Variational Principles on Metric and Uniform Spaces

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Chapter 1

Introduction

DEAE IN AETERNUM INCOGNITAE.

The main goal of the present work is to give most general formulations of **Ekeland's Variational Principle** as well as of so-called **Minimal Element Theorems** on metric and uniform spaces.

A minimal element theorem gives conditions for the existence of minimal elements of an ordered set X or $X \times Y$ with respect to certain order relations. Ekeland's variational principle ensures the existence of minimal points for a (small) perturbation of a function $f : X \to Y$, where Y is supplied with an order relation.

We call both kinds of theorems simply **Variational Principles** since they have a fundamental idea in common: to vary a certain point to obtain another one, not so far away, with some useful extremality properties. Moreover, in several situations a minimal element theorem turns out to be an equivalent formulation of a suitable Ekeland's principle and vice versa. A further object of this work is to find the right equivalent formulation in each situation.

¿From a historical point of view, the story began with X being a topological linear space (Lemma 1 in Phelps' paper [101] from 1963) and a complete metric space (the variational principle, see Ekeland's papers [28], [29], [30] from the beginning of the seventhies), respectively, and $Y = \mathbb{R}$ in both cases. Since the topology of metric spaces as well as of topological linear spaces can be generated by a uniform structure, it is a natural idea to look for a common formulation in uniform spaces. Such a formulation has already been given by Brønsted in the paper [8] from 1974.

However, it turned out that there are two different approaches to the proof: The first one is to assume that X is a complete uniform space and to work with nets instead of sequences. As a rule, Zorn's lemma (or a transfinite induction argument) has to be involved in this case and the assumptions are stronger than in the metric case. Compare Chapter 6 for this approach which is also the basic idea of the work of Nemeth [92], [93],

[94].

The second one is to find assumptions which allows to work with sequences even in uniform spaces. Such assumptions essentially involve a scalarization, i.e., a real valued function linking topological properties and properties of the order relation in question. This approach is presented in Chapter 7 and it is shown that it yields a link between Brønsted's results [8] (he also used a scalarization technique) and recent results of Göpfert, Tammer and Zălinescu [114], [47], [44] and even corresponding set valued variants as in [50].

Using the latter approach, it is also possible to leave the framework of uniform spaces and to work only on ordered sets. This has been done by Brézis and Browder in the influential paper [6]. Subsequent generalizations can be found e.g. in [1], [67] and in several papers by Turinici such as [119], [120], [121], [122], [123]. Results of this type are out of the scope of this work, since it is restricted to the case in which the existence of minimal elements essentially follows from completeness.

Of course, a minimal element theorem on an ordered set $(X, \preceq_X)$ can be applied to a product set $(X \times Y, \preceq_{X\times Y})$ provided the corresponding assumptions are satisfied by $X \times Y$ and $\preceq_{X\times Y}$. Results of this type can be found e.g. in Section 4.7. But in many applications it is desirable to have different sets of assumptions for X and Y. Remaining in the framework of the present thesis, i.e., X is assumed to be a complete metric or a uniform space, the question is what assumptions are essential for Y to obtain a minimal element theorem on $X \times Y$ and an Ekeland type theorem for functions $f : X \to Y$, respectively. The answer is: Algebraic and topological assumptions to Y, e.g., Y is assumed to be a topological linear space, are not essential. To the authors opinion, this is one of the more surprising results of the present investigation. Assumptions have to be put only on the order relation $\preceq_{X\times Y}$ which are satisfied even in cases in which Y is neither a topological nor a linear space. The crucial assumption deals with decreasing sequences: If $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ is decreasing with respect to $\preceq_{X\times Y}$ and $\{x_n\}_{n\in\mathbb{N}}$ is convergent to $x \in X$, then there must be $y \in Y$ such that $(x, y) \preceq_{X\times Y} (x_n, y_n)$ for all $n \in \mathbb{N}$. The importance of this assumption has been figured out in [47], but it is also strongly related to assumption (2) of Brézis–Browder's Theorem 1 in [6].

This allows to obtain minimal element and Ekeland type theorems for example if Y is the power set of a linear space. In fact, so called set valued Ekeland's principles as recently established by Chen, Huang and Hou in [11], [12], [13], Truong in [117] and Hamel and Löhne in [50] are the main motivation to look for minimal element theorems on $X \times Y$ with Y being more general than a linear space (compare [44]) or a topological Abelian group (compare [93], [94]).

Following this path, it was possible to prove variational principles on $(X \times Y, \preceq_{X\times Y})$ and for $f : X \to (Y, \preceq_Y)$, respectively, under very mild assumptions concerning Y. Especially, Y can be assumed to be an ordered monoid. Since the power set of an ordered monoid is an ordered monoid as well (with suitable order relations), this covers also set valued variational principles.

This is the reason for the investigations of Chapter 2 of this thesis: The topic is the structure of ordered monoids with special emphasis to those properties being invariant under passing to power sets. Several details of this chapter are not new: For example, it is well–known that the power set of a monoid is a monoid with respect to the Minkowski operation. Also, the order relations for power sets of ordered sets are not new. But the author is not aware of a thorough presentation of algebraic and order theoretic properties of ordered monoids and their power sets together with their interrelations.

The present work contains five main chapters.

Chapter 2 deals with basic structures: algebraic, order and topological structures used in the subsequent chapters. Mainly, the concepts of this chapter are developed in order to formulate the variational principles.

However, there are several results not used in the subsequent chapters. They remained in this text since they shed some light on basic ideas or illustrate the difference to widely used concepts or may serve as a starting point for future developments: For example, this applies for the results about the interrelations between the infimum and the set of minimal points of a subset of an ordered set $(W, \preceq)$ on one side and the infima with respect to the two canonical extensions of $\preceq$ to the power set of W on the other side. Compare Section 2.2. The so–called domination property (lower external stability, cf. [85] in the context of linear spaces) plays a pivotical role and the relationships between vector and set valued optimization problems are turned up side down in some sense.

On the other hand, there are some concepts without an explicit definition like *group* or *complete metric space*. In this cases, the definitions are very standard and the terms are used with the same meaning in almost all text books on corresponding topics.

The basic algebraic structure is a commutative monoid, i.e., a commutative semigroup with a neutral element. This seems to be a natural starting point since the power set of a group is a monoid (with respect to the corresponding Minkowski operation) as well as order completion of a group leads to an ordered monoid whereas the power set of a monoid is a monoid again as well as the order completion of an ordered monoid. This means: The monoidal structure is stable under passing to power sets and under order completion.

A new concept is that of a **conlinear space** introduced in Section 2.1. This concept is more general than that of a linear space and it turns out that the power set as well as the order completion of a linear space is a conlinear space. Moreover, a convex cone in the classical sense (i.e., a subset of a linear space invariant under multiplication with nonnegative real numbers and under addition) is also a conlinear space. On the other hand, there are conlinear spaces which can not be identified with a cone as a subset of a linear space.

It appears to me that the concept of a conlinear space might be a natural framework to define and investigate convexity. Several initial clues supporting this idea can be found in Section 2 of this thesis. Some elementary concepts and facts carry over from Convex Analysis in linear spaces to a Convex Analysis in conlinear spaces, a topic under development. Some more results in this direction can be found in the PhD thesis of A. Löhne [83] and the diploma thesis of C. Schrage [109].

Section 2.2 contains order theoretic concepts, especially the definition and basic properties of the two canonical extensions $\preccurlyeq$, $\curlyeqprec$ of a quasiorder $\preceq$ on a set W to the power set $\widehat{\mathcal{P}}(W)$ (including $\emptyset$). These extensions are widely used in theoretical information sciences. A thorough survey is the 1993 paper by C. Brink [7]. Our exposition emphasizes on formulas for the infimum and supremum of subsets of $\widehat{\mathcal{P}}(W)$ with respect to $\preccurlyeq$ and $\curlyeqprec$, respectively. As mentioned above, there are close relationships between these extrema and the sets of infimal and minimal points of W with respect to $\preceq$. This shed some new light on inherent difficulties of vector optimization and provokes a surprising answer to the question what we shall understand by a solution of an optimization problem with a set valued objective function. However, this is not a main topic of this thesis.

The concepts connected with topological and uniform structures are introduced in order to have as much freedom as possible to define order relations on uniform spaces satisfying the regularity assumption of the minimal element theorem. This is motivated by the fact that there are at least three different types of Ekeland type results on uniform spaces concerning the order relation. Mizoguchi [86] used pseudometrics and Fang [33] quasimetrics whereas Nemeth's results in several papers involve so called cone valued metrics. A few attempts have been made to unify these approaches, e.g. in [10] and [51]. The observation that all these approaches apply for different order relations, but for the same class of spaces seems to be new.

