Banach Space Valued Stochastic Integral Equations and Their Optimal Control

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Contents

CHAPTER 1

Introduction

The stochastic analysis in infinite dimensional spaces and its application belong to the modern research of stochastic. The essential part of stochastic analysis is the stochastic integration theory. Stochastic integral is introduced in the case of separable Hilbert spaces for example by [10, 35, 34] based on the fundamental ideas of Itô [14] and Gikhman [9]. In general the integrands are stochastic processes with values in the spac of Hillbert-Schmidt operators. Often the integrals are defined with respect to Hilbert space valued cylindrical Wiener processes for example [8, 34]. The important property of an Itô integral is so-called Itô isometry. Roughly speaking the Itô isometry says that the expectation of the norm square of the Itô integral in the Hilbert space is equal to expectation of a usual Lebesgue integral.
Generalizing stochastic integrals for Banach valued processes demanded some subtle changes in framework of Hilbert-valued processes. By development of unconditional martingale differences spaces (UMD) the stochastic integral could be defined for E-valued progressively valued processes, where E is an UMD Banach space [24, 30].
The goal of this thesis is to consider E-Valued processes which satisfy forward or backward integral equations in the form of Itô Volterra type. Backward integral equations in the form of Itô Volterra type are used to consider optimal control problem for controlled forward equations of Itô Volterra type by means of a stochastic maximum principle method.

1.1. Previous Research

Stochastic partial differential equations are part of the research of infinite dimensional stochastic analysis and these equations can be interpreted as stochastic evolution equations and the solutions are defined in a generalized sense. There are following three main approaches for dealing with such equations :

- The mild solution: The problem contains a linear operator which generates a semigroup of operators in a Hilbert space and the problem is considered as stochastic integral equation (Ito-Volterra type), which contains a stochastic convolution (see for example [34] and the literature cited therein).
- The generalized weak solution (weak solution, analytically weak solution): The solution of the partial differential equation satisfies a scalar product equation (see [34], [35]).
- The variational solution (generalized solution, (V, H)-solution): The problem is defined by a stochastic evolution equation over a triplet of rigged Hilbert spaces (V, H, V^*) (see for example [35], [36]).

In stochastic analysis if the solution is defined on a given probability space, then the solution is called (probabilistic) strong solution, and if they are defined by constructing the probability space, then the solution is called (probabilistic) weak solution. in this thesis we will consider problems on a given complete probability space $(\Omega, \mathcal{F}, P)$ and we investigate mild solutions (as well as generalized weak solutions).
In some applications, it is necessary also to consider the Banach space case regarding of Banach spaced valued stochastic integrals. For example, Banach spaced valued stochastic partial differential equation are discussed in the mild solution sense (Ito-Volterra type equations) in [30, 31, 27, 15, 28, 23, 37, 38, 5].
Two typical methods are common to solve optimal control problems:

- Using of dynamical optimization (Bellman principle).
- Using necessary optimality conditions (Maximum principle).

In [40] the Bellman principle is used to solve an optimal control problem for a stochastic Banach spaces valued differential equation in the case of cylindrical Wiener processes are appleid as noise process. A maximum principle is proved in [39] for an optimal control of stochastic partial differential equations in Banach Spaces with finite dimensional Wiener process noise.
If optimal control problems for stochastic Itô-Volterra equations should be solved by using of a maximum principle then a theory of stochastic backward equations of Itô-Volterra type must be used.

A linear backward stochastic differential (BSDE) equation was first introduced by J. Bismut [4] in the finite dimensional case as the equation for the adjoint variable in the stochastic version of the Pontryagin maximum principle. A general nonlinear backward stochastic differential equation in the finite dimensional case (see for example E. Pardoux and S. G. Peng [20]) which appears in the optimal stochastic control problem is the following:

$$Y(t) = \xi + \int_t^1 f(s, Y(s), Z(s))ds - \int_t^1 Z(s)dW(s), \text{ for } 0 \leq t \leq 1,$$

where $(W(t))_{t\in[0,1]}$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$ with the natural filtration $(\mathcal{F}_t)_{t\in[0,1]}$ and ξ is a given $\mathcal{F}_1$-measurable random variable such that $\mathbf{E}|\xi|^{\mathbf{2}} < \infty$. In [20] E. Pardoux and S. G. Peng considered an adapted solution as a pair of real valued adapted processes $(Y(\cdot), Z(\cdot))$ which satisfy almost surely the above equation. They proved the existence and uniqueness of the adapted solution by assuming the Lipschitz continuity for the generating function f. The interest on backward stochastic differential equations has grown rapidly in regards to the connections of this subject with computational finance, stochastic control problem, and partial differential equations. These equations also provide probabilistic interpretation for solutions to both elliptic and parabolic nonlinear partial differential equations. Indeed, coupled with a forward stochastic differential equations, such BSDEs give an extension of the Feynman-Kac formula to nonlinear case. Numerous authors recently investigated various BSDEs and properties of their solutions, see for instance V. Anh [1], E. Essaky, K. Bahlali and Ouknine Y.

[7], Y. Hu [12], [13], J. Lin [18] or R. Negrea and C. Preda [22] and the references specified therein. In particular, many efforts have been made to relax the assumptions on the coefficient functions. For instance, several papers treat BSDEs with continuous or local drift. In one dimensional case, the essential tool is the comparison-technique. In multidimensional case, the improvements of the Lipschitz condition on the generator, mainly concern only the Y variable and the conditions considered are global. One of the first works treating multidimensional BSDEs with both local conditions on the drift and only square integrable terminal data is the work of K. Bahlali [3]. This author considered BSDEs with locally Lipschitz coefficients in both variables Y and Z. Backward stochastic nonlinear integral equations have been studied by J. Lin [18] under global Lipschitz conditions on the drift term. More precisely, in this work J. Lin proved an existence and uniqueness result for the following nonlinear BSDE of Volterra type:

$$Y(t) + \int_t^T f(t,s,Y(s),Z(t,s))ds + \int_t^T [g(t,s,Y(s)) + Z(t,s)dW(s)] = \xi.$$

In 2006 J. Yong [32] started a detailed analysis of backward stochastic integral equations. The author first proved an existence and uniqueness result for general integral equation of Volterra type given by

$$Y(t) = f(t) + \int_t^T h(t,s,y(s),z(t,s),z(s,t))ds - \int_t^T Z(t,s)dW(s), \tag{1.1}$$

where $h : [0,T] \times [0,T] \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d} \times \Omega \to \mathbb{R}^m$ and $f : [0,T] \times \Omega \to \mathbb{R}^m$ are given. The difficulties that arise in (1.1) are mainly given by the fact that the generator h depends simultaneously on t and s which imply that the equation cannot be reduced to a BSDE in general and the process f is allowed to be only $\mathcal{F}_T$ measurable (not necessarily $\mathcal{F}$-adapted). Equations of the above form often occur in various models of the modern mathematical finance. In the cited work stability statements were also formulated and a duality principle for linear forward stochastic and linear backward stochastic differential equations was obtained. As an application of the duality principle, the author presented a comparison theorem for one dimensional backward stochastic integral equations and a Pontryagin type maximum principle for optimal control of stochastic integral equations. He continued his research and the work in 2008 [33] improves the previously obtained results. This work treats various general BSDEs and new concepts of solutions. The regularity of these solutions is studied by means of Malliavin derivate. Conditions for the Malliavin differentiability are stated and one of the most important result states that the Malliavin derivative of the process $Y(\cdot)$ can be obtained by solving a BSDE.

As an extension, stochastic backward integral equations with values in a separable Hilbert space were recently intensively investigated by V. Anh, W. Grecksch and J. Yong [1], [2].

In the above mentioned literature, all processes were assumed to be real-valued or Hilbert-space valued. Here we are interested in the study of backward stochastic differential equations for Banach-space valued processes.

During the last years the stochastic analysis in Banach spaces have been developed vastly. It has only recently been realized that many results can be generalized under certain circumstances beyond the Hilbert space case. First of all one has to replace the well-known orthogonality by unconditionality. This is the essence of the UMD Banach spaces, where UMD is the abbreviation for unconditional martingale difference. In 2007, J. van Neerven, M. Veraar and L. Weis [26] gave a complete integration theory for Banach-space valued processes. The UMD property is essential to obtain a two-sided estimate of the stochastic integral. In the absence of the famous Itô-isometry such estimates combined with the operator-valued version of the Burkholder-Davis-Gundy inequalities are crucial. In the cited work, martingale representation theorems are also proved. For this thesis such results are usefull to solve BSDEs. Mallivian calculus is also calculated in [21]in UMD Banach spaces.

However there are major differences between these statements and the well-known results from the Hilbert space case. In this setting we cannot expect to get a Banach-space valued process whose stochastic integral represents a given E-valued martingale. But we always obtain an operator-valued random variable with this property. Recently, M. Ondrejat and M. Veraar [19] studied weak characterizations of stochastical integrability and its connection to martingale representation theorems.

Existence, uniqueness and smoothness properties are discussed in [17] for the solution of a Banach space valued stochastic backward Itô Volterra equation in the case of UMD Banach space valued stochastic integrals with respect to a one dimensianal real Wiener process. The stochastic analysis in UMD Banach spaces and the ideas in [1], [2] are used.

1.2. Structure of Thesis

In this thesis we consider stochastic processes particylarly in Banach space $E = L^q(\mathbf{S}, \Sigma, \mu)$ where μ is a σ-finite or finite measure.

In chapter 2 an existence and uniqueness theorem (see Theorem 2.1) is proved for a forward stochastic Volterra integral equation with respect to a H-cylindrical Brownian motion in Banach space E where μ is a σ-finite measure and $q \geq 2$. The Banach fixed-point theorem is used. In general the stochastic convolution has not continuous paths. In Theorem 2.2 conditions are given such that a solution process has with probability one continuous paths in E in which μ is a finite measure. The case of a σ-finite measure is given in Theorem 2.3. Some similarities between assumption in Theorem 2.1 and assuptions that for example given by [41] are discussed in 2.2. As examples, some forms of stochastic heat equations are given. A linear heat equation with additive noise and with multiplicative noise are considered in Equations (2.5) and (2.7), the unique solutions are found in according to Theorem 2.1. The same method is used also in 2.3.3 for a stochastic heat equation with Lipschitz nonlinearities and smoothness of its solution is also considered.

In chapter 3, first a simple backward stochastic integral Equation (3.1) in Banach space E with respect to a H-cylindrical Brownian motion are going to be introduced. The unique solution will be derived. Throughout this chapter μ is a σ-finite measure and $q \leq 2$. Then a more general backward stochastic integral Equation (3.4) discussed. The martingale representation theorem in Banach spaces and Banach fixed-point theorem are our important tools. The general backward stochastic Volttera type integral equation are given by Equation (3.7). To find unique adapted solution processes it is needed to introduce M-adapted solution see Definition 3.2 and using some forms of stochastic Fredholm integral equations (3.9). Finally by suitable assumptions, a unique adapted M-solution are found in Theorem 3.1.

In chapter 4 the duality principles between forward and backward stochastic Volterra integral equations are discussed. The forward stochastic Volterra integral equation (4.1) is defined in a σ-finite space where $q \geq 2$ and correspondingly by using its adjoint operators, two types of backward Volterra integral equations in σ-finite space with $q \leq 2$ are introduced. First the duality principle between Forward and bachward Volltera Equations (4.1), (4.2) are proved in Theorem 4.1. The second and more general duality principle are also given in Theorem 4.2 that is between Equations (4.1) and (4.3). The Itô formula in Banach spaces in [5] are very useful to compute duality principles and as in Remark 4.1 is explained these computation for duality principle hold for every UMD-spaces if forward stochastic Volterra integral equations and corresponding backward stochastic Volterra integral equations are well-defined and more importantly martingale representation theorem can be used efficiently, for example in co-type(2) spaces.

In chapter 5, the stochastic optimal controls in Banach space E with a finite measure is introduced and a maximum principle is used to deal with the stochastic control problem. The E-valued state process is defined through a forward stochastic Volterra integral equation and initially, real valued state process is considered in a bounded closed interval. The cost function is given as Bolza form. The concepts of Nemytskii operators and Fréchet derivatives are employed to assumptions. Regarding to controlled forward stochastic Volterra equation, backward stochastic Volterra equation is derived by using adjoint operators. By using the results of previous chapters, Theorem 5.1 is proved for a stochastic control problem. It is interested sometimes that the control process is also F-valued process where F is a Banach space. Then the stochastic problem is generalized for a F-valued control process. This problem is proved in Theorem 5.2 where a special stochastic forward Volterra equation is used and in Theorem 5.3 it is proved for general one. At the case that the F-valued control process is applied, for getting well-defined corresponding backward equations, the cost function without terminal term (5.3) is employed. Finally in this chapter for application some forms of controlled stochastic heat equations are used as stochastic forward Volterra equations and maximum principle is used to solve a relevant stochastic control problem.

CHAPTER 2

Forward Stochastic Volterra Integral Equation

In this chapter we are going to find unique adapted solution for forward stochastic Volterra integral equation in Banach space L^q.

2.1. Unique Adapted Solution

Consider following forward stochastic Volterra integral equation(FSVIE) in Banach space $E = L^q(\mathbf{S}, \Sigma, \mu)$ where $q \geq 2$ and μ is a σ-finite measure

$$X(t) = \varphi(t) + \int_0^t b(t, s, X(s))\, ds + \int_0^t \rho(t, s, X(s))\, dW^H(s), \quad t \in [0, T] \tag{2.1}$$

The stochastic integral defined with respect to H-cylindrical Brownian motion $W^H(\cdot)$, Definition A.2. Our goal is to find unique adapted E-valued process $X(t)$. Our approach will be to use Banach fixed-point theorem in complete spaces and we use the definition of L^p-stochastic integrability in Banach space E, Definition A.3 and Theorem A.1, specially we use the so-called L^p-stochastic integrability theorem at the case of $E = L^q(\mathbf{S}, \Sigma, \mu)$, Theorem A.2. We use the concept of Nemytskii operator for b and ρ and by using L^p-stochastic integrability it is assumed that there is $\varrho(\cdot,\cdot,\cdot,\cdot) : \Omega \times [0, T] \times [0, T] \times \mathbf{S} \longrightarrow H$ such that $(\rho(t, s, x)h)(\cdot) = [\varrho(t, s, x, \cdot), h]$, for every $h \in H$ and $x \in \mathbb{R}$. In most of this work we drop $\eta \in \mathbf{S}$ and $\omega \in \Omega$ for easiness. We are going to set some assumptions on E-valued process $\varphi(t)$, $b(t, s, \cdot)$ and E-valued operator process $\rho(t, s, \cdot)$. Let $\Delta = \{(t, s) : 0 \leq s \leq t \leq T\}$ and we set following assumptions

(H 0) For each $t \in [0, T]$, $\varphi(t) \in L^p(\Omega; E)$ is $\mathcal{F}_t$-adapted and moreover

$$\mathbb{E}\left\| \left(\int_0^T |\varphi(t)|^2 dt \right)^{\frac{1}{2}} \right\|_E^p < \infty$$

(H 1)

$$b : \Omega \times \Delta \times E \times \mathbf{S} \longrightarrow E$$
$$\rho : \Omega \times \Delta \times E \times \mathbf{S} \longrightarrow \mathcal{L}(H; E)$$

are progressively measurable for each $x \in E$ and $t \in [0, T]$. We use the concept of H-strongly measurability for $\rho(\cdot,\cdot,\cdot)$, i.e. for every $h \in H$, $\rho(\cdot,\cdot,\cdot)h$ is strongly measurable.

(H 2) There exist some positive constants K_1 and K_2 such that for every $t, s \in [0, T]$ and $x, y \in \mathbb{R}$

$$|b(t, s, x) - b(t, s, y)| \leq K_1|x - y|$$
$$\|\varrho(t, s, x) - \varrho(t, s, y)\|_H \leq K_2|x - y|$$

(*H* 3)

$$\mathbb{E}\left\|\sup_{t\in[0,T]}\left(\int_0^t |b(t,s,0)|^2 ds\right)^{\frac{1}{2}}\right\|_E^p < \infty$$

$$\mathbb{E}\left\|\sup_{t\in[0,T]}\left(\int_0^t \|\varrho(t,s,0)\|_H^2\, ds\right)^{\frac{1}{2}}\right\|_E^p < \infty$$

In this thesis we consider the case $2 \le q \le p$. Let define the following Banach space

$$\mathcal{M}_T = \left\{ X(\cdot) \in L^p_{\mathbb{F}}(\Omega; E) : \|X(\cdot)\|^p_{\mathcal{M}_T} = \mathbb{E}\left\|\left(\int_0^T |X(t)|^2 dt\right)^{\frac{1}{2}}\right\|_E^p < \infty \right\}$$

First we are going to prove some lemmas and finally the existence and uniqueness theorem will be given.

LEMMA 2.1. *If assumptions (H 0), (H 1), (H 2) and (H 3) hold and* $x(\cdot) \in \mathcal{M}_T$ *then* $X(\cdot) \in \mathcal{M}_T$ *where* $X(\cdot)$ *is defined by the operator* $\mathcal{A} : \mathcal{M}_T \longrightarrow \mathcal{M}_T$ *as following*

$$X(t) = \mathcal{A}(x(\cdot))(t) = \varphi(t) + \int_0^t b(t,s,x(s))ds + \int_0^t \rho(t,s,x(s))dW^H(s)\,, t \in [0,T]$$

PROOF. Adaptedness is clearly resulted by definition. Let $x(t) \in L^p_{\mathbb{F}}(\Omega; E)$ then we have

$$\begin{aligned}
\mathbb{E}\|X(t)\|^p &= \mathbb{E}\left\|\varphi(t) + \int_0^t b(t,s,x(s))ds + \int_0^t \rho(t,s,x(s))dW^H(s)\right\|^p \\
&\le c\left\{\mathbb{E}\,\|\varphi(t)\|^p + \mathbb{E}\left\|\int_0^t b(t,s,x(s))ds\right\|^p \right.\\
&\qquad \left. + \mathbb{E}\left\|\int_0^t \rho(t,s,x(s))dW^H(s)\right\|^p\right\}
\end{aligned}$$

where we used the norm property and Young's inequality. In this thesis we assume $c \ge 0$ is a positive universal constant and it could change its values. Now we consider summands from above equation separately. First term is bounded by assumption (H 0). We can write for the second term by using the norm property, Hölder's inequality and assumptions in (H2)

$$\begin{aligned}
\mathbb{E}\left\|\int_0^t b(t,s,x(s))ds\right\|^p &\le \mathbb{E}\left\|\sqrt{t}\left(\int_0^t |b(t,s,x(s))|^2 ds\right)^{\frac{1}{2}}\right\|^p \\
&\le t^{\frac{p}{2}}\mathbb{E}\left\|\left(\int_0^t |b(t,s,x(s)) + b(t,s,0) - b(t,s,0)|^2\, ds\right)^{\frac{1}{2}}\right\|^p \\
&\le c\left\{\mathbb{E}\left\|\left(\int_0^t |b(t,s,0)|^2 ds\right)^{\frac{1}{2}}\right\|^p + K_1^p\mathbb{E}\left\|\left(\int_0^t |x(s)|^2 ds\right)^{\frac{1}{2}}\right\|^p\right\} \\
&\le c\left\{\mathbb{E}\left\|\sup_{t\in[0,T]}\left(\int_0^t |b(t,s,0)|^2 ds\right)^{\frac{1}{2}}\right\|^p + \mathbb{E}\left\|\left(\int_0^T |x(s)|^2 ds\right)^{\frac{1}{2}}\right\|^p\right\}
\end{aligned} \tag{2.2}$$

and it results that $\mathbb{E}\left\|\int_0^t b(t,s,x(s))ds\right\|^p$ is bounded for every $t \in [0,T]$.

For the third part we use the Theorem A.2 of L^p-stochastic integrability. So we have

$$\begin{aligned}
\mathbb{E}\left\|\int_0^t \rho(t,s,x(s))dW^H(s)\right\|^p &\le c\mathbb{E}\left\|\left(\int_0^t \|\varrho(t,s,x(s))\|_H^2\, ds\right)^{\frac{1}{2}}\right\|^p \\
&= c\mathbb{E}\left\|\left(\int_0^t |\varrho(t,s,x(s))+\varrho(t,s,0)-\varrho(t,s,0)\|_H^2\, ds\right)^{\frac{1}{2}}\right\|^p \\
&\le c\left\{\mathbb{E}\left\|\left(\int_0^t \|\varrho(t,s,0))\|_H^2\, ds\right)^{\frac{1}{2}}\right\|^p + K_2^p\mathbb{E}\left\|\left(\int_0^t |x(s)|^2 ds\right)^{\frac{1}{2}}\right\|^p\right\} \\
&\le c\left\{\mathbb{E}\left\|\sup_{t\in[0,T]}\left(\int_0^t \|\varrho(t,s,0))\|_H^2\, ds\right)^{\frac{1}{2}}\right\|^p + \mathbb{E}\left\|\left(\int_0^T |x(s)|^2 ds\right)^{\frac{1}{2}}\right\|^p\right\}
\end{aligned}$$

similarly it yields that $\mathbb{E}\left\|\int_0^t \rho(t,s,x(s))dW^H(s)\right\|^p$ is bounded for every $t\in[0,T]$. By using above calculations it yields that $\forall t\in[0,T], X(t)\in L^p(\Omega;E)$.
Now it remains to show $\mathbb{E}\left\|\left(\int_0^T |X(t))|^2dt\right)^{\frac{1}{2}}\right\|_E^p$. For simplicity we set $\forall\eta\in\mathbf{S}$

$$\|X(\cdot)\|_{L^2(0,T)} = \|X(\cdot,\eta)\|_{L^2(0,T)} = \left(\int_0^T |X(t,\eta))|^2dt\right)^{\frac{1}{2}}$$

then we can write by using the norm property

$$\begin{aligned}
\mathbb{E}\left\|\|X(\cdot)\|_{L^2(0,T)}\right\|_E^p \le c\Bigg\{&\mathbb{E}\left\|\|\varphi(\cdot)\|_{L^2(0,T)}\right\|_E^p + \mathbb{E}\left\|\left\|\int_0^\cdot b(\cdot,s,x(s))ds\right\|_{L^2(0,T)}\right\|_E^p \\
&+\mathbb{E}\left\|\left\|\int_0^\cdot \rho(\cdot,s,x(s))dW^H(s)\right\|_{L^2(0,T)}\right\|_E^p\Bigg\}
\end{aligned}$$

Now we consider again above summands separately. First part by assumption is bounded, since

$$\mathbb{E}\left\|\|\varphi(\cdot)\|_{L^2(0,T)}\right\|_E^p = \mathbb{E}\left\|\left(\int_0^T |\varphi(t)|^2dt\right)^{\frac{1}{2}}\right\|_E^p < \infty$$

For the second part, it yields

$$\begin{aligned}
\mathbb{E}\left\|\left\|\int_0^\cdot b(\cdot,s,x(s))ds\right\|_{L^2(0,T)}\right\|_E^p &= \mathbb{E}\left\|\left(\int_0^T \left|\int_0^t b(t,s,x(s))ds\right|^2 dt\right)^{\frac{1}{2}}\right\|_E^p \\
&\le \mathbb{E}\left\|\left(\int_0^T \sup_{t\in[0,T]}\left|\int_0^t b(t,s,x(s))ds\right|^2 dt\right)^{\frac{1}{2}}\right\|_E^p
\end{aligned}$$

$$\leq c\mathbb{E}\left\|\sup_{t\in[0,T]}\left(\left|\int_0^t b(t,s,x(s))ds\right|^2\right)^{\frac{1}{2}}\right\|_E^p$$

$$\leq c\left\{\mathbb{E}\left\|\sup_{t\in[0,T]}\left(\left|\int_0^t b(t,s,0)ds\right|^2\right)^{\frac{1}{2}}\right\|_E^p+\mathbb{E}\left\|\left(\int_0^T |x(s)|^2ds\right)^{\frac{1}{2}}\right\|^p\right\}<\infty$$

For the third part which contains a stochastic integration, we can write

$$\mathbb{E}\left\|\left\|\int_0^{\cdot}\rho(\cdot,s,x(s))dW^H(s)\right\|_{L^2(0,T)}\right\|_E^p$$

$$=\mathbb{E}\left\|\left(\int_0^T\left|\int_0^t\rho(t,s,x(s))dW^H(s)\right|^2dt\right)^{\frac{1}{2}}\right\|_{L^q(S)}^p$$

$$=\mathbb{E}\left(\int_{\mathbf{S}}\left(\int_0^T\left|\int_0^t\rho(t,s,x(s))dW^H(s)\right|^2T\frac{dt}{T}\right)^{\frac{q}{2}}d\mu\right)^{\frac{p}{q}}$$

since $2\leq q$ we use Jensen's inequality for lebesgue measure dt in finite interval $[0,t]$, therefore above last equation is less or equal than

$$\leq T^{(\frac{q}{2}-1)\frac{p}{q}}\mathbb{E}\left(\int_{\mathbf{S}}\left(\int_0^T\left|\int_0^t\rho(t,s,x(s))dW^H(s)\right|^q dt\right)d\mu\right)^{\frac{p}{q}}$$

by applying Fubini's theorem to the left term, it yields

$$=T^{(\frac{q}{2}-1)\frac{p}{q}}\mathbb{E}\left(\int_0^T\int_{\mathbf{S}}\left|\int_0^t\rho(t,s,x(s))dW^H(s)\right|^q d\mu dt\right)^{\frac{p}{q}}$$

$$\leq T^{(\frac{q}{2}-1)\frac{p}{q}}T^{\frac{p}{q}-1}\mathbb{E}\left(\int_0^T\left(\int_{\mathbf{S}}\left|\int_0^t\rho(t,s,x(s))dW^H(s)\right|^q d\mu\right)^{\frac{p}{q}}dt\right)$$

$$=c\int_0^T\mathbb{E}\left(\int_{\mathbf{S}}\left|\int_0^t\rho(t,s,x(s))dW^H(s)\right|^q d\mu\right)^{\frac{p}{q}}dt$$

$$=c\int_0^T\mathbb{E}\left\|\int_0^t\rho(t,s,x(s))dW^H(s)\right\|_E^p dt$$

Since $2\leq q\leq p$ we used again Jensen's inequality for dt and Fubini's theorem for above calculations and by using L^p-stochastic integrability Theorem A.2 and assumption $(H\ 2)$, the last term can be estimated by

$$
\begin{aligned}
&\leq c\int_0^T c\mathbb{E}\left\|\left(\int_0^t \|\rho(t,s,x(s))\|_H^2\,ds\right)^{\frac{1}{2}}\right\|_E^p dt\\
&\leq \int_0^T c\left\{\mathbb{E}\left\|\left(\int_0^t |\varrho(t,s,0))|^2 ds\right)^{\frac{1}{2}}\right\|^p + K_2^p\mathbb{E}\left\|\left(\int_0^t |x(s)|^2 ds\right)^{\frac{1}{2}}\right\|^p\right\} dt\\
&\leq \int_0^T c\left\{\mathbb{E}\left\|\sup_{t\in[0,T]}\left(\int_0^t |\varrho(t,s,0))|^2 ds\right)^{\frac{1}{2}}\right\|^p + \mathbb{E}\left\|\left(\int_0^T |x(s)|^2 ds\right)^{\frac{1}{2}}\right\|^p\right\} dt\\
&< \infty
\end{aligned}
$$

by combining above calculations it results $\mathbb{E}\left\|\left(\int_0^T |X(t)|^2 ds\right)^{\frac{1}{2}}\right\|_E^p < \infty$. □

LEMMA 2.2. *For small enough value $\tau > 0$ the map $\mathcal{A} : \mathcal{M}_\tau \to \mathcal{M}_\tau$ is contractive where $\mathcal{A}(x(\cdot))(t) = X(t)$ given in Lemma 1 for $t \in [0,\tau]$.*

PROOF. Let $x(\cdot), y(\cdot) \in \mathcal{M}_\tau$ and $X(t), Y(t)$ are the processes defined by $\mathcal{A}(\cdot)$ then it can be written

$$
\|X(\cdot) - Y(\cdot)\|_{\mathcal{M}_\tau}^p = \mathbb{E}\left\|\left(\int_0^\tau |X(t) - Y(t)|^2\,dt\right)^{\frac{1}{2}}\right\|_E^p
$$

and by using definition of $X(\cdot), Y(\cdot)$, we have

$$
\begin{aligned}
X(t) - Y(t) &= \varphi(t) + \int_0^t b\,(t,s,x(s))\,ds + \int_0^t \rho\,(t,s,x(s))\,dW^H(s)\\
&\quad - \left(\varphi(t) + \int_0^t b\,(t,s,y(s))\,ds + \int_0^t \rho\,(t,s,y(s))\,dW^H(s)\right)\\
&= \int_0^t (b\,(t,s,x(s)) - b\,(t,s,y(s)))\,ds\\
&\quad + \int_0^t (\rho\,(t,s,x(s)) - \rho\,(t,s,y(s)))\,dW^H(s)
\end{aligned}
$$

and using the norm property it results

$$
\begin{aligned}
\|X(\cdot) - Y(\cdot)\|_{\mathcal{M}_\tau}^p \leq c\Bigg\{&\mathbb{E}\left\|\left\|\int_0^\cdot (b\,(\cdot,s,x(s)) - b\,(\cdot,s,y(s)))\,ds\right\|_{L^2(0,\tau)}\right\|_E^p\\
&+ \mathbb{E}\left\|\left\|\int_0^\cdot (\rho\,(\cdot,s,x(s)) - \rho\,(\cdot,s,y(s)))\,dW^H(s)\right\|_{L^2(0,\tau)}\right\|_E^p\Bigg\}
\end{aligned}
$$

Now we consider every term separately. Hölder's inequality and (H 2) assumption are used for following equations

$$
\begin{aligned}
&\mathbb{E}\left\|\left(\int_0^\tau \left|\int_0^t \left(b\left(t,s,x(s)\right)-b\left(t,s,y(s)\right)\right)ds\right|^2 dt\right)^{\frac{1}{2}}\right\|_E^p\\
\leq&\mathbb{E}\left\|\left(\int_0^\tau t\int_0^t \left|b\left(t,s,x(s)\right)-b\left(t,s,y(s)\right)\right|^2 dsdt\right)^{\frac{1}{2}}\right\|_E^p\\
\leq&K_1^p\tau^{\frac{p}{2}}\mathbb{E}\left\|\left(\int_0^\tau \int_0^t \left|x(s)-y(s)\right|^2 dsdt\right)^{\frac{1}{2}}\right\|_E^p \qquad (2.3)\\
\leq&K_1^p\tau^{\frac{p}{2}}\mathbb{E}\left\|\left(\int_0^\tau \int_0^\tau \left|x(s)-y(s)\right|^2 dsdt\right)^{\frac{1}{2}}\right\|_E^p\\
\leq&K_1^p\tau^{p}\mathbb{E}\left\|\left(\int_0^\tau \left|x(s)-y(s)\right|^2 ds\right)^{\frac{1}{2}}\right\|_E^p
\end{aligned}
$$

Similarly for the second term we can write

$$
\begin{aligned}
&\mathbb{E}\left\|\left(\int_0^\tau \left|\int_0^t \left(\rho\left(t,s,x(s)\right)-\rho\left(t,s,y(s)\right)\right)dW^H(s)\right|^2 dt\right)^{\frac{1}{2}}\right\|_E^p\\
=&\mathbb{E}\left(\int_{\mathbf{S}}\left|\int_0^\tau \left|\int_0^t \left(\rho\left(t,s,x(s)\right)-\rho\left(t,s,y(s)\right)\right)dW^H(s)\right|^2 dt\right|^{\frac{q}{2}} d\mu\right)^{\frac{p}{q}}\\
\leq&\tau^{(\frac{q}{2}-1)\frac{p}{q}}\mathbb{E}\left(\int_{\mathbf{S}}\int_0^\tau \left|\int_0^t \left(\rho\left(t,s,x(s)\right)-\rho\left(t,s,y(s)\right)\right)dW^H(s)\right|^q dtd\mu\right)^{\frac{p}{q}}\\
=&\tau^{(\frac{q}{2}-1)\frac{p}{q}}\mathbb{E}\left(\int_0^\tau\int_{\mathbf{S}} \left|\int_0^t \left(\rho\left(t,s,x(s)\right)-\rho\left(t,s,y(s)\right)\right)dW^H(s)\right|^q d\mu dt\right)^{\frac{p}{q}}\\
\leq&\tau^{(\frac{q}{2}-1)\frac{p}{q}}\tau^{\frac{p}{q}-1}\mathbb{E}\int_0^\tau\left(\int_{\mathbf{S}} \left|\int_0^t \left(\rho\left(t,s,x(s)\right)-\rho\left(t,s,y(s)\right)\right)dW^H(s)\right|^q d\mu\right)^{\frac{p}{q}} dt\\
=&\tau^{\frac{p}{2}-1}\int_0^\tau \mathbb{E}\left\|\int_0^t \left(\rho\left(t,s,x(s)\right)-\rho\left(t,s,y(s)\right)\right)dW^H(s)\right\|_E^p dt
\end{aligned}
$$

Several times Jensen's inequality for Lebesgue measure in finite interval and Fubini's theorem were used for above relations and we use now the L^p-stochastic integrablity Theorem A.2 for stochastic integral, so it results that, the last term is smaller than

$$
\tau^{\frac{p}{2}-1}\int_0^\tau c\,\mathbb{E}\left\|\left(\int_0^t \left\|\varrho\left(t,s,x(s)\right)-\varrho\left(t,s,y(s)\right)\right\|_H^2 ds\right)^{\frac{1}{2}}\right\|_E^p dt
$$

and by using assumption (H 2) it leads that it is smaller or equal than

$$\tau^{\frac{p}{2}-1}\int_0^\tau cK_2^p\,\mathbb{E}\left\|\left(\int_0^t |x(s)-y(s)|_H^2\,ds\right)^{\frac{1}{2}}\right\|_E^p dt$$
$$= c\tau^{\frac{p}{2}}K_2^p\,\mathbb{E}\left\|\left(\int_0^\tau |x(s)-y(s)|^2\,ds\right)^{\frac{1}{2}}\right\|_E^p$$

Now by combining above results, it holds

$$\|X(\cdot)-Y(\cdot)\|_{\mathcal{M}_\tau}^p \le \left(K_1^p\tau^p + cK_2^p\tau^{\frac{p}{2}}\right)\|x(\cdot)-y(\cdot)\|_{\mathcal{M}_\tau}^p$$

by taking τ small enough it results that $\mathcal{A}:\mathcal{M}_\tau\to\mathcal{M}_\tau$ is contractive.

