], Bao and Mordukhovich [6], and Durea, Strugariu and Tammer [18].

Chapter 9

Conclusion and Outlook

In this chapter we present some conclusions and some potential open problems for furture research found during the work.

9.1 Conclusion

In this thesis, we presented the relationships between Lipschitz continuity and convexity of functions. These relationships were studied systematically for vector-valued functions in Chapter 4 and for set-valued functions in Chapter 5. In Chapter 4, after introducing the concepts of C-convex and C-bounded functions, we proved that a C-convex vector-valued function is locally Lipschitz around a given point if it is C-bounded from above on a neighborhood of this point, where C is a normal cone. Obviously, this assertion is a significantly general form of the result given in [52], in which the function is considered only in finite-dimensional spaces. Compared with the similar result of Borwein [9], we observed that the boundedness in Theorem 4.2.7 is clearly weaker than the one in [9, Corollary 2.4]. Therefore, we obtained a result slightly stronger than Borwein [9]. Moreover, we presented an accurate Lipschitz constant in the first proof of Theorem 4.2.7.

The relationships between Lipschitz continuity and convexity for set-valued functions are abundant since there are many approaches to define them in the literature. In the first section of Chapter 5, we presented six types of convex set-valued functions. Then we derived scalarizing functions to investigate the properties of the convex set-valued functions. Using these scalarizing functions, we proved the C-Lipschitzianity of convex set-valued functions in Section 5.2.

We represented an alternative concept of C-Lipschitzianity given by Kuwano and Tanaka [50]. By means of the nonlinear scalarizing functional, we obtained a result similar to one in [50] with milder assumptions.

Section 5.4 showed the relationships between the upper (lower) G-Lipschitzianity and the $\mathfrak{Cs}$-convexity of set-valued functions.

The obtained results are applied in order to derive the necessary optimality conditions for vector- and set-valued optimization problems. In particular, the objective functions are considered to be either Lipschitz or convex. We considered the Lagrangian necessary conditions for (weakly) Pareto efficient solutions of vector optimization problems in both solid and non-solid cases in Chapter 7.
In Chapter 8, we established necessary optimality conditions for minimizers of the set-valued optimization problem based on the primal-space approach and the dual-space approach.

9.2 Outlook

There are many possible open research problems related to the work in this thesis. In the following, we list some potential problems and research directions, which may be of interest in the future.
The Lipschitz continuity of C-convex vector-valued functions. In order to deal with this problem in our work, we have to assume that C is normal cone, and the function is a mapping between two normed vector spaces. We expect to extend this result for general spaces and alternative conditions on the cone C.
The relationships between Lipschitzianities and convexities of set-valued functions. Although there are so many types of Lipschitzianities and convexities of set-valued functions in the literature, one continues to find more approaches to define them. Therefore, the relationships between them will need further exploration.
Variable ordering structures. We want to derive new concepts of convexity and Lipschitzianity of set-valued functions in spaces which are equipped with variable ordering structures. We also expect to extend all necessarry optimality conditions for solutions of set-valued optimization with variable ordering structures.

Appendix

A Optimality conditions for scalar optimization problems

For the convenience of the reader we shall call to mind scalar optimization problems and show separately the necessary and sufficient conditions for their optimal solutions. They are useful in Chapter 7 and 8, because we shall transfer the vector- and set-valued optimization problems to the corresponding scalar optimization problems by using an appropriate scalar function to scalarize objective functions.

Let X be a Banach space, and D be a subset of X. We consider the scalar optimization problem of minimizing a function $f : X \to \overline{\mathbb{R}}$ over the set D, or briefly,

$$\min_{x \in D} \to f(x). \tag{P}$$

Here we call f an objective function, D the constraint domain, and (P) the optimization problem with constraints. A solution of the problem (P) is called a global minimum point. We say that $\bar{x} \in D$ is a local minimum point if there exists a neighborhood U of $\bar{x}$ such that $f(\bar{x}) \leq f(x)$ for every $x \in D \cap U$.