Therefore, we collect four possibilities to introduce a uniform structure and show their equivalence. Only two of them are quite standard.

In Chapter 4 variational principles on metric spaces are presented.

Although most of the results, especially the main Theorem 16, are special cases of results of corresponding theorems on uniform spaces, we prefered to give direct proofs in metric spaces. This is for several reasons: First, the proofs are in many cases simpler, more direct, in some sense constructive and already contain the essential ideas. Secondly, for the vast majority of applications especially the metric case is interesting and most of the papers on variational principles deal with this case. And thirdly, the metric case served as a blueprint for the sequential analysis in uniform spaces of Chapter 7.

The leading questions of this chapter (and also for the two subsequent chapters) are the following: What are the indispensable ingredients for a proof of a variational principle? What are the mathematical concepts lying at the bottom of the theory? Is it possible to find a general sceme for all proofs of Ekeland type results?

The answer is as follows. The indispensable ingredients are a (complete) metric space, a quasiorder with lower closed sections and a further link between topological properties and the properties of the order. This linking assumption is called **regularity** of the order: Decreasing sequences have to be asymptotic. The basic result, Theorem 16, contains just these things. To the opinion of the author, all Ekeland type theorems on metric spaces including vector and set valued variants can be proven by verifying the assumptions of the basic theorem for a suitable order relation. This program is carried out in the remaining part of the chapter producing almost all known results in the field – among them Ekeland's original result, the Kirk-Caristi fixed point theorem and the drop theorem as well as a lot of new results especially for functions with values in ordered monoids and its power sets.

Let us note that several authors try to avoid using order relations explicitely while proving Ekeland type results. For example, Ekeland itself in [31] gave such a proof. However, the order relation is still present (and is sometimes called *dynamical system* as in [4] and several papers of Isac) and therefore it seems to be adequate to start with an order relation on a metric space. The idea for a proof of Ekeland's principle using a minimal element theorem on metric spaces can be traced back to the 1983 paper [22] of S. Dancs et al. There are many authors who used the Dancs–Hegedüs–Medvegyev theorem for proving one or another variant of Ekeland's principle, see for example [59], [60], [61], but the central importance of such a theorem seems to be new knowlegde as well as its far reaching applicability and its equivalence to the other theorems in Section 4.1.

Concerning the meaning of the equivalence between different variational principles we refer the reader to the discussion in Section 4.1. There are several papers presenting lists of theorems being equivalent to Ekeland's principle in some sense, e.g. [24], [98], [38], [96], [97] and also the book [58] to mention a few. In Section 4.1 we present some theorems being equivalent to the basic minimal element principle on metric spaces, Theorem 16. In Section 4.2 results are presented being equivalent to Ekeland's principle for functions with values in ordered monoids and Section 4.5 contains a series of theorems being equivalent to Ekeland's original result (Theorem 1.1 of [30]) involving realvalued functions. Of course, for each image space Y a corresponding list is possible but we do not focus on such equivalence assertions for each type of Y.

In Chapter 5, a minimal element theorem is presented for a subset of a product set $X \times Y$, where X is a (complete) metric space. Similar results for X a (sequentially) complete uniform space are contained in Section 6.3 and 7.2. The main new feature of these results is again that Y is merely assumed to be a nonempty set. Only assumptions to the order relation on $X \times Y$ appear. Therefore, Y can be chosen as the power set of a linear space for example. This leads to a minimal element theorems on $X \times \mathcal{P}(V)$, where $\mathcal{P}(V)$ is the power set of a linear space V.

Chapter 6 deals with variational principles on complete uniform spaces without scalarization. The development in this direction originates from I. Vályi [124] and A. B. Németh [92]. We start with a basic minimal element theorem on complete uniform spaces, Theorem 24 and apply it in order to obtain a series of corollaries for various single and set valued situations. Considering functions on uniform spaces with values in ordered monoids we establish generalizations of the most recent results of Németh [93], [94].

In Chapter 7, a systematic treatment of situations is given in which a scalarization function is present. Under this assumption we need to consider only sequentially complete uniform spaces. A series of corollaries is presented involving more complicated scalarization functions from step to step starting from continuous linear function (Y has to be a locally convex space), going to sublinear functions of Tammer-Weidner type on linear spaces and even on power sets of linear spaces. The starting point of these developments is Brønsted's paper [8], but we also obtain generalizations of results collected in the book [44] as well as those of [51] and [50].

This thesis does not deal with applications of the presented theorems, not even applications in "pure" mathematics such as ABB theorems, existence for solutions of vector optimization problems, necessary optimality and approximate optimality conditions, fuzzy metric space theory, geometry of Banach spaces, economical fixed point theory, to mention a few main fields. We only remark that variational principles in the spirit of Phelps and Ekeland turned out to be undispensable tools for recent developments in various fields of mathematics. The history of necessary optimality conditions for nonsmooth optimal control problems since Clarke's pioneering work [17] may serve as a prominent example.

Chapter 2

Basic Framework

2.1 Algebraic structures

In this section, several algebraic structures are introduced forming the framework for the theory of the next chapters. The goal is to replace the concept of a linear space by a more general one. This is motivated on the one hand by the structure of the power set $\mathcal{P}(V)$, V denoting a real linear space and on the other hand by the algebraic properties of $\mathbb{R} \cup \{+\infty\}$, $\mathbb{R} \cup \{-\infty\}$ and $\mathbb{R} \cup \{\pm\infty\}$, respectively. The elementwise addition (Minkowski sum) of two subsets of V does not satisfy the axiom of the existence of an inverse element. The same phenomenon appears in $\mathbb{R} \cup \{\pm\infty\}$, for example: It does not make sense to define $(+\infty) + (-\infty) = 0$ in most cases. Depending on the purpose, definitions like $(+\infty)+(-\infty) = +\infty$ and $(+\infty)+(-\infty) = -\infty$ occur, called *inf-addition* and *sup-addition*, respectively, in [106], Section 1.E.

2.1.1 Monoids

Let Y be a nonempty set and $Y \times Y$ the set of all ordered pairs of elements of Y. A **binary operation** on Y is a mapping of $Y \times Y$ into Y.

Definition 1 *Let Y be a nonempty set and $\circ$ a binary operation on Y. The pair $(Y, \circ)$ is called a* **monoid** *iff*
(M1) $\forall y_1, y_2, y_3 \in Y : \; y_1 \circ (y_2 \circ y_3) = (y_1 \circ y_2) \circ y_3$;
(M2) $\exists \theta \in Y \; \forall y \in Y : \; y \circ \theta = \theta \circ y = y$.
A monoid is called **commutative** *iff the relation $\circ$ is commutative, i.e.*
(M3) $\forall y_1, y_2 \in Y : \; y_1 \circ y_2 = y_2 \circ y_1$.

A monoid is nothing else than a semigroup with a neutral element; hence all results on semigroups apply also on monoids. The neutral element of a monoid is unique.

In this note, we only consider commutative monoids, even though several results may be formulated in a more general framework.

Example 1 *(i) A set consisting of three elements, say $Y = \{U, L, \theta\}$, can be provided with a monoidal structure by defining $L \circ L = L$, $U \circ U = U$, $\theta \circ \theta = \theta$, $L \circ U = U \circ L = U$, $L \circ \theta = \theta \circ L = L$, $U \circ \theta = \theta \circ U = U$. The axioms (M1), (M2) are easy to check by noting that all expressions involving U produce U and expressions not containing U, but L produce L. Thus the three elements are in a certain hierarchical order with respect to the operation $\circ$: U dominates the two others, L dominates θ. Of course, $(Y, \circ)$ is not a group. This example will be of some importance later on.*
(ii) The set $Y = \{0, 1, 2\}$, together with the operation $y_1 \circ y_2 = \min\{y_1, y_2\}$ can be identified with the monoid in (i) by setting $U = 0$, $L = 1$, $\theta = 2$. The neutral element is $\theta = 2$. In the same way, a monoidal structure on $\{-\infty, 0, +\infty\}$ is obtaind by identifying $U = +\infty$, $L = -\infty$ (inf-addition) and vice versa $U = -\infty$, $L = +\infty$ (sup-addition).
(iii) The set $\mathbb{R}^n \cup \{+\infty\}$ as well as $\mathbb{R}^n \cup \{-\infty\}$ can be made to an commutative monoid by defining

$$x + (+\infty) = +\infty + x = (+\infty) \quad \textit{and} \quad x + (-\infty) = -\infty + x = (-\infty),$$

respectively, for all $x \in \mathbb{R}^n$ and $+\infty + (+\infty) = +\infty$ in the first case and $-\infty + (-\infty) = -\infty$ in the second one.