□

THEOREM 2.1. *By assumptions* (*H* 0), (*H* 1), (*H* 2) *and* (*H* 3), *FSVIE (2.1) has a unique adapted solution* $X(\cdot)\in\mathcal{M}_T$.

PROOF. By using Lemma 1 and Lemma 2, it is clear the map $\mathcal{A}$ defined in Lemma 1 contractive in the interval $[0,\tau]$ then by using Banach fixed-point theorem in complete space $\mathcal{M}_\tau$, there exists a unique solution in the interval $[0,\tau]$ and by induction to the whole interval $[0,T]$ we can find the unique solution $X(t)$ in $[0,T]$. □

2.2. Path Continuity

In this section we want to consider path continuity of the solution derived in theorem (2.1). For this reason let the following assumptions hold:

(G 1) there exists positive constants K_3, K_4, K_5 such that for every $t_1, t_2 \in [0,T]$

$$|\varphi(t_1)-\varphi(t_2)| \le K_3|t_1-t_2|$$
$$|b(t_1,s,x)-b(t_2,s,x)| \le K_4|t_1-t_2|$$
$$\|\varrho(t_1,s,x)-\varrho(t_2,s,x)\|_H \le K_5|t_1-t_2|$$

(G 2)

$$\mathbb{E}\left\|\left(\int_0^T |\varphi(t)|^4 ds\right)^{\frac{1}{4}}\right\|_E^p < \infty$$
$$\mathbb{E}\left\|\left(\int_0^T |X(t)|^4 ds\right)^{\frac{1}{4}}\right\|_E^p < \infty$$
$$\mathbb{E}\left\|\sup_{t\in[0,T]}\left(\int_0^t |b(t,s,0)|^4 ds\right)^{\frac{1}{4}}\right\|_E^p < \infty$$
$$\mathbb{E}\left\|\sup_{t\in[0,T]}\left(\int_0^t \|\varrho(t,s,0)\|_H^4\, ds\right)^{\frac{1}{4}}\right\|_E^p < \infty$$

In the following we write $\|\cdot\|$ instead of $\|\cdot\|_E$.

LEMMA 2.3. *If* $q \geq 4$ *and assumptions* $(H\ 0)$, $(H\ 1)$, $(H\ 2)$ *and* $(H\ 3)$ *hold, then the assumptions (G 2) hold.*

PROOF. By using Jensen's inequality and Fubini's theorem, we can write for example

$$\mathbb{E}\left\|\left(\int_0^T |X(t)|^4 dt\right)^{\frac{1}{4}}\right\|_E^p = \mathbb{E}\left(\int_{\mathbf{S}} \left(\int_0^T |X(t)|^4 dt\right)^{\frac{q}{4}} d\mu\right)^{\frac{p}{q}}$$

$$\leq T^{(\frac{q}{4}-1)\frac{p}{q}}\mathbb{E}\left(\int_{\mathbf{S}}\int_0^T |X(t)|^q dt\, d\mu\right)^{\frac{p}{q}} = T^{(\frac{p}{4}-\frac{p}{q})}\mathbb{E}\left(\int_0^T \int_{\mathbf{S}} |X(t)|^q d\mu\, dt\right)^{\frac{p}{q}}$$

$$\leq T^{(\frac{p}{4}-\frac{p}{q})}T^{(\frac{p}{q}-1)}\mathbb{E}\left(\int_0^T \left(\int_{\mathbf{S}} |X(t)|^q d\mu\right)^{\frac{p}{q}} dt\right)$$

$$= T^{(\frac{p}{4}-1)}\int_0^T \mathbb{E}\left(\int_{\mathbf{S}} |X(t)|^q d\mu\right)^{\frac{p}{q}} dt = c\int_0^T \mathbb{E}\left\|X(t)\right\|^p dt < \infty$$

since from Lemma(1) $X(\cdot) \in L^p(\Omega; E)$. By using similar calculations, assumptions (G 2) hold. □

THEOREM 2.2. *Let* μ *is a finite measure over* $(\mathbf{S}, \Sigma)$ *and assumptions* $(H\ 0)$, $(H\ 1)$, $(H\ 2)$, $(H\ 3)$, $(G\ 1)$ *and* $(G\ 2)$ *hold then the solution of FSVIE equation(2.1) has continuous path in* $E = L^q(S, \mathcal{S}, \mu)$ *for some* p.

PROOF. We use Kolmogrov's continuity theorem and show that for all $t, s \in [0, T]$, $\mathbb{E}\|X_t - X_s\|^\alpha \leq D\,|t-s|^{1+\beta}$ holds for some constants $\alpha, D, \beta > 0$.
Let $t_1, t_2 \in [0, T]$ and $t_1 < t_2$.

$$\mathbb{E}\left\|X(t_2) - X(t_1)\right\|_E^p =$$
$$\mathbb{E}\Bigg\|\varphi(t_2) + \int_0^{t_2} b\left(t_2, s, X(s)\right) ds + \int_0^{t_2} \rho\left(t_2, s, X(s)\right) dW^H(s)$$
$$-\left(\varphi(t_1) + \int_0^{t_1} b\left(t_1, s, X(s)\right) ds + \int_0^{t_1} \rho\left(t_1, s, X(s)\right) dW^H(s)\right)\Bigg\|_E^p$$

by using the norm property this is smaller or equal than

$$c\mathbb{E}\left\|\varphi(t_2) - \varphi(t_1)\right\|_E^p$$
$$+\, c\mathbb{E}\left\|\int_0^{t_1} \left(b\left(t_2, s, X(s)\right) - b\left(t_1, s, X(s)\right)\right) ds + \int_{t_1}^{t_2} b\left(t_2, s, X(s)\right) ds\right\|_E^p$$
$$+\, c\mathbb{E}\left\|\int_0^{t_1} \left(\rho(t_2, s, X(s)) - \rho(t_1, s, X(s))\right) dW^H(s) + \int_{t_1}^{t_2} \rho(t_2, s, X(s)) dW^H(s)\right\|^p$$

Now we consider every summands separately. For the first part by using the assumptions

$$\mathbb{E}\left\|\varphi(t_2) - \varphi(t_1)\right\|^p \leq K_3^p |t_2 - t_1|^p$$

Again by using the norm property we can split the second part in two terms as follows

$$
\begin{aligned}
&\mathbb{E}\left\|\int_0^{t_1}\left(b(t_2,s,X(s))-b(t_1,s,X(s))\right)ds\right\|_E^p\\
\leq& K_4^p\mathbb{E}\left\|\int_0^{t_1}(t_2-t_1))\,ds\right\|_E^p = K_4^p|t_2-t_1|^p
\end{aligned}
\tag{2.4}
$$

and similar for second term of second part it can be written

$$
\begin{aligned}
&\mathbb{E}\left\|\int_{t_1}^{t_2}b(t_2,s,X(s))ds\right\|^p\leq\mathbb{E}\left\|\left(\int_{t_1}^{t_2}ds\right)^{\frac{3}{4}}\left(\int_{t_1}^{t_2}|b(t_2,s,X(s))|^4ds\right)^{\frac{1}{4}}\right\|_E^p\\
\leq& c(t_2-t_1)^{\frac{3p}{4}}\left\{\mathbb{E}\left\|\left(\int_{t_1}^{t_2}|b(t_2,s,0)|^4\right)^{\frac{1}{4}}\right\|_E^p+\mathbb{E}\left\|\left(\int_{t_1}^{t_2}|X(s)|^4\right)^{\frac{1}{4}}\right\|_E^p\right\}\\
\leq& c(t_2-t_1)^{\frac{3p}{4}}\left\{\mathbb{E}\left\|\left(\sup_{t\in[0,T]}\int_0^t|b(t,s,0)|^4\right)^{\frac{1}{4}}\right\|_E^p+\mathbb{E}\left\|\left(\int_0^T|X(s)|^4\right)^{\frac{1}{4}}\right\|_E^p\right\}\\
\leq& c(t_2-t_1)^{\frac{3p}{4}}
\end{aligned}
$$

For the stochastic integral part, after using the norm property, it yields to two parts and we consider these terms one by one. L^p-stochastic integrability Theorem A.2 and Hölder's inequality will be applied.

$$
\begin{aligned}
&\mathbb{E}\left\|\int_0^{t_1}\left(\rho(t_2,s,X(s))-\rho(t_1,s,X(s))\right)dW^H(s)\right\|_E^p\\
&\leq c\mathbb{E}\left\|\left(\int_0^{t_1}\|\varrho(t_2,s,X(s))-\varrho(t_1,s,X(s))\|_H^2\,ds\right)^{\frac{1}{2}}\right\|_E^p\\
&\leq cK_5^p\mathbb{E}\left\|\left(\int_0^{t_1}(t_2-t_1)^2ds\right)^{\frac{1}{2}}\right\|^p\\
&\leq cK_5^p t_1^{\frac{p}{2}}(t_2-t_1)^p
\end{aligned}
$$

and similar for another term, it yields

$$
\begin{aligned}
&\mathbb{E}\left\|\int_{t_1}^{t_2}\rho(t_2,s,X(s))W^H(s)\right\|_E^p\\
\leq& c\mathbb{E}\left\|\left(\int_{t_1}^{t_2}\|\varrho(t_2,s,X(s))\|_H^2\,ds\right)^{\frac{1}{2}}\right\|_E^p\\
\leq& c\mathbb{E}\left\|\left(\int_{t_1}^{t_2}ds\right)^{\frac{1}{4}}\left(\int_{t_1}^{t_2}\|\varrho(t_2,s,X(s))\|_H^4\,ds\right)^{\frac{1}{4}}\right\|_E^p\\
\leq& c|t_2-t_1|^{\frac{p}{4}}\mathbb{E}\left\|\left(\int_{t_1}^{t_2}\|\varrho(t_2,s,X(s))\|_H^4\,ds\right)^{\frac{1}{4}}\right\|_E^p
\end{aligned}
$$

$$\leq c|t_2 - t_1|^{\frac{p}{4}} \left\{ \mathbb{E} \left\| \left(\int_{t_1}^{t_2} \|\varrho(t_2, s, 0)\|_H^4 \right)^{\frac{1}{4}} \right\|_E^p + \mathbb{E} \left\| \left(\int_{t_1}^{t_2} |X(s)|^4 \right)^{\frac{1}{4}} \right\|_E^p \right\}$$

$$\leq c|t_2 - t_1|^{\frac{p}{4}} \left\{ \mathbb{E} \left\| \left(\sup_{t\in[0,T]} \int_0^t \|\varrho(t, s, 0)\|_H^4 \right)^{\frac{1}{4}} \right\|_E^p + \mathbb{E} \left\| \left(\int_0^T |X(s)|^4 \right)^{\frac{1}{4}} \right\|_E^p \right\}$$

$$\leq c|t_2 - t_1|^{\frac{p}{4}}$$

by gathering all above results, it yields

$$\mathbb{E}\,\|X(t_2) - X(t_1)\|_E^p$$

$$\leq c_1(t_2 - t_1)^p + c_2 t_1^p (t_2 - t_1)^p + c_3(t_2 - t_1)^{\frac{3p}{4}} + c_4 t_1^{\frac{p}{2}} (t_2 - t_1)^p + c_5(t_2 - t_1)^{\frac{p}{4}}$$

$$\leq c|t_2 - t_1|^{\frac{p}{4}}$$

By taking $\alpha = p > 4$ the continuity property in Kolmogrov's theorem is satisfied.

□

REMARK 2.1. *It is essential for path continuity in theorem 2.2 that the μ be finite measure. But in the case of only σ-finite measure we could have also path continuity by changing our assumptions in (G 1). It is needed that our positive constants will be elements of E, as given followingly.*

Let following assumption holds:

(G′ 1) there exist functions $k_3(\cdot), k_4(\cdot), k_5(\cdot) \in E$ such that for every $t_1, t_2 \in [0, T]$ and $\eta \in \mathbf{S}$,

$$|\varphi(t_1, \eta) - \varphi(t_2, \eta)| \leq k_3(\eta)|t_1 - t_2|$$

$$|b(t_1, s, x, \eta) - b(t_2, s, x, \eta)| \leq k_4(\eta)|t_1 - t_2|$$

$$\|\varrho(t_1, s, x, \eta) - \varrho(t_2, s, x, \eta)\|_H \leq k_5(\eta)|t_1 - t_2|$$

THEOREM 2.3. *If assumptions (H 0), (H 1), (H 2), (H 3) and (G′ 1), (G 2) hold and μ is a σ-finite measure then the solution of FSVIE Equation(2.1) has continuous path in E for some $p > 4$.*

PROOF. By using similar method in proof of Theorem 2.2, the theorem can be easily proved. For example if we assume the Formula (2.4) and apply assumption (G′1), it yields

$$\mathbb{E} \left\| \int_0^{t_1} (b(t_2, s, X(s)) - b(t_1, s, X(s)))\, ds \right\|_E^p$$

$$\leq \mathbb{E} \left\| \int_0^{t_1} k_4(\cdot)\, |t_2 - t_1|\, ds \right\|_E^p \leq \|k_4(\cdot)\|_E^p |t_2 - t_1|^p \leq c|t_2 - t_1|^p$$

and so on the goal is resulted by Kolgomrov's continuity theorem. □

REMARK 2.2. *Theorem 2.1 is also true in the case of $\mu(\mathbf{S}) < \infty$ if we introduce the following assumptions:*

let $\mathcal{H}_0$ *be the set of all measurable functions* $\kappa : \Delta \longrightarrow \mathbb{R}^+$ *with* $t \longrightarrow \int\limits_0^t \kappa(t,s)ds \in L^\infty(0,T)$ *and* $\limsup\limits_{\varepsilon\downarrow 0} \left\| \int\limits_{\cdot}^{\cdot+\varepsilon} \kappa(\cdot + t, s)ds \right\|_{L^\infty(0,T)} = 0$

(H 0 ′) *we assume for* $\kappa_1, \kappa_2 \in \mathcal{H}_0$ *and*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int\limits_0^t \kappa_i^2(t,s)ds < \infty$$

and

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int\limits_0^t (\kappa_1^2(t,s) + \kappa_2^2(t,s))\mathbb{E}\|\varphi(s)\|_E^2 < \infty$$

(H 1 ′) *for all* $t, s \in [0,T]$ *and* $x, y \in \mathbb{R}$

$$|b(t,s,x) - b(t,s,y)| \leq \kappa_1(t,s)|x-y|,$$

$$\|\varrho(t,s,x) - \varrho(t,s,y)\|_H \leq \kappa_2(t,s)|x-y|$$

(H 2 ′) *for all* $t, s \in [0,T]$ *and* $x \in \mathbb{R}$

$$|b(t,s,x)| \leq \kappa_1(t,s)(|x|+1),$$

$$\|\varrho(t,s,x)\|_H \leq \kappa_2(t,s)(|x|+1)$$

These conditions are similar to Zhang [41].
If we substitute (H 2 ′) by

(H 2 ″) *for all* $t, s \in [0,T]$ *and* $x \in \mathbb{R}$

$$|b(t,s,x)| \leq \kappa_1(t,s)(|x|),$$

$$\|\varrho(t,s,x)\|_H \leq \kappa_2(t,s)(|x|)$$

then Theorem 2.1 holds also for a σ*-finite measure space* $(\mathbf{S}, \Sigma, \mu)$.
If $x, y \in E$ *then we get from (H 1 ′) and (H 2 ′) for all* $t, s \in [0,T]$

(H 1 ″)

$$\|b(t,s,x) - b(t,s,y)\|_E \leq \kappa_1(t,s)\|x-y\|_E,$$

$$\|\|\varrho(t,s,x) - \varrho(t,s,y)\|_H\|_E \leq \kappa_2(t,s)\|x-y\|_E$$

(H 2 ″)

$$\|b(t,s,x)\|_E \leq \kappa_1(t,s)(\|x\|_E + 1),$$

$$\|\|\varrho(t,s,x)\|_H\|_E \leq \kappa_2(t,s)(\|x\|_E + 1).$$

Exemplary we consider the second term (2.2) *in the proof of Lemma 2.1*

$$
\begin{aligned}
&\mathbb{E}\left\|\int_0^t b(t,s,x(s))ds\right\|^p \le \mathbb{E}\left\|\int_0^t \kappa_1(t,s)(|x(s)|+1)^2 ds\right\|^p \\
&\le \mathbb{E}\left\|\left(\int_0^t \kappa_1^2(t,s)ds\right)^{\frac{1}{2}}\left(\int_0^t |x(s)+1|^2\, ds\right)^{\frac{1}{2}}\right\|^p \\
&\le \left(\int_0^t \kappa_1^2(t,s)ds\right)^{\frac{p}{2}}\left\{c\sqrt{t}\mu(\mathbf{S}) + \mathbb{E}\left\|\left(\int_0^t \kappa_1^2(t,s)ds\int_0^t |x(s)|^2 ds\right)^{\frac{1}{2}}\right\|^p\right\} \\
&\le \left(\operatorname*{ess\,sup}_{t\in[0,T]}\int_0^t \kappa_1^2(t,s)ds\right)^{\frac{p}{2}}\left\{c\sqrt{T}\mu(\mathbf{S}) + \mathbb{E}\left\|\left(\int_0^T |x(s)|^2 ds\right)^{\frac{1}{2}}\right\|^p\right\}
\end{aligned}
$$

and in the proof of Lemma 2.2 for (2.3) *we have*

$$
\begin{aligned}
&\mathbb{E}\left\|\left(\int_0^\tau \left|\int_0^t \left(b\left(t,s,x(s)\right) - b\left(t,s,y(s)\right)\right) ds\right|^2 dt\right)^{\frac{1}{2}}\right\|_E^p \\
&\le \mathbb{E}\left\|\left(\int_0^\tau \left|\int_0^t \kappa_1(t,s)\left|x(s)-y(s)\right| ds\right|^2 dt\right)^{\frac{1}{2}}\right\|_E^p \\
&\le \mathbb{E}\left\|\left(\int_0^\tau \left(\int_0^t \kappa_1^2(t,s)ds\int_0^t \left|x(s)-y(s)\right|^2 ds\right) dt\right)^{\frac{1}{2}}\right\|_E^p \\
&\le \left(\operatorname*{ess\,sup}_{t\in[0,T]}\int_0^t \kappa_1^2(t,s)ds\right)^{\frac{p}{2}}\mathbb{E}\left\|\left(\int_0^\tau \left(\int_0^t \left|x(s)-y(s))\right|^2 ds\right) dt\right)^{\frac{1}{2}}\right\|_E^p \\
&\le \tau^{\frac{p}{2}}\left(\operatorname*{ess\,sup}_{t\in[0,T]}\int_0^t \kappa_1^2(t,s)ds\right)^{\frac{p}{2}}\mathbb{E}\left\|\left(\int_0^\tau \left|x(s)-y(s)\right|^2 ds\right)^{\frac{1}{2}}\right\|_E^p
\end{aligned}
$$

2.3. Examples

In this section as examples, we consider some forms of stochastic heat equation with homogeneous Dirichlet boundary conditions in Banach space $E = L^q(0,1)$, $q \ge 2$.

2.3.1. Linear Heat Equation with Additive Noise. In first example consider the following stochastic heat equation with additive noise

$$
\begin{cases}
dX(t,\xi) = \Delta X(t,\xi)dt + \phi dW^H(t), & t\in[0,T], \xi\in(0,1)\\
X(0,\xi) = X_0(\xi), & \xi\in(0,1), X_0(\cdot)\in L^p_{\mathcal{F}_0}(\Omega;E)\\
X(t,0) = X(t,1) = 0, & t\in[0,T]
\end{cases}
\tag{2.5}
$$

where Δ is Laplacian and $H = L^2(0,1)$. We also assume that $\phi \in \gamma(L^2(0,1);L^q(0,1))$ is γ-Radonifying operator.

By these assumptions we define above given heat equation as following stochastic evolution equation in Theorem 1.1 [24]

$$\begin{cases} dX(t) = AX(t)dt + \phi dW^H(t), & t \in [0,T], \\ X(0) = X_0 \end{cases}$$

where A is Dirichlet Laplacian operator on E and it generates analytic C_0-semigroup $S(t)$ By using Theorem 1.1 [24] we can find an unique mild solution which satisfies following equation

$$X(t) = S(t)X_0 + \int_0^t S(t-s)\phi dW^H(s),\ t \in [0,T] \tag{2.6}$$

We can reformulate Equation (2.6) as following FSVIE

$$X(t) = \varphi(t) + \int_0^t \rho\,(t,s,X(s))\,dW^H(s),\ t \in [0,T]$$

where $\rho\,(t,s,X(s)) \equiv S(t-s)\phi$, $\varphi(t) = S(t)X_0$ and obviously $b(t,s,X(s)) \equiv 0$. Correspondingly Equation (2.6) is an example of a FSVIE and Theorem 2.1 shows, that (2.6) has a unique solution process in $\mathcal{M}_T$.

2.3.2. Linear Heat Equation with Multiplicative Noise. Let the following stochastic heat equation with multiplicative noise is given

$$\begin{cases} dX(t,\xi) = \Delta X(t,\xi) + \psi X(t,\xi)dW^H(t), & t \in [0,T], \xi \in (0,1) \\ X(0,\xi) = X_0(\xi), & \xi \in (0,1), X_0(\cdot) \in E \\ X(t,0) = X(t,1) = 0, & t \in [0,T] \end{cases} \tag{2.7}$$

where Δ is Laplacian and $\psi \in \mathcal{L}(E;\gamma(H;E))$. Similar previous section it can be reformulated as

$$\begin{cases} dX(t) = AX(t)dt + B(t,X(t))dW^H(t), & t \in [0,T], \\ X(0) = X_0 \end{cases}$$

where A is Dirichlet Laplacian operator on E and it generates analytic C_0-semigroup $S(t)$, and $B(t,X(t)) := \psi X(t)$.

The assumptions in Theorem 1.1 [24] are again satisfied and there exist a unique mild solution which satisfies following equation

$$X(t) = S(t)X_0 + \int_0^t S(t-s)\psi X(s)dW^H(s),\ t \in [0,T] \tag{2.8}$$

Equation (2.8) can be written as following FSVIE

$$X(t) = \varphi(t) + \int_0^t b\,(t,s,X(s))\,ds + \int_0^t \rho\,(t,s,X(s))\,dW^H(s),\ t \in [0,T]$$

where $\varphi(t) = S(t)X_0,\ b\,(t,s,X(s))) \equiv 0$ and $\rho\,(t,s,X(s))) \equiv S(t-s)\psi X(s)$. It follows from Theorem 2.1, that (2.8) has a unique solution process in $\mathcal{M}_T$.

2.3.3. Stochastic Heat Equation with Lipschitz Nonlinearities. We introduce

$$\begin{cases} dX(t,\xi) = \Delta X(t,\xi) + F(X(t,\xi))dt + G(X(t,\xi))dW^H(t), & t \in [0,T], \xi \in (0,1) \\ X(0,\xi) = X_0(\xi), & \xi \in (0,1), X_0(\cdot) \in E \\ X(t,0) = X(t,1) = 0, & t \in [0,T] \end{cases}$$

and define the solution in the sence of mild solution, that is

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)G(X(s))dW^H(s),\ t \in [0,T] \tag{2.9}$$

where $S(t)$ ($t \in [0,T]$) is the C_0-semigroup of the last examples.
We assume that there is a constant $L > 0$ and $F : \mathbb{R} \longrightarrow \mathbb{R}$ with $F(0) = 0$ and $|F(x) - F(y)| \leq L|x-y|$ for all $x, y \in \mathbb{R}$, moreover $G : \mathbb{R} \longrightarrow H$ with $G(0) = 0$, $\|G(x) - G(y)\|_H \leq L|x-y|$ for all $x, y \in \mathbb{R}$. Obviously the assumption of Theorem 2.1 are fulfilled if we set $\varphi(t) = S(t)X_0$, $b(t,s,X(s) = S(t-s)F(X(s))$ and $\rho(t,s,X(s) = S(t-s)G(X(s))$.

2.4. More Smoothness of the Solution Process of 2.3.3

In general the mild solution is not a strong solution, where a strong solution of (2.9) is defined by $X(t) \in D(A)$ and

$$X(t) = X_0 + \int_0^t (AX(s) + F(X(s)))ds + \int_0^t G(X(s))dW^H(s),\ t \in [0,T]$$

for all $t \in [0,T]$ with the probability one.
Let A be the generator of an analytical C_0-semigroup $(S(t))_{t\geq 0}$ in a Banach space E. Then it is possible to define the fractional power[1] $A^{-\alpha}$ for $\alpha > 0$ and $A^{\alpha} = (A^{-\alpha})^{-1}$ for $\alpha > 0$[2]. If $\alpha \in (0,1)$ then explicit formula for $A^{\alpha}x$ ($x \in D(A) \subset D(A^{\alpha})$) is known[3]. The operator norm of $A^{-\alpha}$ is bounded[4]. A^{α} is a closed operator with following domain[5]

$$D(A^{\alpha}) = R(A^{-\alpha}),\ \overline{D(A^{\alpha})} = E. \tag{2.10}$$

If additionally $0 \in \rho(A)$, then[6] the operator $A^{\alpha}S(t)$ is bounded for every $t > 0$ and $\|A^{\alpha}S(t)\| \leq M_{\alpha}t^{-\alpha}e^{-\theta t}$.
We consider F and G as defined in previous section 2.3.3 and want to choose conditions such that at least $X(t) \in D(A^{\alpha})$. We introduce for $\alpha \in (0, \frac{1}{2})$ and $X_0 \in D(A^{\alpha})$ the following FSVIE

$$\begin{aligned} Y(t) =& A^{\alpha}S(t)X_0 + \int_0^t A^{\alpha}S(t-s)F(A^{-\alpha}Y(s))ds \\ &+ \int_0^t A^{\alpha}S(t-s)G(A^{-\alpha}Y(s))dW^H(s), t \in [0,T] \end{aligned} \tag{2.11}$$

[1] see formula (6.9) page 70 in [29]
[2] Definition 6.7 page 72 in [29]
[3] formula (6.16) page 72 in [29]
[4] Lemma 6.3 page 71 in [29]
[5] Theorem 6.8 page 72 in [29]
[6] Theorem 6.13 page 74 in [29]

Obviously the condition of Remark 2.2 are fulfilled with $\kappa_1(t,s) = \kappa_2(t,s) = \frac{1}{(t-s)^\alpha}$ for $\alpha \in (0, \frac{1}{2})$. If we apply $A^{-\alpha}$ on both sides of (2.11) then we have following equation since $A^{-\alpha}$ is linear and bounded

$$A^{-\alpha}Y(t) = S(t)X_0 + \int_0^t S(t-s)F(A^{-\alpha}Y(s))ds + \int_0^t S(t-s)G(A^{-\alpha}Y(s))dW^H(s), \quad t \in [0,T]. \tag{2.12}$$

Here we set $X(t) := A^{-\alpha}Y(t)$ then we get with (2.10) that $X(t) \in D(A^\alpha)$.

CHAPTER 3

Backward Stochastic Volterra Integral Equation

In this chapter we introduce backward stochastic Volterra integral equation (BSVIE) in Banach space $E = L^q(\mathbf{S}, \Sigma, \mu)$ with respect to a H-cylindrical Brownian motion. We are going to find unique adapted solution.