It is well known that minimizing the function f over D is equivalent to minimizing the following function $h : X \to \overline{\mathbb{R}}$

$$h(x) := f(x) + \delta_D(x), \tag{1}$$

over all of the space X, where δ_D is the indicator function of D. Hence (P) is equivalent to

$$\min_{x \in X} \to h(x), \tag{P1}$$

(P1) is called unconstrained optimization problem.

We recall the well-known Euler's equation about the first-order necessary optimality conditions, where the objective function f is Gâteaux differentiable..

Proposition A.1. *([38, Theorem 3.17]) Let X be a Banach space, and D be an open subset of X. We consider the problem (P), where the objective function $f : X \to \mathbb{R}$ is Gâteaux differentiable. If $\bar{x} \in D$ is a local minimal solution of the problem (P), then $f'(\bar{x}) = 0$.*

When D is not an open set, the proposition above is not true. We can take an simple example: consider $f(x) = x$ and $D = [0, 1]$. Clearly 0 is a minimum of f on D, but $f'(0) = 1$.

In the special case that f is a convex function, D is a convex subset of X and $D \cap \operatorname{dom} f \neq \emptyset$, then (P) is called convex optimization problem, hence the function h, determined by (1), is also convex on X. We recall the two fundamental properties of the convex optimization problem. Firstly, any locally optimal point of convex optimization problem is also (globally) optimal (see [74, Proposition 2.5.8]). This is a reason why we look only for global minimum points in a convex optimization problem. The second one is that the necessary conditions for optimality become sufficient. Now we shall show the necessary conditions for optimality of the convex optimization problem.

Proposition A.2. *([74, Theorem 2.5.7]) Let X be a Banach space. We consider the problem (P1), where the objective function $h : X \to \overline{\mathbb{R}}$ is proper and convex on X, then $\bar{x} \in \operatorname{dom} h$ is a minimal solution of the problem (P1) if and only if $0 \in \partial h(\bar{x})$.*

Applying the calculus for convex functions (see Proposition 6.2.4), we have the necessary and sufficient conditions for minimal solutions of optimization problem with constraints (P).

Proposition A.3. *([74, Theorem 2.9.1]) Let X be a Banach space, $f : X \to \overline{\mathbb{R}}$ be a proper convex function and D be a convex set. Suppose that either $\operatorname{dom} f \cap \operatorname{int} D \neq \emptyset$, or there exists $x_0 \in \operatorname{dom} f \cap D$, where f is continuous. Then, $\bar{x} \in D$ is a minimal solution of the problem (P) if and only if $0 \in \partial f(\bar{x}) + N(\bar{x}, D)$.*

Now we consider the problems (P) and (P1), where the objective functions are locally Lipschitz. The following propositions present the necessary conditions of locally minimal solutions using the generalized gradient given by (6.11).

Proposition A.4. *([13, Proposition 2.3.2]) Let X be a Banach space. We consider the problem (P1), where the objective function $h : X \to \mathbb{R}$ is locally Lipschitz. If $\bar{x}$ is a locally minimal solution of the problem (P1) then $0 \in \partial_C h(\bar{x})$.*

Applying the calculus to locally Lipschitz functions (see Proposition 6.3.3), we have the necessary conditions for locally minimal solutions of optimization problem with constraints (P).

Proposition A.5. *([13, Corollary 2.4.3]) Let X be a Banach space and $f : X \to \mathbb{R}$ be a locally Lipschitz function. If $\bar{x} \in D$ is a locally minimal solution of the problem (P) then $0 \in \partial_C f(\bar{x}) + N_C(\bar{x}, D)$.*