Considering $\mathbb{R}^n \cup \{+\infty, -\infty\}$ there are two main possibilities to extend the operation $+$ to the case when both summands are non finite elements of $\mathbb{R}^n$, namely,

$$\begin{aligned}(-\infty) + (+\infty) &= (+\infty) + (-\infty) = +\infty;\\ (-\infty) + (+\infty) &= (+\infty) + (-\infty) = -\infty.\end{aligned}$$

Each of these possibilities leads to an commutative monoid. Later on, we shall discuss some applications. The definition $(-\infty) + (+\infty) = (+\infty) + (-\infty) = 0$, at the first glance more natural, does not produce a monoid since the associative law (M1) is violated.
(iv) The set of all nonempty subsets of the real line with respect to elementwise addition or multiplication is a commutative monoid. The neutral elements are $\{0\}$ and $\{1\}$, respectively.

The last example can be generalized in order to produce new monoids. Let Y be a nonempty set. We denote by $\mathcal{P}(Y)$ the set of all nonempty subsets of Y and by $\widehat{\mathcal{P}}(Y)$ the set of all subsets of Y including the empty set $\emptyset$, i.e. $\widehat{\mathcal{P}}(Y) = \mathcal{P}(Y) \cup \{\emptyset\}$.

Let $(Y, \circ)$ be a monoid. We define an operation $\odot$ on $\mathcal{P}(Y)$ by

$$\forall M_1, M_2 \in \mathcal{P}(Y): \quad M_1 \odot M_2 := \{y_1 \circ y_2 : \ y_1 \in M_1, \, y_2 \in M_2\}.$$

The operation $\odot$ can be extended to $\widehat{\mathcal{P}}(Y)$ by

$$\forall M \in \widehat{\mathcal{P}}(Y): \quad M \odot \emptyset = \emptyset \odot M = \emptyset.$$

This means, $\emptyset \in \widehat{\mathcal{P}}(Y)$ is defined to be a zero element in the sense of [19], p. 3. A zero element of a commutative monoid is always unique.

The property of being a monoid is stable under passing to power sets.

Proposition 1 *Let $(Y, \circ)$ be a monoid. Then $(\mathcal{P}(Y), \odot)$ and $\left(\widehat{\mathcal{P}}(Y), \odot\right)$ are monoids as well. In each case, the neutral element is $\Theta = \{\theta\}$. If $(Y, \circ)$ is commutative, so are $(\mathcal{P}(Y), \odot)$, $\left(\widehat{\mathcal{P}}(Y), \odot\right)$.*

Proof. Immediately from the definition. ■

Example 2 *Let $(Y, \circ)$ be a commutative monoid. For $M \subseteq Y$ we define the (plus)-indicator function belonging to M by*

$$I_M^+(y) := \begin{cases} 0 & : \ y \in M \\ +\infty & : \ y \notin M \end{cases}.$$

Denote by $I^+(Y)$ the set of all functions f on Y such that $f(Y) \subseteq \{0, +\infty\}$, i.e., $I^+(Y)$ is the set of (plus)-indicator functions for subsets of Y. Defining

$$\left(I_{M_1}^+ \Box I_{M_2}^+\right)(y) := \inf \left\{ I_{M_1}^+(y_1) + I_{M_2}^+(y_2) : \ y_1 \circ y_2 = y \right\}$$

for $M_1, M_2 \in \widehat{\mathcal{P}}(Y)$ one may see that $(I^+(Y), \Box)$ is a commutative monoid. Since

$$M = M_1 \odot M_2 \quad \Longleftrightarrow \quad I_M^+ = I_{M_1}^+ \Box I_{M_2}^+,$$

there is an isomorphism between $\left(\widehat{\mathcal{P}}(Y), \odot\right)$ and $(I^+(Y), \Box)$.

Of course, a similar consideration is possible with $I^-(Y)$ replacing $+\infty$ by $-\infty$ and inf *by* sup*.*

We introduce further notation considering elements of monoids with special properties.

Definition 2 *Let $(Y, \circ)$ be a commutative monoid. An element $y \in Y$ is called* **invertible** *iff there exists an $y' \in Y$ such that*

$$y \circ y' = y' \circ y = \theta.$$

The set of all invertible elements of $(Y, \circ)$ is denoted by Y_{in}.

Clearly, $\theta \in Y$ is always invertible. Moreover, $(Y_{in}, \circ)$ is a subgroup of the given monoid being maximal in the sense that there is no other subgroup of $(Y, \circ)$ containing all invertibles and at least one more element. Therefore, the set $Y_{in} \subset Y$ is called the **maximal subgroup** of the given monoid. Of course, $(Y, \circ)$ is a group iff $Y = Y_{in}$.

Note that in several textbooks on semigroups, e.g. [19], p. 21ff, invertible elements are called *units*. Passing to power sets, the maximal subgroup of a monoid is invariant.

Proposition 2 *Let $(Y, \circ)$ be a commutative monoid with the maximal subgroup $(Y_{in}, \circ)$. Then it is also the maximal subgroup of $(\mathcal{P}(Y), \odot)$ and $\left(\widehat{\mathcal{P}}(Y), \odot\right)$, respectively, in the sense that $y \in Y_{in}$ is identified with $\{y\} \in \mathcal{P}(Y)$.*

Proof. Let $Y_1, Y_2 \in \mathcal{P}(Y)$ be invertible such that $Y_1 \odot Y_2 = \{\theta\}$. Then

$$\forall y_1 \in Y_1,\ y_2 \in Y_2 :\ y_1 \circ y_2 = \theta$$

contradicting the uniqueness of inverse elements in groups if at least one of Y_1, Y_2 contains more than one element. Concerning $\widehat{\mathcal{P}}(Y)$, it suffices to note that, by definition of $\odot$, $\emptyset$ is not invertible. ■

Example 3 *(i) The set* $Y := \mathbb{R}^2_+$ *of all elements of* $\mathbb{R}^2$ *with nonnegative components, together with the usual vector addition, forms a commutative monoid with* $Y_{in} = \left\{(0,0)^T\right\}$.

(ii) The set $Y := \left\{y = (y_1, y_2)^T \in \mathbb{R}^2 :\ y_2 \geq 0\right\}$, *together with the usual vector addition, forms a commutative monoid with* $Y_{in} = \left\{(y_1, 0)^T \in \mathbb{R}^2 :\ y_1 \in \mathbb{R}\right\}$.

Proposition 3 *Let* $(Y, \circ)$ *be a commutative monoid and* $Y_{nin} := Y \backslash Y_{in} \cup \{\theta\}$ *the set of all noninvertible elements and* θ. *Then* $(Y_{nin}, \circ)$ *is a monoid as well.*

Proof. Let $y, y' \in Y_{nin}$. Then $y \circ y'$ is a noninvertible since otherwise there would be a $u \in Y$ such that $(y \circ y') \circ u = y \circ (y' \circ u) = \theta$. Hence y is invertible contradicting the assumption. ■

Definition 3 *Let* $(Y, \circ)$ *be a monoid. An element* $y \in Y$ *is said to be* **idempotent** *iff*

$$y = y \circ y.$$

An idempotent element $y \neq \theta$ *is called* **nontrivial**.

Of course, an idempotent element is an element coinciding with all of its n-**powers**, i.e.,

$$\forall n = 1, 2, \ldots :\quad y = y^n := \underbrace{y \circ \ldots \circ y}_{n\ times}.$$

Proposition 4 *Let* $(Y, \circ)$ *be a commutative monoid and* $Y_{id} \subseteq Y$ *the set of all idempotent elements. Then* $(Y_{id}, \circ)$ *is a commutative monoid as well.*

Proof. Let $y, y' \in Y$ be idempotent elements. Then

$$\left(y \circ y'\right) \circ \left(y \circ y'\right) = (y \circ y) \circ \left(y' \circ y'\right) = y \circ y',$$

i.e., the operation $\circ$ transfers idempotent elements into idempotent elements. ■

Proposition 5 *Let* $(Y, \circ)$ *be a commutative monoid and* $y \in Y$ *be an idempotent element. Then* $\{y\}$ *is an idempotent element of* $\mathcal{P}(Y)$ *and* $\widehat{\mathcal{P}}(Y)$.

PROOF. Obvious. ■

The following proposition shows the difficulties connected with nontrivial idempotent elements.