3.1. A Special BSVIE

First we consider the following simple BSVIE in Banach space $E = L^q(\mathbf{S}, \Sigma, \mu)$ where $1 < q \leq 2$ and μ is a σ-finite measure,

$$Y(t) = \varphi(t) - \int_t^T Z(t,s)dW^H(s)\,,\, t \in [0,T] \tag{3.1}$$

where $\varphi(\cdot)$ is given with $\mathbb{E}\left(\sup_{t\in[0,T]} \|\varphi(t)\|^p\right) < \infty$, $1 < p \leq q \leq 2$ and $\varphi(t)$ is $\mathcal{F}_T^H$-measurable, $\mathcal{F}_t^H$ is the σ-algebra generated by $W^H(s)$ (H-cylindrical Brownian motion) $s \leq t$. We are interested in finding unique adapted solution $(Y(t), Z(t,s))$ where $Z(t,s)$ is linear bounded operator which operates followingly $Z(t,\cdot) : \Omega \times [0,T] \longrightarrow \mathcal{L}(H;E)$ and $Z(t,s)$ is $\mathcal{F}_s^H$-measurable for $s \leq t$. Since $\mathbb{E}\left(\sup_{t\in[0,T]} \|\varphi(t)\|^p\right) < \infty$, we have especially $\mathbb{E}\|\varphi(t)\|^p < \infty, \forall t \in [0,T]$ and $\mathbb{E}\left(\int_0^T \|\varphi(t\|^p dt\right) < \infty$. By using Hölder inequality it yields that $\mathbb{E}\|\varphi(t)\| < \infty, \forall t \in [0,T]$. Therefore the conditional expectation could be defined as

$$\psi_t(r) := \mathbb{E}\left(\varphi(t)|\mathcal{F}_r^H\right) \quad \forall t \in [0,T],\, 0 \leq r \leq T \tag{3.2}$$

$\psi_t(r)$ is only $\mathcal{F}_r$-measurable and is a L^p-martingale with respect to r (for simplicity we set $\mathcal{F}_r := \mathcal{F}_r^H$), because

$$\mathbb{E}\|\psi_t(r)\|^p = \mathbb{E}\|\mathbb{E}\left(\varphi(t)|\mathcal{F}_r\right)\|^p \leq \mathbb{E}(\mathbb{E}\{\|\varphi(t)\|^p|\mathcal{F}_r\}) = \mathbb{E}\|\varphi(t)\|^p < \infty$$

and for $s < r$

$$\mathbb{E}(\psi_t(r)|\mathcal{F}_s) = \mathbb{E}(\mathbb{E}(\varphi(t)|\mathcal{F}_r)|\mathcal{F}_s) = \mathbb{E}(\varphi(t)|\mathcal{F}_s) = \psi_t(s).$$

Now we can use martingale representation theorem in Banach spaces, Theorem A.6. Then we can find a unique $X_t \in L^p_{\mathbb{F}}\left(\Omega; \gamma\left(L^2(0,T;H), E\right)\right)$ such that

$$\psi_t(r) = \mathbb{E}(\psi_t(r)) + I^{W_H}(\xi_{X_t}(r))$$

where for $f \in H$, $\xi_{X_t}(r,\omega)f := (X(t,\omega))(\mathbb{1}_{[0,r]}f)$ and I^{W_H} is the integral process for ξ_{X_t}, $I^{W_H} : r \longrightarrow I^{W_H}(\xi_{X_t}(r))$. In especial case, for example when E is cotype-2 Definition A.4 then this integral process is represented by a L^p-stochastically integrable process $Z_t(\cdot)$ that is unique and we have

$$I^{W_H}(\xi_{X_t}(r)) = \int_0^r Z_t(s)dW^H(s).$$

For simplicity we denote $Z_t(s) := Z(t,s)$. In our case $E = L^q(\mathbf{S},\Sigma,\mu)$, $1 < q \leq 2$, this space is cotype-2 then we can find unique process $Z(t,s) \in L^p(\Omega; L^2(0,T;\gamma(H,E)))$ and moreover we can use Theorem A.3, Theorem A.2, we can find $\mathfrak{Z} : \Omega \times [0,T] \times \mathbf{S} \longrightarrow H$ such that $\forall h \in H$, $(Z_t(s)h)(\cdot) = [\mathfrak{Z}_t(s,\cdot),h]_H$ is the element in E ($[\cdot,\cdot]_H$ is inner product in Hilbert space H) and $\mathbb{E}\left\|\left(\int_0^T \|\mathfrak{Z}_t(s,\cdot)\|_H^2 ds\right)^{\frac{1}{2}}\right\|_E^p < \infty$. The Theorem A.2 indicates that the Z_t is L^p-stochastically integrable with respect to W^H if and only if

$$\mathbb{E}\left\|\left(\int_0^T \|\mathfrak{Z}_t(s,\cdot)\|_H^2 ds\right)^{\frac{1}{2}}\right\|_E^p < \infty$$

and we have

$$\mathbb{E}\left\|\int_0^T Z_t(s)dW^H(s)\right\|_E^p \simeq \mathbb{E}\left\|\left(\int_0^T \|\mathfrak{Z}_t(s,\cdot)\|_H^2 ds\right)^{\frac{1}{2}}\right\|_E^p,$$

where $\simeq$ is defined by

Definition 3.1. *Let $A \simeq B$ if and only there exists positive constants c, C which depend on E, p, such that $cB \leq A \leq CB$.*

Again for simplicity we write $\mathfrak{Z}_t(s) := \mathfrak{Z}(t,s)$ similar to $Z_t(s) := Z(t,s)$. Therefore we have now

$$\psi_t(r) = \mathbb{E}(\varphi(t)) + \int_0^r Z(t,s)dW^H(s)$$

by letting $r = T$ it yields

$$\mathbb{E}(\varphi(t)|\mathcal{F}_T) = \mathbb{E}(\varphi(t)) + \int_0^T Z(t,s)dW^H(s)$$

and since $\varphi(t)$ is $\mathcal{F}_T$ measurable

$$\varphi(t) = \mathbb{E}(\varphi(t)) + \int_0^T Z(t,s)dW^H(s).$$

We can write

$$\mathbb{E}(\varphi(t)) + \int_0^t Z(t,s)dW^H(s) = \varphi(t) - \int_t^T Z(t,s)dW^H(s)$$

by defining $Y(t)$ as $Y(t) = \mathbb{E}(\varphi(t)) + \int_0^t Z(t,s)dW^H(s)$ for $t \in [0,T]$ it results that $Y(t)$ is $\mathcal{F}_t$-adapted process and $(Y(\cdot), Z(\cdot,\cdot))$ is an adapted solution for BSVIE. This pair has following properties

$$\int_0^T Z(t,s)dW^H(s) = \varphi(t) - \mathbb{E}(\varphi(t))$$

$$\mathbb{E}\left\|\int_0^T Z(t,s)dW^H(s)\right\|^p = \mathbb{E}\left\|\varphi(t) - \mathbb{E}(\varphi(t))\right\|^p \leq c(\mathbb{E}\|\varphi(t)\|^p + \mathbb{E}\|\mathbb{E}(\varphi(t)\|^p)$$

by Jensen's inequality it yields

$$\mathbb{E}\left\|\int_0^T Z(t,s)dW^H(s)\right\|^p \leq c(\mathbb{E}\|\varphi(t)\|^p + \mathbb{E}\mathbb{E}(\|\varphi(t)\|^p) \leq c\mathbb{E}\|\varphi(t)\|^p$$

and we have

$$\mathbb{E}\left\|\left(\int_0^T \|\mathfrak{Z}(t,s)\|_H^2 ds\right)^{\frac{1}{2}}\right\|^p \leq c\mathbb{E}\|\varphi(t)\|^p.$$

By construction $Y(t)$ is $\mathcal{F}_t$ adapted then we can write

$$\mathbb{E}(Y(t)|\mathcal{F}_t) = \mathbb{E}\left(\left(\varphi(t) - \int_t^T Z(t,s)dW^H(s)\right)\Big|\mathcal{F}_t\right)$$

$$Y(t) = \mathbb{E}(\varphi(t)|\mathcal{F}_t) - \mathbb{E}\left(\int_t^T Z(t,s)dW^H(s)\right) = \mathbb{E}(\varphi(t)|\mathcal{F}_t)$$

and

$$\mathbb{E}\|Y(t)\|^p = \mathbb{E}\left\|\mathbb{E}(\varphi(t)|\mathcal{F}_t)\right\|^p \leq \mathbb{E}\left(\mathbb{E}\left\{\|\varphi(t)\|^p|\mathcal{F}_t\right\}\right) = \mathbb{E}\|\varphi(t)\|^p < \infty$$

it yields by using of assumptions on $\varphi(\cdot)$

$$\mathbb{E}\left(\int_0^T \|Y(t)\|^p dt\right) \leq \mathbb{E}\left(\sup_{t\in[0,T]} \|\varphi(t)\|^p\right) < \infty$$

then $Y(\cdot) \in L^p_{\mathbb{F}}(\Omega\times[0,T];E)$, $Z(\cdot,\cdot) \in L^p_{\mathbb{F}}\left(\Omega\times[0,T];\gamma\left(L^2(0,T;H),E\right)\right)$ and $\mathfrak{Z}(\cdot,\cdot) \in L^p_{\mathbb{F}}\left(\Omega\times[0,T];L^q\left(\mathbf{S},L^2(0,T;H)\right)\right)$. Also we can get

$$\sup_{t\in[0,T]} \|Y(t)\|^p \leq \mathbb{E}\left(\sup_{t\in[0,T]} \|\varphi(t)\|^p|\mathcal{F}_t\right)$$

and

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|Y(t)\|^p\right) \leq \mathbb{E}\left(\sup_{t\in[0,T]} \|\varphi(t)\|^p\right) < \infty.$$

By putting these results together, it yields

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[0,T]} \|Y(t)\|^p\right) + \mathbb{E}\left(\int_0^T \left\|\int_0^T Z(t,s)dW^H(s)\right\|^p dt\right) \\
&\leq \mathbb{E}\left(\sup_{t\in[0,T]} \|Y(t)\|^p + c\int_0^T \left\|\left(\int_0^T \|\mathfrak{Z}(t,s)ds\|_H^2\right)^{\frac{1}{2}}\right\|^p dt\right) \\
&\leq c\mathbb{E}\left(\sup_{t\in[0,T]} \|\varphi(t)\|^p\right).
\end{aligned} \tag{3.3}$$

Following inequalities can be also resulted

$$
\begin{aligned}
&\mathbb{E}\left(\int_0^T \|Y(t)\|^p dt\right) + \mathbb{E}\left(\int_0^T \left\|\int_0^T Z(t,s)dW^H(s)\right\|^p dt\right) \\
&\leq \mathbb{E}\left(\int_0^T \|Y(t)\|^p dt + c\int_0^T \left\|\left(\int_0^T \|\mathfrak{z}(t,s)ds\|_H^2\right)^{\frac{1}{2}}\right\|^p dt\right) \\
&\leq c\mathbb{E}\left(\int_0^T \|\varphi(t)\|^p dt\right) \leq c\mathbb{E}\left(\sup_{t\in[0,T]} \|\varphi(t)\|^p\right)
\end{aligned}
$$

We show now the uniqueness of the solution. Let $(Y(\cdot), Z(\cdot,\cdot))$ and $(\bar{Y}(\cdot), \bar{Z}(\cdot,\cdot))$ be solutions of BSVIE (3.1), then we have

$$
Y(t) - \bar{Y}(t) = -\int_t^T \left(Z(t,s) - \bar{Z}(t,s)\right) dW^H(s)
$$

by letting $\delta^Y(\cdot) := Y(\cdot) - \bar{Y}(\cdot)$ and $\delta^Z(\cdot,\cdot) := Z(\cdot,\cdot) - \bar{Z}(\cdot,\cdot)$ and $\delta^{\mathfrak{z}}(\cdot,\cdot) := \mathfrak{z}(\cdot,\cdot) - \bar{\mathfrak{z}}(\cdot,\cdot)$, and we get the following BSVIE

$$
\delta^Y(t) = -\int_t^T \delta^Z(t,s)dW^H(s)
$$

and we set for this problem the process $\varphi(t)$ is 0, then from Equation(3.3)

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[0,T]} \|\delta^Y(t)\|^p\right) + \mathbb{E}\left(\int_0^T \left\|\int_0^T \delta^Z(t,s)dW^H(s)\right\|^p dt\right) \\
&\leq \mathbb{E}\left(\sup_{t\in[0,T]} \|\delta^Y(t)\|^p + c\int_0^T \left\|\left(\int_0^T \left\|\delta^{\mathfrak{z}}(t,s)ds\right\|_H^2\right)^{\frac{1}{2}}\right\|^p dt\right) \\
&\leq 0
\end{aligned}
$$

subsequently $\delta^Y(\cdot) = 0$ a.s. , $\delta^Z(\cdot,\cdot) = 0$ a.s and $\mathfrak{z}(\cdot,\cdot) \equiv \bar{\mathfrak{z}}(\cdot,\cdot)$ a.s.

3.2. A More General BSDE

We consider a more general case but important backward stochastic differential equation (BSDE) defined by the following equation

$$
Y(t) = X + \int_t^T f(s, Y(s), Z(s))ds - \int_t^T Z(s)dW^H(s),\ t \in [0,T]
$$

where X is $\mathcal{F}_T$-measurable is given with $\mathbb{E}\|X\|_E^p < \infty$,

$$
\begin{aligned}
&Z : \Omega \times [0,T] \longrightarrow \mathcal{L}(H,E) \\
&Y : \Omega \times [0,T] \longrightarrow E
\end{aligned}
$$

and the generator function f is given as follows

$$
f : \Omega \times [0,T] \times E \times E \times \mathbf{S} \longrightarrow E
$$

by using the stochastic integrability Theorems A.3, A.2 we can let

$$\mathfrak{Z} : \Omega \times [0,T] \times \mathbf{S} \longrightarrow H$$

is strongly measurable function such that for all $h \in H$ and $t \in [0,T]$ $(Z(t)h)(\cdot) = [\mathfrak{Z}(t,\cdot), h]_H$, and by this definition f could be defined as

$$f : \Omega \times [0,T] \times E \times H \times \mathbf{S} \longrightarrow E$$

and the BSDE can be written following

$$Y(t) = X + \int_t^T f(s, Y(s), \mathfrak{Z}(s))ds - \int_t^T Z(s)dW^H(s),\ t \in [0,T] \tag{3.4}$$

We assume following assumptions

$(H1)$ f is $\mathcal{F}_t$-adapted and $\mathbb{E}\left\|\int_0^T |f(t,0,0,\cdot)|dt\right\|^p < \infty$

(H2) $\forall t \in [0,T], y, \bar{y} \in \mathbb{R}, \mathfrak{z}, \bar{\mathfrak{z}} \in H$,

$$|f(t,y,\mathfrak{z},.) - f(t,\bar{y},\bar{\mathfrak{z}},.)| \le L_y(t)|y-\bar{y}| + L_{\mathfrak{z}}(t)\|\mathfrak{z}-\bar{\mathfrak{z}}\|_H$$

where $L_y(\cdot) \in L^\infty(0,T)$ and $L_{\mathfrak{z}}(\cdot)$ are positive deterministic functions such that $\int_0^T L_{\mathfrak{z}}(t)^{2+\varepsilon}dt < \infty$ for $\varepsilon > 0$.

We define following space

$$H^p[R,S] := L^p_{\mathbb{F}}\left([R,S]\times\Omega; E\right) \times L^p_{\mathbb{F}}\left(\Omega; L^q\left(\mathbf{S}; L^2([R,S]; H)\right)\right)$$

this space is a Banach space with the equipped norm

$$\|(Y(\cdot), \mathfrak{Z}(\cdot))\|_{H^p[R,S]} := \left\{\mathbb{E}\left(\sup_{t\in[R,S]} \|Y(t)\|_E^p\right) + \mathbb{E}\left\|\left(\int_R^S \|\mathfrak{Z}(s,\cdot)\|_H^2 ds\right)^{\frac{1}{2}}\right\|_E^p\right\}^{\frac{1}{p}}$$

Proposition 3.1. *Let $(H1)$ and $(H2)$ hold, if $X \in L^p(\Omega; E)$ then BSDE* (3.4) *admits an unique solution $(Y(\cdot), \mathfrak{Z}(\cdot)) \in H^p[0,T]$ correspondingly $(Y(\cdot), Z(\cdot))$, where $\forall h \in H$, $(Z(t)h)(\cdot) = [\mathfrak{Z}(t,\cdot), h]_H$ in E, and we have following estimate*

$$\begin{aligned}
\mathbb{E}\left(\sup_{t\in[0,T]} \|Y(t)\|^p\right) &+ \mathbb{E}\left\|\left(\int_0^T \|\mathfrak{Z}(s,\cdot)\|_H^2 ds\right)^{\frac{1}{2}}\right\|^p \\
&\le c\mathbb{E}\left\{\|X\|^p + \left\|\int_0^T |f(s, Y(s), \mathfrak{Z}(s), \cdot)|ds\right\|^p\right\} \\
&\le c\mathbb{E}\left\{\|X\|^p + \left\|\int_0^T |f(s,0,0,\cdot)|ds\right\|^p\right\}
\end{aligned}$$

Proof. Let $(y(\cdot), \mathfrak{z}(\cdot)) \in H^p[0,T]$, we want to find the unique solution $(Y(\cdot), Z(\cdot,\cdot))$ for the following special backward equation

$$Y(t) = X + \int_t^T f(s, y(s), \mathfrak{z}(s))ds - \int_t^T Z(s)dW^H(s), \quad t \in [0,T] \tag{3.5}$$

Then first we have to show $\mathbb{E}\left\|\int_0^T f(s,y(s),\mathfrak{z}(s),\cdot)ds\right\|^P < \infty$. We can write by using the Lipschitz condition in $(H2)$

$$
\begin{aligned}
&\mathbb{E}\left\|\int_0^T f(s,y(s),\mathfrak{z}(s),\cdot)ds\right\|^p \\
&\leq \mathbb{E}\left\|\int_0^T |f(s,y(s),\mathfrak{z}(s),\cdot) - f(s,0,0,\cdot) + f(s,0,0,\cdot)|ds\right\|^p \\
&\leq c\left\{\mathbb{E}\left\|\int_0^T |f(s,0,0,\cdot)|ds\right\|^p + \mathbb{E}\left\|\int_0^T L_y(s)|y(s)|ds\right\|^p\right. \\
&\qquad\left. +\mathbb{E}\left\|\int_0^T L_{\mathfrak{z}}(s)\|\mathfrak{z}(s)\|_H ds\right\|^p\right\}
\end{aligned}
$$

the norm property and Young inequality were used. Now consider every part of the last terms of above inequality, by assumptions $\mathbb{E}\left\|\int_0^T |f(s,0,0,\cdot)|ds\right\|^p < \infty$, for the second part we have

$$
\begin{aligned}
\mathbb{E}\left\|\int_0^T L_y(s)|y(s)|ds\right\|^p \leq & c\mathbb{E}\left\|\int_0^T |y(s)|ds\right\|^p \\
=& c\mathbb{E}\left(\int_{\mathbf{S}}\left(\int_0^T |y(s)|ds\right)^q d\mu\right)^{\frac{p}{q}} \\
\leq & cT^{(q-1)\frac{p}{q}}\mathbb{E}\left(\int_{\mathbf{S}}\int_0^T |y(s)|^q ds d\mu\right)^{\frac{p}{q}} \\
=& cT^{(q-1)\frac{p}{q}}\mathbb{E}\left(\int_0^T\int_{\mathbf{S}} |y(s)|^q d\mu ds\right)^{\frac{p}{q}} \\
\leq & cT^p\mathbb{E}\left(\sup_{s\in[0,T]}\int_{\mathbf{S}} |y(s)|^q d\mu\right)^{\frac{p}{q}} \\
=& cT^p\mathbb{E}\left(\sup_{s\in[0,T]}\|y(s)\|_E^p\right) < \infty
\end{aligned}
$$

we used the Jensen's inequality to the integral with respect to ds and Fubini's theorem. For third part by using Hölder's inequality it can be written

$$
\begin{aligned}
&\mathbb{E}\left\|\int_0^T L_{\mathfrak{z}}(s)\|\mathfrak{z}(s)\|_H ds\right\|^p \\
&\leq \mathbb{E}\left\|\left(\int_0^T |L_{\mathfrak{z}(s)}|^2 ds\right)^{\frac{1}{2}}\left(\int_0^T \|\mathfrak{z}(s)\|_H^2 ds\right)^{\frac{1}{2}}\right\|^p \\
&= \left(\int_0^T |L_{\mathfrak{z}(s)}|^2 ds\right)^{\frac{p}{2}}\mathbb{E}\left\|\left(\int_0^T \|\mathfrak{z}(s)\|_H^2 ds\right)^{\frac{1}{2}}\right\|^p
\end{aligned}
$$

$$\leq \left(\int_0^T ds\right)^{\frac{p\varepsilon}{4+2\varepsilon}} \left(\int_0^T \left(|L_{\mathfrak{z}(s)}|^2\right)^{\frac{2+\varepsilon}{2}} ds\right)^{\frac{p}{2+\varepsilon}} \mathbb{E}\left\|\left(\int_0^T \|\mathfrak{z}(s)\|_H^2 ds\right)^{\frac{1}{2}}\right\|^p$$

$$< \infty.$$

In according to above calculations and $X \in L^p(\Omega; E)$ it yields that $\mathbb{E}\left\|X + \int_0^T f(s, y(s), \mathfrak{z}(s), \cdot)ds\right\|^p < \infty$. Now we can define E-Valued L^p-martingale similar to Equation(3.2) by

$$\psi(r) = \mathbb{E}\left(\left(X + \int_0^T f(s, y(s), \mathfrak{z}(s))ds\right) \mid \mathcal{F}_r\right), \quad r \in [0, T].$$

By martingale representation Theorem A.6, there exists a unique process $Z(\cdot) \in L^p_{\mathbb{F}}\left(\Omega, \gamma(L^2((0,T), H); E)\right)$ where

$$\psi(r) = \mathbb{E}\left(X + \int_0^T f(s, y(s), \mathfrak{z}(s))ds\right) + \int_0^r Z(s)dW^H(s)$$

let $r = T$ and by $\mathcal{F}_T$-measurability of $X + \int_0^T f(s, y(s), \mathfrak{z}(s))ds$, we have

$$\begin{aligned} X+&\int_0^T f(s, y(s), \mathfrak{z}(s))ds \\ &= \mathbb{E}\left(X + \int_0^T f(s, y(s), \mathfrak{z}(s))ds\right) + \int_0^T Z(s)dW^H(s) \end{aligned}$$

in other words

$$\begin{gathered} X + \int_0^t f(s, y(s), \mathfrak{z}(s)ds + \int_t^T f(s, y(s), \mathfrak{z}(s))ds = \\ \mathbb{E}\left(X + \int_0^T f(s, y(s), \mathfrak{z}(s))ds\right) + \int_0^t Z(s)dW^H(s) + \int_t^T Z(s)dW^H(s) \end{gathered}$$

now by taking

$$\begin{aligned} Y(t) =&\mathbb{E}\left(X + \int_0^T f(s, y(s), \mathfrak{z}(s))ds\right) \\ &- \int_0^t f(s, y(s), \mathfrak{z}(s))ds + \int_0^t Z(s)dW^H(s) \end{aligned}$$

we can find analogous to simple BSVIE (3.1), unique $(Y(\cdot), Z(\cdot))$ such that Equation(3.5) holds. By L^p-Martingale property and adaptedness of generator function, we can write

$$Y(t) = \mathbb{E}\left(Y(t) \mid \mathcal{F}_t\right) = \mathbb{E}\left(\left(X + \int_t^T f(s, y(s), \mathfrak{z}(s))ds\right) \mid \mathcal{F}_t\right)$$

and it yields

$$\begin{aligned} |Y(t)| \leq&\mathbb{E}\left(\left|X + \int_t^T f(s, y(s), \mathfrak{z}(s))ds\right| \mid \mathcal{F}_t\right) \\ \leq&\mathbb{E}\left(\left(|X| + \int_0^T |f(s, y(s), \mathfrak{z}(s))| \, ds\right) \mid \mathcal{F}_t\right) \end{aligned}$$

and consequently

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|Y(t)\|^p\right)<\infty$$

further we get

$$\begin{aligned}
&\mathbb{E}\left\|\int_0^T Z(s)dW^H(s)\right\|^p\\
&=\mathbb{E}\left\|X+\int_0^T f(s,y(s),\mathfrak{z}(s))ds-\mathbb{E}\left(X+\int_0^T f(s,y(s),\mathfrak{z}(s))ds\right)\right\|^p\\
&\leq 2^{(p-1)}\left\{\mathbb{E}\left\|X+\int_0^T f(s,y(s),\mathfrak{z}(s))ds\right\|^p\right.\\
&\qquad\left.+\,\mathbb{E}\left\|\mathbb{E}\left(X+\int_0^T f(s,y(s),\mathfrak{z}(s))ds\right)\right\|^p\right\}\\
&\leq 2^{(p-1)}\left\{\mathbb{E}\left\|X+\int_0^T f(s,y(s),\mathfrak{z}(s))ds\right\|^p\right.\\
&\qquad\left.+\,\mathbb{E}\left(\mathbb{E}\left\|X+\int_0^T f(s,y(s),\mathfrak{z}(s))ds\right\|^p\right)\right\}\\
&\leq 2^p\mathbb{E}\left\|X+\int_0^T f(s,y(s),\mathfrak{z}(s))ds\right\|^p<\infty.
\end{aligned}$$

By using the theorem of stochastic integrability of stochastic process in $L^q(\mathbf{S},\Sigma,\mu)$, Theorem A.2, $1<q\leq 2$, it results

$$\mathbb{E}\left\|\left(\int_0^T \|\mathfrak{Z}(s)\|_H^2 ds\right)^{\frac{1}{2}}\right\|^p\leq c\mathbb{E}\left\|X+\int_0^T f(s,y(s),\mathfrak{z}(s))ds\right\|^p$$

and finally putting together, we have

$$\begin{aligned}
\mathbb{E}\left(\sup_{t\in[0,T]}\|Y(t)\|^p\right)&+\mathbb{E}\left\|\left(\int_0^T \|\mathfrak{Z}(s,\cdot)\|_H^2 ds\right)^{\frac{1}{2}}\right\|^p\\
&\leq c\mathbb{E}\left\{\|X\|^p+\left\|\int_0^T |f(s,y(s),\mathfrak{z}(s),\cdot)|ds\right\|^p\right\}
\end{aligned}$$

Here $\mathfrak{Z}(s)$ is correspond to $Z(s)$ and is the solution. Therefore for each $S\in[0,T]$ we can define a map $\Phi : H^p[S,T]\longrightarrow H^p[S,T]$ by $\Phi(y(\cdot),\mathfrak{z}(\cdot))=(Y(\cdot),\mathfrak{Z}(\cdot))$ where $(Z(t)h)(\cdot)=[\mathfrak{Z}(t,\cdot),h]_H$, in E. For finding unique solution in $[S,T]$ we use the Banach fixed-point theorem and we show that Φ is contractive in $[S,T]$ for some S. Now let $(\bar{y}(\cdot),\bar{\mathfrak{z}}(\cdot))\in H^p[S,T]$ in according to above calculations $(\bar{Y}(\cdot),\bar{\mathfrak{Z}}(\cdot))\in H^p[S,T]$ be the unique solution of related BSDE then we have for $t \in [S,T]$

$$Y(t) - \bar{Y}(t) = \int_t^T \left(f(s, y(s), \mathfrak{z}(s)) - f(s, \bar{y}(s), \bar{\mathfrak{z}}(s))\right) ds \\ - \int_t^T \left(Z(s) - \bar{Z}(s)\right) dW^H(s)$$

This is a BSDE and by defining $\delta_Y(t) = Y(t) - \bar{Y}(t)$, $\delta_Z(t) = Z(t) - \bar{Z}(t)$, $\delta_{\mathfrak{Z}}(t) = \mathfrak{Z}(t) - \bar{\mathfrak{Z}}(t)$ and by using again previous calculations we have

$$\begin{aligned}
&\mathbb{E}\left\{\sup_{t\in[0,T]} \|\delta_Y(t)\|^p + \left\|\left(\int_0^T \|\delta_{\mathfrak{Z}}(s)\|_H^2 ds\right)^{\frac{1}{2}}\right\|^p\right\} \\
&= \left\|(Y(\cdot), \mathfrak{Z}(\cdot)) - (\bar{Y}(\cdot), \bar{\mathfrak{Z}}(\cdot))\right\|^p_{H^p[S,T]} \\
&\leq c\mathbb{E}\left\{\left\|\int_S^T f(s, y(s), \mathfrak{z}(s)) - f(s, \bar{y}(s), \bar{\mathfrak{z}}(s)) ds\right\|^P\right\} \\
&\leq c\mathbb{E}\left\|\int_S^T L_y(s)|y(s) - \bar{y}(s)| ds + \int_S^T L_{\mathfrak{z}}(s)\|\mathfrak{z}(s) - \bar{\mathfrak{z}}(s)\|_H ds\right\|^p \\
&\leq c\left\{\mathbb{E}\left\|\int_S^T L_y(s)|y(s) - \bar{y}(s)| ds\right\|^p + \left\|\int_S^T L_{\mathfrak{z}}(s)\|\mathfrak{z}(s) - \bar{\mathfrak{z}}(s)\|_H ds\right\|^p\right\}
\end{aligned}$$

consider first part

$$\begin{aligned}
&\mathbb{E}\left\|\int_S^T L_y(s)|y(s) - \bar{y}(s)| ds\right\|^p \\
&\qquad\leq K^p\mathbb{E}\left\|\int_S^T |y(s) - \bar{y}(s)| ds\right\|^p \\
&\qquad\leq K^p(T-S)^p\mathbb{E}\left\{\sup_{s\in[S,T]} \|y(s) - \bar{y}(s)\|^p\right\}
\end{aligned}$$

where $K = \sup_{s\in[S,T]} L_y(s)$ and for the second part

$$\begin{aligned}
&\mathbb{E}\left\|\int_S^T L_{\mathfrak{z}}(s)\|\mathfrak{z}(s) - \bar{\mathfrak{z}}(s)\|_H ds\right\|^p \\
&\leq (T-S)^{\frac{p}{2}\cdot\frac{\varepsilon}{2+\varepsilon}} \cdot (K')^{\frac{p}{2}\cdot\frac{2}{2+\varepsilon}}\mathbb{E}\left\|\left(\int_S^T \|\mathfrak{z}(s) - \bar{\mathfrak{z}}(s)\|_H^2 ds\right)^{\frac{1}{2}}\right\|^p
\end{aligned}$$

where $K' = \left(\int_S^T L_{\mathfrak{z}}(s)^{2+\varepsilon} ds\right)^{\frac{p}{2+\varepsilon}}$ and finally it results

$$\begin{aligned}
&\left\|(Y(\cdot), \mathfrak{Z}(\cdot)) - (\bar{Y}(\cdot), \bar{\mathfrak{Z}}(\cdot))\right\|^p_{H^p[S,T]} \\
&\quad\leq cF(T-S)\left\|(y(\cdot), \mathfrak{z}(\cdot)) - (\bar{y}(\cdot), \bar{\mathfrak{z}}(\cdot))\right\|^p_{H^p[S,T]}
\end{aligned}$$

where $F(S-T) = K^p(T-S)^p + K'^{\frac{p}{p+\varepsilon}}(T-S)^{\frac{p\varepsilon}{2(2+\varepsilon)}}$ $F(T-S)$ is a polynomial function of $(T-S)$ and $F(0) = 0$, then we can choose S such that $F(T-S) < \frac{1}{c}$, it means the map Φ is contractive and we can find unique fixed-point $(Y(\cdot), \mathfrak{Z}(\cdot))$ for $t \in [S, T]$, correspondingly $(Y(\cdot), Z(\cdot))$ and by induction we can find unique solution in $[0, T]$. $\square$

3.3. The General BSVIE

In this section we consider the following generalized BSVIE

$$Y(t) = \varphi(t) + \int_t^T f(t, s, Y(s), Z(t, s), Z(s, t))ds - \int_t^T Z(t, s)dW^H(s), \quad \forall t \in [0, T] \tag{3.6}$$

or with similar arguments that led to Equation(3.4), it can be rewritten as

$$Y(t) = \varphi(t) + \int_t^T f(t, s, Y(s), \mathfrak{Z}(t, s), \mathfrak{Z}(s, t))ds - \int_t^T Z(t, s)dW^H(s), \quad \forall t \in [0, T] \tag{3.7}$$

where $f : \Omega \times \Delta^c \times E \times H \times H \times \mathbf{S} \longrightarrow E$, $\varphi : \Omega \times [0, T] \longrightarrow E$ are given functions and $\Delta = \{(t, s) : 0 < s < t < T\}$. Before generalizing the solution to (3.7), it must be mentioned that without defining some additional assumptions on f, φ and especially on solutions for this equation, we could not expect the unique solution. The most important restriction on solutions is to restrict them to adapted M-solutions.

Definition 3.2. *$(Y(\cdot), Z(\cdot, \cdot))$ is called adapted M-solution of Equation (3.6) with respect to cylindrical Brownian motion W^H if $(Y(\cdot), Z(\cdot, \cdot))$ is adapted solution of BSVIE (3.6) and the following equation holds for $0 \le S' \le T$*

$$Y(t) = \mathbb{E}(Y(t)|\mathcal{F}_{S'})) + \int_{S'}^t Z(t, s)dW^H(s), \quad t \in [S', T]. \tag{3.8}$$

Remark 3.1. *A contradiction example for non-uniqueness of solutions if the above condition (3.8) does not hold, could be given followingly in according to Yong [33] in a special case $E = \mathbb{R}$*

$$Y(t) = \int_t^T Z(s, t)ds - \int_t^T Z(t, s)dW(s)$$

by defining

$$\begin{cases} Y(t) = (T - t)\psi(t) & t \in [0, T] \\ Z(t, s) = \mathbb{1}_{[0,t]}\psi(s) & (t, s) \in [0, T] \times [0, T] \end{cases}$$

$(Y(\cdot), Z(\cdot, \cdot))$ is an adapted solution of given equation and it holds for any $\psi(\cdot) \in L^2_{\mathbb{F}}(0, T)$. (Here $Y(t) = (T - t)\psi(t) \neq \mathbb{E}(Y(t)|\mathcal{F}_{S'}) + \int_{S'}^t Z(t, s)dW(s)$)

Remark 3.2. *For previous BSDE (3.4) in Proposition 3.1, since $Y(\cdot)$ is adapted process, we can find unique adapted M-solution by using martingale representation theorem as follows*

$$Y(t) = \mathbb{E}(Y(t)) + \int_0^t \zeta(t, s)dW^H(s)$$

$$Z(t, s) = \begin{cases} \zeta(t, s) & (t, s) \in \Delta \\ Z(s) & (t, s) \in \Delta^c \end{cases}$$

Remark 3.3. *Another important fact is that we can not easily find the solution by induction similar to Proposition 3.1. For example in Proposition 3.1 first we found the solution by fixed-point theorem in $[T-\delta, T]$ for some $\delta \in [0,T]$ then we could write*

$$\begin{aligned} Y(t) &= X + \int_t^T f(s, Y(s), \mathfrak{Z}(s))ds - \int_t^T Z(s)dW^H(s) \\ &= X + \int_{T-\delta}^T f(s, Y(s), \mathfrak{Z}(s))ds - \int_{T-\delta}^T Z(s)dW^H(s) \\ &\quad + \int_t^{T-\delta} f(s, Y(s), \mathfrak{Z}(s))ds - \int_t^{T-\delta} Z(s)dW^H(s),\ t \in [0, T-\delta] \end{aligned}$$

we could define

$$Y(T-\delta) = X + \int_{T-\delta}^T f(s, Y(s), \mathfrak{Z}(s))ds - \int_{T-\delta}^T Z(s)dW^H(s)$$

where $(Y(s), Z(s))$, $s \in [T-\delta, T]$ is unique solution and according to $Y(T-\delta) \in L^p_{\mathcal{F}_{T-\delta}}(\Omega \times [0,T]; E)$, assumptions on the generator function and fixed-point theorem, we could find the unique solution for the following equation

$$Y(t) = Y(T-\delta) + \int_t^{T-\delta} f(s, Y(s), \mathfrak{Z}(s))ds - \int_t^{T-\delta} Z(s)dW^H(s),\ t \in [0, T-\delta]$$

and by induction we could find the unique solution in $[0,T]$. But for Equation(3.7) let that we can find the solution in $[T-\delta, T]$ or $(Y(t), Z(t,s))$ for $(t,s) \in [T-\delta, T] \times [T-\delta, T]$ and let

$$Y^t(T-\delta) = \varphi(t) + \int_{T-\delta}^T f(t, s, Y(s), \mathfrak{Z}(t,s), \mathfrak{Z}(s,t))ds - \int_{T-\delta}^T Z(t,s)dW^H(s)$$

here we have to first know whether $Y^t(T-\delta)$ is $\mathcal{F}_{T-\delta}$-measurable, and we could not proceed like Proposition 3.1, since we can not easily find solution for the following equation in $t \in [0, T-\delta]$

$$Y(t) = Y^t(T-\delta) + \int_t^{T-\delta} f(t, s, Y(s), \mathfrak{Z}(t,s), \mathfrak{Z}(s,t))ds - \int_t^{T-\delta} Z(t,s)dW^H(s)$$

For this equation, because of $Y^t(T-\delta)$, not only we need the values of $Y(t)$, $Z(t,s)$ for $(t,s) \in [0, T-\delta] \times [T-\delta, T]$ also we need these values in $(t,s) \in [T-\delta, T] \times [0, T-\delta]$, and correspondingly $Y^t(T-\delta)$ are being well defined.