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Index of notation

$\mathbb{Z}$	set of integers
$\mathbb{N}, \mathbb{N}^*$	set of nonnegative integers, $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}^n_+$	nonnegative orthant of $\mathbb{R}^n$
$\overline{\mathbb{R}}$	$\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$
$X, Y, Z, \ldots$	real linear spaces or topological linear spaces
C, K	cones (in X, Y, Z)
$\geq_C$, $\leq_C$	a partial ordering relation generated by C
C-convex	cone-convex (function)
$+\infty_C$	a maximal element w.r.t. $\geq_C$
$[A]_C$	$[A]_C := (A + C) \cap (A - C)$
C^+	positive dual cone of C
C_1^+	$C_1^+ := C^+ \cap U_{Y^*}$
C^0	polar cone of a nonempty cone C
$C^\#$	quasi-interior of the dual cone C^+
$X^\bullet$	$X^\bullet := X \cup \{+\infty_C\}$
X^*	the topological dual space of X
$\lVert\cdot\rVert_X, \lVert\cdot\rVert_*$	norm in X, norm in X^*
$\operatorname{cl} A$	closure of a set A
cl_{w^*}	closure of a set A w.r.t weak* topology
$\operatorname{int} A$	(topological) interior of the set A
$\operatorname{bd} A$	(topological) boundary of the set A
$\operatorname{conv} A$	convex hull of the set A
$B(x, r)$	closed ball centered at x with radius $r > 0$
U_Y, S_Y	closed unit ball and unit sphere in a space Y
$F : X \rightrightarrows Y$	set-valued function
$f : X \to Y^\bullet$	vector-valued function
$\operatorname{epi} f$	epigraph of a function f
$\operatorname{gph} f$	graph of a function f

$\operatorname{dom} f$	domain of a vector-valued function $f : X \to Y^\bullet$
$\operatorname{dom} F$	domain of a set-valued function $F : X \rightrightarrows Y$
$\mathcal{E}_f : X \rightrightarrows Y$	epigraphical multifunction of f
$\mathcal{E}_F : X \rightrightarrows Y$	epigraphical multifunction of F
δ_A	(convex) indicator function of a set A
$\delta^*(\cdot, A)$	support function of a set A
$d(x, A)$	distance from x to A
$\ominus_A, \ominus_D, \ominus_G, \ominus_M$	algebraic, Demyanov, geometric and metric differences
$\mathcal{K}(\mathbb{R}^n)$	set of nonempty compact subsets of $\mathbb{R}^n$
$\mathcal{C}(\mathbb{R}^n)$	set of nonempty convex compact subsets of $\mathbb{R}^n$
$\varphi_{A,e}$	$\varphi_{A,e}(y) := \inf\{\lambda \in \mathbb{R} \mid \lambda \cdot e \in y + A\}$
$\Delta_A(\cdot)$	oriented distance function w.r.t. a set A
$T(A, x)$	*contingent cone* to A at x
$N(x; A)$	*normal cone* in the sense of convex analysis
$N_C(x; A)$	Clarke's *normal cone* to A at $x \in A$
$\hat{N}_\epsilon(x; A)$	ϵ-*normal cone*
$\hat{N}(x; A)$	Fréchet *normal cone*
$N_L(x; A)$	(basic, limiting or Mordukhovich) *normal cone*
$\partial f(x)$	Fenchel *subdifferential* of $f : X \to \overline{\mathbb{R}}$ at $x \in X$
$\partial_C f(x)$	Clarke *subdifferential* of $f : X \to \overline{\mathbb{R}}$ at $x \in X$
$\partial_L f(x)$	(basic, normal, Mordukhovich) *subdifferential* of $f : X \to \overline{\mathbb{R}}$ at $x \in X$
$\partial^{\leq} f(x)$	*subdifferential* of $f : X \to Y^\bullet$ at $x \in X$
$L(X, Y)$	set of linear continuous function from X to Y
$\operatorname{Min}(A; C)$	set of Pareto minimal points of A w.r.t. C
$\operatorname{WMin}(A; C)$	set of weakly Pareto minimal points of A w.r.t. C
$\hat{D}^*_\epsilon F(x, y)$	ϵ-*coderivative* of F at (x, y)
$\hat{D}^* F(x, y)$	precoderivative or Fréchet *coderivative* of F at (x, y)
$D^* F(x, y)$	(basic, normal, Mordukhovich) *coderivative* of F at (x, y)
$\langle y^*, \cdot \rangle, (y^*, \cdot), y^*(\cdot)$	linear continuous functional $y^* : Y \to \mathbb{R}$

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Selbständigkeitserklärung

Hiermit erkläre ich Vu Anh Tuan an Eides statt, dass ich die vorliegende Dissertation selbständig und ohne fremde Hilfe angefertigt habe. Ich habe keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und die den benutzten Werken wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht.

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