Proposition 6 *A monoid with a nontrivial idempotent element can not be embedded in a group.*

PROOF. Let $(Y, \circ)$ be a monoid and $(G, \diamond)$ be a group such that $Y \subset G$ and $\diamond$ coincides with $\circ$ on Y. Let $y \in Y$ be a nontrivial idempotent element. Then there exists $g \in G$ such that $y \diamond g = \theta$. This implies

$$\theta = y \diamond g = (y \circ y) \diamond g = y \diamond (y \diamond g) = y \circ \theta = y$$

contradicting the nontriviality of y. ■

An commutative monoid with the unique idempotent element θ can be embedded in a group if and only if the cancellation property holds true, i.e. for three elements y, y_1, y_2 we have

$$y_1 \circ y = y_2 \circ y \qquad \Longrightarrow \qquad y_1 = y_2.$$

A nontrivial idempotent element destroys the cancellation property, compare [19], p. 6, 1. (b) and p. 34ff.

In Example 1, (i), (iii) we have seen that the monoid operation can be dominated by certain elements. We give a precise definition of this property which is essentially due to A. Löhne [82].

Definition 4 *Let $(Y, \circ)$ be a monoid. The subset $Y_1 \subseteq Y$ is said to* **dominate** *the subset $Y_2 \subseteq Y$, shortly $Y_1 \succ Y_2$, iff*

$$y_1 \in Y_1,\ y_2 \in Y_2 \qquad \Longrightarrow \qquad y_1 \circ y_2 \in Y_1.$$

An element $\widehat{y} \in Y$ is called the **dominant element** *of Y with respect to $\circ$ iff $\{\widehat{y}\} \succ Y$.*

Proposition 7 *Let $(Y, \circ)$ be a commutative monoid and $\hat{y} \in Y$ be a dominant element different from θ. Then it is unique and an idempotent element.*

PROOF. The uniqueness is obvious. By definition of dominant elements, $\hat{y} \circ y = \hat{y}$ for all $y \in Y$. Setting $y = \hat{y}$, the result follows. ■

Example 4 *(i) Easy to check examples for monoides with dominant elements are given in Example 1, (i) and (ii).*

(ii) Considering Example 1, (iii), we denote by $(\mathbb{R}^n)^\vartriangle$ the monoid $(\mathbb{R}^n \cup \{\pm\infty\}, +)$ where the element $+\infty$ is dominant. Likewise, $(\mathbb{R}^n)^\triangledown$ is $(\mathbb{R}^n \cup \{\pm\infty\}, +)$ where $-\infty$ is dominant. In case $n = 1$ we write $\mathbb{R}^\vartriangle$ and $\mathbb{R}^\triangledown$, respectively.

(iii) If $(Y, \circ)$ *is a monoid, the monoid* $\left(\widehat{\mathcal{P}}(Y), \odot\right)$ *contains the dominant element* $\emptyset$*.*

(iv) Let $\mathbb{N} = \{1, 2, \ldots\}$ *be the set of positive integers and* $a \circ b := \max\{a, b\}$*. Then* $(\mathbb{N}, \circ)$ *is a commutative monoid with neutral element 1 consisting only of idempotent elements.*

Likewise, the set $\mathbb{N} \cup \{+\infty\}$ *with the operation* $a \circ b := \min\{a, b\}$ *is a commutative monoid with neutral element* $+\infty$ *and the nontrivial dominant element* 1*.*

2.1.2 Conlinear spaces

The concept of a conlinear space generalizes the concept of a real linear (vector) space. In this section, we start with monoids with a binary operation called addition and denoted by $+$.

Definition 5 *A set* Y*, together with an addition* $+$*, is said to be a* **(real) conlinear space** $(Y, +)$ *iff the following axioms are satisfied:*
(C1) $(Y, +)$ *is a commutative monoid with neutral element* θ*.*
(C2) There is mapping from $\mathrm{I\!R}_+ \times Y$ *into* Y*, assigning* $t \geq 0$ *and* $y \in Y$ *the product* $ty := t \cdot y \in Y$ *such that the following conditions are satisfied:*
(i) $\forall y \in Y, \forall s, t \geq 0 : \; s \cdot (t \cdot y) = (st) \cdot y$*;*
(ii) $\forall y \in Y : \; 1 \cdot y = y$*;*
(iii) $\forall y \in Y : \; 0 \cdot y = \theta$*;*
(iv) $\forall t \geq 0, \forall y_1, y_2 \in Y : \; t \cdot (y_1 + y_2) = (t \cdot y_1) + (t \cdot y_2)$*.*

Note that the validity of the second distributive law $(s + t)\, y = (sy) + (ty)$ is not required, not even for $s, t \geq 0$. Instead, we impose (C2, (iii)). This is the main difference to the concept of (ordered) cones in [71], Section 1.1. As a consequence, a conlinear structure is stable under passing to power sets whereas a cone in the sense of [71] is not. See Proposition 10 below.

The following properties are easy to prove directly from the axioms.

Proposition 8 *Let* $(Y, +)$ *be a conlinear space. Then:*
(i) If $t \geq 0$*, then* $t\theta = \theta$*.*
(ii) If $t > 0$*,* $y_1, y_2 \in Y$ *and* $ty_1 = ty_2$*, then* $y_1 = y_2$*.*
(iii) If $y \in Y \backslash \{\theta\}$ *and* $t > 0$*, then* $ty \neq \theta$*.*

Proof. (i) We have $t\theta = t\,(0 \cdot y) = (t0) \cdot y = 0 \cdot y = \theta$.

(ii) Multiplying the equality $ty_1 = ty_2$ by t^{-1} and using (C2, (i)) we obtain the result.

(iii) $ty = \theta$ would imply $t^{-1}(ty) = t^{-1} \cdot \theta = \theta$ which contradicts $t^{-1}(ty) = \left(t^{-1}t\right) y = 1 \cdot y = y \neq \theta$. ■

Note that $t_1 y = t_2 y$ for $y \in Y \backslash \{\theta\}$ does not imply $t_1 = t_2$. An example is given below.

Let $(Y, +)$ be a conlinear space. If $Y' \subseteq Y$ and $(Y', +)$ is itself a conlinear space, then it is called a **conlinear subspace** of Y. A subset $Y' \subseteq Y$ is a conlinear subspace if and only if $t \geq 0$, $y, y_1, y_2 \in Y'$ imply $ty \in Y'$ and $y_1 + y_2 \in Y'$.

Example 5 *(i)* $\mathbb{R}\cup\{+\infty\}$ *is a conlinear space if the usual multiplication with nonnegative real numbers is extended as follows:*

$$\forall t > 0: \ t \cdot (+\infty) = +\infty \quad \text{and} \quad 0 \cdot (+\infty) = 0.$$

Likewise, $\mathbb{R}\cup\{-\infty\}$ *can be supplied with a conlinear structure. These two conlinear spaces can be decomposed into the linear space* $\mathbb{R}$ *(see Definition 9 below) and the conlinear spaces* $\{0, +\infty\}$ *and* $\{0, -\infty\}$*, respectively.*
(ii) $(\mathbb{R}^n)^\vartriangle$ *and* $(\mathbb{R}^n)^\triangledown$ *are conlinear spaces using the same conventions dealing with* $\pm\infty$*, i.e.,*

$$\forall t > 0: \ t \cdot (+\infty) = +\infty, \ t \cdot (-\infty) = -\infty$$

and

$$0 \cdot (+\infty) = 0 \cdot (-\infty) = 0.$$

Similarly, $\{-\infty, 0, +\infty\}$ *can be supplied with a conlinear structure in two different ways. Compare (ii) of Example 1.*

Proposition 9 *Let* X *be a nonempty set and* $(Y, +)$ *be a conlinear space. Then the set* $R(X, Y)$ *of all functions mapping* X *into* Y *is a conlinear space with respect to the pointwise operations*

$$(f_1 \oplus f_2)(x) := f_1(x) + f_2(x), \ x \in X \tag{2.1}$$
$$(t \cdot f)(x) := tf(x), \ t \geq 0, \, x \in X. \tag{2.2}$$

Proof. By (2.1), (2.2) the expressions $f_1 \oplus f_2$ and $t \cdot f$ are well-defined for $f, f_1, f_2 \in R(X, Y)$, $t \geq 0$. Defining the neutral element θ in $R(X, Y)$ by

$$\forall x \in X: \ \theta(x) = \theta_Y$$

where θ_Y is the neutral element of Y the axioms (C1) and (C2) of Definition 5 are easy to check. ■

Let X be a nonempty set. With the definitions of Example 5, the following spaces can be recognized as examples of conlinear spaces with respect to the corresponding pointwise operations by means of Proposition 9: Since $\{0, +\infty\}$ and $\{0, -\infty\}$ are conlinear, the sets

$$I^+(X) := R(X, \{0, +\infty\}) \quad \text{and} \quad I^-(X) := R(X, \{0, -\infty\})$$

can be supplied with a conlinear structure according to (2.1) and (2.2). The same is true for

$$R^+(X) := R(X, \mathbb{R} \cup \{+\infty\}) \quad \text{and} \quad R^-(X) := R(X, \mathbb{R} \cup \{-\infty\})$$

since $\mathbb{R} \cup \{+\infty\}$ and $\mathbb{R} \cup \{-\infty\}$ are conlinear and also for

$$R^\vartriangle(X) := R\left(X, \mathbb{R}^\vartriangle\right) \quad \text{and} \quad R^\triangledown(X) := R\left(X, \mathbb{R}^\triangledown\right)$$

since $\mathbb{R}^\vartriangle$ and $\mathbb{R}^\triangledown$ are conlinear.