Now consider $\forall R, S \in [0,T]$

$$\lambda(t,r) = \varphi(t) + \int_r^T f(t, s, \varrho(t,s))ds - \int_r^T \rho(t,s)dW^H(s),\ \forall r \in [R,T],\ \forall t \in [S,T] \tag{3.9}$$

where $(\rho(t,s)h)(\cdot) = [\varrho(t,s,\cdot), h]_H$ in E and f is given. This equation is stochastic Fredholm integral equation (SFIE) on $[S,T]$, parametrised by $r \in [R,T]$. We want to find unique process $(\lambda(\cdot,\cdot), \rho(\cdot,\cdot))$ in which $(\lambda(t,\cdot), \rho(t,\cdot))$ be adapted for each $t \in [S,T]$. Let following assumption hold:

- (F1) $R, S \in [0,T]$ and $f : \Omega \times [S,T] \times [R,T] \times H \times \mathbf{S} \longrightarrow E$ be $\mathcal{B}([S,T] \times [R,T]) \otimes \mathcal{B}(H) \otimes \mathcal{B}(E) \times \mathcal{F}_T$-measurable such that $s \longrightarrow f(t, s, \mathfrak{z}, \cdot)$ is progressively measurable

$\forall(t,\mathfrak{z}) \in [S,T] \times H$ and

$$\int_S^T \mathbb{E}\left\| \int_R^T |f(t,s,0,\cdot)|ds \right\|_E^p dt < \infty$$

- (F2) $\forall(t,s) \in [S,T] \times [R,T]$ and $\mathfrak{z}, \bar{\mathfrak{z}} \in H$

$$|f(t,s,\mathfrak{z},\cdot) - f(t,s,\bar{\mathfrak{z}},\cdot)| \leq L(t,s)\|\mathfrak{z}(\cdot) - \bar{\mathfrak{z}}(\cdot)\|_H$$

where $L : [S,T] \times [R,T] \longrightarrow \mathbb{R}^+$ is deterministic function such that

$$\sup_{t\in[S,T]} \int_R^T L(t,s)^{2+\varepsilon} ds < \infty$$

for some $\varepsilon > 0$

Proposition 3.2. *If (F1), (F2) hold and $\varphi(t) \in L^p_{\mathcal{F}_T}(\Omega\times[S,T];E)$ such that $\mathbb{E}\left(\sup_{t\in[S,T]} \|\varphi(t)\|^p\right) < \infty$ then Equation(3.9) has for every $t \in [S,T]$ a unique adapted solution $(\lambda(t,\cdot),\varrho(t,\cdot)) \in H^p[R,T]$ and following estimate holds*

$$\begin{aligned}
\|(\lambda(t,\cdot),\varrho(t,\cdot))\|^p_{H^p[R,T]} &= \mathbb{E}\left\{ \sup_{r\in[R,T]} \|\lambda(t,r))\|^p + \left\| \left(\int_R^T \|\varrho(t,s,\cdot)\|_H^2 ds \right)^{\frac{1}{2}} \right\|^p \right\} \\
&\leq c\mathbb{E}\left\{ \|\varphi(t)\|^p + \left\| \int_R^T |f(t,s,\varrho(t,s),\cdot)|ds \right\|^p \right\} \\
&\leq c\mathbb{E}\left\{ \|\varphi(t)\|^p + \left\| \int_R^T |f(t,s,0,\cdot)|ds \right\|^p \right\}
\end{aligned}$$

Proof. This is derived by Proposition 3.1, if we consider Equation(3.9) for every fixed $t \in [0,T]$. □

At first let us consider for our purpose some especial types of Equation(3.9). First let $r = S$ and we define $\psi^S(t) = \lambda(t,S)$, $Z(t,s) = \rho(t,s)$, $\mathfrak{Z}(t,s) = \varrho(t,s)$ $\forall t \in [R,S]$, $s \in [S,T]$, Equation(3.9) yields

$$\psi^S(t) = \varphi(t) + \int_S^T f(t,s,\mathfrak{Z}(t,s))ds - \int_S^T Z(t,s)dW^H(s), \quad t \in [R,S],\ R,\ S \in [0,T] \tag{3.10}$$

Following proposition is the consequence of Proposition 3.2:

Proposition 3.3. *If assumptions (F1) and (F2) hold and $\varphi(t) \in L^p_{\mathcal{F}_S}(\Omega \times [R,S];E)$ such that $\mathbb{E}\left(\sup_{t\in[R,S]} \|\varphi(t)\|^p\right) < \infty$ then Equation(3.10) has a unique adapted solution $(\psi^S(t),\mathfrak{Z}(t,\cdot)) \in$*

$H^p[R,T]$, $\forall t \in [R,S]$ *and following estimate holds*

$$\begin{aligned}
\left\|(\psi^S(t), \mathfrak{Z}(t,\cdot))\right\|^p_{H^p[R,T]} &= \mathbb{E}\left\{ \|\psi^S(t))\|^p + \left\| \left(\int_R^T \|\mathfrak{Z}(t,s,\cdot)\|^2_H ds \right)^{\frac{1}{2}} \right\|^p \right\} \\
&\le c\mathbb{E}\left\{ \|\varphi(t)\|^p + \left\| \int_R^T |f(t,s,\mathfrak{Z}(t,s),\cdot)| ds \right\|^p \right\} \\
&\le c\mathbb{E}\left\{ \|\varphi(t)\|^p + \left\| \int_R^T |f(t,s,0,\cdot)| ds \right\|^p \right\}, \\
&\qquad \forall t \in [R,S]
\end{aligned}$$

Here it must be mentioned again that $\psi^S(t)$ for each t is $\mathcal{F}_S$-measurable. Another representation of Equation(3.10) can be as following , let $S = R$ and

$$\begin{cases} Y(t) = \lambda(t,t) & t \in [S,T] \\ Z(t,s) = \rho(t,s) & (t,s) \in \Delta^c[S,T] = \{(t,s) : S < t < s < T\}) \\ \mathfrak{Z}(t,s) = \varrho(t,s) & (t,s) \in \Delta^c[S,T] \end{cases}$$

then Equation(3.9) yields

$$Y(t) = \varphi(t) + \int_t^T f(t,s,\mathfrak{Z}(t,s))ds - \int_t^T Z(t,s)dW^H(s),\ t \in [S,T] \tag{3.11}$$

We confine ourselves to M-solutions as $Y(t) = \mathbb{E}(Y(t)|\mathcal{F}_S) + \int_S^t Z(t,s)dW^H(s)$ where $(t,s) \in \Delta[S,T](S < s \le t < T)$. Notice that in the case $(t,s) \in \Delta[S,T]$, $Z(t,s)$ and $\rho(t,s)$ could be different. Now we can give the following proposition:

Proposition 3.4. *Suppose the assumptions (F1), (F2) hold and let $\varphi(t)$ be $\mathcal{F}_T$ measurable and $\mathbb{E}\left(\sup_{t\in[S,T]} \|\varphi(t)\|^p\right) < \infty$. Then Equation*(3.11) *has a unique adapted M-solution*

$$(Y(\cdot), \mathfrak{Z}(\cdot,\cdot)) \in L^p_{\mathbb{F}}(\Omega \times [S,T]; E) \times L^p_{\mathbb{F}}(\Omega \times [S,T]; L^q(\mathbf{S}; L^2(S,T;H))$$

and following inequalities hold

$$\begin{aligned}
&\mathbb{E}\left\{ \|Y(t)\|^p + \left\| \left(\int_t^T \|\mathfrak{Z}(t,s,\cdot)\|^2_H ds \right)^{\frac{1}{2}} \right\|^p \right\} \\
\le& c\mathbb{E}\left\{ \|\varphi(t)\|^p + \left\| \int_t^T |f(t,s,\mathfrak{Z}(t,s),\cdot)| ds \right\|^p \right\} \\
\le& c\mathbb{E}\left\{ \|\varphi(t)\|^p + \left\| \int_t^T |f(t,s,0,\cdot)| ds \right\|^p \right\}, \quad t \in [S,T]
\end{aligned}$$

and

$$\mathbb{E}\left\{\int_S^T \|Y(t)\|^p dt + \int_S^T \left\|\left(\int_S^T \|\mathfrak{Z}(t,s,\cdot)\|_H^2 ds\right)^{\frac{1}{2}}\right\|^p dt\right\}$$
$$\leq c\mathbb{E}\left\{\int_S^T \|\varphi(t)\|^p dt + \int_S^T \left\|\int_S^T |f(t,s,0,\cdot)|ds\right\|^p dt\right\}$$

Proof. First let $(y(\cdot),\mathfrak{z}(\cdot,\cdot)) \in H^p[S,T]$ such that $y(t) = \mathbb{E}(y(t)|\mathcal{F}_S) + \int_S^t z(t,s)dW^H(s)$ then similar to Proposition 3.1 and using Banach fixed-point theorem we can find unique adapted M-solution. □

Remark 3.4. *Without the assumption of* $\mathbb{E}\left(\sup\limits_{t\in[S,T]} \|\varphi(t)\|^p\right) < \infty$, *Propositions 3.2, 3.3, 3.4 are satisfied also and the only assumption of* $\varphi(t) \in L^p_{\mathcal{F}_T}(\Omega \times [S,T]; E)$ *is sufficient.*

Now we can deal with generalized BSVIE(3.7). First we put some assumptions

- (G1) let $f : \Omega \times \Delta^c \times E \times H \times H \times \mathbf{S} \longrightarrow E$ be the $\mathcal{B}(\Delta^c) \otimes \mathcal{B}(E \times H \times H) \otimes \Sigma \otimes \mathcal{F}_T$ -measurable such that $s \longrightarrow f(t,s,y,\mathfrak{z},\varrho,\zeta)$ is $\mathbb{F}$-progressively for $(t,y,\mathfrak{z},\varrho,\zeta) \in [0,T] \times E \times H \times H \times \mathbf{S}$ and
$$\mathbb{E}\int_0^T \left\|\int_t^T |f(t,s,0,0,0,\cdot)|ds\right\|^p dt < \infty$$
for simplicity we define $f_0(t,s,\cdot) := f(t,s,0,0,0,\cdot)$
- (G2) The following Lipschitz condition for every $(t,s) \in \Delta^c$, $y,\bar{y} \in \mathbb{R}$, $\mathfrak{z},\bar{\mathfrak{z}},\varrho,\bar{\varrho} \in H$ holds
$$|f(t,s,y,\mathfrak{z},\varrho) - f(t,s,\bar{y},\bar{\mathfrak{z}},\bar{\varrho})| \leq L_1(t,s)|y-\bar{y}| + L_2(t,s)\|\mathfrak{z}-\bar{\mathfrak{z}}\|_H + L_3(t,s)\|\varrho-\bar{\varrho}\|_H$$
where $L_i(t,s), i = 1,2,3$ are positive deterministic functions such that $\sup\limits_{t\in[0,T]} L_1(t,s) \in L^\infty[0,T]$ and $\sup\limits_{t\in[0,T]} \int_t^T L_i(t,s)^{2+\varepsilon}ds < \infty, i = 2,3$ for some $\varepsilon > 0$.

We define the Banach space

$$\mathcal{H}^p[S,T] = \left\{(y(\cdot),\mathfrak{z}(\cdot,\cdot)) : \|(y(\cdot),\mathfrak{z}(\cdot,\cdot))\|_{\mathcal{H}^p[S,T]} < \infty\right\}$$

with following norm

$$\|(y(\cdot),\mathfrak{z}(\cdot,\cdot))\|^p_{\mathcal{H}^p[S,T]} = \mathbb{E}\left\{\int_S^T \|y(t)\|^p dt + \int_S^T \left\|\left(\int_S^T \|\mathfrak{z}(t,s)\|_H^2 ds\right)^{\frac{1}{2}}\right\|^p dt\right\}$$

Theorem 3.1. *If (G1) , (G2) hold and* $\varphi(t) \in L^p_{\mathcal{F}_T}(\Omega\times[0,T];E)$ *then there exists unique adapted* M*-solution* $(Y(\cdot),\mathfrak{Z}(\cdot,\cdot)) \in \mathcal{H}[0,T]$ *for BSVIE Equation*(3.7)*, such that following estimate holds*

$$\|(Y(\cdot),\mathfrak{Z}(\cdot,\cdot))\|_{\mathcal{H}^p[S,T]} \leq c\mathbb{E}\left\{\int_0^T \|\varphi(t)\|^p dt + \int_0^T \left\|\int_0^T |f_0(t,s,\cdot)|ds\right\|^p dt\right\}$$

Proof. We prove this theorem in several steps:
Step(1)- First we find the solution in $[S,T]$ for some $S \in [0,T]$, we define the space $\mathcal{M}^p[S,T]$

of all $(y(\cdot),\mathfrak{z}(\cdot,\cdot))\in\mathcal{H}^p[0,T]$ such that

$$Y(t)=\mathbb{E}(Y(t)|\mathcal{F}_S))+\int_S^t Z(t,s)dW^H(s)\quad t\in[S,T]$$

and we consider the following equation

$$Y(t)=\varphi(t)+\int_t^T f(t,s,y(s),\mathfrak{Z}(t,s),\mathfrak{z}(s,t))ds-\int_t^T Z(t,s)dW^H(s),\qquad \forall t\in[0,T]$$

where $(y(\cdot),\mathfrak{z}(\cdot,\cdot))\in\mathcal{M}^p[S,T]$ is given. By considering this equation, it is similar to Equation (3.11) and we can use Proposition 3.4 to find the unique adapted M-solution $(Y(\cdot),\mathfrak{Z}(\cdot,\cdot))$. Now we define the following map

$$\Upsilon:\mathcal{M}^p[S,T]\longrightarrow\mathcal{M}^p[S,T]$$

by

$$\Upsilon(y(\cdot),\mathfrak{z}(\cdot,\cdot))=(Y(\cdot),\mathfrak{Z}(\cdot,\cdot)),\quad \forall(y(\cdot),\mathfrak{z}(\cdot,\cdot))\in\mathcal{M}^p[S,T].$$

We show that this map is contractive and we use Banach fixed-point theorem for the existence of a unique solution in $[S,T]$. Let $(\bar{y}(\cdot),\bar{\mathfrak{z}}(\cdot,\cdot))\in\mathcal{M}^p[S,T]$ such that $\Upsilon(\bar{y}(\cdot),\bar{\mathfrak{z}}(\cdot,\cdot))=(\bar{Y}(\cdot),\bar{\mathfrak{Z}}(\cdot,\cdot))$, then by using Proposition 3.4

$$\begin{aligned}
&\|(Y(\cdot),\mathfrak{Z}(\cdot,\cdot))-(\bar{Y}(\cdot),\bar{\mathfrak{Z}}(\cdot,\cdot))\|^p_{\mathcal{H}^p[S,T]}\\
&\le\mathbb{E}\left\{\int_S^T\left\|Y(t)-\bar{Y}(t)\right\|^p dt+\int_S^T\left\|\left(\int_S^T\left\|\mathfrak{Z}(t,s)-\bar{\mathfrak{Z}}(t,s)\right\|^2ds\right)^{\frac{1}{2}}\right\|^p dt\right\}\\
&\le c\mathbb{E}\left\{\int_S^T\left\|\int_t^T|f(t,s,y(t),0,\mathfrak{z}(s,t))-f(t,s,\bar{y}(t),0,\bar{\mathfrak{z}}(s,t))|ds\right\|^p dt\right\}\\
&\le c\mathbb{E}\left\{\int_S^T\left\|\int_t^T L_1(t,s)|y(t)-\bar{y}(t)|ds+\int_t^T L_2(t,s)\|\mathfrak{z}(t,s)-\bar{\mathfrak{z}}(t,s)\|_H ds\right\|^p dt\right\}\\
&\le c\left\{\mathbb{E}\left(\int_S^T\left\|\int_t^T L_1(t,s)|y(t)-\bar{y}(t)|ds\right\|^p dt\right)\right.\\
&\qquad\left.+\mathbb{E}\left(\int_S^T\left\|\int_t^T L_3(t,s)\|\mathfrak{z}(t,s)-\bar{\mathfrak{z}}(t,s)\|_H ds\right\|^p dt\right)\right\}.
\end{aligned}$$

For first part of the last inequality we have

$$\begin{aligned}
&\mathbb{E}\left\{\int_S^T\left\|\int_t^T L_1(t,s)|y(t)-\bar{y}(t)|ds\right\|^p dt\right\}\\
&\quad\le\mathbb{E}\left\{\int_S^T\left\|\int_t^T K_1|y(t)-\bar{y}(t)|ds\right\|^p dt\right\}\\
&\quad\le\mathbb{E}\left\{\int_S^T\left\|\int_S^T K_1|y(t)-\bar{y}(t)|ds\right\|^p dt\right\}
\end{aligned}$$

$$\leq (T-S)^{p-1}\mathbb{E}\left\{\int_S^T\int_S^T \|K_1|y(t)-\bar{y}(t)|\|^p\, ds\, dt\right\}$$
$$\leq (T-S)^p K_1^p \mathbb{E}\left\{\int_S^T \|y(t)-\bar{y}(t)\|^p\, ds\right\}$$

and for the second part

$$\mathbb{E}\left\{\int_S^T \left\|\int_t^T L_3(t,s)\|\mathfrak{z}(t,s)-\bar{\mathfrak{z}}(t,s)\|_H ds\right\|^p dt\right\}$$
$$\leq \mathbb{E}\left\{\int_S^T \left\|\int_S^T L_3(t,s)\|\mathfrak{z}(t,s)-\bar{\mathfrak{z}}(t,s)\|_H ds\right\|^p dt\right\}$$
$$\leq \mathbb{E}\left\{\int_S^T \left\|\left(\int_S^T (L_3(t,s))^2 ds\right)^{\frac{1}{2}}\left(\int_S^T \|\mathfrak{z}(t,s)-\bar{\mathfrak{z}}(t,s)\|_H^2 ds\right)^{\frac{1}{2}}\right\|^p dt\right\}$$
$$=(T-S)^{\frac{p}{2}\cdot\frac{\varepsilon}{2+\varepsilon}}\left(\int_S^T (L_3(t,s))^{2+\varepsilon} ds\right)^{\frac{p}{2}\cdot\frac{2}{2+\varepsilon}}\mathbb{E}\left\|\left(\int_S^T \|\mathfrak{z}(t,s)-\bar{\mathfrak{z}}(t,s)\|_H^2 ds\right)^{\frac{1}{2}}\right\|^p$$

therefore we have

$$\|(Y(\cdot),\mathfrak{Z}(\cdot,\cdot))-(\bar{Y}(\cdot),\bar{\mathfrak{Z}}(\cdot,\cdot))\|^p_{\mathcal{H}^p[S,T]}$$
$$\leq c\left((T-S)^p k_1+(T-S)^{\frac{p}{2}\cdot\frac{\varepsilon}{2+\varepsilon}}k_3\right)\|(y(\cdot),\mathfrak{z}(\cdot,\cdot))-(\bar{y}(\cdot),\bar{\mathfrak{z}}(\cdot,\cdot))\|^p_{\mathcal{H}^p[S,T]}$$

where k_1, and k_3 are positive constants. By taking $T-S>0$ small enough the contraction property holds. It means that the unique adapted M-solution $(Y(t),Z(t,s))$, for $(t,s)\in[S,T]\times[S,T]$ are found by Banach fixed-point theorem.

Step(2)- Now we want to find the values $Z(t,s)$ for $(t,s)\in[S,T]\times[R,S]$. As in the previous step we found $Y(\cdot)\in L_{\mathbb{F}}(\Omega\times[S,T];E)$, it results

$$\forall t\in[S,T]\quad \psi_t(r)=\mathbb{E}(Y(t)|\mathcal{F}_r),\quad 0<r<T$$

is L^p-Martingale. By Martingale representation theorem in UMD-spaces ($1<q\leq 2$, E is cotype-2), we can find unique adapted process $Z(t,s)$ such that

$$\mathbb{E}(Y(t)|\mathcal{F}_S)=\mathbb{E}(Y(t))+\int_0^S Z(t,s)dW^H(s)$$

or by conditional expectation on $\mathcal{F}_R$, $R<S$ we have

$$\mathbb{E}(Y(t)|\mathcal{F}_S)=\mathbb{E}(Y(t)|\mathcal{F}_R)+\int_R^S Z(t,s)dW^H(s),\quad t\in[S,T]$$

in other words we could find the values $Z(t,s)$ for $(t,s)\in[S,T]\times[R,S]$, $R\in[0,S]$. By combining these values with step 1 we have $Z(t,s)$ for $(t,s)\in[S,T]\times[R,T]$.

Step(3)- It follows from Step(2) that we have the values $Y(s)$, $Z(s,t)$ for $t\in[R,S]$ and $s\in[S,T]$. Let $\mathfrak{z}\in H$ and define $f^S(t,s,\mathfrak{z})=f(t,s,Y(s),\mathfrak{z},\mathfrak{Z}(s,t))$, for $(t,s,\mathfrak{z})\in[R,S]\times[S,T]\times H$, then we can write

$$\int_S^T f(t,s,Y(s),\mathfrak{z},\mathfrak{Z}(s,t))ds = \int_S^T f^S(t,s,\mathfrak{z})ds$$

and we consider the following SFIE:

$$\varphi^S(t) = \varphi(t) + \int_S^T f^S(t,s,\mathfrak{z})ds - \int_S^T Z(t,s)dW^H(s)$$

by Proposition 3.4 we can find unique solution of above SFIE,

$$\begin{cases} \varphi(\cdot)^S \in L^p_{\mathcal{F}_S}(\Omega \times [R,S];E) \\ \mathcal{Z}(\cdot,\cdot) \in L^p_{\mathbb{F}}\big(R,S;L^q(\mathbf{S};L^2(S,T;H))\big) \end{cases}$$

It means that we found $Z(t,s)$ for $(t,s) \in [R,S] \times [S,T]$ and by combining the previous results we found the unique adapted M-solution $(Y(\cdot),Z(\cdot,\cdot))$, for $Y(t)$, $t \in [S,T]$ and $Z(t,s)$ for $(t,s) \in [S,T] \times [R,T] \bigcup [R,S] \times [S,T]$.

Step(4)- By using SFIE is step(3), we can write our BSVIE as follows

$$Y(t) = \varphi^S(t) + \int_t^S f(t,s,Y(s),\mathfrak{Z}(t,s),\mathfrak{Z}(s,t))ds - \int_t^S Z(t,s)dW^H(s), \qquad t \in [0,S].$$

From step(3) we know that $\varphi^S(t)$ is $\mathcal{F}_S$-measurable for each $t \in [0,S]$ and this equation is BSVIE and could be easily solved as steps 1, 2, 3 in interval $[S',S]$. Finally by induction we can find unique adapted M-solution for whole of the interval. $\square$

CHAPTER 4

Duality Principles

In this chapter, duality principle between linear forward stochastic Volterra integral equation and linear backward stochastic Volterra integral equation are being derived. The Itô formula in UMD-Banach spaces where derived by Brzezniak, Neerven, Veraar and Weis [5] is very crucial tool for the proof of duality principle.

4.1. A First Duality Principle

We consider the following FSVIE in Banach space $E = L^q(\mathbf{S}, \Sigma, \mu)$ where $q \geq 2$ and μ is a σ-finite measure

$$X(t) = \varphi(t) + \int_0^t A_0(t,s)X(s)ds + \int_0^t A_1(t,s)X(s)dW^H(s), \quad t \in [0,T] \tag{4.1}$$

where $W^H(\cdot)$ is cylindrical Brownian motion, $\varphi(t) \in L^p_{\mathbb{F}}([0,T] \times \Omega; E)$ and for each $(t,s) \in [0,T] \times [0,T]$ $A_i(t,s)$, $i = 0,1$ be linear bounded operators defined as follows

$$\begin{cases} A_0(t,s) : \Omega \times E \longrightarrow E \\ A_1(t,s) : \Omega \times E \longrightarrow \mathcal{L}(H;E) \end{cases}$$

$$\begin{cases} A_0(\cdot,\cdot) \in L^\infty\left(0,T; L^\infty_{\mathbb{F}}(0,T;\mathcal{L}(E;E)\right) \\ A_1(\cdot,\cdot) \in L^\infty\left(0,T; L^\infty_{\mathbb{F}}(0,T;\mathcal{L}(E;\mathcal{L}(H;E))\right) \end{cases}$$

For each $t \in [0,T]$ $A_i(t,\cdot)$ is $\mathcal{F}_s$-adapted $s \geq 0$, $\sup\limits_{t\in[0,T]} \sup\limits_{s\in[0,T]} \|A_i(t,s)\| < \infty$, $i = 0,1$. In according to these assumptions FSVIE (4.1) because of Theorem 2.1 admits an unique adapted $X(\cdot) \in L^p_{\mathbb{F}}([0,T] \times \Omega; E)$, where $q \leq p$.

Now we consider following BSVIE in dual space of E, i.e $E^* = L^{q'}(\mathbf{S}, \Sigma, \mu)$, $1 < q' \leq 2$ and $\frac{1}{q} + \frac{1}{q'} = 1$, let $A_0^*(t,s)$ be the adjoint operator of $A_0(t,s)$ and $A_1^*(t,s)h := (A_1(t,s)h)^*$ be the adjoint operator of $A_1(t,s)h$ for every $(t,s) \in [0,T] \times [0,T]$ and $h \in H$;

$$\begin{aligned} Y(t) =& \psi(t) + \int_t^T \left(A_0^*(s,t)Y(s) + \sum_{n\geq 1} A_1^*(s,t)h_n Z(s,t)h_n \right) ds \\ & - \int_t^T Z(t,s)dW^H(s), \quad t \in [0,T] \end{aligned} \tag{4.2}$$

The cylindrical Wiener process $W^H(\cdot)$ is the same process for two equations and is defined in Hilbert space H.

$\{h_n\}_{n\geq 1}$ is orthonormal basis in Hilbert space H and for each $\omega \in \Omega$ and $(t,s) \in [0,T] \times [0,T]$

$$\begin{aligned} &Z(t,s) \in \mathcal{L}(H;E^*) \\ &A_0^* \in \mathcal{L}(E^*;E^*) \\ &A_1^* h \in \mathcal{L}\left(E^*;E^*\right) \end{aligned}$$

where $Z(t,s)$, A_0^* and A_1^*h are bounded linear operators and for each $h \in H$, $Z(t,s)h \in E^*$ and $A_1^*(t,s)h \in \mathcal{L}(E^*;E^*)$ then $A_1^*(t,s)hZ(t,s)h := A_1^*(t,s)h\,(Z(t,s)h) \in E^*$. We can find $\mathfrak{Z} : [0,T] \times [0,T] \times \Omega \times \mathbf{S} \longrightarrow H$ such that $(Z(t,s)h)(\cdot) = [\mathfrak{Z}(t,s,\cdot),h]_H$ in E^*, $\forall h \in H$. For well-definition we assume also that $\sum\limits_{n\geq 1} \|A_1^*(s,t)h_n\|$ is bounded for every $(t,s) \in [0,T] \times [0,T]$. Now if $\psi(\cdot) \in L^p_{\mathcal{F}_T}([0,T] \times \Omega; E^*)$ by considering

$$f(t,s,Y(s),\mathfrak{Z}(t,s),\mathfrak{Z}(s,t)) = A_0^*(s,t)Y(s) + \sum_{n\geq 1} A_1^*(s,t)h_n Z(s,t)h_n$$

we can find a unique adapted M-solution $(Y(\cdot), Z(\cdot,\cdot))$ of (4.2).
For every $x^* \in E^*$ let the duality pairing be given by $x^*(x) = \langle x, x^*\rangle$.

Theorem 4.1. *If FSVIE Equation*(4.1) *and BSVIE Equation*(4.2) *are fulfilled then following duality principle holds*

$$\mathbb{E}\left\{\int_0^T \langle X(t),\psi(t)\rangle\, dt\right\} = \mathbb{E}\left\{\int_0^T \langle \varphi(t),Y(t)\rangle\, dt\right\}$$

Proof. Since $Y(\cdot)$ is $\mathcal{F}_t$-adapted and $1 < q' \leq 2$, we can use martingale representation theorem in Banach spaces Theorem A.6, and there exists a unique adapted process $Z(\cdot,\cdot)$ such that $Y(t) = \mathbb{E}(Y(t)) + \int_0^t Z(t,s)dW^H(s)$ and we see

$$\begin{aligned} &\mathbb{E}\int_0^T \langle \varphi(t),Y(t)\rangle = \\ &\mathbb{E}\int_0^T \left\langle X(t) - \int_0^t A_0(t,s)X(s)ds - \int_0^t A_1(t,s)X(s)dW^H(s), Y(t)\right\rangle dt \\ &= \mathbb{E}\int_0^T \langle X(t),Y(t)\rangle\, dt - \mathbb{E}\int_0^T \left\langle \int_0^t A_0(t,s)X(s)ds, Y(t)\right\rangle dt \\ &- \mathbb{E}\int_0^T \left\langle \int_0^t A_1(t,s)X(s)dW^H(s), Y(t)\right\rangle dt \\ &= \mathbb{E}\int_0^T \langle X(t),Y(t)\rangle dt - \mathbb{E}\int_0^T \int_0^t \langle X(s), A_0^*(t,s)Y(t)\rangle ds dt \\ &- \mathbb{E}\int_0^T \left\langle \int_0^t A_1(t,s)X(s)dW^H(s), \mathbb{E}(Y(t)) + \int_0^t Z(t,s)dW^H(s)\right\rangle dt \\ &= \mathbb{E}\int_0^T \langle X(t),Y(t)\rangle dt - \mathbb{E}\int_0^T \int_s^T \langle X(s), A_0^*(t,s)Y(t)\rangle dt ds \\ &- \mathbb{E}\int_0^T \left\langle \int_0^t A_1(t,s)X(s)dW^H(s), \mathbb{E}(Y(t))\right\rangle dt \end{aligned}$$

$$
\begin{aligned}
&- \mathbb{E}\int_0^T \left\langle \int_0^t A_1(t,s)X(s)dW^H(s), \int_0^t Z(t,s)dW^H(s) \right\rangle dt \\
&= \mathbb{E}\int_0^T \langle X(t), Y(t)\rangle dt - \mathbb{E}\int_0^T \int_t^T \langle X(t), A_0^*(s,t)Y(s)\rangle dsdt \\
&- \int_0^T \left\langle \mathbb{E}\left(\int_0^t A_1(t,s)X(s)dW^H(s)\right), \mathbb{E}(Y(t)) \right\rangle dt \\
&- \mathbb{E}\int_0^T \left\langle \int_0^t A_1(t,s)X(s)dW^H(s), \int_0^t Z(t,s)dW^H(s) \right\rangle dt
\end{aligned}
$$

For above calculations we used properties of adjoint operators and stochastic differential equations with respect to cylindrical Brownian motion. Now for the last term we use the Itô-Formula in Banach spaces by Brzezniak, Neerven, Veraar and Weis, Theorem A.8.
Let for each (fixed) $t \in [0,T]$, $Q(r) = \int_0^r A_1(t,s)X(s)dW^H(s)$ in E and $V(r) = \int_0^r Z(t,s)dW^H(s)$ in E^* where $r \in [0,t]$ then we observe by Theorem A.8

$$
\begin{aligned}
&\langle Q(r), V(r)\rangle - \langle Q(0), V(0)\rangle = \int_0^r \left(\langle Q(s), 0\rangle + \langle 0, V(s)\rangle \right) ds \\
&+ \int_0^r \left(\langle Q(s), Z(t,s)\rangle + \langle A_1(t,s)X(s), V(s)\rangle \right) dW^H(s) \\
&+ \int_0^r \sum_{n\geq 1} \langle (A_1(t,s)X(s))(h_n), (Z(t,s))(h_n)\rangle \, ds.
\end{aligned}
$$

By taking expectation of above equations for $r = t$ and by knowing that expectation of stochastic integral is zero we have

$$
\begin{aligned}
&\mathbb{E}\left\langle \int_0^t A_1(t,s)X(s)dW^H(s), \int_0^t Z(t,s)dW^H(s) \right\rangle \\
&\quad = \mathbb{E}\int_0^t \sum_{n\geq 1} \langle (A_1(t,s)X(s))h_n, (Z(t,s))h_n\rangle \, ds
\end{aligned}
$$

We write $(A_1(t,s)X(s))(h_n) = (A_1(t,s)h_n)(X(s)) = A_1(t,s)h_nX(s)$ since $X(\cdot) \in E$ and $A_1(t,s) \in \mathcal{L}(E; \mathcal{L}(H;E))$. By integrating above result over $[0,T]$ with respect to Lebesgue measure dt and using Fubini's theorem and property of adjoint operator it yields

$$
\begin{aligned}
&\mathbb{E}\int_0^T \left\langle \int_0^t A_1(t,s)X(s)dW^H(s), \int_0^t Z(t,s)dW^H(s) \right\rangle dt \\
&\qquad = \mathbb{E}\int_0^T \int_0^t \sum_{n\geq 1} \langle A_1(t,s)h_nX(s), Z(t,s)h_n\rangle \, dsdt \\
&\qquad = \mathbb{E}\int_0^T \int_0^t \sum_{n\geq 1} \langle X(s), A_1^*(t,s)h_nZ(t,s)h_n\rangle \, dsdt
\end{aligned}
$$

Therefore by substitution, adaptedness of $X(\cdot)$ and knowing that expectation of stochastic integral is zero and by Fubini's Theorem, Equation (4.2) and using elementary calculus, it yields

$$\begin{aligned}
&\mathbb{E}\int_0^T \langle \varphi(t), Y(t)\rangle = \\
&\mathbb{E}\int_0^T \langle X(t), Y(t)\rangle dt - \mathbb{E}\int_0^T\int_t^T \langle X(t), A_0^*(s,t)Y(s)\rangle dsdt \\
&-\mathbb{E}\int_0^T\int_0^s \sum_{n\geq 1}\langle X(t), A_1^*(s,t)h_n Z(s,t)h_n\rangle\, dtds \\
&= \mathbb{E}\int_0^T \langle X(t), Y(t)\rangle dt - \mathbb{E}\int_0^T\int_t^T \langle X(t), A_0^*(s,t)Y(s)\rangle dsdt \\
&-\mathbb{E}\int_0^T\int_t^T \sum_{n\geq 1}\langle X(t), A_1^*(s,t)h_n Z(s,t)h_n\rangle\, dsdt \\
&= \int_0^T \mathbb{E}\left\langle X(t), Y(t) - \int_t^T A_0^*(s,t)Y(s)ds - \int_t^T \sum_{n\geq 1} A_1^*(s,t)h_n Z(s,t)h_n ds\right\rangle dt \\
&= \int_0^T \mathbb{E}\left\langle X(t), \psi(t) - \int_t^T Z(t,s)dW^H(s)\right\rangle dt \\
&= \int_0^T \mathbb{E}\left\langle X(t), \psi(t)\right\rangle dt - \int_0^T \mathbb{E}\left(\mathbb{E}\left\langle X(t), \int_t^T Z(t,s)dW^H(s)\right\rangle \middle| \mathcal{F}_t\right) dt \\
&= \int_0^T \mathbb{E}\left\langle X(t), \psi(t)\right\rangle dt - \int_0^T \mathbb{E}\left\langle X(t), \mathbb{E}\left(\int_t^T Z(t,s)dW^H(s)\right)\right\rangle dt \\
&= \int_0^T \mathbb{E}\left\langle X(t), \psi(t)\right\rangle dt
\end{aligned}$$

□

4.2. A More General Duality Principle

Now we want to show another duality principle between FSVIE Equation(4.1) and another general BSVIE. Therefore consider following BSVIE in dual space E^*

$$\begin{aligned}
Y(t) =&\psi(t) + A_0^*(T,t)\eta + \sum_{n\geq 1} A_1^*(T,t)h_n\rho(t)h_n \\
&+\int_t^T \left(A_0^*(s,t)Y(s) + \sum_{n\geq 1} A_1^*(s,t)h_n Z(s,t)h_n\right) ds \\
&-\int_t^T Z(t,s)dW^H(s), \quad t\in[0,T]
\end{aligned} \tag{4.3}$$

where $\eta \in L^{p'}_{\mathcal{F}_T}(\Omega; E^*)$, $\frac{1}{p}+\frac{1}{p'}=1$, $q\leq p$ and ρ is a unique adapted process that is defined by martingale representation theorem as follows:
since η is $\mathcal{F}_T$-measurable and $1 < q' \leq 2$, E^* is co-type(2), in according to Theorem A.6, we

have the following representation theorem

$$\eta = \mathbb{E}(\eta) + \int_0^T \rho(t) dW^H(t)$$

It means that we can find unique $\rho(\cdot) \in L^{p'}_{\mathbb{F}}\left(\Omega; \gamma\left(L^2(0,T;H);E^*\right)\right)$ and correspondingly $\varrho(\cdot) \in L^{p'}_{\mathbb{F}}\left(\Omega; L^{q'}\left(\mathbf{S}; L^2(0,T;H)\right)\right)$ such that $(\rho(t)h)(\cdot) = [\varrho(t,\cdot),h]_H$ in E^*. Therefore ρ is well defined and also the Equation(4.3) and by assumptions it has a unique adapted M-solution

$$(Y(\cdot),Z(\cdot)) \in L^{p'}_{\mathbb{F}}\left([0,T]\times\Omega;E^*\right) \times L^{p'}_{\mathbb{F}}\left([0,T];\gamma\left(L^2(0,T;H);E^*\right)\right)$$

or correspondingly

$$(Y(\cdot),\mathfrak{Z}(\cdot)) \in L^{p'}_{\mathbb{F}}\left([0,T]\times\Omega;E^*\right) \times L^{p'}_{\mathbb{F}}\left([0,T];L^{q'}\left(\mathbf{S};L^2(0,T;H)\right)\right)$$

where $(Z(t,s)h)(\cdot) = [\mathfrak{Z}(t,s,\cdot),h]_H$ in E^*

THEOREM 4.2. *If FSVIE Equation*(4.1) *and BSVIE Equation*(4.3) *are fulfilled then following duality holds for every* $\eta \in L^{p'}_{\mathcal{F}_T}(\Omega;E^*)$

$$\mathbb{E}\left\{\langle X(T),\eta\rangle + \int_0^T \langle X(t),\psi(t)\rangle\, dt\right\} = \mathbb{E}\left\{\langle \varphi(T),\eta\rangle + \int_0^T \langle \varphi(t),Y(t)\rangle\, dt\right\}$$

PROOF. We can write the BSVIE Equation(4.3) as follows

$$\begin{aligned} Y(t) =& \hat{\psi}(t) + \int_t^T \left(A_0^*(s,t)Y(s) + \sum_{n\geq 1} A_1^*(s,t)h_n Z(s,t)h_n\right) ds \\ & - \int_t^T Z(t,s)dW^H(s) \end{aligned}$$

where

$$\hat{\psi}(t) = \psi(t) + A_0^*(T,t)\eta + \sum_{n\geq 1} A_1^*(T,t)h_n\rho(t)h_n$$

Now we can use Theorem 4.1 and it results

$$\begin{aligned} &\mathbb{E}\int_0^T \langle\varphi(t),Y(t)\rangle\, dt = \mathbb{E}\int_0^T \left\langle X(t),\hat{\psi}(t)\right\rangle dt \\ &= \mathbb{E}\int_0^T \left\langle X(t),\psi(t) + A_0^*(T,t)\eta + \sum_{n\geq 1} A_1^*(T,t)h_n\rho(t)h_n\right\rangle dt \\ &= \mathbb{E}\int_0^T \langle X(t),\psi(t)\rangle\, dt + \mathbb{E}\int_0^T \langle A_0(T,t)X(t),\eta\rangle\, dt \\ &+ \mathbb{E}\int_0^T \sum_{n\geq 1} \langle A_0(T,t)h_nX(t),\rho(t)h_n\rangle\, dt \end{aligned} \tag{4.4}$$

Now we show that following equation holds

$$
\begin{aligned}
&\mathbb{E}\left\langle X(T)-\varphi(T),\eta\right\rangle = \\
&\quad \mathbb{E}\int_0^T \left\{ \left\langle A_0(T,t)X(t),\eta\right\rangle + \sum_{n\geq 1}\left\langle A_1(T,t)h_nX(t),\rho(t)h_n\right\rangle \right\} dt
\end{aligned}
$$

Consider

$$
\begin{aligned}
&\mathbb{E}\left\langle X(T)-\varphi(T),\eta\right\rangle = \\
&\mathbb{E}\left\langle \varphi(T)+\int_0^T A_0(T,s)X(s)ds+\int_0^T A_1(T,s)X(s)dW^H(s)-\varphi(T),\eta\right\rangle \\
&= \mathbb{E}\left\langle \int_0^T A_0(T,s)X(s)ds,\eta\right\rangle + \mathbb{E}\left\langle \int_0^T A_1(T,s)X(s)dW^H(s),\eta\right\rangle
\end{aligned}
$$

Now by replacing $\eta=\mathbb{E}(\eta)+\int_0^T \rho(s)dW^H(s)$ and setting

$$
\begin{cases} K(t)=\int_0^t A_1(T,s)X(s)dW^H(s) \\ \nu(t)=\int_0^t \rho(s)dW^H(s) \end{cases}, \quad t\in[0,T]
$$

we can use Itô Formula in UMD Banach spaces Theorem A.8 as follows

$$
\begin{aligned}
&\mathbb{E}\left\langle \int_0^T A_1(T,s)X(s)dW^H(s),\mathbb{E}(\eta)+\int_0^T \rho(s)dW^H(s)\right\rangle \\
&= \mathbb{E}\left\langle \int_0^T A_1(T,s)X(s)dW^H(s),\mathbb{E}(\eta)\right\rangle \\
&+ \mathbb{E}\left\langle \int_0^T A_1(T,s)X(s)dW^H(s),\int_0^T \rho(s)dW^H(s)\right\rangle \\
&= \mathbb{E}\left\langle K(T),\nu(T)\right\rangle \\
&= \mathbb{E}\Bigg(\int_0^T \left\{ \left\langle K(s),0\right\rangle + \left\langle 0,\nu(s)\right\rangle \right\} ds \\
&+ \int_0^T \left\{ \left\langle K(s),\rho(s)\right\rangle + \left\langle A_1(T,s)X(s),\nu(s)\right\rangle \right\} dW^H(s) \\
&+ \int_0^T \sum_{n\geq 1}\left\langle A_1(T,s)h_nX(s),\rho(s)h_n\right\rangle ds \Bigg)
\end{aligned}
$$

The first equation is clearly zero and the second term is stochastic integral, by taking expectation it is zero too. Only it remains the third term, therefore it yields

$$
\begin{aligned}
&\mathbb{E}\left\langle X(T)-\varphi(T),\eta\right\rangle = \\
&\mathbb{E}\left\{ \int_0^T \left\langle A_0(T,s)X(s),\eta\right\rangle ds + \int_0^T \sum_{n\geq 1}\left\langle A_1(T,s)h_nX(s),\rho(s)h_n\right\rangle ds \right\}
\end{aligned}
$$

Now by replacing it in equation(4.4) it results

$$\begin{aligned}&\mathbb{E}\int_0^T \langle \varphi(t), Y(t)\rangle \, dt = \mathbb{E}\int_0^T \left\langle X(t), \hat{\psi}(t)\right\rangle dt \\ &= \mathbb{E}\int_0^T \langle X(t), \psi(t)\rangle \, dt + \mathbb{E}\left\langle X(T) - \varphi(T), \eta\right\rangle\end{aligned}$$

and this finally yields

$$\begin{aligned}&\mathbb{E}\int_0^T \langle \varphi(t), Y(t)\rangle \, dt + \mathbb{E}\left\langle \varphi(T), \eta\right\rangle \\ &= \mathbb{E}\int_0^T \langle X(t), \psi(t)\rangle \, dt + \mathbb{E}\left\langle X(T), \eta\right\rangle\end{aligned}$$

□

REMARK 4.1. *It must be mentioned that martingale representation theorem can be used since $L^{q'}$ spaces, $q' \leq 2$ are co-type(2). Our proof can be resulted for every FSVIE* (4.1) *in every UMD space E, at the case that first we can find unique solutions for FSVIE* (4.1). *Second to consider BSVIE* (4.3) *in dual space E^* we need a martingale representation theorem. That is known only at the case that E^* is of co-type(2). If E is so, that this property holds, then we get also duality principles.*

CHAPTER 5

Maximum Principle

We consider in this chapter an optimal control problem for a FSVIE in Banach space E where $E = L^q(\mathbf{S}, \Sigma, \mu)$, μ is a finite measure and the stochastic integral is defined with respect to a H-cylindrical Brownian motion. We prove an optimality condition of maximum principle type. At first the control process is a real valued process and next it will be generalized to a Banach valued control process.

5.1. Stochastic Optimal Problem

Consider the following FSVIE in Banach space $E = L^q(\mathbf{S}, \Sigma, \mu)$, $q \geq 2$

$$X(t) = \varphi(t) + \int_0^t b\,(t, s, X(s), u(s))\, ds + \int_0^t \rho\,(t, s, X(s), u(s))\, dW^H(s),\ t \in [0, T]$$

where $X(\cdot)$, $u(\cdot)$ are the state and the control processes respectively and $W^H(\cdot)$ is H-cylindrical Brownian motion. We set following assumptions $\varphi(\cdot) \in L^p_{\mathbb{F}}(\Omega \times [0, T]; E)$ where $q \leq p$ and b, ρ are defined measurably (measurability defined similarly to chapter 2) as

$$\begin{aligned} &b : \Omega \times [0, T] \times [0, T] \times E \times U \times \mathbf{S} \longrightarrow E \\ &\rho : \Omega \times [0, T] \times [0, T] \times E \times U \times \mathbf{S} \longrightarrow \mathcal{L}(H; E) \end{aligned}$$

$\varphi(\cdot)$ and $X(\cdot)$ are E-valued random variables and $u(\cdot)$ is real valued process as

$$u : [0, T] \times \Omega \times \mathbf{S} \longrightarrow U$$

where U is a bounded closed interval of $\mathbb{R}$ and we define

$$\mathcal{U} = \{u : [0, T] \times \Omega \times \mathbf{S} \longrightarrow U \mid u(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable}\}$$

clearly we can consider $\mathcal{U} \subset E$.
We define the following cost function as Bolza form

$$J(u(\cdot)) = \mathbb{E} \int_0^T \int_{\mathbf{S}} h(t, X(t), u(t)) d\mu dt + \mathbb{E} \int_{\mathbf{S}} g(X(T)) d\mu$$

where

$$\begin{aligned} &h : \Omega \times [0, T] \times E \times U \times \mathbf{S} \longrightarrow E \\ &g : \Omega \times E \times \mathbf{S} \longrightarrow E \end{aligned}$$

We use the concept of Nemytskii operator for b, h, g and we assume they have first continuous bounded derivatives with respect to x and u, in other words we assume $\left|\frac{\partial b(t,s,x,u)}{\partial x}\right| := |b_x(t, s, x, u)| \leq K_x$, $\left|\frac{\partial b(t,s,x,u)}{\partial u}\right| := |b_u(t, s, x, u)| \leq K_u$ for all $t, s \in [0, T]$, $\eta \in \mathbf{S}$, for all

$x \in \mathbb{R}, u \in U$ almost sure, where K_x, K_u are positive constants and so on. Since $\rho \in \mathcal{L}(H;E)$ is a linear bounded operator $\forall t, s \in [0,T]$, $\eta \in \mathbf{S}$, $w \in \Omega$, $x \in E$, $u \in \mathcal{U}$, we assume it has first continuous Fréchet derivative with respect to x, $\rho_x(t,s,x,u) : E \longrightarrow \mathcal{L}(H;E)$ and $\rho_x(t,s,x,u)$ is a linear continuous operator from E to $\mathcal{L}(H;E)$ such that for each $h \in \{h_n\}_{n\geq 1}$ where $\{h_n\}_{n\geq 1}$ is orthonormal basis of Hilbert space H, $\|\rho_x(t,s,x,u)h\|_{\mathcal{L}(E;E)}$ is bounded almost sure. We also assume it has first continuous Fréchet derivative with respect to u, $\rho_u(t,s,x,u) : \mathcal{U} \longrightarrow \mathcal{L}(H;E)$ such that for each $h \in \{h_n\}_{n\geq 1}$; $\|\rho_u(t,s,x,u)h\|_{\mathcal{L}(\mathcal{U};E)}$ is bounded almost sure.
Since $\mathcal{U}$ is convex, for each $\varepsilon \in (0,1)$, $u_\varepsilon(\cdot) \in \mathcal{U}$ where $u_\varepsilon(\cdot) := \overline{u}(\cdot) + \varepsilon(u(\cdot) - \overline{u}(\cdot))$ and let $(\overline{X}(\cdot), \bar{u}(\cdot))$ are optimal state and control processes. Let $\overline{X}_\varepsilon(\cdot)$ be the solution of the following FSVIE when $\overline{u}_\varepsilon(\cdot)$ is chosen

$$\overline{X}_\varepsilon(t) = \varphi(t) + \int_0^t b\left(t,s,\overline{X}_\varepsilon(s),\overline{u}_\varepsilon(s)\right) ds + \int_0^t \rho\left(t,s,\overline{X}_\varepsilon(s),\overline{u}_\varepsilon(s)\right) dW^H(s),\ t \in [0,T]$$

and we define $\xi_\varepsilon(t) := \frac{X_\varepsilon(t) - \overline{X}(t)}{\varepsilon}$ for each $t \in [0,T]$

Lemma 5.1. *$\xi_\varepsilon(t)$ is uniformly bounded in $L^p(\Omega;E)$ with respect to t and ε and correspondingly $\xi_\varepsilon(\cdot)$ is uniformly bounded in $L^p(\Omega \times [0,T];E)$ with respect to ε for $2 \leq q \leq p$.*

Proof. By using the definition of $\xi_\varepsilon(\cdot)$ and substituting corresponding FSVIE for $X(\cdot)$ and $\overline{X}(\cdot)$, it yields

$$\begin{aligned}
&\mathbb{E}\left\|\xi_\varepsilon(t)\right\|^p = \mathbb{E}\left\|\frac{X_\varepsilon(t) - \overline{X}(t)}{\varepsilon}\right\|^p = \\
&\frac{1}{\varepsilon^p}\mathbb{E}\left\|\varphi(t) + \int_0^t b\left(t,s,X_\varepsilon(s),u_\varepsilon(s)\right) ds + \int_0^t \rho\left(t,s,X_\varepsilon(s),u_\varepsilon(s)\right) dW^H(s)\right.\\
&\left.-\varphi(t) - \int_0^t b\left(t,s,\overline{X}(s),\overline{u}(s)\right) ds - \int_0^t \rho\left(t,s,\overline{X}(s),\overline{u}(s)\right) dW^H(s)\right\|^p \\
&= \frac{1}{\varepsilon^p}\mathbb{E}\left\|\int_0^t \left\{b\left(t,s,X_\varepsilon(s),u_\varepsilon(s)\right) - b\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right\} ds\right.\\
&\quad\left. + \int_0^t \left\{\rho\left(t,s,X_\varepsilon(s),u_\varepsilon(s)\right) - \rho\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right\} dW^H(s)\right\|^p
\end{aligned}$$

By using triangle inequality and Yong inequality it yields

$$\begin{aligned}
\mathbb{E}\left\|\xi_\varepsilon(t)\right\|^p \leq \frac{c}{\varepsilon^p}&\left\{\mathbb{E}\left\|\int_0^t \left\{b\left(t,s,X_\varepsilon(s),u_\varepsilon(s)\right) - b\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right\} ds\right\|^p\right.\\
&\left.+ \mathbb{E}\left\|\int_0^t \left\{\rho\left(t,s,X_\varepsilon(s),u_\varepsilon(s)\right) - \rho\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right\} dW^H(s)\right\|^p\right\}
\end{aligned}$$

Now we consider every summand from above equation separately. For the first summand we have

$$\begin{aligned}
A =& \frac{c}{\varepsilon^p}\mathbb{E}\left\|\int_0^t \left\{\left(b\left(t,s,X_\varepsilon(s),u_\varepsilon(s)\right) - b\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\right)\right.\right.\\
&\left.\left.+b\left(t,s,\overline{X}(s),u_\varepsilon(s)\right) - b\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right\} ds\right\|^p
\end{aligned}$$

since there exists first derivative of $b(t,s,x,u)$ with respect to x,u for every $\eta \in \mathbf{S}$ and it is bounded a.e μ, there exist $r_1 =: r_1(t,s,x,u) \in (0,1)$ and $r_2 =: r_2(t,s,x,u) \in (0,1)$ such that following equations hold

$$\begin{aligned} &b\left(t,s,X_\varepsilon(s),u_\varepsilon(s)\right) - b\left(t,s,\overline{X}(s),u_\varepsilon(s)\right) = \\ &\qquad b_x\left(t,s,\overline{X}(s)+r_1(X_\varepsilon(s)-\overline{X}(s)),u_\varepsilon(s)\right)\left(X_\varepsilon(s)-\overline{X}(s)\right) \end{aligned}$$

and

$$\begin{aligned} &b\left(t,s,\overline{X}(s),u_\varepsilon(s)\right) - b\left(t,s,\overline{X}(s),\overline{u}(s)\right) = \\ &\qquad b_x\left(t,s,\overline{X}(s),\overline{u}(s)+r_2(u_\varepsilon(s)-\overline{u}(s))\right)\left(u_\varepsilon(s)-\overline{u}(s)\right) \end{aligned}$$

and for simplicity we set $\widetilde{X}_\varepsilon(s) := \overline{X}(s)+r_1(X_\varepsilon(s)-\overline{X}(s))$ and $\widetilde{u}_\varepsilon(s) := \overline{u}(s)+r_2(u_\varepsilon(s)-\overline{u}(s))$, then we can have

$$\begin{aligned} A = \frac{c}{\varepsilon^p}\Bigg\{\mathbb{E}\Bigg\|&\int_0^t b_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)\left(X_\varepsilon(s)-\overline{X}(s)\right)ds \\ &+\int_0^t b_u\left(t,s,\overline{X}(s),\widetilde{u}_\varepsilon(s)\right)\left(u_\varepsilon(s)-\overline{u}(s)\right)ds\Bigg\|^p\Bigg\} \end{aligned}$$

By boundedness of derivatives we can write

$$\begin{aligned} A &\le \frac{c}{\varepsilon^p}\left\{\mathbb{E}\left\|\int_0^t K_x\left|X_\varepsilon(s)-\overline{X}(s)\right|ds + \int_0^t K_u\left|u_\varepsilon(s)-\overline{u}(s)\right|ds\right\|^p\right\} \\ &\le c\left\{\mathbb{E}\left\|\int_0^t K_x\frac{\left|X_\varepsilon(s)-\overline{X}(s)\right|}{\varepsilon}ds + \int_0^t K_u\left|u(s)-\overline{u}(s)\right|ds\right\|^p\right\} \end{aligned}$$

By assumptions K_x, K_u are positive upper bounds for derivatives and do not depend on η, t, s, w.

$$\begin{aligned} A &\le c\left\{\mathbb{E}\left\|\int_0^t \frac{\left|X_\varepsilon(s)-\overline{X}(s)\right|}{\varepsilon}ds\right\|^p + \mathbb{E}\left\|\int_0^t \left|u(s)-\overline{u}(s)\right|ds\right\|^p\right\} \\ &\le c\left\{\mathbb{E}\left(\int_0^t \frac{\left\|X_\varepsilon(s)-\overline{X}(s)\right\|}{\varepsilon}ds\right)^p + \mathbb{E}\left(\int_0^t \left\|u(s)-\overline{u}(s)\right\|ds\right)^p\right\} \end{aligned}$$

$\mathcal{U}$ is bounded then second part of above equations is bounded and since $p>1$ we can use Jensen's inequality for measure ds and it results

$$A \le ct^{p-1}\left\{\mathbb{E}\int_0^t \left\|\frac{X_\varepsilon(s)-\overline{X}(s)}{\varepsilon}\right\|^p ds + \mathbb{E}\int_0^t \left\|u(s)-\overline{u}(s)\right\|^p ds\right\}$$

Now we consider the stochastic integral part

$$B = \frac{c}{\varepsilon^p}\mathbb{E}\left\|\int_0^t \left\{\rho\left(t,s,X_\varepsilon(s),u_\varepsilon(s)\right)-\rho\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right\}dW^H(s)\right\|^p$$

By using L^p-stochastically integrability in L^q spaces Theorem A.2 we have

$$B \le \frac{c}{\varepsilon^p}\mathbb{E}\left\|\left(\int_0^t \left\|\varrho\left(t,s,X_\varepsilon(s),u_\varepsilon(s)\right)-\varrho\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right\|_H^2 ds\right)^{\frac{1}{2}}\right\|^p$$

we have $(\rho(t,s,x,u)h)(\cdot) = [\varrho(t,s,x,u,\cdot),h]_H$ by Theorem A.3 and Theorem A.2. Since we assumed $2 \le q \le p$, we can apply Jensen's inequality and it yields

$$\begin{aligned}
B &\le \frac{c}{\varepsilon^p}\mathbb{E}\left(\int\limits_{\mathbf{S}}\left(\int_0^t \left\|\varrho\left(t,s,X_\varepsilon(s),u_\varepsilon(s)\right)-\varrho\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right\|_H^2 ds\right)^{\frac{q}{2}} d\mu\right)^{\frac{p}{q}} \\
&\le \frac{c}{\varepsilon^p}t^{\frac{p}{2}-\frac{p}{q}}\mathbb{E}\left(\int\limits_{\mathbf{S}}\int_0^t \left\|\varrho\left(t,s,X_\varepsilon(s),u_\varepsilon(s)\right)-\varrho\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right\|_H^q ds d\mu\right)^{\frac{p}{q}} \\
&= \frac{c}{\varepsilon^p}t^{\frac{p}{2}-\frac{p}{q}}\mathbb{E}\left(\int_0^t\int\limits_{\mathbf{S}} \left\|\varrho\left(t,s,X_\varepsilon(s),u_\varepsilon(s)\right)-\varrho\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right\|_H^q d\mu ds\right)^{\frac{p}{q}} \\
&\le \frac{c}{\varepsilon^p}t^{\frac{p}{2}-1}\mathbb{E}\int_0^t\left(\int\limits_{\mathbf{S}} \left\|\varrho\left(t,s,X_\varepsilon(s),u_\varepsilon(s)\right)-\varrho\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right\|_H^q d\mu\right)^{\frac{p}{q}} ds \\
&= \frac{c}{\varepsilon^p}t^{\frac{p}{2}-1}\mathbb{E}\int_0^t \left\|\left\|\varrho\left(t,s,X_\varepsilon(s),u_\varepsilon(s)\right)-\varrho\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right\|_H\right\|^p ds.
\end{aligned}$$

By definition of $\widetilde{X}_\varepsilon(s)$ and $\widetilde{u}_\varepsilon(s)$ there exist $r_1, r_2 \in (0,1)$ such that we can write

$$\begin{aligned}
&\varrho\left(t,s,X_\varepsilon(s),u_\varepsilon(s)\right)-\varrho\left(t,s,\overline{X}(s),\overline{u}(s)\right) \\
&= \left(\varrho\left(t,s,X_\varepsilon(s),u_\varepsilon(s)\right)-\varrho\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\right) \\
&\qquad + \varrho\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)-\varrho\left(t,s,\overline{X}(s),\overline{u}(s)\right) \\
&= \varrho_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)\left(X_\varepsilon(s)-\overline{X}(s)\right)+\varrho_u\left(t,s,\overline{X}(s),\widetilde{u}_\varepsilon(s)\right)\left(u_\varepsilon(s)-\overline{u}(s)\right).
\end{aligned}$$

By substitution this equation in B, it results

$$\begin{aligned}
B \le& \frac{c}{\varepsilon^p}\mathbb{E}\int_0^t \Bigg\| \Big\|\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)\left(X_\varepsilon(s)-\overline{X}(s)\right) \\
&+ \varrho_u\left(t,s,\overline{X}(s),\widetilde{u}_\varepsilon(s)\right)\left(u_\varepsilon(s)-\overline{u}(s)\right)\Big\|_H\Bigg\|^p ds \\
\le& \frac{c}{\varepsilon^p}\Bigg\{\mathbb{E}\int_0^t \left\|\left\|\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)\left(X_\varepsilon(s)-\overline{X}(s)\right)\right\|_H\right\|^p \\
&+ \left\|\left\|\varrho_u\left(t,s,\overline{X}(s),\widetilde{u}_\varepsilon(s)\right)\left(u_\varepsilon(s)-\overline{u}(s)\right)\right\|_H\right\|^p ds\Bigg\} \\
\le& c\Bigg\{\mathbb{E}\int_0^t \left\|\left\|\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)\frac{\left(X_\varepsilon(s)-\overline{X}(s)\right)}{\varepsilon}\right\|_H\right\|^p ds \\
&+ \mathbb{E}\int_0^t \left\|\left\|\varrho_u\left(t,s,\overline{X}(s),\widetilde{u}_\varepsilon(s)\right)\frac{\left(u_\varepsilon(s)-\overline{u}(s)\right)}{\varepsilon}\right\|_H\right\|^p ds\Bigg\}
\end{aligned}$$

and

$$B \leq c\Bigg\{\mathbb{E}\int_0^t \left\|\left\|\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)\xi_\varepsilon(s)\right\|_H\right\|^p ds$$
$$+\,\mathbb{E}\int_0^t \left\|\left\|\varrho_u\left(t,s,\overline{X}(s),\widetilde{u}_\varepsilon(s)\right)(u(s)-\overline{u}(s))\right\|_H\right\|^p ds\Bigg\}$$

Now we consider $\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)$. This term is Fréchet derivative with respect to x and it holds for every $\eta \in \mathbf{S}$, in the following relation

$$\left(\rho_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)\xi_\varepsilon(s)h\right)(\eta) = \left[\left(\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)\xi_\varepsilon(s)\right)(\eta),h\right]_H.$$

It defines a linear operator from E to H and it is also an element of E. We can consider this operator as Nemytskii operator for every $\eta \in \mathbf{S}$, it means
$\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s,\eta),u_\varepsilon(s,\eta),\eta\right)\xi_\varepsilon(s,\eta)$ for every fixed t,s and η is an element of $\mathbb{R}$, therefore we can have following norm inequality

$$\left\|\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s,\eta),u_\varepsilon(s,\eta),\eta\right)\xi_\varepsilon(s,\eta)\right\|_H \leq \left\|\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s,\eta),u_\varepsilon(s,\eta),\eta\right)\right\|_{\mathcal{L}(\mathbb{R};H)}|\xi_\varepsilon(s,\eta)|$$

By similar explanation we have following norm inequality for $\varrho_u\left(t,s,\overline{X}(s),\widetilde{u}_\varepsilon(s)\right)$

$$\left\|\varrho_u\left(t,s,\overline{X}(s,\eta),\widetilde{u}_\varepsilon(s,\eta),\eta\right)(u(s,\eta)-\overline{u}(s,\eta))\right\|_H$$
$$\leq \left\|\varrho_u\left(t,s,\overline{X}(s,\eta),\widetilde{u}_\varepsilon(s,\eta),\eta\right)\right\|_{\mathcal{L}(\mathbb{R};H)}|(u(s,\eta)-\overline{u}(s,\eta))|$$

It must be mentioned that $\xi_\varepsilon(s)$, $(u(s)-\overline{u}(s))$ are elements of E. By replacing above calculations in B, it yields

$$B \leq c\Bigg\{\mathbb{E}\int_0^t \left\|\left\|\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s,\cdot),u_\varepsilon(s,\cdot),\cdot\right)\right\|_{\mathcal{L}(\mathbb{R};H)}|\xi_\varepsilon(s,\cdot)|\right\|^p ds$$
$$+\,\mathbb{E}\int_0^t \left\|\left\|\varrho_u\left(t,s,\overline{X}(s,\cdot),\widetilde{u}_\varepsilon(s,\cdot),\cdot\right)\right\|_{\mathcal{L}(\mathbb{R};H)}|(u(s,\cdot)-\overline{u}(s,\cdot))|\right\|^p ds\Bigg\}$$

By assumptions, $\left\|\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s,\eta),u_\varepsilon(s,\eta),\eta\right)\right\|_{\mathcal{L}(\mathbb{R};H)}$ and $\left\|\varrho_u\left(t,s,\overline{X}(s,\eta),\widetilde{u}_\varepsilon(s,\eta),\eta\right)\right\|_{\mathcal{L}(\mathbb{R};H)}$ are uniformly a.s bounded then finally for part B we have the following inequality

$$B \leq c\Bigg\{\mathbb{E}\int_0^t \|\xi_\varepsilon(s)\|^p ds + \mathbb{E}\int_0^t \|(u(s)-\overline{u}(s))\|^p ds\Bigg\}$$

By combining inequalities for parts A and B, it yields

$$\mathbb{E}\,\|\xi_\varepsilon(t)\|^p \leq c\Bigg\{\mathbb{E}\int_0^t \|\xi_\varepsilon(s)\|^p ds + \mathbb{E}\int_0^t \|u(s)-\overline{u}(s)\|^p ds\Bigg\}, \quad \forall t \in [0,T]$$

where again c is a universal constant. Here we can use the Gronwall's inequality by taking $v(t) = \mathbb{E}\,\|\xi_\varepsilon(t)\|^p$, $\alpha(t) = c\mathbb{E}\int_0^t \|u(s)-\overline{u}(s)\|^p ds$, $F = c$ and we can write

$$v(t) \leq \alpha(t) + F\int_0^t v(s)ds, \quad 0 \leq t \leq T$$

then we have $v(t) \leq \alpha(t)\exp(Ft), \quad 0 \leq t \leq T$ since $u(\cdot), \overline{u}(\cdot) \in \mathcal{U}$ and $\mathcal{U}$ is bounded then $\alpha(t)$ is bounded by some positive constant C too. In other words it results

$$\mathbb{E}\left\|\frac{X_\varepsilon(t)-\overline{X}(t)}{\varepsilon}\right\|^p = \mathbb{E}\left\|\xi_\varepsilon(t)\right\|^p \leq C\exp ct \leq K, \quad 0 \leq t \leq T$$

where K is positive constant, moreover it yields

$$\mathbb{E}\int_0^T\left\|\frac{X_\varepsilon(t)-\overline{X}(t)}{\varepsilon}\right\|^p dt = \mathbb{E}\int_0^T \left\|\xi_\varepsilon(t)\right\|^p dt \leq KT, \quad \forall\varepsilon \geq 0$$

and it results that $\mathbb{E}\int_0^T \|\xi_\varepsilon(t)\|^p\,dt$ is uniformly bounded with respect to ε. □