Remark 1 *Let $f \in R^+(X)$. Then there are $f_V \in V(X) := \{f \in R^+(X) : f(X) \subseteq \mathrm{I\!R}\}$ and $f_{I^+} \in I^+(X)$ such that $f = f_V \oplus f_{I^+}$, f coincides with f_V on $\mathrm{dom}\, f := \{x \in X : f(x) \in \mathrm{I\!R}\}$ and f_{I^+} is uniquely determined, namely, $f_{I^+} = I^+_{\mathrm{dom}\, f}$.*

Both of $(I^+(X), \oplus)$ and $(V(X), \oplus)$ are conlinear spaces, the latter one is even linear (see Definition 9 below).

Of course, an analogous consideration can be done for $R^-(X)$.

As it is the case for monoids, the property of being a conlinear space is stable under passing to power sets. We define the product of $\alpha \geq 0$ and $M \in \mathcal{P}(Y)$ by $\alpha M := \{\alpha y : y \in M\}$. Concerning $\widehat{\mathcal{P}}(Y)$, we define $\alpha \cdot \emptyset = \emptyset$ for $\alpha > 0$ and $0 \cdot \emptyset = \{\theta\}$.

Proposition 10 *Let $(Y, +)$ be a conlinear space. Then $(\mathcal{P}(Y), \oplus)$ and $\left(\widehat{\mathcal{P}}(Y), \oplus\right)$ are conlinear spaces as well.*

Proof. We know from Proposition 1 that $(\mathcal{P}(Y), \oplus)$ and $\left(\widehat{\mathcal{P}}(Y), \oplus\right)$ are commutative monoids with neutral element $\Theta = \{\theta\}$, hence axiom (C1) is satisfied. The properties (C2, (i)) to (C2, (iv)) are easy to check. ■

Definition 6 *Let $(Y, +)$ be a conlinear space. An element $y \in Y$ is said to be a* **cone** *iff*

$$\forall t > 0 : \quad ty = y.$$

A cone $y \neq \theta$ is called **nontrivial**. *The set of all cones of Y is denoted by Y_c.*

This definition looks somehow unusual. Setting $Y = \mathcal{P}(\mathrm{I\!R}^n)$ for example, we rediscover cones as subsets of the linear space $\mathrm{I\!R}^n$, see [105], p. 13. There are further objects being cones in the sense of the above definition. For example, $+\infty$ is a cone of $\mathrm{I\!R}^\vartriangle$, compare (ii) of Example 5. Note that a cone of a conlinear space is not necessarily an idempotent element of the underlying monoid. This is since $2y \neq y + y$ in general.

Proposition 11 *Let $(Y, +)$ be a conlinear space. Then $(Y_c, +)$ is a conlinear space as well.*

Proof. It already suffices to show that $y_1, y_2 \in Y_c$ implies $y_1 + y_2 \in Y_c$. This follows by (C2, iv). ■

If $y \in Y$ is a cone, then $\{y\} \in \mathcal{P}(Y)$ is a cone, too.

The concept of a conlinear space is sufficient to define convexity. In fact, it seems to be the natural framework for convexity rather than linear spaces. Here, we only give the definition of convex elements and convex subsets of a conlinear space as well as some elementary facts.

Definition 7 *Let* $(Y, +)$ *be a conlinear space. An element* $y \in Y$ *is said to be* **convex** *iff*

$$\forall t_1, t_2 > 0: \quad (t_1 + t_2)\, y = t_1 y + t_2 y. \tag{2.3}$$

The set of all convex elements of Y *is denoted by* Y_{co}.

A subset $M \subset Y$ *is called a* **convex subset** *of* Y *iff*

$$\forall t \in (0,1): \quad tM \oplus (1-t)\, M \subseteq M. \tag{2.4}$$

The set of all nonempty convex subsets of Y *is denoted by* $Co\,(Y)$.

Of course, $\theta \in Y$ is always a convex element.

Proposition 12 *Let* $(Y, +)$ *be a conlinear space. Then* $(Y_{co}, +)$, $(Co\,(Y), \oplus)$ *and* $\left(\widehat{Co}\,(Y), \oplus\right)$ *are conlinear spaces as well. Thereby,* $\widehat{Co}\,(Y) = Co\,(Y) \cup \{\emptyset\}$.

Proof. Concerning Y_{co}, it suffices to show that $ty, y_1 + y_2 \in Y_{co}$ whenever $y, y_1, y_2 \in Y_{co}$ and $t \geq 0$. This is straightforward using (C2, i, iv) and (2.3).

Concerning $Co\,(Y)$, we have to show that $M, M_1, M_2 \in Co\,(Y)$ implies $tM \in Co\,(Y)$ whenever $t \geq 0$ and $M_1 \oplus M_2 \in Co\,(Y)$. This is straightforward as well as to check the axioms (C1), (C2).

The extension to $\widehat{Co}\,(Y)$ is obvious. ■

Note that Y_{co} is a conlinear subspace of $(Y, +)$, while $Co\,(Y)$ and $\left(\widehat{Co}\,(Y), \oplus\right)$ are conlinear subspaces of $(\mathcal{P}\,(Y), \oplus)$.

Putting Proposition 11 and 12 together, the following result is obtained.

Proposition 13 *Let* $(Y, +)$ *be a conlinear space. Then* $(Y_c \cap Y_{co}, +)$ *is a conlinear space as well.*

Proof. Immediately by Propositions 11 and 12. ■

The following two propositions answer the question for the relationships between convex subsets of $(Y, +)$ and convex elements of $(\mathcal{P}\,(Y), \oplus)$. In our general framework, the situation is a bit more complicated than in the linear case, i.e. $Y = V$ is a linear space (see [105], Theorem 3.2.). This is due to the fact that a convex subset of a conlinear space may contain nonconvex elements.

Proposition 14 *Let* $(Y, +)$ *be a conlinear space. Then, every convex element of* $(\mathcal{P}\,(Y), \oplus)$ *is a convex subset of* $(Y, +)$.

Proof. Let $M \subseteq Y$ be a convex element of $(\mathcal{P}\,(Y), \oplus)$, i.e., for all $t_1, t_2 > 0$,

$$(t_1 + t_2)\, M = t_1 M \oplus t_2 M. \tag{2.5}$$

We have to show (2.4). Take $t \in (0,1)$. Set $t_1 = t$, $t_2 = 1-t$. By (2.5), we have $tM \oplus (1-t)\, M \subseteq M$. ■

The most simple condition for a convex subset $M \subseteq Y$ to be a convex element of $(\mathcal{P}(Y), \oplus)$ is of course $M \subseteq tM \oplus (1-t)\, M$ whenever $t \in [0,1]$, or equivalently, $(t_1 + t_2)\, M = t_1 M + t_2 M$ whenever $t_1, t_2 > 0$. An important special case gives the following proposition.

Proposition 15 *Let $(Y, +)$ be a conlinear space. Then, a convex subset $M \subseteq Y$ containing only convex elements is a convex element of $(\mathcal{P}(Y), \oplus)$.*

PROOF. Let $M \subseteq Y$ be a convex subset consisting of convex elements only. We have to show that (2.3) holds true. By (2.4), we have $tM \oplus (1-t)\, M \subseteq M$ for $t \in (0,1)$. Since $t_1 + t_2 > 0$, we can replace t by $\frac{t_1}{t_1+t_2}$ and multiply by $t_1 + t_2$. This gives $t_1 M \oplus t_2 M \subseteq (t_1 + t_2)\, M$. Conversely, take $y \in M$. Then $(t_1 + t_2)\, y = t_1 y + t_2 y$, since M consists of convex elements only. Hence $(t_1 + t_2)\, M \subseteq t_1 M \oplus t_2 M$ completing the proof. ■

Remark 2 *There are subsets of conlinear spaces satisfying (2.4) but do not consist of convex elements only. Moreover, a convex subset of a conlinear space Y is not necessarily a convex element of $\mathcal{P}(Y)$.*

For example, take $Y = \mathcal{P}(\mathrm{I\!R})$ and $M = \mathcal{P}([0,1])$. M is a convex subset of Y, but neither it consists only of convex elements nor is it a convex element of $\mathcal{P}(Y)$. Observe that for $y := \{0,1\} \in M$ we do not have $y \in \frac{1}{2}M \oplus \frac{1}{2}M$, hence $M \neq \frac{1}{2}M \oplus \frac{1}{2}M$. To see this, assume $y = \frac{1}{2}y_1 + \frac{1}{2}y_2$, $y_1, y_2 \in M$. Then $1 \in y_1, y_2$ as well as $0 \in y_1, y_2$. This implies $\frac{1}{2} \in y$, a contradiction.