LEMMA 5.2. *If ε tends to zero then $\xi_\varepsilon(\cdot)$ tends to $\xi(\cdot)$ in $L^p([0,T]\times\Omega; E)$ and $\xi_\varepsilon(t)$ tends to $\xi(t)$ in $L^p(\Omega; E)$ for all $t \in [0,T]$, where $\xi(t)$ satisfies the following FSVIE*

$$\xi(t) = \int_0^t \left\{b_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\xi(s) + b_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)\left(u(s)-\overline{u}(s)\right)\right\}ds$$
$$+\int_0^t \left\{\rho_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\xi(s) + \rho_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)\left(u(s)-\overline{u}(s)\right)\right\}dW^H(s),$$
$$0 \leq t \leq T.$$

PROOF. We see by using the definition of $\xi_\varepsilon(\cdot)$

$$\mathbb{E}\left\|\xi_\varepsilon(t)-\xi(t)\right\|^p =$$
$$\mathbb{E}\left\|\int_0^t\left\{\frac{b\left(t,s,X_\varepsilon(s),u_\varepsilon(s)\right)-b\left(t,s,\overline{X}(s),\overline{u}(s)\right)}{\varepsilon}\right.\right.$$
$$\left.- \; b_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\xi(s) - b_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)\left(u(s)-\overline{u}(s)\right)\right\}ds$$
$$+\int_0^t\left\{\frac{\rho\left(t,s,X_\varepsilon(s),u_\varepsilon(s)\right)-\rho\left(t,s,\overline{X}(s),\overline{u}(s)\right)}{\varepsilon}\right.$$
$$\left.\left.- \; \rho_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\xi(s) - \rho_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)\left(u(s)-\overline{u}(s)\right)\right\}dW^H(s)\right\|^p$$

By similar calculations that we done in Lemma 5.1, it results

$$\leq c\mathbb{E}\left\|\int_0^t\left\{b_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)\frac{X_\varepsilon(s)-\overline{X}(s)}{\varepsilon}\right.\right.$$
$$+ \; b_u\left(t,s,\overline{X}(s),\widetilde{u}_\varepsilon(s)\right)\left(u(s)-\overline{u}(s)\right)$$
$$\left.\left.- \; b_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\xi(s) - b_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)\left(u(s)-\overline{u}(s)\right)\right\}ds\right\|^p$$
$$+ \; c\mathbb{E}\left\|\int_0^t\left\{\frac{\rho\left(t,s,X_\varepsilon(s),u_\varepsilon(s)\right)-\rho\left(t,s,\overline{X}(s),\overline{u}(s)\right)}{\varepsilon}\right.\right.$$
$$\left.\left.- \; \rho_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\xi(s) - \rho_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)\left(u(s)-\overline{u}(s)\right)\right\}dW^H(s)\right\|^p$$

Now we consider first summand and we denote it by A_1

$$\begin{aligned}
A_1 = \mathbb{E}\Bigg\|\int_0^t \Big\{ b_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)\xi_\varepsilon(s) - b_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\xi(s) \\
+ \left(b_u\left(t,s,\overline{X}(s),\widetilde{u}_\varepsilon(s)\right) - b_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right)\left(u(s)-\overline{u}(s)\right)\Big\} ds\Bigg\|^p \\
\leq c\mathbb{E}\Bigg\|\int_0^t \Big\{ b_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)\xi_\varepsilon(s) - b_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\xi(s)\Big\} ds\Bigg\|^p \\
+ c\mathbb{E}\Bigg\|\int_0^t \left(b_u\left(t,s,\overline{X}(s),\widetilde{u}_\varepsilon(s)\right) - b_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right)\left(u(s)-\overline{u}(s)\right) ds\Bigg\|^p
\end{aligned}$$

Again we consider part one by one, we denote the first and second summands as A_{11} and A_{12} respectively,

$$\begin{aligned}
A_{11} \leq c\mathbb{E}\int_0^t \Big\|\Big\{ b_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)\xi_\varepsilon(s) - b_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\xi_\varepsilon(s) \\
+ b_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\xi_\varepsilon(s) - b_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\xi_\varepsilon(s) \\
+ b_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\xi_\varepsilon(s) - b_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\xi(s)\Big\|^p ds \\
\leq c\mathbb{E}\int_0^t \Big\|\Big\{ b_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right) - b_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\Big\}\xi_\varepsilon(s)\Big\|^p ds \\
+ c\mathbb{E}\int_0^t \left\|\left\{ b_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right) - b_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right\}\xi_\varepsilon(s)\right\|^p ds \\
+ c\mathbb{E}\int_0^t \left\| b_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\left\{\xi_\varepsilon(s)-\xi(s)\right\}\right\|^p ds
\end{aligned}$$

Now consider part by part, similar to our previous notations we set every summands as A_{111}, A_{112} and A_{113} receptively,

$$\begin{aligned}
A_{111} = \mathbb{E}\int_0^t \Big\|\Big\{ b_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right) - b_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\Big\}\left(\xi_\varepsilon(s)-\xi(s)\right) \\
+ \Big\{ b_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right) - b_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\Big\}\xi(s)\Big\|^p ds \\
\leq c\mathbb{E}\int_0^t \Big\|\Big\{ b_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right) - b_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\Big\}\left(\xi_\varepsilon(s)-\xi(s)\right)\Big\|^p ds \\
+ c\mathbb{E}\int_0^t \Big\|\Big\{ b_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right) - b_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\Big\}\xi(s)\Big\|^p ds
\end{aligned}$$

Since $b_x\left(t,s,X(s),u(s)\right)$ is bounded a.s, we have

$$\left| b_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right) - b_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\right| \leq K$$

and

$$\begin{aligned}
A_{111} \leq & Kc\mathbb{E}\int_0^t \left\|\xi_\varepsilon(s)-\xi(s)\right\|^p ds \\
& + cE\int_0^t \Big\|\Big\{ b_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right) - b_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\Big\}\xi(s)\Big\|^p ds
\end{aligned}$$

we can also get $\left|b_x\left(t,s,\widetilde{X}_\varepsilon(s,\eta),u_\varepsilon(s,\eta),\eta\right)-b_x\left(t,s,\overline{X}(s,\eta),u_\varepsilon(s,\eta),\eta\right)\right|$ tends to zero in measure. It results from dominant convergence theorem that

$$\lim_{\varepsilon\to 0} E\int_0^t \left\|\left\{b_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)-b_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\right\}\xi(s)\right\|^p ds=0$$

for every $t\in[0,T]$.
With similar calculations for second part we have

$$\begin{aligned}A_{112} \leq & Kc\mathbb{E}\int_0^t \|\xi_\varepsilon(s)-\xi(s)\|^p\,ds\\ &+cE\int_0^t \left\|\left\{b_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)-b_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right\}\xi(s)\right\|^p ds\end{aligned}$$

and

$$\lim_{\varepsilon\to 0} E\int_0^t \left\|\left\{b_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)-b_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right\}\xi(s)\right\|^p ds=0$$

for every $t\in[0,T]$. For the third part A_{113}, it yields

$$A_{113}\leq Kc\mathbb{E}\int_0^t \|\xi_\varepsilon(s)-\xi(s)\|^p\,ds$$

Now we consider the stochastic part A_2, by using the theorem of L^p-stochastic integrability by Theorem A.3 and Theorem A.2, it yields

$$\begin{aligned}A_2 \leq & c\mathbb{E}\left\|\left(\int_0^t \left\|\frac{\varrho\left(t,s,X_\varepsilon(s),u_\varepsilon(s)\right)-\varrho\left(t,s,\overline{X}(s),\overline{u}(s)\right)}{\varepsilon}\right.\right.\right.\\ &-\left.\left.\left.\varrho_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\xi(s)-\varrho_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)\left(u(s)-\overline{u}(s)\right)\right\|_H^2 ds\right)^{\frac{1}{2}}\right\|^p\end{aligned}$$

since $2\leq q\leq p$ by using Jensen's inequality we have

$$\begin{aligned}A_2 \leq & c\mathbb{E}\int_0^t \left\|\left\|\frac{\varrho\left(t,s,X_\varepsilon(s),u_\varepsilon(s)\right)-\varrho\left(t,s,\overline{X}(s),\overline{u}(s)\right)}{\varepsilon}\right.\right.\\ &-\left.\left.\varrho_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\xi(s)-\varrho_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)\left(u(s)-\overline{u}(s)\right)\right\|_H\right\|_E^p ds\end{aligned}$$

By using Fréchet's derivatives and similar calculations in the proof of Lemma 5.1 it yields

$$\begin{aligned}A_2 \leq & c\mathbb{E}\int_0^t \left\|\left\|\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)\frac{X_\varepsilon(s)-\overline{X}(s)}{\varepsilon}\right.\right.\\ &+\varrho_u\left(t,s,\overline{X}(s),\widetilde{u}_\varepsilon(s)\right)\left(u(s)-\overline{u}(s)\right)\\ &-\left.\left.\varrho_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\xi(s)-\varrho_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)\left(u(s)-\overline{u}(s)\right)\right\|_H\right\|_E^p ds\end{aligned}$$

$$\leq c\mathbb{E}\int_0^t \left\|\left\|\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)\xi_\varepsilon(s)-\varrho_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\xi(s)\right\|_H\right\|_E^p ds$$
$$+\,c\mathbb{E}\int_0^t \|\|\varrho_u\left(t,s,\overline{X}(s),\widetilde{u}_\varepsilon(s)\right)(u(s)-\overline{u}(s))$$
$$-\varrho_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)(u(s)-\overline{u}(s))\|_H\|_E^p ds$$

Consider part by part, set A_{21} and A_{22} respectively for every summands

$$A_{21}\leq c\mathbb{E}\int_0^t \left\|\left\|\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)\xi_\varepsilon(s)-\varrho_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\xi_\varepsilon(s)\right.\right.$$
$$+\varrho_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\xi_\varepsilon(s)-\varrho_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\xi_\varepsilon(s)$$
$$+\left.\left.\varrho_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\xi_\varepsilon(s)-\varrho_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\xi(s)\right\|_H\right\|_E^p ds$$
$$\leq c\mathbb{E}\int_0^t \left\|\left\|\left\{\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)-\varrho_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\right\}\xi_\varepsilon(s)\right\|_H\right\|_E^p ds$$
$$+c\mathbb{E}\int_0^t \|\|\{\varrho_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)-\varrho_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\}\xi_\varepsilon(s)\|_H\|_E^p ds$$
$$+\,c\mathbb{E}\int_0^t \|\|\varrho_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\{\xi_\varepsilon(s)-\xi(s)\}\|_H\|_E^p ds$$

Again consider one by one, set A_{211}, A_{212} and A_{213} respectively, for example for the first part A_{211} we can write

$$A_{211}=c\mathbb{E}\int_0^t \left\|\left\|\left\{\varrho\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)-\varrho_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\right\}(\xi_\varepsilon(s)-\xi(s))\right.\right.$$
$$+\left.\left.\left\{\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)-\varrho_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\right\}\xi(s)\right\|_H\right\|_E^p ds$$
$$\leq c\mathbb{E}\int_0^t \left\|\left\|\left\{\varrho\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)-\varrho_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\right\}(\xi_\varepsilon(s)-\xi(s))\right\|_H\right\|_E^p ds$$
$$+c\mathbb{E}\int_0^t \left\|\left\|\left\{\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)-\varrho_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\right\}\xi(s)\right\|_H\right\|_E^p ds$$

By considering pointwise as we have done in proof of Lemma 5.1, we can use for example following norm inequality for each part of above equation

$$\left\|\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s,\eta),u_\varepsilon(s,\eta),\eta\right)\xi_\varepsilon(s,\eta)\right\|_H\leq\left\|\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s,\eta),u_\varepsilon(s,\eta),\eta\right)\right\|_{\mathcal{L}(\mathbb{R};H)}|\xi_\varepsilon(s,\eta)|$$

and by assumptions $\|\varrho_x(t,s,X(s,\eta),u(s,\eta),\eta)\|_{\mathcal{L}(\mathbb{R};H)}$ is uniformly a.s bounded then it yields

$$A_{211}$$
$$\leq c\mathbb{E}\int_0^t \left\|\left\|\varrho\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)-\varrho_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\right\|_{\mathcal{L}(\mathbb{R};H)}|\xi_\varepsilon(s)-\xi(s)|\right\|_E^p ds$$
$$+\,c\mathbb{E}\int_0^t \left\|\left\|\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)-\varrho_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\right\|_{\mathcal{L}(\mathbb{R};H)}|\xi(s)|\right\|_E^p ds$$

$$\begin{aligned}
\leq & cK\mathbb{E}\int_0^t \|\xi_\varepsilon(s)-\xi(s)\|_E^p\,ds \\
&+ c\mathbb{E}\int_0^t \left\| \left\|\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)-\varrho_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\right\|_{\mathcal{L}(\mathbb{R};H)} |\xi(s)|\right\|_E^p ds
\end{aligned}$$

If ε tends to zero then by the dominant convergence theorem

$$\mathbb{E}\int_0^t \left\| \left\|\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)-\varrho_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\right\|_{\mathcal{L}(\mathbb{R};H)} |\xi(s)|\right\|_E^p ds$$

tends to zero too. By similar calculations we have following results for other parts too,

$$\begin{aligned}
A_{212} \leq & cK\mathbb{E}\int_0^t \|\xi_\varepsilon(s)-\xi(s)\|_E^p\,ds \\
&+ c\mathbb{E}\int_0^t \left\| \left\|\varrho_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)-\varrho_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right\|_{\mathcal{L}(\mathbb{R};H)} |\xi(s)|\right\|_E^p ds
\end{aligned}$$

and

$$A_{213} \leq cK\mathbb{E}\int_0^t \|\xi_\varepsilon(s)-\xi(s)\|_E^p\,ds$$

where

$$E\int_0^t \left\| \left\|\varrho_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)-\varrho_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right\|_{\mathcal{L}(\mathbb{R};H)} |\xi(s)|\right\|_E^p ds$$

tends to zero if ε tends to zero. Since $u(\cdot)\in\mathcal{U}$, it is almost sure bounded, then by dominant convergence theorem it results that

$$A_{12} = c\mathbb{E}\left\|\int_0^t \left(b_u\left(t,s,\overline{X}(s),\widetilde{u}_\varepsilon(s)\right)-b_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right)(u(s)-\overline{u}(s))\,ds\right\|^p$$

and

$$\begin{aligned}
A_{22} = & c\mathbb{E}\int_0^t \Big\| \Big\|\varrho_u\left(t,s,\overline{X}(s),\widetilde{u}_\varepsilon(s)\right)(u(s)-\overline{u}(s)) \\
&\qquad -\varrho_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)(u(s)-\overline{u}(s))\Big\|_H\Big\|_E^p ds \\
\leq & c\mathbb{E}\int_0^t \left\| \left\|\varrho_u\left(t,s,\overline{X}(s),\widetilde{u}_\varepsilon(s)\right)-\varrho_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right\|_{\mathcal{L}(\mathbb{R};H)} |u(s)-\overline{u}(s)|\right\|_E^p ds
\end{aligned}$$

tend to zero.

Now by taking all above results, we have

$$\mathbb{E}\|\xi_\varepsilon(t)-\xi(t)\|^p \leq cK\mathbb{E}\int_0^t \|\xi_\varepsilon(s)-\xi(s)\|^p\,ds + cF_\varepsilon(t)$$

where

$$F_\varepsilon(t) = E\left\|\int_0^t \left(b_u\left(t,s,\overline{X}(s),\widetilde{u}_\varepsilon(s)\right) - b_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right)(u(s)-\overline{u}(s))\,ds\right\|^p$$
$$+\,\mathbb{E}\int_0^t \left\|\left\|\varrho_u\left(t,s,\overline{X}(s),\widetilde{u}_\varepsilon(s)\right) - \varrho_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right\|_{\mathcal{L}(\mathbb{R};H)} |u(s)-\overline{u}(s)|\right\|_E^p ds$$
$$+\,\mathbb{E}\int_0^t \left\|\left\{b_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right) - b_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\right\}\xi(s)\right\|^p ds$$
$$+\,\mathbb{E}\int_0^t \left\|\left\{b_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right) - b_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right\}\xi(s)\right\|^p ds$$
$$+\,\mathbb{E}\int_0^t \left\|\left\|\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right) - \varrho_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right)\right\|_{\mathcal{L}(\mathbb{R};H)} |\xi(s)|\right\|_E^p ds$$
$$+\,\mathbb{E}\int_0^t \left\|\left\|\varrho_x\left(t,s,\overline{X}(s),u_\varepsilon(s)\right) - \varrho_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right\|_{\mathcal{L}(\mathbb{R};H)} |\xi(s)|\right\|_E^p ds$$

Here we can use the Gronwall's inequality by taking $v(t) = \mathbb{E}\left\|\xi_\varepsilon(t)-\xi(t)\right\|^p$, $\alpha(t) = cF_\varepsilon(t)$, $A = cK$ and

$$v(t) \le \alpha(t) + A\int_0^t v(s)ds, \quad 0 \le t \le T$$

then we have

$$v(t) \le F_\varepsilon(t)\exp(\int_0^t cK ds), \quad 0 \le t \le T$$

or

$$\mathbb{E}\left\|\xi_\varepsilon(t)-\xi(t)\right\|^p \le F_\varepsilon(t)e^{cKt}$$

for each $t \in [0,T]$, by knowing that if ε tends to zero then $F_\varepsilon(t)$ tends to zero, then it results that $\xi_\varepsilon(\cdot)$ tends to $\xi(\cdot)$ in $L^p\left([0,T]\times\Omega;E\right)$. □

Now we can consider our control problem. It is necessary to mention, since $E = L^q(\mathbf{S},\Sigma,\mu)$ is reflexive then it is Radon-Nikodyn space, see [6] and dual space of $L^p\left([0,T]\times\Omega;L^q\left(\mathbf{S},\Sigma,\mu\right)\right)$ is $L^{p'}\left([0,T]\times\Omega;L^{q'}\left(\mathbf{S},\Sigma,\mu\right)\right)$, where $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q}+\frac{1}{q'}=1$ and for every $f \in L^p\left([0,T]\times\Omega;E\right)$ and $g \in L^{p'}\left([0,T]\times\Omega;E^*\right)$ where E^* is dual space of E, here the duality paring is given as follows

$$\langle f(\cdot),g(\cdot)\rangle_{L^{p'}([0,T]\times\Omega;E^*)} = \mathbb{E}\int_0^T \langle f(t),g(t)\rangle_{E^*}\,dt = \mathbb{E}\int_0^T\int_{\mathbf{S}} f(t,\cdot)g(t,\cdot)d\mu dt$$

LEMMA 5.3. *If ε tends to zero, then*

$$\lim_{\varepsilon\to 0}\left|\frac{J(u_\varepsilon(\cdot))-J(\overline{u}(\cdot))}{\varepsilon}\right.$$
$$-\mathbb{E}\int_0^T\int_{\mathbf{S}}\left\{h_x\left(t,\overline{X}(t),\overline{u}(t)\right)\xi(t)+h_u\left(t,\overline{X}(t),\overline{u}(t)\right)(u(t)-\overline{u}(t))\right\}d\mu dt$$
$$\left.-\mathbb{E}\int_{\mathbf{S}}g_x\left(\overline{X}(T)\right).\xi(T)d\mu\right|=0$$

PROOF. By considering the definition of the cost function, we can write

$$\frac{J(u_\varepsilon(\cdot))-J(\overline{u}(\cdot))}{\varepsilon}=$$
$$\mathbb{E}\int_0^T\int_{\mathbf{S}}\frac{h(t,X_\varepsilon(t),u_\varepsilon(t))-h\left(t,\overline{X}(t),\overline{u}(t)\right)}{\varepsilon}d\mu dt$$
$$+\mathbb{E}\int_{\mathbf{S}}\frac{g(X_\varepsilon(T))-g\left(\overline{X}(T)\right)}{\varepsilon}d\mu$$

For proving it, we consider every summand separately. First we show that

$$\mathsf{B}_\varepsilon=\mathbb{E}\int_0^T\int_{\mathbf{S}}\frac{h(t,X_\varepsilon(t),u_\varepsilon(t))-h\left(t,\overline{X}(t),\overline{u}(t)\right)}{\varepsilon}d\mu dt$$

tends to

$$\mathsf{B}=\mathbb{E}\int_0^T\int_{\mathbf{S}}\left\{h_x\left(t,\overline{X}(t),\overline{u}(t)\right)\xi(t)\;+h_u\left(t,\overline{X}(t),\overline{u}(t)\right)(u(t)-\overline{u}(t))\right\}d\mu dt$$

By using the derivatives of $h(t,s,x,u)$, adding and subtracting $h\left(t,\overline{X}(t),u_\varepsilon(t)\right)$ it yields

$$\mathsf{B}_\varepsilon=\mathbb{E}\left\{\int_0^T\int_{\mathbf{S}}h_x\left(t,\widetilde{X}_\varepsilon(t),u_\varepsilon(t)\right)\frac{X_\varepsilon(t)-\overline{X}(t)}{\varepsilon}d\mu dt\right.$$
$$\left.+\int_0^T\int_{\mathbf{S}}h_u\left(t,\overline{X}(t),\widetilde{u}_\varepsilon(t)\right)\frac{u_\varepsilon(t)-\overline{u}(t)}{\varepsilon}d\mu dt\right\}$$

We want to use the Lemma 5.1, by using the norm properties we can write

$$\lim_{\varepsilon\to 0}|\mathsf{B}_\varepsilon-\mathsf{B}|=\lim_{\varepsilon\to 0}\left|\mathbb{E}\int_0^T\int_{\mathbf{S}}\left\{h_x\left(t,\widetilde{X}_\varepsilon(t),u_\varepsilon(t)\right)\frac{X_\varepsilon(t)-\overline{X}(t)}{\varepsilon}\right.\right.$$
$$\left.\left.+h_u\left(t,\overline{X}(t),\widetilde{u}_\varepsilon(t)\right)\frac{u_\varepsilon(t)-\overline{u}(t)}{\varepsilon}d\mu dt\right\}-\mathsf{B}\right|$$

$$
\begin{aligned}
&\leq \lim_{\varepsilon\to 0}\left\{\mathbb{E}\int_0^T\int\limits_{\mathbf{S}}\left|h_x\left(t,\widetilde{X}_\varepsilon(t),u_\varepsilon(t)\right)\xi_\varepsilon(t)-h_x\left(t,\overline{X}(t),\overline{u}(t)\right)\xi(t)\right|d\mu dt\right.\\
&\quad\left.+\mathbb{E}\int_0^T\int\limits_{\mathbf{S}}\left|\left(h_u\left(t,\overline{X}(t),\widetilde{u}_\varepsilon(t)\right)-h_u\left(t,\overline{X}(t),\overline{u}(t)\right)\right)(u(t)-\overline{u}(t))\right|d\mu dt\right\}
\end{aligned}
$$

Now we consider every summand in above equation denoted by B_1 and B_2. For the first one B_1 by adding and subtracting $h_x\left(t,\overline{X}(t),u_\varepsilon(t)\right)\xi(t)$ and $h_x\left(t,\overline{X}(t),\overline{u}(t)\right)\xi_\varepsilon(t)$ it yields

$$
\begin{aligned}
B_1 \leq&\mathbb{E}\int_0^T\int\limits_{\mathbf{S}}\left|h_x\left(t,\widetilde{X}_\varepsilon(t),u_\varepsilon(t)\right)-h_x\left(t,\overline{X}(t),u_\varepsilon(t)\right)\right|\left|\xi_\varepsilon(t)\right|d\mu dt\\
&+\mathbb{E}\int_0^T\int\limits_{\mathbf{S}}\left|h_x\left(t,\overline{X}(t),u_\varepsilon(t)\right)-h_x\left(t,\overline{X}(t),\overline{u}(t)\right)\right|\left|\xi_\varepsilon(t)\right|d\mu dt\\
&+\mathbb{E}\int_0^T\int\limits_{\mathbf{S}}\left|h_x\left(t,\overline{X}(t),\overline{u}(t)\right)\right|\left|\xi_\varepsilon(t)-\xi(t)\right|d\mu dt
\end{aligned}
$$

We consider again every summand separately as B_{11}, B_{12} and B_{13}. Since $h_x(t,x,u)$ is bounded then $\left|h_x\left(t,\widetilde{X}_\varepsilon(t),u_\varepsilon(t)\right)-h_x\left(t,\overline{X}(t),u_\varepsilon(t)\right)\right|$ is bounded a.s too and tends to zero in measure. Now we can use dominant convergence theorem and we use Hölder's inequality. It results

$$
\begin{aligned}
B_{11} \leq \mathbb{E}\int_0^T&\left\{\left(\int\limits_{\mathbf{S}}\left|h_x\left(t,\widetilde{X}_\varepsilon(t),u_\varepsilon(t)\right)-h_x\left(t,\overline{X}(t),u_\varepsilon(t)\right)\right|^{q'}d\mu\right)^{\frac{1}{q'}}\right.\\
&\left.\cdot\left(\int\limits_{\mathbf{S}}\left|\xi_\varepsilon(t)\right|^q d\mu\right)^{\frac{1}{q}}\right\}dt
\end{aligned}
$$

Now we apply Hölder's inequality for the product measure $d\nu = dt\cdot dP$,

$$
\begin{aligned}
B_{11} \leq &\left(\mathbb{E}\int_0^T\left(\int\limits_{\mathbf{S}}\left|h_x\left(t,\widetilde{X}_\varepsilon(t),u_\varepsilon(t)\right)-h_x\left(t,\overline{X}(t),u_\varepsilon(t)\right)\right|^{q'}d\mu\right)^{\frac{p'}{q'}}dt\right)^{\frac{1}{p'}}\\
&\cdot\left(\mathbb{E}\int_0^T\left(\int\limits_{\mathbf{S}}\left|\xi_\varepsilon(t)\right|^q d\mu\right)^{\frac{p}{q}}dt\right)^{\frac{1}{p}}
\end{aligned}
$$

where $\frac{1}{q'}+\frac{1}{q}=1$ and $\frac{1}{p}+\frac{1}{p'}=1$. By dominant convergence theorem the first factor tends to zero if ε tends to zero and the second part is bounded with respect to $\varepsilon>0$. The similar calculations show that B_{12} tends to zero too. for the B_{13} it results by boundedness of $h_x\left(t,\overline{X}(t),\overline{u}(t)\right)$

$$B_{13} \le \left(\mathbb{E}\int_0^T \left(\int\limits_{\mathbf{S}} \left|h_x\left(t,\overline{X}(t),\overline{u}(t)\right)\right|^{q'} d\mu\right)^{\frac{p'}{q'}} dt\right)^{\frac{1}{p'}}$$

$$\cdot\left(\mathbb{E}\int_0^T \left(\int\limits_{\mathbf{S}} \left|\xi_\varepsilon(t)-\xi(t)\right|^{q} d\mu\right)^{\frac{p}{q}} dt\right)^{\frac{1}{p}}$$

Since $\mathbb{E}\int_0^T \|\xi_\varepsilon(t)-\xi(t)\|^P\, dt$ tends to zero, it yields that $B_{13}\to 0$ as $\varepsilon\to 0$. For B_2, we can get also by similar calculations to previous ones, for example

$$B_2 \le \left(\mathbb{E}\int_0^T \left(\int\limits_{\mathbf{S}} \left|h_u\left(t,\overline{X}(t),\widetilde{u}(t)\right)-h_u\left(t,\overline{X}(t),\overline{u}(t)\right)\right|^{q'} d\mu\right)^{\frac{p'}{q'}} dt\right)^{\frac{1}{p'}}$$

$$\cdot\left(\mathbb{E}\int_0^T \left(\int\limits_{\mathbf{S}} \left|u(t)-\overline{u}(t)\right|^{q} d\mu\right)^{\frac{p}{q}} dt\right)^{\frac{1}{p}}$$

since $\mathcal{U}$ is bounded, then second part is bounded too and we can apply dominant convergence theorem, and it yields that $B_2\to 0$ as $\varepsilon\to 0$.

Now we show that

$$\mathsf{D}_\varepsilon = \mathbb{E}\int\limits_{\mathbf{S}} \frac{g\left(X_\varepsilon(T)\right)-g\left(\overline{X}(T)\right)}{\varepsilon} d\mu$$

tends to $\mathsf{D} = \mathbb{E}\int\limits_{\mathbf{S}} g_x\left(\overline{X}(T)\right) d\mu$. Note that since $g_x(\cdot)$ is bounded, then $g_x(T)\in L^{p'}(\Omega;E^*)$, where $\frac{1}{p}+\frac{1}{p'}=1$ and $E^*=L^{q'}(\mathbf{S},\Sigma,d\mu)$, $\frac{1}{q}+\frac{1}{q'}=1$.

$$|\mathsf{D}_\varepsilon-\mathsf{D}| = \left|\mathbb{E}\int\limits_{\mathbf{S}} \frac{g\left(X_\varepsilon(T)\right)-g\left(\overline{X}(T)\right)}{\varepsilon} d\mu - \mathbb{E}\int\limits_{\mathbf{S}} g_x\left(\overline{X}(T)\right) d\mu\right|$$

Now by adding and subtracting $g_x\left(\overline{X}(T)\right)\xi_\varepsilon(T)$, it yields

$$|\mathsf{D}_\varepsilon-\mathsf{D}| \le \mathbb{E}\int\limits_{\mathbf{S}} \left|g_x\left(\widetilde{X}_\varepsilon(T)\right)-g_x\left(\overline{X}(T)\right)\right| |\xi_\varepsilon(T)|\, d\mu$$

$$-\mathbb{E}\int\limits_{\mathbf{S}} \left|g_x\left(\overline{X}(T)\right)\right| |\xi_\varepsilon(T)-\xi(T)|\, d\mu$$

By using Hölder's inequality we have

$$
\begin{aligned}
&|\mathsf{D}_\varepsilon - \mathsf{D}| \leq \\
&\left(\mathbb{E}\left(\int_{\mathbf{S}} \left|g_x\left(\widetilde{X}_\varepsilon(T)\right) - g_x\left(\overline{X}(T)\right)\right|^{q'} d\mu\right)^{\frac{p'}{q'}}\right)^{\frac{1}{p'}} \left(\mathbb{E}\left(\int_{\mathbf{S}} |\xi_\varepsilon(T)|^q \, d\mu\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} \\
&+ \left(\mathbb{E}\left(\int_{\mathbf{S}} \left|g_x\left(\overline{X}(T)\right)\right|^{q'} d\mu\right)^{\frac{p'}{q'}}\right)^{\frac{1}{p'}} \left(\mathbb{E}\left(\int_{\mathbf{S}} |\xi_\varepsilon(T) - \xi(T)|^q \, d\mu\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}
\end{aligned}
$$

above calculations tend to zero as ε tends to zero, since $\xi_\varepsilon(\cdot) \to \xi(\cdot)$ in $L^p([0,T] \times \Omega; E)$ and $\xi_\varepsilon(T) \to \xi(T)$ in $L^p(\Omega; E)$(See Lemma 5.2). □

We know that $\xi(t)$ solves FSVIE in $E = L^q(\mathbf{S}, \Sigma, \mu)$ and it has the property $\xi(\cdot) \in L^p([0,T] \times \Omega; E)$. Since $h_x\left(t, \overline{X}(\cdot), \overline{u}(\cdot)\right)$ is bounded then for every $\overline{u}(\cdot) \in \mathcal{U}$, $h_x(\cdot, \cdot, \overline{u}(\cdot))$ is in $L^{p'}([0,T] \times \Omega; E^*)$ and with similar arguments it results that $g_x(\cdot), h_u(\cdot, \cdot, \overline{u}(\cdot)) \in L^{p'}([0,T] \times \Omega; E^*)$. Since $(\overline{u}(\cdot), \overline{X}(\cdot))$ is an optimal pair and Lemma 5.3 we can write

$$
\begin{aligned}
0 \leq & \frac{J(u_\varepsilon(\cdot)) - J(\overline{u}(\cdot))}{\varepsilon} \xrightarrow{\varepsilon \to 0} \mathbb{E}\left\langle \xi(T), g_x\left(\overline{X}(T)\right)\right\rangle \\
&+ \mathbb{E}\left\{\int_0^T \left(\left\langle \xi(t), h_x\left(t, \overline{X}(t), \overline{u}(t)\right)\right\rangle + \left\langle u(t) - \overline{u}(t), h_u\left(t, \overline{X}(t), \overline{u}(t)\right)\right\rangle\right) dt\right\}
\end{aligned}
$$

$\xi(t)$ satisfies following FSVIE

$$
\begin{aligned}
\xi(t) =& \overline{\varphi}(t) + \int_0^t b_x\left(t, s, \overline{X}(s), \overline{u}(s)\right) \xi(s) ds \\
&+ \int_0^t \rho_x\left(t, s, \overline{X}(s), \overline{u}(s)\right) \xi(s) dW^H(s), \qquad t \in [0,T]
\end{aligned}
$$

where

$$
\begin{aligned}
\overline{\varphi}(t) =& \int_0^t b_u\left(t, s, \overline{X}(s), \overline{u}(s)\right) (u(s) - \overline{u}(s)) \, ds \\
&+ \int_0^t \rho_u\left(t, s, \overline{X}(s), \overline{u}(s)\right) (u(s) - \overline{u}(s)) \, dW^H(s)
\end{aligned}
\tag{5.1}
$$

By boundedness assumptions we can define following bounded linear operators

$$
\begin{aligned}
B_u(t,s)u(s) =& b_u\left(t, s, \overline{X}(s), \overline{u}(s)\right) u(s) \\
B_x(t,s)\xi(s) =& b_x\left(t, s, \overline{X}(s), \overline{u}(s)\right) \xi(s) \\
A_u(t,s)u(s) =& \rho_u\left(t, s, \overline{X}(s), \overline{u}(s)\right) u(s) \\
A_x(t,s)\xi(s) =& \rho_x\left(t, s, \overline{X}(s), \overline{u}(s)\right) \xi(s)
\end{aligned}
$$

where for every $(t,s) \in [0,T] \times [0,T]$

$$\begin{aligned}
&B_u(t,s) \in \mathcal{L}(E;E)\\
&B_x(t,s) \in \mathcal{L}(E;E)\\
&A_u(t,s) \in \mathcal{L}(E;\mathcal{L}(H;E))\\
&A_x(t,s) \in \mathcal{L}(E;\mathcal{L}(H;E))
\end{aligned}$$

So we can define adjoint operators and similar to chapter 3 we let $A_x^*(\cdot,\cdot)\,h_n := (A_x(\cdot,\cdot)\,h_n)^*$ and $A_x^*(\cdot,\cdot)\,h_n := (A_u(\cdot,\cdot)\,h_n)^*$.

THEOREM 5.1. *If $(\overline{X}(\cdot),\overline{u}(\cdot))$ be an optimal solution and $\sum_{n\geq 1} \|A_x(t,s)\,h_n\|_{\mathcal{L}(E;E)}$, $\sum_{n\geq 1} \|A_u(t,s)\,h_n\|_{\mathcal{L}(E;E)}$ are bounded a.s for every $t,s \in [0,T]\times[0,T]$, then there exists a unique adapted M-solution $(Y(\cdot),Y_0(\cdot),\upsilon(\cdot);Z(\cdot,\cdot),Z_0(\cdot,\cdot),\zeta(\cdot))$ of the following BSVIEs:*

$$\begin{aligned}
Y(t) =& h_x\left(t,\overline{X}(t),\overline{u}(t)\right) + B_x^*(T,t)\,g_x\left(\overline{X}(T)\right) + \sum_{n\geq 1} A_x^*(T,t)\,h_n\zeta(t)h_n\\
&+\int_t^T \left\{B_x^*(s,t)\,Y(s) + \sum_{n\geq 1} A_x^*(s,t)\,h_n Z(s,t)h_n\right\} ds\\
&-\int_t^T Z(t,s)dW^H(s),\\
\upsilon(t) =& g_x\left(\overline{X}(T)\right) - \int_t^T \zeta(s)dW^H(s),\\
Y_0(t) =& B_u^*(T,t)\,g_x\left(\overline{X}(T)\right) + \sum_{n\geq 1} A_u^*(T,t)\,h_n\zeta(t)h_n\\
&+\int_t^T \left\{B_u^*(s,t)\,Y(s) + \sum_{n\geq 1} A_u^*(s,t)\,h_n Z(s,t)h_n\right\} ds\\
&-\int_t^T Z_0(t,s)dW^H(s), \qquad t\in[0,T]
\end{aligned}$$

where $Y(\cdot),Y_0(\cdot),\upsilon(\cdot) \in L^p_{\mathbb{F}}(\Omega\times[0,T];E)$ and $Z(\cdot,\cdot),Z_0(\cdot,\cdot),\zeta(\cdot) \in L^p_{\mathbb{F}}\left(\Omega\times[0,T];\gamma\left(L^2(0,T;H),E\right)\right)$ such that

$$\left\langle u(t)-\overline{u}(t), Y_0(t) + h_u\left(t,\overline{X}(t),\overline{u}(t)\right)\right\rangle \geq 0,\ \forall u(\cdot)\in\mathcal{U}, \forall t\in[0,T]\ a.s$$

PROOF. By boundedness of $\mathcal{U}$, it results that $\mathbb{E}\,\|\overline{\varphi}(t)\|^p$ is bounded with respect to $t\in[0,T]$ where $\overline{\varphi}(t)$ is defined by Equation (5.1). Since $g_x(\cdot)$ is bounded we have $\mathbb{E}\left\|g_x\left(\overline{X}(T)\right)\right\|_{E^*} < \infty$, and by assumptions we know that $g_x\left(\overline{X}(T)\right)$ is $\mathcal{F}_T$-measurable, therefore we use martnigale representation theorem in Banach spaces, Theorem A.6. ($E = L^q(\mathbf{S},\Sigma,\mu)$, $2\leq q\leq p$ then $E^* = L^{q'}(\mathbf{S},\Sigma,\mu)$, $1 < p' \leq q' \leq 2$ and E^* is cotype(2)). We can find unique adapted process $\zeta(\cdot)$ such that for every $r\in[0,T]$

$$\gamma(r) = \mathbb{E}\left(g_x\left(\overline{X}(T)\right)|\mathcal{F}_r\right) = \mathbb{E}\left(g_x\left(\overline{X}(T)\right)\right) + \int_0^r \zeta(s)dW^H(s)$$

and from above equation it results

$$\gamma(T) = g_x\left(\overline{X}(T)\right) = \mathbb{E}\left(g_x\left(\overline{X}(T)\right)\right) + \int_0^T \zeta(s)dW^H(s)$$

or

$$g_x\left(\overline{X}(T)\right) - \int_t^T \zeta(s)dW^H(s) = \mathbb{E}\left(g_x\left(\overline{X}(T)\right)\right) + \int_0^t \zeta(s)dW^H(s)$$

Let $\upsilon(t) =: \mathbb{E}\left(g_x\left(\overline{X}(T)\right)\right) + \int_0^t \zeta(s)dW^H(s)$ or

$$\upsilon(t) = g_x\left(\overline{X}(T)\right) - \int_t^T \zeta(s)dW^H(s)$$

So we can define BSVIE in $E^* = L^{q'}(\mathbf{S}, \Sigma, \mu)$ as follows

$$\begin{aligned} Y(t) =& h_x\left(t, \overline{X}(t), \overline{u}(t)\right) + B_x^*(T,t)\, g_x\left(\overline{X}(T)\right) + \sum_{n\geq 1} A_x^*(T,t)\, h_n \zeta(t) h_n \\ &+ \int_t^T \left\{ B_x^*(s,t)\, Y(s) + \sum_{n\geq 1} A_x^*(s,t)\, h_n Z(s,t) h_n \right\} ds \\ &- \int_t^T Z(t,s) dW^H(s), \qquad t \in [0,T] \end{aligned}$$

Let

$$\hat{\psi}(t) = h_x\left(t, \overline{X}(t), \overline{u}(t)\right) + B_x^*(T,t)\, g_x\left(\overline{X}(T)\right) + \sum_{n\geq 1} A_x^*(T,t)\, h_n \zeta(t) h_n, \qquad t \in [0,T]$$

By assumptions it holds $\mathbb{E}\left\|\hat{\psi}(t)\right\|_{E^*}^{p'} < \infty$, for each t in $[0,T]$ then BSVIE have the unique M-adapted solution

$$(Y(\cdot), Z(\cdot,\cdot)) \in L_{\mathbb{F}}^{p'}([0,T] \times \Omega; E^*) \times L_{\mathbb{F}}^{p'}\left([0,T]; L^{q'}\left(\mathbf{S}; L^2(0,T;H)\right)\right)$$

Now we can apply duality principle Theorem 4.2 and we have following relation

$$\begin{aligned} &\mathbb{E}\left\{ \left\langle \xi(T), g_x\left(\overline{X}(T)\right)\right\rangle + \int_0^T \left\langle \xi(t), h_x\left(t, \overline{X}(t), \overline{u}(t)\right)\right\rangle dt \right\} \\ &\quad = \mathbb{E}\left\{ \left\langle \overline{\varphi}(T), g_x\left(\overline{X}(T)\right)\right\rangle + \int_0^T \left\langle \overline{\varphi}(t), Y(t)\right\rangle dt \right\} \end{aligned}$$

Now we replace $\overline{\varphi}(T)$ and $\overline{\varphi}(t)$ from FSVIE, and we denote the summands in the last term by F_1 and F_2. It yields for F_1

$$F_1 = \mathbb{E}\left\langle \xi(T), g_x\left(\overline{X}(T)\right)\right\rangle =$$

$$
\begin{aligned}
&\mathbb{E}\Big\langle \int_0^T b_u\left(T,s,\overline{X}(s),\overline{u}(s)\right)(u(s)-\overline{u}(s))\,ds \\
&+\int_0^T \rho_u\left(T,s,\overline{X}(s),\overline{u}(s)\right)(u(s)-\overline{u}(s))\,dW^H(s), g_x\left(\overline{X}(T)\right)\Big\rangle \\
&=\mathbb{E}\Big\langle \int_0^T b_u\left(T,s,\overline{X}(s),\overline{u}(s)\right)(u(s)-\overline{u}(s))\,ds, g_x\left(\overline{X}(T)\right)\Big\rangle \\
&+\mathbb{E}\Big\langle \int_0^T \rho_u\left(T,s,\overline{X}(s),\overline{u}(s)\right)(u(s)-\overline{u}(s))\,dW^H(s), \\
&\qquad\qquad \mathbb{E}\left(g_x\left(\overline{X}(T)\right)\right)+\int_0^T \zeta(s)dW^H(s)\Big\rangle
\end{aligned}
$$

by using adjoint operators and Itô formula in Banach spaces, Theorem A.8, it results

$$
\begin{aligned}
F_1 &= \mathbb{E}\int_0^T \left\langle u(s)-\overline{u}(s), B_u^*(T,s)g_x\left(\overline{X}(T)\right)\right\rangle ds \\
&+\mathbb{E}\int_0^T \sum_{n\geq 1}\left\langle \rho_u\left(T,s,\overline{X}(s),\overline{u}(s)\right)(u(s)-\overline{u}(s))\,h_n, \zeta(s)h_n\right\rangle ds \\
&= \mathbb{E}\int_0^T \left\langle u(s)-\overline{u}(s), B_u^*(T,s)g_x\left(\overline{X}(T)\right)\right\rangle ds \\
&+\mathbb{E}\int_0^T \sum_{n\geq 1}\left\langle A_u(T,s)h_n\left(u(s)-\overline{u}(s)\right), \zeta(s)h_n\right\rangle ds \\
&= \mathbb{E}\int_0^T \left\langle u(s)-\overline{u}(s), B_u^*(T,s)g_x\left(\overline{X}(T)\right)\right\rangle ds \\
&+\mathbb{E}\int_0^T \left\langle u(s)-\overline{u}(s), \sum_{n\geq 1} A_u^*(T,s)h_n\zeta(s)h_n\right\rangle ds
\end{aligned}
$$

and finally it yields

$$
F_1 = \mathbb{E}\int_0^T \left\langle u(t)-\overline{u}(t), B_u^*(T,t)g_x\left(\overline{X}(T)\right)+\sum_{n\geq 1} A_u^*(T,t)h_n\zeta(t)h_n\right\rangle dt
$$

Now consider the term F_2. For the following calculations, we used the adjoint operators properties, Martingale representation theorem for $Y(t)$ Theorem A.6, Itô's formula in Banach spaces Theorem A.8 and the properties of stochastic Integral.

$$
\begin{aligned}
F_2 =& \mathbb{E}\int_0^T \langle \overline{\varphi}(t), Y(t)\rangle\, dt = \\
&\mathbb{E}\int_0^T \Big\langle \int_0^t b_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)(u(s)-\overline{u}(s))\,ds \\
&\qquad +\int_0^t \rho_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)(u(s)-\overline{u}(s))\,dW^H(s), Y(t)\Big\rangle\, dt
\end{aligned}
$$

$$
\begin{aligned}
=&\mathbb{E}\int_0^T \left\langle \int_0^t b_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)\left(u(s)-\overline{u}(s)\right)ds, Y(t)\right\rangle dt \\
&+\mathbb{E}\int_0^T \left\langle \int_0^t \rho_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)\left(u(s)-\overline{u}(s)\right)dW^H(s), Y(t)\right\rangle dt \\
=&\mathbb{E}\int_0^T \left\langle \int_0^t b_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)\left(u(s)-\overline{u}(s)\right)ds, Y(t)\right\rangle dt \\
&+\mathbb{E}\int_0^T \left\langle \int_0^t \rho_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)\left(u(s)-\overline{u}(s)\right)dW^H(s),\right. \\
&\qquad\qquad\qquad\qquad \left. \mathbb{E}(Y(t)+\int_0^t Z(t,s)dW^H(s)\right\rangle dt \\
=&\mathbb{E}\int_0^T\int_0^t \left\langle u(s)-\overline{u}(s), B_u^*(t,s)Y(t)\right\rangle dsdt \\
+&\mathbb{E}\int_0^T\int_0^t \left\langle \sum_{n\geq 1}\rho_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)h_n\left(u(s)-\overline{u}(s)\right), Z(t,s)h_n\right\rangle dsdt
\end{aligned}
$$

Moreover we can write

$$
\begin{aligned}
F_2 =&\mathbb{E}\int_0^T\int_s^T \left\langle u(s)-\overline{u}(s), B_u^*(t,s)Y(t)\right\rangle dtds \\
&+\int_0^T\int_s^T \left\langle u(s)-\overline{u}(s), \sum_{n\geq 1} A_u^*(t,s)h_n Z(t,s)h_n\right\rangle dtds \\
=&\mathbb{E}\int_0^T\int_t^T \left\langle u(t)-\overline{u}(t), B_u^*(s,t)Y(s)\right\rangle dsdt \\
&+\int_0^T\int_t^T \left\langle u(t)-\overline{u}(t), \sum_{n\geq 1} A_u^*(s,t)h_n Z(s,t)h_n\right\rangle dsdt \\
=&\ \mathbb{E}\int_0^T \left\langle u(t)-\overline{u}(t), \int_t^T \left\{B_u^*(s,t)Y(s)+\sum_{n\geq 1} A_u^*(s,t)h_n Z(s,t)h_n\right\} ds\right\rangle dt
\end{aligned}
$$

By combining above two results F_1 and F_2, we have

$$
\begin{aligned}
&\mathbb{E}\left\{\left\langle \overline{\varphi}(T), g_x\left(\overline{X}(T)\right)\right\rangle + \int_0^T \left\langle \overline{\varphi}(t), Y(t)\right\rangle dt\right\} \\
&= \mathbb{E}\int_0^T \left\langle u(t)-\overline{u}(t), B_u^*(T,t)g_x\left(\overline{X}(T)\right)+\sum_{n\geq 1} A_u^*(T,t)h_n\zeta(t)h_n\right. \\
&\qquad\qquad \left. +\int_t^T \left\{B_u^*(s,t)Y(s)+\sum_{n\geq 1} A_u^*(s,t)h_n Z(s,t)h_n\right\} ds\right\rangle dt
\end{aligned}
$$

Here we define another simple BSVIE in E^* as follows

$$
\begin{aligned}
Y_0(t) =& B_u^*(T,t)\, g_x\left(\overline{X}(T)\right) + \sum_{n\geq 1} A_u^*(T,t)\, h_n \zeta(t) h_n \\
& + \int_t^T \left\{ B_u^*(s,t)\, Y(s) + \sum_{n\geq 1} A_u^*(s,t)\, h_n Z(s,t) h_n \right\} ds \\
& - \int_t^T Z_0(t,s) dW^H(s), \qquad t \in [0,T]
\end{aligned}
$$

It is well defined and there exists unique M-solution

$$
(Y_0(\cdot), Z_0(\cdot,\cdot)) \in L_{\mathbb{F}}^{p'}([0,T]\times\Omega; E^*) \times L_{\mathbb{F}}^{p'}\left([0,T]; L^{q'}\left(\mathbf{S}; L^2(0,T;H)\right)\right)
$$

Finally consider again following relation by minimality of cost function with respect to control process $\overline{u}(\cdot)$

$$
\begin{aligned}
0 \leq & \frac{J(u_\varepsilon(\cdot)) - J(\overline{u}(\cdot))}{\varepsilon} \xrightarrow{\varepsilon\to 0} \mathbb{E}\Big\{ \left\langle \xi(T), g_x\left(\overline{X}(T)\right)\right\rangle \\
& + \int_0^T \left(\left\langle \xi(t), h_x\left(t, \overline{X}(t), \overline{u}(t)\right)\right\rangle + \left\langle u(t)-\overline{u}(t), h_u\left(t,\overline{X}(t),\overline{u}(t)\right)\right\rangle \right) dt \Big\} \\
=& \mathbb{E}\left\{ \int_0^T \left\langle u(t)-\overline{u}(t), Y_0(t) + \int_t^T Z_0(t,s) dW^H(s) + h_u\left(t,\overline{X}(t),\overline{u}(t)\right) \right\rangle dt \right\} \\
=& \mathbb{E}\left\{ \int_0^T \left\langle u(t)-\overline{u}(t), Y_0(t) + h_u\left(t,\overline{X}(t),\overline{u}(t)\right)\right\rangle dt \right\}, \ \forall u(\cdot) \in \mathcal{U}
\end{aligned}
$$

notice that since $u(t)-\overline{u}(t)$ is $\mathcal{F}_t$-measurable, we used following equations

$$
\begin{aligned}
& \mathbb{E}\left\{ \int_0^T \left\langle u(t)-\overline{u}(t), \int_t^T Z_0(t,s) dW^H(s) \right\rangle dt \right\} \\
&= \int_0^T \mathbb{E}\left\{ \mathbb{E}\left(\left\langle u(t)-\overline{u}(t), \int_t^T Z_0(t,s) dW^H(s) \right\rangle \middle| \mathcal{F}_t \right) \right\} dt \\
&= \int_0^T \mathbb{E}\left\{ \left\langle u(t)-\overline{u}(t), \mathbb{E}\left(\int_t^T Z_0(t,s) dW^H(s) \middle| \mathcal{F}_t \right) \right\rangle \right\} dt \\
&= \int_0^T \mathbb{E}\left\{ \left\langle u(t)-\overline{u}(t), \mathbb{E}\left(\int_t^T Z_0(t,s) dW^H(s) \right) \right\rangle \right\} dt \\
&= \int_0^T \mathbb{E}\left\{ \left\langle u(t)-\overline{u}(t), 0\right\rangle \right\} dt = 0.
\end{aligned}
$$

Therefore it results

$$
\left\langle u(t)-\overline{u}(t), Y_0(t) + h_u\left(t,\overline{X}(t),\overline{u}(t)\right)\right\rangle \geq 0, \ \forall u(\cdot) \in \mathcal{U}, t \in [0,T] \text{ a.s}
$$

It is easy to see that above result holds. Let $d\nu = dt \cdot dP$ if there is some $u(\cdot)$ such that above equations is not hold for some $\mathcal{A} \subset \Omega \times [0,T]$ with $\nu(\mathcal{A}) > 0$. Now if we set

$$
u^\sharp = \begin{cases} u(t) & (t,\omega) \in \mathcal{A} \\ 0 & (t,\omega) \notin \mathcal{A} \end{cases}
$$

then

$$\mathbb{E}\int_0^T \left\langle u^\sharp(t) - \overline{u}(t), Y_0(t) + h_u\left(t, \overline{X}(t), \overline{u}(t)\right)\right\rangle dt < 0,$$

and this contradicts the optimality of $\overline{u}(\cdot)$. □

5.2. Generalization of the Set Admissible Solutions

In this section we generalize our stochastic control problem. First we consider the following FSVIE in Banach space $E = L^q(\mathbf{S}, \Sigma, \mu)$, where here μ is finite measure, $q \geq 2$

$$\begin{aligned} X(t) = \varphi(t) + &\int_0^t \{b_1(t, s, X(s)) + b_2(t, s)u(s)\}\, ds \\ + &\int_0^t \rho(t, s, X(s), u(s))\, dW^H(s), \quad t \in [0, T] \end{aligned} \tag{5.2}$$

At this case our control process is defined in another separable Banach space F and we let $b_2(t, s)$ is a linear bounded operator from F to E. In other words we consider for each $t \in [0, T]$, $u(t)$ is F-valued strongly $\mathcal{F}_t$-adapted process, moreover we assume that there exist a constant c_F such that $\|u(t)\|_F \leq c_F$ for each $t \in [0, T]$ a.s.. Now we define

$$\mathcal{U} = \{u : [0, T] \times \Omega \longrightarrow F \mid \|u(t)\|_F \leq c_F \text{ and } u(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable}\}$$

we see that if $u_1, u_2 \in F$ then for every $\varepsilon > 0$, $u_1\varepsilon + (1 - \varepsilon)u_2 \in F$, since $\|\varepsilon u_1 + (1 - \varepsilon)u_2\| \leq \varepsilon\|u_1\|_F + (1 - \varepsilon)\|u_2\|_F \leq \varepsilon c_F + (1 - \varepsilon)c_F = c_F$, and it results that $\mathcal{U}$ is convex. Analogous to previous section we assume similar assumptions for well-definition of Equation(5.2), for example, $\varphi(\cdot) \in L^p_{\mathbb{F}}([0, T] \times \Omega; E)$ where $q \leq p$ and b_1, b_2, ρ are defined measurably as

$$\begin{aligned} &b_1 : [0, T] \times [0, T] \times E \times \Omega \times \mathbf{S} \longrightarrow E \\ &b_2 : [0, T] \times [0, T] \times \Omega \longrightarrow \mathcal{L}(F; E) \\ &\rho\ : [0, T] \times [0, T] \times E \times F \times \Omega \times \mathbf{S} \longrightarrow \mathcal{L}(H; E) \end{aligned}$$

We consider the following cost function

$$J(u(\cdot)) = \mathbb{E}\int_0^T \int_{\mathbf{S}} h(t, X(t), u(t)) d\mu dt \tag{5.3}$$

where

$$h : [0, T] \times E \times F \times \Omega \times \mathbf{S} \longrightarrow E$$

We use the concept of Nemystkii operator for b_1 and h and we let they have first continuous bounded derivative with respect to x, i.e. we assume $\left|\frac{\partial b_1(t,s,x)}{\partial x}\right| := |b_{1,x}(t, s, x)| \leq K'_x$, $\left|\frac{\partial h(t,x,u)}{\partial u}\right| := |h_x(t, s, x, u)| \leq c_{h,x}$ for all $t, s \in [0, T]$, $x \in \mathbb{R}$, $\eta \in \mathbf{S}$, $u \in F$ almost sure, where K'_x, $c_{u,x}$ are positive constants. Similar for ρ we let that $\rho_x(t, s, x, u) : E \longrightarrow \mathcal{L}(H; E)$ and $\rho_u(t, s, x, u) : F \longrightarrow \mathcal{L}(H; E)$ are first continuous Fréchet derivative with respect to x and u such that for each $h \in \{h_n\}_{n\geq 1}$; $\|\rho_x(t, s, x, u)h\|_{\mathcal{L}(E;E)}$ and $\|\rho_u(t, s, x, u)h\|_{\mathcal{L}(F;E)}$ are bounded almost sure, and moreover we assume that $h_u(t, x, u)$ Fréchet derivative of h with respect to u exist and is a.s. bounded.

As we have seen $\mathcal{U}$ is convex and for each $\varepsilon \in (0,1)$, $u_\varepsilon(\cdot) \in \mathcal{U}$ where $u_\varepsilon(\cdot) := \overline{u}(\cdot) + \varepsilon(u(\cdot) - \overline{u}(\cdot))$ and let $(\overline{X}(\cdot), \bar{u}(\cdot))$ are optimal state and control processes. Let $\overline{X}_\varepsilon(\cdot)$ be the solution of FSVIE Equation(1) when $\overline{u}_\varepsilon(\cdot)$ is chosen and we define $\xi_\varepsilon(t) := \frac{X_\varepsilon(t) - \overline{X}(t)}{\varepsilon}$ for each $t \in [0,T]$.
With similar calculations and techniques that we used to prove lemmas and theorem in previous section we can easily prove following lemmas and theorems. For not repeating the same procedure we overlook bringing similar straightforward proofs, only here it must be mentioned that finiteness of μ is crucial and we used following equations for proving:

$$\|b_2(t,s)(u_\varepsilon(s) - \overline{u}(s))\|_E \leq \|b_2(t,s)\|_{\mathcal{L}(F;E)} \|u_\varepsilon(s) - \overline{u}(s)\|_F$$

$$\begin{aligned}
&\left\|\varrho\left(t,s,X_\varepsilon(s),u_\varepsilon(s)\right) - \varrho\left(t,s,\overline{X}(s),\overline{u}(s)\right)\right\|_H \\
&= \left\|\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)\left(X_\varepsilon(s) - \overline{X}(s)\right) + \varrho_u\left(t,s,\overline{X}(s),\widetilde{u}(s)\right)\left(u_\varepsilon(s) - \overline{u}(s)\right)\right\|_H \\
&\leq \left\|\varrho_x\left(t,s,\widetilde{X}_\varepsilon(s),u_\varepsilon(s)\right)\right\|_{\mathcal{L}(R;H)} \left|\left(X_\varepsilon(s) - \overline{X}(s)\right)\right| \\
&\qquad + \left\|\varrho_u\left(t,s,\overline{X}(s),\widetilde{u}(s)\right)\right\|_{\mathcal{L}(F;H)} \|u_\varepsilon(s) - \overline{u}(s)\|_F
\end{aligned}$$

Lemma 5.4. *$\xi_\varepsilon(t)$ is uniformly bounded in $L^p(\Omega;E)$ with respect to t and ε and correspondingly $\xi_\varepsilon(\cdot)$ is uniformly bounded in $L^p(\Omega \times [0,T];E)$ with respect to ε for $2 \leq q \leq p$.*

Lemma 5.5. *If ε tends to zero then $\xi_\varepsilon(\cdot)$ tends to $\xi(\cdot)$ in $L^p([0,T] \times \Omega;E)$ and $\xi_\varepsilon(t)$ tends to $\xi(t)$ in $L^p(\Omega;E)$ for all $t \in [0,T]$, where $\xi(t)$ satisfies the following FSVIE*

$$\begin{aligned}
\xi(t) = &\int_0^t \left\{b_{1,x}\left(t,s,\overline{X}(s)\right)\xi(s) + b_2\left(t,s\right)\left(u(s) - \overline{u}(s)\right)\right\} ds \\
&+ \int_0^t \left\{\rho_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\xi(s) + \rho_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)\left(u(s) - \overline{u}(s)\right)\right\} dW^H(s), \\
&\qquad 0 \leq t \leq T.
\end{aligned}$$

By defining

$$\overline{\varphi}(t) = \int_0^t b_2\left(t,s\right)\left(u(s) - \overline{u}(s)\right) ds + \int_0^t \rho_u\left(t,s,\overline{X}(s),\overline{u}(s)\right)\left(u(s) - \overline{u}(s)\right) dW^H(s)$$

we can rearrange the FSVIE for $\xi(t)$ as following

$$\xi(t) = \overline{\varphi}(t) + \int_0^t b_{1,x}\left(t,s,\overline{X}(s)\right)\xi(s)ds + \int_0^t \rho_x\left(t,s,\overline{X}(s),\overline{u}(s)\right)\xi(s)dW^H(s), \qquad 0 \leq t \leq T.$$

Lemma 5.6. *If ε tends to zero, then*

$$\lim_{\varepsilon \to 0} \left| \frac{J\left(u_\varepsilon(\cdot)\right) - J\left(\overline{u}(\cdot)\right)}{\varepsilon} - \mathbb{E}\int_0^T \int_{\mathbf{S}} \left\{h_x\left(t,\overline{X}(t),\overline{u}(t)\right)\xi(t) + h_u\left(t,\overline{X}(t),\overline{u}(t)\right)\left(u(t) - \overline{u}(t)\right)\right\} d\mu dt \right| = 0$$

Now we define following BSVIE in $E^* = L^{q'}(\mathbf{S}, \Sigma, \mu)$

$$
\begin{aligned}
Y(t) = & - h_x\left(t, \overline{X}(t), \overline{u}(t)\right) \\
& + \int_t^T \left\{ B_x^*\left(s,t\right) Y(s) + \sum_{n\geq 1} A_x^*\left(s,t\right) h_n Z(s,t) h_n \right\} ds \\
& - \int_t^T Z(t,s) dW^H(s),
\end{aligned}
$$

where $B_x^*\left(s,t\right)$ and $A_x^*\left(s,t\right) h := \left(A_x\left(s,t\right) h\right)^*$ are adjoint operators of following defined operators

$$
\begin{aligned}
B_x(t,s)\xi(s) = & b_{1,x}\left(t, s, \overline{X}(s)\right) \xi(s) \\
A_x(t,s)\xi(s) = & \rho_x\left(t, s, \overline{X}(s), \overline{u}(s)\right) \xi(s)
\end{aligned}
$$

moreover we let

$$
A_u(t,s)u(s) = \rho_u\left(t, s, \overline{X}(s), \overline{u}(s)\right) u(s)
$$

and $A_u^*\left(s,t\right) h := \left(A_u\left(s,t\right) h\right)^*$and for every $(t,s) \in [0,T] \times [0,T]$

$$
\begin{aligned}
& B_x(t,s) \in \mathcal{L}\left(E; E\right) \\
& A_x(t,s) \in \mathcal{L}\left(E; \mathcal{L}\left(H; E\right)\right) \\
& A_u(t,s) \in \mathcal{L}\left(F; \mathcal{L}\left(H; E\right)\right)
\end{aligned}
$$

By using the duality Theorem 4.1 we can write

$$
\mathbb{E} \int_0^T \left\langle \xi(t), -h_x\left(t, \overline{X}(t), \overline{u}(t)\right) \right\rangle dt = \mathbb{E} \int_0^T \left\langle \overline{\varphi}(t), Y(t) \right\rangle dt
$$

by substituting $\overline{\varphi}(t)$ and with similar calculations in the proof of Theorem 5.1. we can have the following result

$$
\begin{aligned}
& \mathbb{E} \int_0^T \left\langle \xi(t), h_x\left(t, \overline{X}(t), \overline{u}(t)\right) \right\rangle dt \\
& = -\mathbb{E} \int_0^T \int_t^T \left\langle b_2(s,t)\left(u(t) - \overline{u}(t)\right), Y(s) \right\rangle ds dt \\
& \qquad\qquad + \int_0^T \int_t^T \sum_{n\geq 1} \left\langle A_u(s,t) h_n \left(u(t) - \overline{u}(t)\right), Z(s,t) h_n \right\rangle ds dt \\
& = -\mathbb{E} \int_0^T \Bigg\langle u(t) - \overline{u}(t), \\
& \qquad\qquad \int_t^T \left\{ b_2^*(s,t) Y(s) + \sum_{n\geq 1} A_u^*(s,t) h_n Z(s,t) h_n \right\} ds \Bigg\rangle dt
\end{aligned}
$$

and finally we can write $0 \leq \frac{J(u_\varepsilon(\cdot))-J(u(\cdot))}{\varepsilon}$ tends to

$$-\mathbb{E}\int_0^T \Bigg\langle u(t)-\overline{u}(t),$$
$$\int_t^T \left\{ b_2^*(s,t)Y(s)+\sum_{n\geq 1} A_u^*(s,t)h_n Z(s,t)h_n \right\} ds - h_u^*(t,\overline{X}(t),\overline{u}(t)) \Bigg\rangle dt$$

and we can have following theorem

THEOREM 5.2. *If $(\overline{X}(\cdot),\overline{u}(\cdot))$ be an optimal solution which state process is given by Equation(5.2) and $\sum_{n\geq 1}\|A_x(t,s)h_n\|_{\mathcal{L}(E;E)}$, $\sum_{n\geq 1}\|A_u(t,s)h_n\|_{\mathcal{L}(E;E)}$ are bounded a.s for every $t,s\in[0,T]\times[0,T]$, then there exists a unique adapted M-solution $(Y(\cdot),Z(\cdot,\cdot))$ of the following BSVIE:*

$$\begin{aligned} Y(t) =& -h_x\left(t,\overline{X}(t),\overline{u}(t)\right) \\ &+\int_t^T \left\{ B_x^*(s,t)\,Y(s)+\sum_{n\geq 1} A_x^*(s,t)\,h_n Z(s,t)h_n \right\} ds \\ &-\int_t^T Z(t,s)dW^H(s), \end{aligned}$$

where $Y(\cdot)\in L^p_{\mathbb{F}}(\Omega\times[0,T];E)$ and $Z(\cdot,\cdot)\in L^p_{\mathbb{F}}\left(\Omega\times[0,T];\gamma\left(L^2(0,T;H),E\right)\right)$ such that

$$\Bigg\langle u(t)-\overline{u}(t)\ ,$$
$$\int_t^T \left\{ b_2^*(s,t)Y(s)+\sum_{n\geq 1} A_u^*(s,t)h_n Z(s,t)h_n \right\} ds - h_u^*(t,\overline{X}(t),\overline{u}(t)) \Bigg\rangle \leq 0,$$
$$\forall u(\cdot)\in\mathcal{U}, \forall t\in[0,T]\ a.s$$

REMARK 5.1. *We used in this section Bolza cost function without terminal cost Equation (5.3). If we assume the general Bolza cost function, then the process $(Y_0(\cdot))_{t\in[0,T]}$ has values in F^* and we need assumptions for F^*, so that the BSVIE is defined. If F is a separable Hilbert space, the equations could be well defined. It is an open question to define BSVIE if F is not the type of E. Especially a corresponding martingale representation theorem will be needed.*

REMARK 5.2. *This procedure can be written also for general following FSVIE*

$$X(t)=\varphi(t)+\int_0^t b\left(t,s,X(s),u(s)\right)ds+\int_0^t \rho\left(t,s,X(s),u(s)\right)dW^H(s), \qquad t\in[0,T] \tag{5.4}$$

where at this time

$$\begin{aligned} b &: [0,T]\times[0,T]\times E\times F\times\Omega\times\mathbf{S}\longrightarrow E \\ \rho &: [0,T]\times[0,T]\times E\times F\times\Omega\times\mathbf{S}\longrightarrow \mathcal{L}(H;E) \end{aligned}$$

and we let the same assumptions for ρ in this section hold and moreover we assume $b(t,s,x(s),u(s))$ has first continuous Fréchet derivative $b_u(t,s,x(s),u(s))$ with respect to u and first continuous bounded derivative $b_x(t,s,x,u)$ with respect to x as Nemytskii operator, and every results in this section hold only in lemmas and theorem we have to replace $b_{1,x}(t,s,x(s))$ with $b_x(t,s,x(s),u(s))$ and $b_2(t,s)$ with $b_u(t,s,x(s),u(s))$.