Some important facts about convex subsets of conlinear spaces carry over from the linear theory. Compare [105], §2.

Theorem 1 *The intersection of an arbitrary collection of convex subsets of a conlinear space is a convex subset.*

PROOF. Elementary. ■

Again, there is an additional assumption necessary for convex elements of $\mathcal{P}(Y)$.

Corollary 1 *Let $(Y, +)$ be a conlinear space. Let $M_\alpha \subseteq Y$, $\alpha \in A$ be a family of convex subsets of $(\mathcal{P}(Y), \oplus)$. If the intersection*

$$M := \bigcap_{\alpha \in A} M_\alpha$$

contains only convex elements of Y, then M is a convex element of $(\mathcal{P}(Y), \oplus)$.

Proof. By Theorem 1, M is a convex subset. Since M contains only convex elements by assumption, Proposition 15 gives the result. ■

Let n be a positive integer. We call a sum

$$t_1 y_1 + t_2 y_2 + \ldots + t_n y_n$$

a **convex combination** of the elements $y_i \in Y$, $i = 1, \ldots, n$, whenever $t_i \geq 0$, $i = 1, \ldots, n$ and $\sum_{i=1}^{n} t_i = 1$.

Theorem 2 *Let $(Y, +)$ be a conlinear space. A subset $M \subseteq Y$ is a convex subset if and only if it contains all the convex combinations of its elements.*

Proof. The if-part is obvious, the only-if-part by induction. ■

Definition 8 *Let $(Y, +)$ be a conlinear space and $M \subseteq Y$ a subset. The* **convex hull** *$co\, M$ of M is the intersection of all convex subsets of Y containing M.*

By Theorem 1, $co\, M$ is always a convex subset of Y.

Theorem 3 *Let $(Y, +)$ be a conlinear space and $M \subseteq Y$ a subset. Then $co\, M$ coincides with the set of all convex combinations of elements of M.*

Proof. By Theorem 2, the set of all convex combination of elements of M is contained in $co\, M$. Conversely, let

$$u = \sum_{i=1}^{n} t_i u_i, \qquad v = \sum_{j=1}^{m} s_j v_j$$

convex combinations of elements $u_i, v_j \in M$. Take $t \in (0, 1)$. Then

$$y := tu + (1-t)\, v = \sum_{i=1}^{n} (tt_i)\, u_i + \sum_{j=1}^{m} ((1-t)\, s_j)\, v_j$$

is a convex combination of elements of M, too. Hence the set of all convex combinations of elements of M is a convex subset and contains M. Hence it coincides with $co\, M$. ■

Corollary 2 *Let $(Y, +)$ be a conlinear space. Then $y \in Y$ is a convex element if and only if*

$$co\, \{y\} = \{y\}\,.$$

Proof. If y is a convex element, then every convex combination of y with itself coincides with y. Conversely, we have for $t \in (0, 1)$

$$y = ty + (1-t)\, y.$$

Let $t_1, t_2 > 0$. Substituting $t = \frac{t_1}{t_1 + t_2}$ and multiplying by $t_1 + t_2$ we obtain

$$(t_1 + t_2)\, y = t_1 y + t_2 y$$

as desired. ■

Remark 3 *The convex hull of $\{y\}$ may happen to contain more than one element. In general,*

$$co\ \{y\} = \left\{ \sum_{i=1}^{n} t_i y : \ t_i \geq 0, \ \sum_{i=1}^{n} t_i = 1, \ n \in \mathbb{N}, \ n \geq 1 \right\}.$$

A convex element of the conlinear space Y being a cone at the same time is called a **convex cone** in Y.

Proposition 16 *Let $(Y, +)$ be a conlinear space. A cone $y \in Y$ is a convex element if and only if $y + y = y$, i.e. y is an idempotent element of the monoid constituting Y.*

PROOF. (1) Let $y \in Y$ be a cone and a convex element. Then for $t_1 = t_2 = 1$ we obtain from (2.3) $y = 2y = y + y$.

(2) Let $y \in Y$ be a convex cone. For $t_1, t_2 > 0$ equality (2.3) reduces to $y = y$. ■

It turns out that a convex cone of $\mathcal{P}(Y)$ is itself a conlinear space if it contains $\theta \in Y$. Hence, the terms conlinear subspace of Y and convex cone of $\mathcal{P}(Y)$ containing $\theta \in Y$ are synonyms in the framework of conlinear spaces.

Moreover, a cone in $\mathcal{P}(Y)$ being a convex subset of Y is almost a convex element.

Proposition 17 *Let $C \in \mathcal{P}(Y)$ be a cone containing $\theta \in Y$. Then C is a convex element of $\mathcal{P}(Y)$ if and only if it is a convex subset of Y.*

PROOF. Since every convex element of $\mathcal{P}(Y)$ is a convex subset of Y (Proposition 14), it remains to show the converse. The cone property and (2.4) for $t \in (0, 1)$ imply $tC \oplus (1 - t) C = C \oplus C \subseteq C$. Since $\theta \in C$, we have $C \subseteq C \oplus C$, hence $C = C \oplus C$. Proposition 16 gives the result. ■

Example 6 *Set $Y = \mathcal{P}(\mathbb{R})$, $C = \mathcal{P}(\mathbb{R}_+) \setminus \{0\}$. Then C is a convex subset of Y, a cone, but not a convex element of $\mathcal{P}(Y)$. To see this, take $c = \{0, 1\} \in C$ and assume $c = c_1 + c_2$, $c_1, c_2 \in C$. Then $0 \in c_1, c_2$ and $\gamma_1 \in c_1$, $\gamma_2 \in c_2$ such that $\gamma_1, \gamma_2 \geq 0$, $\gamma_1 + \gamma_2 = 1$. Hence $c = \{0, 1\} = c_1 + c_2 \supseteq \{0, \gamma_1, \gamma_2, \gamma_1 + \gamma_2\}$. Without loss of generality, we must have $\gamma_1 = 0$, $\gamma_2 = 1$. This implies $c_1 = \{0\}$ which is not possible.*

Let $(Y, +)$ be conlinear space. According to Definition 2, we denote the set of invertible elements of Y with respect to $+$ by Y_{in}. We finish this section by defining a linear space.

Definition 9 *A conlinear space $(Y, +)$ is said to be a (real)* **linear space** *iff it consists only of elements being convex and invertible at the same time.*

We shall show that this definition is consistent with the usual definition of a linear (vector) space. A definition of linear spaces can be found e.g. in [75] vol. I, §. We state the fact in a more convenient form.

Theorem 4 *Let $(Y,+)$ be a conlinear space and $Y_l := Y_{in} \cap Y_{co} \subseteq Y$. Then $(Y_l,+)$ is a linear space, and it is the largest one contained in Y.*

Proof. For $y \in Y_l$, we define a multiplication with negative reals by

$$(-\alpha)\, y := \alpha y'$$

where $\alpha > 0$ and $y+y' = \theta$. It remains to show the following properties for all $y, y_1, y_2 \in Y_l$, $\alpha, \beta \in \mathrm{I\!R}$:
(1) $(Y_l,+)$ is a commutative group.
(2) $\alpha\,(y_1 + y_2) = \alpha y_1 + \alpha y_2$.
(3) $(\alpha + \beta)\, y = \alpha y + \beta y$.
(4) $\alpha\,(\beta y) = (\alpha\beta)\, y$.
Let's start with (1). We have to show that $y_1, y_2 \in Y_l$ implies $y_1 + y_2 \in Y_l$. Since the set of all invertible elements of a monoid forms a group, $y_1 + y_2$ is invertible. Since inverse elements in groups are unique, we have $(y_1 + y_2) + (y_1' + y_2') = (y_1 + y_1') + (y_2 + y_2') = \theta$, hence $(y_1 + y_2)' = y_1' + y_2'$. Applying (C2, iv) the convexity of y_1, y_2 implies

$$\begin{aligned}
(\alpha+\beta)\,(y_1+y_2) &\overset{(C2,\,iv)}{=} (\alpha+\beta)\,y_1 + (\alpha+\beta)\,y_2 \\
&\overset{y_1,y_2\ convex}{=} \alpha y_1 + \beta y_1 + \alpha y_2 + \beta y_2 \\
&\overset{(C1),(C2,\,iv)}{=} \alpha\,(y_1+y_2) + \beta\,(y_1+y_2)\,,
\end{aligned}$$

hence $y_1 + y_2$ is convex. Hence $(Y_l,+)$ is a commutative group.

(2) has to be proven for $\alpha < 0$. This follows from (1) and the convexity of the y_1', y_2'.

(3) is obvious for $\alpha, \beta > 0$ and $\alpha, \beta < 0$. Without loss of generality, consider the case $\alpha < 0$, $\beta > 0$ and $\alpha + \beta < 0$. Then

$$\begin{aligned}
(\alpha+\beta)\,y &= \beta y + \beta y' + (-1)\,(\alpha+\beta)\,y' \\
&\overset{\beta,-(\alpha+\beta)>0}{=} \beta y + (\beta + (-1)\,(\alpha+\beta))\,y' \\
&= (-1)\,\alpha y' + \beta y \\
&= \alpha y + \beta y.
\end{aligned}$$

(4) can be proven by a case study with respect to α, β. Exemplary, we check the case $\alpha > 0$, $\beta < 0$. Then

$$\begin{aligned}
\alpha\,(\beta y) &= \alpha\,(-|\beta|\,y) = \alpha\left(|\beta|\,y'\right) \\
&\overset{(C2,\,i)}{=} (\alpha\,|\beta|)\,y' = (\alpha\beta)\,y.
\end{aligned}$$

■

The set $Y_l := Y_{in} \cap Y_{co}$ is called the **lineality space** of Y.

Corollary 3 *A conlinear space $(Y,+)$ is linear if and only if $Y = Y_{co} \cap Y_{in}$.*

Every element of a linear space V is a convex element, hence every subset of V consists of convex elements only. Hence a subset $M \subseteq V$ is convex if and only if M is a convex element of $(\mathcal{P}(V), \oplus)$. For cones, something more can be said.

Corollary 4 *Let $(V, +)$ be a linear space and $C \subseteq V$ a cone of $(\mathcal{P}(V), \oplus)$. Then the following facts are equivalent:*
(i) C is a convex element of $(\mathcal{P}(V), \oplus)$.
(ii) C is a convex subset of $(V, +)$.
(iii) $C \oplus C \subseteq C$.

PROOF. The equivalence of (i) and (ii) is clear from the remark above. We have $C \subseteq C \oplus C$, since $c \in C$ implies $\frac{1}{2}c \in C$ and consequently $c = \frac{1}{2}c + \frac{1}{2}c \in C \oplus C$. Hence $C \oplus C = C$. The equivalence of (i) and (iii) follows from Proposition 16. ■

¿From the results above, one may see that every convex subset of a linear space V containing $\theta \in V$ and being a cone in $(\mathcal{P}(V), \oplus)$ is a conlinear space. However, it is not possible to reduce the investigation of conlinear spaces to convex cones as subsets of linear spaces.

Theorem 5 *A conlinear space with a nontrivial cone can not be embedded into a linear space.*

PROOF. Let $(Y, +)$ be a conlinear space and $(V, +)$ a linear space such that $Y \subseteq V$ and $+$ coincides on Y. Let $y \in Y$, $y \neq \theta$ be a nontrivial cone. Then there is $v \in V$, $v \neq \theta$ such that $y + v = \theta$. Since V is linear, we have $2y = y + y$ and therefore $\theta = v + y = v + 2y = v + y + y = y$, a contradiction. ■

Example 7 *Let $(V, +)$ be a linear space. (i) Since $(V, +)$ is especially conlinear, by Proposition 10 $(\mathcal{P}(V), \oplus)$ and $\left(\widehat{\mathcal{P}}(V), \oplus\right)$ are conlinear spaces as well.*

(ii) The set of all convex cones of $(\mathcal{P}(V), \oplus)$ containing $\theta \in V$ form a conlinear space consisting only of idempotent elements. This follows from Proposition 13.

2.1.3 Semilinear spaces

In the last paragraph, we have seen that a linear space can be understood as the subset of a conlinear space contaning those elements which are invertible and convex at the same time. In this case, the definition of negative multiples was possible. Conversely, considering e.g. the power set of a linear space, it seems to be a natural idea to have a multiplication with negative real numbers, even though inverse elements with respect to the addition do not exists.

Note that not all conlinear spaces admit such an operation. For example, a pointed convex cone of a linear space is a conlinear space, but does not contain the negative of any of its elements beside zero.

We call a conlinear space with a (-1)-multiplication a semilinear space. This concept is very close to that of an almost linear space introduced by G. Godini [42] around 1985.

Definition 10 *A set Y, together with an addition $+$, is said to be a* **(real) semilinear space** $(Y,+)$ *iff the following axioms are satisfied:*
(S1) $(Y,+)$ *is a commutative monoid with neutral element θ;*
(S2) For any two elements $y \in Y$ and $t \in \mathbb{R}$ there exists the product $ty := t \cdot y \in Y$ such that the following conditions are satisfied:
(i) $\forall y \in Y, \forall s,t \in \mathbb{R} : \ s(ty) = (st)\,y$;
(ii) $\forall y \in Y : \ 1 \cdot y = y$;
(iii) $\forall y \in Y : \ 0 \cdot y = \theta$;
(iv) $\forall t \in \mathbb{R}, \forall y_1, y_2 \in Y : \ t(y_1 + y_2) = (ty_1) + (ty_2)$.

Again, the second distributive law $(s+t)\,y = (sy) + (ty)$ does not hold in general, not even for nonnegative numbers. This is a difference to Godini's almost linear spaces [42]. The second distributive law is not valid for the power set of a linear space being a semilinear but not an almost linear space. The following properties can be proven in the same way as Proposition 8.

Proposition 18 *Let $(Y,+)$ be a semilinear space. Then:*
(i) If $t \in \mathbb{R}$, then $t\theta = \theta$.
(ii) If $t \in \mathbb{R} \setminus \{0\}$, $y_1, y_2 \in Y$ and $ty_1 = ty_2$, then $y_1 = y_2$.
(iii) If $y \in Y \setminus \{\theta\}$ and $t \in \mathbb{R} \setminus \{0\}$, then $ty \neq \theta$.

Starting from a semilinear space we are able to generate new semilinear spaces by passing to power sets.

Proposition 19 *Let $(Y,+)$ be a semilinear space. Defining the product of $t \in \mathbb{R}$ and $M \in \mathcal{P}(Y)$ by $tM := \{ty : \ y \in M\}$ and agreeing on $t \cdot \emptyset = \emptyset$ for $t \neq 0$ and $0 \cdot \emptyset = \{\theta\}$, the spaces $(\mathcal{P}(Y), \oplus)$ and $\left(\widehat{\mathcal{P}}(Y), \oplus\right)$ are semilinear spaces as well.*

Proof. We know from Proposition 1 that $(\mathcal{P}(Y), \oplus)$ and $\left(\widehat{\mathcal{P}}(Y), \oplus\right)$ are commutative monoids with neutral element $\Theta = \{\theta\}$, hence axiom (S1) is satisfied. The properties (S2, (i)) to (S2, (iv)) are easy to check. ■

Let $(Y,+)$ be a semilinear space. If $Y' \subseteq Y$ and $(Y',+)$ is itself a semilinear space with the same multiplication with real numbers as Y, then it is called a **semilinear subspace** of Y. A subset $Y' \subseteq Y$ is a semilinear subspace if and only if $t \in \mathbb{R}$, $y, y_1, y_2 \in Y'$ imply $ty \in Y'$ and $y_1 + y_2 \in Y'$.

Let $(Y,+)$ be a semilinear space. We define the set of invertible, convex and symmetric elements and the set of cones of Y, respectivly, by

$$\begin{aligned}
Y_{in} &:= \left\{y \in Y : \ \exists y' \in Y : \ y + y' = \theta\right\}, \\
Y_{co} &:= \{y \in Y : \ \forall t_1, t_2 \geq 0 : \ (t_1 + t_2)\,y = t_1 y + t_2 y\}, \\
Y_{sy} &:= \{y \in Y : \ y = (-1)\,y\}, \\
Y_c &:= \{y \in Y : \ \forall t > 0 : \ ty = y\}.
\end{aligned}$$

As in the case of conlinear spaces, we denote $Y_l := Y_{in} \cap Y_{co}$.

Proposition 20 *Let $(Y,+)$ be a semilinear space. Then $(Y_c,+)$, $(Y_{co},+)$ and $(Y_{sy},+)$ are semilinear spaces as well.*

PROOF. Take $y \in Y_c$. Then $(-1)\,y \in Y_c$ by (S2, (i)): For $t > 0$, we obtain $t\,(-1)\,y = (-1)\,(ty) = (-1)\,y$. Taking $y_1, y_2 \in Y_c$, by (S2, (iv)) it follows $t\,(y_1 + y_2) = ty_1 + ty_2 = y_1 + y_2$, hence $y_1 + y_2 \in Y_c$. Therefore, $(Y_c,+)$ is a semilinear space.