In according to above remark we can use similar notations in Theorem 5.1 and the following theorem can be given

THEOREM 5.3. *If $(\overline{X}(\cdot),\overline{u}(\cdot))$ be an optimal solution which state process is given by Equation(5.4) with the cost function (5.3) and $\sum_{n\geq 1}\|A_x(t,s)\,h_n\|_{\mathcal{L}(E;E)}$, $\sum_{n\geq 1}\|A_u(t,s)\,h_n\|_{\mathcal{L}(E;E)}$ are bounded a.s for every $t,s\in[0,T]\times[0,T]$, then there exists a unique adapted M-solution $(Y(\cdot),Z(\cdot,\cdot))$ of the following BSVIE:*

$$\begin{aligned}Y(t) = &- h_x\left(t,\overline{X}(t),\overline{u}(t)\right)\\ &+\int_t^T\left\{B_x^*(s,t)\,Y(s)+\sum_{n\geq 1}A_x^*(s,t)\,h_nZ(s,t)h_n\right\}ds\\ &-\int_t^T Z(t,s)dW^H(s),\end{aligned}$$

where $Y(\cdot)\in L^p_{\mathbb{F}}(\Omega\times[0,T];E)$ and $Z(\cdot,\cdot)\in L^p_{\mathbb{F}}\left(\Omega\times[0,T];\gamma\left(L^2(0,T;H),E\right)\right)$ such that

$$\Bigg\langle u(t)-\overline{u}(t)\,,$$

$$\int_t^T\left\{B_u^*(s,t)Y(s)+\sum_{n\geq 1}A_u^*(s,t)h_nZ(s,t)h_n\right\}ds-h_u^*(t,\overline{X}(t),\overline{u}(t))\Bigg\rangle\leq 0,$$

$$\forall u(\cdot)\in\mathcal{U},\forall t\in[0,T]\ a.s$$

REMARK 5.3. *We can apply conditional expectation given σ-algebra $\mathcal{F}_t$ in Theorem 5.3. Since the solutions, control process and operators are $\mathcal{F}_t$ adapted we can have following version of Theorem 5.3 too.*

THEOREM 5.4. *If $(\overline{X}(\cdot),\overline{u}(\cdot))$ be an optimal solution which state process is given by Equation(5.4) with the cost function (5.3) and $\sum_{n\geq 1}\|A_x(t,s)\,h_n\|_{\mathcal{L}(E;E)}$, $\sum_{n\geq 1}\|A_u(t,s)\,h_n\|_{\mathcal{L}(E;E)}$ are bounded a.s for every $t,s\in[0,T]\times[0,T]$, then there exists a unique adapted M-solution $(Y(\cdot),Z(\cdot,\cdot))$ of the following BSVIE:*

$$\begin{aligned}Y(t) = &- h_x\left(t,\overline{X}(t),\overline{u}(t)\right)\\ &+\int_t^T\left\{B_x^*(s,t)\,Y(s)+\sum_{n\geq 1}A_x^*(s,t)\,h_nZ(s,t)h_n\right\}ds\\ &-\int_t^T Z(t,s)dW^H(s),\end{aligned}$$

where $Y(\cdot) \in L^p_{\mathbb{F}}(\Omega \times [0,T]; E)$ *and* $Z(\cdot,\cdot) \in L^p_{\mathbb{F}}\left(\Omega \times [0,T]; \gamma\left(L^2(0,T;H), E\right)\right)$ *such that*

$$\Bigg\langle u(t) - \overline{u}(t)\ ,$$

$$\mathbb{E}\left(\int_t^T \left\{ B^*_u(s,t)Y(s) + \sum_{n\geq 1} A^*_u(s,t)h_n Z(s,t)h_n \right\} ds \middle| \mathcal{F}_t\right) - h^*_u(t, \overline{X}(t), \overline{u}(t))\Bigg\rangle$$
$$\leq 0,\ \forall u(\cdot) \in \mathcal{U}, \forall t \in [0,T]\ a.s$$

REMARK 5.4. *If the control process be a real valued process, then in Theorem 5.3 we use the concept of Nemytskii operator for* h *and* u*, it means* $h_u(t,x,u)$ *must be bounded for every* $t \in [0,t]$*,* $x, u \in \mathbb{R}$ *a.s.*

5.3. Examples

In this section we consider some examples for stochastic optimal control problem by using maximum principle method. We use some forms of stochastic heat equation with homogeneous Dirichlet boundary conditions in Banach space $E = L^q(0,1)$, $q \geq 2$. For example if we assume

5.3.1. Heat Equation with Additive Noise.

In first example we consider a control stochastic heat equation for (2.6)

$$\begin{cases} dX(t,\xi) = \Delta X(t,\xi)dt + u(t,\xi)dt + \phi dW^H(t), & t \in [0,T], \xi \in (0,1) \\ X(0,\xi) = X_0(\xi), & \xi \in (0,1), X_0(\cdot) \in L^p_{\mathcal{F}_0}(\Omega; E) \\ X(t,0) = X(t,1) = 0, & t \in [0,T] \end{cases} \tag{5.5}$$

where Δ is Laplacian and $H = L^2(0,1)$. Let $u(t,\xi)$ is a real valued with respect to t adapted control process and takes values in closed bounded interval of $\mathbb{R}$. We also assume that $\phi \in \gamma(L^2(0,1); L^q(0,1))$ is γ-Radonifying operator.
By these assumptions we can write above given heat equation as following stochastic evolution equation in Theorem 1.1 [24]

$$\begin{cases} dX(t) = AX(t)dt + F(t,X(t))dt + B(t,X(t))dW^H(t), & t \in [0,T], \\ X(0) = X_0 \end{cases}$$

where A is Dirichlet Laplacian on E and it is generated analytic C_0-semigroup $S(t)$, and $F(t,X(t)) := u(t)$, $B(t,X(t)) := \phi$ are defined as

$$\begin{cases} F : [0,T] \times E \longrightarrow E \\ B : [0,T] \times E \longrightarrow \gamma\left(H; E\right) \end{cases}$$

now by these definition it is easily seen that F and B are Lipschitz continuous and L^2_γ-Lipschitz continuous receptively and of linear growth in the second variable uniformly on $[0,T]$, by using

Theorem 1.1 [24] we can find an unique mild solution which satisfies following equation

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(s,X(s))ds + \int_0^t S(t-s)B(s,X(s))dW^H(s), \quad t \in [0,T]$$

We can reformulate Equation(5.5) as following FSVIE

$$X(t) = S(t)X_0 + \int_0^t b\left(t,s,X(s),u(s)\right)ds + \int_0^t \rho\left(t,s,X(s),u(s)\right)dW^H(s), \quad t \in [0,T]$$

where $b\left(t,s,X(s),u(s)\right) := S(t-s)u(s)$ and $\rho\left(t,s,X(s),u(s)\right) := S(t-s)\phi$. Now we consider the following cost function

$$J(u(\cdot)) = \mathbb{E}\int_0^T \int_0^1 h(t,X(t),u(t))d\mu dt$$

where $h : [0,T]\times E\times U\times\Omega\times[0,1] \longrightarrow E$ has first bounded derivative with respect to x and u as Nemytskii operator. By above assumptions it results that $\left|\frac{\partial b(t,s,x,u)}{\partial u}\right| = \left|\frac{dS(t-s)u}{du}\right| = \|S(t-s)\| \leq c$ for every $t,s \in [0,T]$ and $B_u(t,s)$ in Theorem 5.1 is equal to $S(t-s)$, $\left|\frac{\partial b(t,s,x,u)}{\partial x}\right| = 0$ and Fréchet derivative of $\rho\left(t,s,X(s),u(s)\right)$ with respect to x and u exist and is zero. Now we can apply Theorem 5.1 or 5.3 and it yields if $(\overline{X}(\cdot),\overline{u}(\cdot))$ be optimal stochastic solution then there exists an unique adapted M-solution $(Y(\cdot),Z(\cdot,\cdot))$ of following BSVIE in E^*

$$Y(t) = -h_x(t,\overline{X}(t),\overline{u}(t)) - \int_0^T Z(t,s)dW^H(s), \; t \in [0,T]$$

such that

$$\left\langle u(t)-\overline{u}(t), \int_t^T S(t-s)^*Y(s)ds - h_u(t,\overline{X}(t),\overline{u}(t))\right\rangle \leq 0, \quad \forall u(\cdot)\in\mathcal{U}, \forall t\in[0,T] \text{ a.s}$$

or correspondingly

$$\int_0^1 \left\{\int_t^T S(t-s)^*Y(s)ds - h_u(t,\overline{X}(t),\overline{u}(t))\right\}\left(u(t)-\overline{u}(t)\right)d\mu \leq 0, \quad \forall u(\cdot)\in\mathcal{U}, \forall t\in[0,T] \text{ a.s}$$

or by using Theorem 5.4

$$\left\{\mathbb{E}\left(\int_0^1\int_t^T S(t-s)^*Y(s)ds\middle|\mathcal{F}_t\right) - h_u(t,\overline{X}(t),\overline{u}(t))\right\}\left(u(t)-\overline{u}(t)\right)d\mu \leq 0, \quad \forall u(\cdot)\in\mathcal{U}, \forall t\in[0,T] \text{ a.s}$$

If $u(\cdot)\in[-1,1]$ and h does not depend on u, then $\overline{u}(t,\xi) = \mathrm{sign}\left\{\mathbb{E}\int_{\mathrm{t}}^{\mathrm{T}} \mathrm{S(t-s)^*Y(s,\xi)ds}\middle|\mathcal{F}_{\mathrm{t}}\right\}$ for all $t\in[0,T]$ and $\xi\in[0,1]$.

5.3.2. Heat Equation with Multiplicative Noise. Let the following stochastic heat equation from Chapter 2.3.2 controlled by $u(\cdot,\cdot)$

$$\begin{cases} dX(t,\xi) = \Delta X(t,\xi)dt + u(t,\xi)dt + \psi X(t,\xi)dW^H(t), & t\in[0,T], \xi\in(0,1) \\ X(0,\xi) = X_0(\xi), & \xi\in(0,1), X_0(\cdot)\in L^p(E) \\ X(t,0) = X(t,1) = 0, & t\in[0,T] \end{cases}$$

where Δ is Laplacian, $u(t)$ are chosen as in 5.3.1 and $\psi \in \mathcal{L}(E;\gamma(H;E))$. Similar privious section it can be reformulated as

$$\begin{cases} dX(t) = AX(t)dt + F(t,X(t))dt + B(t,X(t))dW^H(t), & t\in[0,T], \\ X(0) = X_0 \end{cases}$$

where A is Dirichlet Laplacian on E with homogenous boundary condition and it is generated analytic C_0-semigroup $S(t)$, and $F(t,X(t)) := u(t)$, $B(t,X(t)) := \psi X(t)$ are defined as

$$\begin{cases} F:[0,T]\times E \longrightarrow E \\ B:[0,T]\times E \longrightarrow \gamma(H;E) \end{cases}$$

The assumptions in Theorem 1.1 [24] are again satisfied and there exist an unique mild solution which satisfies following equation

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(s,X(s))ds + \int_0^t S(t-s)B(s,X(s))dW^H(s), \quad t\in[0,T]$$

above equation can be written as following FSVIE

$$X(t) = S(t)X_0 + \int_0^t b\left(t,s,X(s),u(s)\right)ds + \int_0^t \rho\left(t,s,X(s),u(s)\right)dW^H(s), \quad t\in[0,T]$$

where $b\left(t,s,X(s),u(s)\right) := S(t-s)u(s)$ and $\rho\left(t,s,X(s),u(s)\right) := S(t-s)\psi X(s)$. The cost function with the same assumptions in previous section is also again considered.
We have $\left|\frac{\partial b(t,s,x,u)}{\partial u}\right| = \left|\frac{dS(t-s)u}{du}\right| = \|S(t-s)\| \leq c$ for every $t,s\in[0,T]$, $\left|\frac{\partial b(t,s,x,u)}{\partial x}\right| = 0$ and Fréchet derivative of $\rho\left(t,s,X(s),u(s)\right)$ with respect to u is zero and its Fréchet derivative with respect to x, $\rho_x\left(t,s,X(s),u(s)\right) = S(t-s)\psi$. It means that in Theorem 5.1 $A_x(t,s) = S(t-s)\psi$. By applying Theorem 5.1 or Theorem 5.3, it results if $(\overline{X}(\cdot),\overline{u}(\cdot))$ be optimal stochastic solution and $\sum\limits_{n\geq 1}\|S(t-s)\psi h_n\|_{\mathcal{L}(E;E)}$ is bounded a.s then there exists an unique adapted M-solution $Y(\cdot), Z(\cdot,\cdot)$ of following BSVIE in E^*

$$Y(t) = -h_x(t,\overline{X}(t),\overline{u}(t)) + \int_t^T \sum_{n\geq 1}(S(t-s)\psi h_n)^* Z(s,t)h_n ds - \int_0^T Z(t,s)dW^H(s), \ t\in[0,T]$$

such that such that

$$\left\langle u(t) - \overline{u}(t), \int_t^T S(t-s)^* Y(s) ds - h_u(t, \overline{X}(t), \overline{u}(t)) \right\rangle \leq 0,$$
$$\forall u(\cdot) \in \mathcal{U}, \forall t \in [0,T] \text{ a.s}$$

or correspondingly

$$\int_0^1 \left\{ \int_t^T S(t-s)^* Y(s) ds - h_u(t, \overline{X}(t), \overline{u}(t)) \right\} (u(t) - \overline{u}(t))\, d\mu \leq 0,$$
$$\forall u(\cdot) \in \mathcal{U}, \forall t \in [0,T] \text{ a.s}$$

and by applying Theorem 5.4 it leads

$$\left\{ \mathbb{E} \left(\int_0^1 \int_t^T S(t-s)^* Y(s) ds \middle| \mathcal{F}_t \right) - h_u(t, \overline{X}(t), \overline{u}(t)) \right\} (u(t) - \overline{u}(t))\, d\mu \leq 0,$$
$$\forall u(\cdot) \in \mathcal{U}, \forall t \in [0,T] \text{ a.s}$$

If $h(t, \overline{X}(t,\xi), \overline{u}(t,\xi)) = h_1(t, \overline{X}(t,\xi)) + \overline{u}(t,\xi)$ then it follows from the last inequality $\overline{u}(t,\xi) = \mathrm{sign}\left(\mathbb{E}\left(\int_{\mathrm{t}}^{\mathrm{T}} \mathrm{S(t-s)^*Y(s,}\xi\mathrm{)ds}\middle| \mathcal{F}_{\mathrm{t}}\right) - 1\right)$ for all $t \in [0,T]$ and $\xi \in [0,1]$.

CHAPTER 6

Conclusion and Outlook

Stochastic Itô-Volterra integral equations in unconditional martingale difference Banach spaces were discussed, particularly stochastic processes were defined in Banach space $E = L^q(\mathbf{S}, \Sigma, \mu)$ where μ is σ-finite measure. Especially, we considered H-cylindrical Brownian motion as the noise process for stochastic integrals. We defined suitable conditions such that for the state equation, a unique solution process exists with certain smoothness properties. A stochastic optimal control problem was introduced and necessary optimality conditions of a maximum principle type were proved. We defined adjoint equations using Banach-space-valued backward stochastic Ito-Volterra integral equations (BSVIE) and the uniqueness of the solution process and its properties were proved.

For reaching our goal and finding stochastic variational inequality, the thesis was briefly structured as following. In Chapter 2 we dealt with forward stochastic Volterra integral equations with respect to a H-cylindrical Brownian motion in Banach space E and the unique solution and its some properties were proved. In Chapter 3, backward stochastic Volterra integral equation in Banach space E with respect to a H-cylindrical Brownian motion were introduced. The unique solution was derived. In Chapter 4 the duality principles between Forward and backward stochastic Volterra integral equations were proved. In Chapter 5, we introduced the optimal stochastic controls in Banach space E and we solved it by using maximum principle method and some examples were given there.

The measure μ in Chapter 5 was defined as a finite measure, by considering assumptions similar to the assumptions in Remark 2.1 for the derivatives and control process, such that they be bounded by some $k(\eta)$ where $k(\cdot) \in E$, the Theorem 5.1 can be resulted for a σ-finite measure too. Throughout this thesis the Theorem A.2 were used for handling the stochastic integrals in Banach space $E = L^q(\mathbf{S}, \Sigma, \mu)$. γ-radonifying operators from H into E can be also used in the proofs and stochastic integral equations, only some modifications are necessary, especially in BSVIE, such that the Theorem A.1 (item 4) can be used.

For further research, the study of regularity properties of solutions, especially the solution of BSVIE is an interesting aspect. The application of Mallivian calculus in BSVIE is another interesting research field.

Further interesting research questions are considerations of another Banach spaces and the application of jump processes and processes with memory to noise processes. And it necessitates to use and develop definition and techniques for stochastic integral with respect to different noises in Banach spaces. Developments of numerical methods, especially for finding of solution process of BSVIE are very interesting.

APPENDIX A

L^p-Stochastic Integration

In this appendix we recall some definitions and theorems about stochastic integral in Banach spaces. We bring most important theorems that are used in this thesis. This appendix is due to works of J.M.A.M. van Neerven, M.C. Veraar and L. Weis, for more details about this topics we refer reader to [26],[24], [5] and for further detail J.M.A.M. van Neerven [23]

Definition A.1 (UMD space). *A Banach space E is said to be a UMD_p space with $1 < p < \infty$, if there exists a positive constant C_p, such that for all E-valued L^p-martingale difference sequence $(d_n)_{n=1}^N$ and any choice of signs $\varepsilon_n = \pm 1$ it yields*

$$\left(\mathbb{E}\left\|\sum_{n=1}^{N}\varepsilon_n d_n\right\|^p\right)^{\frac{1}{p}} \leq C_p\left(\mathbb{E}\left\|\sum_{n=1}^{N} d_n\right\|^p\right)^{\frac{1}{p}}$$

Definition A.2 (H-cylindrical Brownian motion). [1] *A family $W^H = \left(W^H(t)\right)_{t\in[0,T]}$ of bounded linear operators from H to $L^2(\Omega)$ is called H-cylindrical Brownian motion if*

(1) $W^H h = \left(W^H(t)h\right)_{t\in[0,T]}$ is real valued Brownian motion for each $h \in H$

(2) $\mathbb{E}\left(W^H(s)g \cdot W^H(t)h\right) = (s \wedge t)[g,h]_H$ for all $s, t \in H$.

Definition A.3 (stochastic integral). [2] *For elementary process $\phi : [0,T] \times \Omega \longrightarrow \mathcal{L}(H,E)$ with the form*

$$\phi(t,w) = \sum_{n=0}^{N}\sum_{m=1}^{M}\mathbf{1}_{(t_{n-1},t_n]\times A_{mn}}\sum_{k=1}^{K} h_k \otimes x_{kmn}$$

where $x_{kmn} \in E$, $0 \leq t_0 < \cdots < t_N \leq T$, the sets $A_{1n}, \ldots, A_{Mn} \in \mathcal{F}_{n-1}^H$ are disjoint for each n and $h_1, \ldots h_K$ are orthonormal elements in H, the L^p-stochastic integral is defined followingly

$$\int_0^T \phi(t)dW^H(t) := \sum_{n=0}^{N}\sum_{m=1}^{M}\mathbf{1}_{A_{mn}}\sum_{k=1}^{K}\left(W^H(t_n)h_k - W^H(t_{n-1})h_k\right)x_{kmn}$$

Definition A.4 (type and cotype spaces). [3] *A Banach space E is type p, $p \in [0,2]$ if there exists a constant $C \geq 0$ such that for all $x_1, \ldots, x_N \in E$ and any Rademacher sequence $(r_n)_{n=1}^N$, it yields*

$$\left(\mathbb{E}\left\|\sum_{i=1}^{N} r_n x_n\right\|_E^2\right)^{\frac{1}{2}} \leq C\left(\left\|\sum_{i=1}^{N} x_n\right\|_E^p\right)^{\frac{1}{p}}$$

[1] Definition, p. 1450 [26]
[2] Definition, p. 1450 [26]
[3] for example refer to [30, 16]

the least possible constant C is called the type p constant of E.
A Banach space E is cotype q, $q \in [2, \infty]$ if there exists a constant $C \geq 0$ such that for all $x_1, \dots, x_N \in E$ and any Rademacher sequence $(r_n)_{n=1}^{N}$, it yields

$$\left(\left\|\sum_{i=1}^{N} x_n\right\|_E^q\right)^{\frac{1}{q}} \leq C\left(\mathbb{E}\left\|\sum_{i=1}^{N} r_n x_n\right\|_E^2\right)^{\frac{1}{2}}$$

the least possible constant C is called the cotype q constant of E.
Every Banach space has type 1 and cotype ∞ with constant 1. Hilbert spaces have type 2 and cotype 2 with constants 1. The L^q-spaces, $q \in [1, \infty)$ have type $\min\{p, 2\}$ *and cotype* $\max\{p, 2\}$.

THEOREM A.1 (L^p-stochastic integrability). [4] *Let E be a UMD space and $1 < p < \infty$. For an H-strongly measurable and adapted process $\phi : [0, T] \times \Omega \to \mathcal{L}(H; E)$ belonging to $L^p\left(\Omega; L^2(0, T; H)\right)$ scalarly, the following assertions are equivalent*

(1) the process ϕ is L^p-stochatically integrable with respect to $W^H(\cdot)$
(2) there exists a sequence of elementary adapted processes $\phi_n : [0, T] \times \Omega \to \mathcal{L}(H; E)$ such that
  (i) *for all $h \in H$ and $x^* \in E^*$,* $\lim_{n \to \infty} \langle \phi_n h, x^* \rangle = \langle \phi h, x^* \rangle$ *in measure on* $[0, T] \times \Omega$
  (ii) *There exists a random variable $\eta \in L^p(\Omega; E)$, such that*

$$\eta = \lim_{n \to \infty} \int_0^T \phi_n(t) dW^H(t) \text{ in } L^p(\Omega; E)$$

(3) there exists a strongly measurable random variable $\eta \in L^p(\Omega; E)$ such that for all $x^* \in E^*$ we have

$$\langle \eta, x^* \rangle = \int_0^T \langle \phi h, x^* \rangle \, dW^H(t) \quad \text{in} \quad L^p(\Omega)$$

(4) ϕ represents an element $X \in L^p\left(\Omega; \gamma\left(L^2(0, T; H), E\right)\right)$ ($\gamma(H; E)$ stands for the γ-radonifying space of all γ-radonifying operators from H into E)

and we have $\eta = I^{W^h}(X) := \int_0^T \phi(t) dW^H(t)$ in $L^p(\Omega; E)$

THEOREM A.2 (L^p-stochastic integrability). [5] *Let E be UMD Banach space over a σ-finite measure space $(\mathbf{S}, \Sigma, \mu)$ and let $p \in (1, \infty)$. Let $\phi : [0, T] \times \Omega \longrightarrow \mathcal{L}(H, E)$ be H-strongly measurable,adapted and assume there exits a strongly measurable function $\varphi : [0, T] \times \Omega \times \mathbf{S} \longrightarrow H$ such that $\forall h \in H$ and $t \in [0, T]$ $(\phi(t)h)(\cdot) = [\varphi(t, \cdot), h]_H$ in E, then ϕ is L^p-stochastically integrable with respect to W^H if and only if*

$$\mathbb{E}\left\|\left(\int_0^T \|\varphi(t, \cdot)\|_H^2 dt\right)^{\frac{1}{2}}\right\|_E^p < \infty.$$

[4] Theorem 3.6, p. 1454 [26]
[5] COROLLARY 3.11, p. 1461 [26]

Further we have

$$\mathbb{E}\left\|\int_0^T \phi(t)dW^H(t)\right\|_E^p \simeq \mathbb{E}\left\|\left(\int_0^T \|\varphi(t,\cdot)\|_H^2 dt\right)^{\frac{1}{2}}\right\|_E^p$$

Theorem A.3. [6] *Let $(\mathbf{S},\Sigma,\mu)$ be a σ-finite measure space and let $1 \le q < \infty$. For an operator $T \in \mathcal{L}(H, L^q(\mathbf{S}))$ the following assertions are equivalent*

(1) $T \in \gamma\left(H, L^q(\mathbf{S})\right)$

(2) for some orthonormal basis $(h_n)_{n=1}^{\infty}$ of H the function $\left(\sum_{n\ge1} |Th_n|^2\right)^{\frac{1}{2}}$ belongs to $L^q(\mathbf{S})$

(3) for all orthonormal basis $(h_n)_{n=1}^{\infty}$ of H the function $\left(\sum_{n\ge1} |Th_n|^2\right)^{\frac{1}{2}}$ belongs to $L^q(\mathbf{S})$

(4) there exists a function $g \in L^q(\mathbf{S})$ such that for all $h \in H$ we have $|Th| \le \|h\|_H \cdot g$ μ-almost everywhere

(5) there exists a function $k \in L^q(S;H)$ such that $Th = [k(\cdot), h]_H$ μ-almost everywhere

moreover, in this situation we may take $k = \left(\sum_{n\ge1} |Th_n|^2\right)^{\frac{1}{2}}$ and have

$$\|T\|_{\gamma(H,L^q(\mathbf{S}))} \simeq \left\|\left(\sum_{n\ge1} |Th_n|^2\right)^{\frac{1}{2}}\right\| \le \|g\|_{L^q(\mathbf{S})}.$$

By considering that $X \in L^p\left(\Omega;\gamma\left(L^2(0,T;H),E\right)\right)$ is the element represented by ϕ and in according to the integral process $t \longrightarrow \int_0^t \phi(s)dW^H(s),\ t\in[0,T]$, the process $\xi:[0,T]\times\Omega \longrightarrow \gamma\left(L^2(0,T;H),E\right)$ can be introduced, it is associated with X and is defined by $\xi(t,\omega)f := (X(w))(\mathbf{1}_{[0,T]}f),\ f \in L^2((0,T);H)$. It must be noted that $\xi_X(T) = X$.

Theorem A.4. [7] *Let E be a UMD space and fix $p \in (1,\infty)$. For all $X \in L^p\left(\Omega;\gamma\left(L^2(0,T;H),E\right)\right)$ the integral process $I^{W_H}(\xi_X)$ is an E-valued L^p-martingale which is continious in p-th moment. It has a continious adapted version which satisfies the maximal inequality*

$$\mathbb{E}\sup_{t\in[0,T]}\left\|I^{W^H}(\xi_X(t))\right\|^p \le q^p\left\|I^{W^H}(X)\right\|^p$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem A.5 (Burkholder-Davis-Gundy inequalities). [8] *Let E be a UMD space and fix $p \in (1,\infty)$.If the H-strongly measurable and adapted process $\phi : [0,T]\times\Omega \longrightarrow \mathcal{L}(H,E)$ is L^p-stochastically integrable, then*

$$\mathbb{E}\sup_{t\in[0,T]}\left\|\int_0^t \phi(s)dW^H(s)\right\|^p \simeq \mathbb{E}\|X\|^p_{\gamma(L^2(0,T;H),E)}$$

[6] Lemma 2.1, p. 945 [24]

[7] Proposition 4.3, p. 1462 [26]

[8] Theorem 4.4, p. 1463 [26]

where $X \in L^p\left(\Omega; \gamma\left(L^2(0,T;H), E\right)\right)$ *is the element represented by* ϕ.

THEOREM A.6 (Martingale representation theorem in UMD spaces). [9] *Let* E *be a UMD space, then every* L^p*-martingale* $M : [0,T] \times \Omega \longrightarrow E$ *adapted to the augmented filtration* $\mathbb{F}^{W^H}$ *has a continuous version, and there exists a unique* $X \in L^p\left(\Omega; \gamma(L^2(0,T;H), E)\right)$ *such that for all* $t \in [0,T]$ *it yields*

$$M(t) = M(0) + I^{W^H}(\xi_X(t)), \quad in \;\; L^p(\Omega; E)$$

For UMD space E *with cotype 2 the above representation takes the form*

$$M(t) = M_0 + \int_0^t \phi(t) dW^H(t).$$

THEOREM A.7 (general Itô formula). [10] *Let* E *and* F *be UMD spaces. Assume that* $f : [0,T] \times E \longrightarrow F$ *is of class* $C^{1,2}$. *Let* $\phi : [0,T] \times \Omega \longrightarrow \mathcal{L}(H,E)$ *be* H*-strongly measurable and adapted process which is stochastically integrable with respect to* W^H *and asuume that paths of* ϕ *belongs to* $L^2(0,T;\gamma(H,E))$ *almost surely. Let* $\psi : [0,T] \times \Omega \longrightarrow E$ *be strongly measurable and adapted with paths in* $L^1(0,T : E)$ *almost surely. Let* $\xi : \Omega \longrightarrow E$ *be strongly* $\mathcal{F}_0$*-measurable. Define* $\zeta : [0,T] \times E \longrightarrow F$ *by*

$$\zeta = \xi + \int_0^\cdot \psi(s) ds + \int_0^\cdot \phi(s) dW^H(s)$$

then $s \longrightarrow D_2 f(s, \zeta(s))\phi(s)$ *is stochastically integrable and almost surely we have, for all* $t \in [0,T]$

$$\begin{aligned} f(t,\zeta(t)) - f(0,\zeta(0)) &= \int_0^t D_1 f(s,\zeta(s)) ds + \int_0^t D_2 f(s,\zeta(s))\psi(s) ds \\ &+ \int_0^t D_2 f(s,\phi(s)) dW^H(s) + \frac{1}{2}\int_0^t \mathrm{Tr}_{\phi(s)}(D_2^2 f(s,\zeta(s))) ds \end{aligned}$$

THEOREM A.8 (Itô formula). [11] *Let* E_1 *and* E_2 *be UMD spaces and let* $f : E_1 \times E_2 \longrightarrow F$ *be a bilinear map. Let* $(h_n)_{n\geq 1}$ *be an orthonormal basis of* H. *For* $i = 1, 2$ *let* $\phi_i : [0,T] \times \Omega \longrightarrow \mathcal{L}(H, E_i)$, $\psi_i : [0,T] \times \Omega \longrightarrow E_i$ *and* $\xi_i : \Omega \longrightarrow E_i$ *satisfy the assumptions of Theorem A.7 and define*

$$\zeta_i = \xi_i + \int_0^\cdot \psi_i(s) ds + \int_0^\cdot \phi_i(s) dW^H(s)$$

then almost surely for all $t \in [0,T]$,

$$\begin{aligned} f(\zeta_1(t),\zeta_2(t)) - f(\zeta_1(0),\zeta_2(0)) &= \int_0^t \left(f(\zeta_1(t),\psi_2(t)) + f(\psi_1(t),\zeta_2(t))\right) ds \\ &+ \int_0^t \left(f(\zeta_1(t),\phi_2(t)) + f(\phi_1(t),\zeta_2(t))\right) dW^H(s) \\ &+ \int_0^t \sum_{n\geq 1} f(\phi_1(s)h_n, \phi_2(s)h_n) ds \end{aligned}$$

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[9] Theorem 5.13, p. 1476 [26]

[10] Theorem 2.4, p. 36 [26]

[11] Corollary 2.6, p. 37 [26]

particularly, for a UMD space E, taking $E_1 = E$, $E_2 = E^*$, $F = \mathbb{R}$ and $f(x, x^*) = \langle x, x^* \rangle$, it follows that almost surely for all $t \in [0, T]$

$$\begin{aligned}
\langle \zeta_1(t), \zeta_2(t) \rangle - \langle \zeta_1(0), \zeta_2(0) \rangle &= \int_0^t \left(\langle \zeta_1(s), \psi_2(s) \rangle + \langle \psi_1(s), \zeta_2(s) \rangle \right) ds \\
&+ \int_0^t \left(\langle \zeta_1(s), \phi_2(s) \rangle + \langle \phi_1(s), \zeta_2(s) \rangle \right) dW^H(s) \\
&+ \int_0^t \sum_{n \geq 1} \langle \phi_1(s) h_n, \phi_2(s) h_n \rangle \, ds.
\end{aligned}$$

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Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die Arbeit selbstständig und ohne fremde Hilfe verfasst, keine anderen als die von mir angegebenen Quellen und Hilfsmittel benutzt und die den benutzten Werken wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht habe.

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