By similar considerations, one can show that $(Y_{co},+)$ and $(Y_{sy},+)$ are semilinear spaces, too. ∎

Proposition 21 *Let $(Y,+)$ be a semilinear space. Then, $(Y_l,+)$ is a linear subspace of $(Y,+)$, and it is the largest one contained in Y.*

PROOF. Every semilinear space is all the more conlinear, hence the result follows by Theorem 4. ∎

Of course, every linear space is almost linear, every almost linear space is semilinear and every semilinear space is conlinear. There exist examples showing that these classes do not coincide. Several examples are listed below.

Example 8 *(i) Let $(V,+)$ be a real linear space. Then it is a semilinear space. We only have to prove that (S1), (S2) imply (C2, (iii)). We omit the proof noting that either the group property or (S2, (iii)) has to be involved.*
(ii) Let $(V,+)$ be a real linear space. Then $(\mathcal{P}(V), \oplus)$ is a semilinear space as well as $\left(\widehat{\mathcal{P}}(V), \oplus\right)$, $(Co(V), \oplus)$ and $\left(\widehat{Co}(V), \oplus\right)$.
(iii) The spaces $\mathbb{R} \cup \{+\infty\}$ and $R^+(X)$ from Example 5, (i) are conlinear, but not semilinear.
(iv) The space $R^\vartriangle(X)$ of all functions $f : X \to \mathbb{R}^\vartriangle$ is a semilinear space as well as $R^\triangledown(X)$.

With the aid of topological properties more examples of semilinear (and conlinear) space may be obtained. Compare Section 2.3.5.

2.2 Order structures

2.2.1 Basic notation

We recall basic order theoretic notation necessary for the following considerations. We refer to [32], [36] and [130].

Let W be a nonempty set. A binary relation on W is understood to be a subset $R \subseteq W \times W$. We say that $w_1 \in W$ **is related to** $w_2 \in W$ iff $(w_1, w_2) \in R$. In this case, we shortly write $w_1 R w_2$. If wRw for all $w \in W$, the relation R is called **reflexive**. If $w_1, w_2, w_3 \in W$, $w_1 R w_2$ and $w_2 R w_3$ implies $w_1 R w_3$, the relation R is called **transitive**. If $w_1 R w_2$, $w_2 R w_1$ for $w_1, w_2 \in W$ implies $w_1 = w_2$, the relation R is called **antisymmetric**.

Definition 11 *Let W be a nonempty set and R a relation on W. R is called a* **quasiorder** *iff it is reflexive and transitive. R is called a* **partial order** *iff it is reflexive, transitive and antisymmetric.*

If R is a quasiorder on W, we write $w_1 \preceq w_2$ instead of $w_1 R w_2$ (or $(w_1, w_2) \in R$) and speak about the quasiorder $\preceq$. The couple $(W, \preceq)$ is called a quasiordered set.

Definition 12 *Let $(W, \preceq)$ be a quasiordered set. The* **lower (upper) section** *$S_l(w)$ $(S_u(w))$ of $w \in W$ are defined by*

$$S_l(w) := \left\{w' \in W : \ w' \preceq w\right\}, \quad S_u(w) := \left\{w' \in W : \ w \preceq w'\right\}.$$

The set of **minimal (maximal) elements** $\min(W)$ $(\max(W))$ *is defined by*

$$\begin{aligned} \min(W) &:= \{w \in W : \ S_l(w) \subseteq S_u(w)\}, \\ \max(W) &:= \{w \in W : \ S_u(m) \subseteq S_l(w)\}. \end{aligned}$$

Of course, an element $\bar{w} \in W$ is **minimal** with respect to $\preceq$ iff

$$w \in W, \ w \preceq \bar{w} \qquad \Longrightarrow \qquad \bar{w} \preceq w.$$

If $\preceq$ is additionally antisymmetric and $\bar{w} \in \min(W)$, then $w \in W$, $w \preceq \bar{w}$ even implies $w = \bar{w}$. Analogous conditions hold true for maximal elements.

Having a quasiordered set $(W, \preceq)$, by a standard procedure an equivalence relation $\sim$ can be defined by

$$w \sim w' \quad \Longleftrightarrow \quad w \preceq w', \ w' \preceq w.$$

Denoting

$$[w] := \left\{w' \in W : \ w' \sim w\right\}$$

and

$$[w] \preceq \left[w'\right] \quad \Longleftrightarrow \quad \forall w \in [w], w' \in \left[w'\right] : \ w \preceq w',$$

the set of all equivalence classes $[W]$ together with $\preceq$ is a partially ordered set. Compare [36], p. 13 or [32], Satz 3.19 for more details.

A subset $M \subseteq W$ is called **bounded from above (below)** in W iff there exist an $w \in W$ such that $m \preceq w$ $(w \preceq m)$ for all $m \in M$. In this case, w is called **upper (lower) bound** of M. A **supremum (infimum)** of M in W is an upper (lower) bound $w \in W$ such that $w \preceq w'$ $(w' \preceq w)$ for any other upper (lower) bound w' of M in W. We use $\sup M$ and $\inf M$, respectively, to denote a supremum and infimum of M. If $(W, \preceq)$ is partially ordered, then $\sup M$ and $\inf M$, if they exist, are unique.

If for every pair of elements $m_1, m_2 \in M \subseteq W$ there exists an upper (lower) bound in M, then M is said to be **directed upwards** (resp. **downwards**).

The quasiordered set $(W, \preceq)$ is called **Dedekind complete** iff every nonempty subset having an upper bound (lower bound) has a supremum (infimum) in W. Note that the two conditions are not independent: $(W, \preceq)$ is Dedekind complete if and only if every nonempty subset having an upper bound has a supremum ([130], Theorem 1.4).

The quasiordered set $(W, \preceq)$ is called **order complete** iff every nonempty subset has an infimum and a supremum in W.

The quasiordered set $(W, \preceq)$ is called a **lattice** iff every subset consisting of two points has an infimum and a supremum in W.

An element $\bar{w} \in W$ is said to be the **largest element** iff $w \preceq \bar{w}$ for all $w \in W$. The **smallest element** is defined analogously. If $(W, \preceq)$ is partially ordered, then the largest and smallest element, if they exist, are unique.

Remark 4 *Let $(W, \preceq)$ be quasiordered. If W has a largest as well as a smallest element, then it is Dedekind complete if and only if it is order complete. If W is order complete, then W has a largest as well as a smallest element. Compare [130], p. 3.*

2.2.2 Ordered product sets

The following definition deals with subsets of a product set supplied with a quasiorder.

Definition 13 *Let X, Y be two nonempty sets and $W = X \times Y$ the set of all ordered pairs (x, y), $x \in X$, $y \in Y$. The quasiorder $\preceq$ on W is called* **partially antisymmetric** *(with respect to X) iff for all $(x, y), (x', y') \in W$*

$$(x, y) \preceq \left(x', y'\right), \ \left(x', y'\right) \preceq (x, y) \quad \Longrightarrow \quad x = x'.$$

It is clear that if $\preceq$ is a partially antisymmetric quasiorder on W, then a point $\bar{w} = (\bar{x}, \bar{y}) \in W$ is a minimal point with respect to $\preceq$ if and only if

$$(x, y) \in W, \ (x, y) \preceq (\bar{x}, \bar{y}) \quad \Longrightarrow \quad x = \bar{x} \text{ and } (\bar{x}, \bar{y}) \preceq (x, y)\,.$$

In some cases, the y-component is not of interest. Therefore, we give the following definition.

Definition 14 *Let X, Y be two nonempty sets and $\preceq$ a partially antisymmetric quasiorder on $W = X \times Y$. A point $\bar{w} = (\bar{x}, \bar{y}) \in W$ is called a* **partial minimal point** *of W iff*

$$(x, y) \in W, \ (x, y) \preceq (\bar{x}, \bar{y}) \quad \Longrightarrow \quad x = \bar{x}.$$

Analogously, partial maximal points are defined.

Of course, if $\preceq$ is a partially antisymmetric quasiorder on $W = X \times Y$, then every minimal point of W is also a partial minimal point while the converse is not true in general.

2.2.3 Power sets of ordered sets

Let $(W, \preceq)$ be quasiordered. We extend the ordering $\preceq$ to the set $\widehat{\mathcal{P}}(W)$, the set of all subsets of W including the empty set, by defining

$$M_1 \preccurlyeq M_2 \quad :\Longleftrightarrow \quad \forall m_2 \in M_2\, \exists m_1 \in M_1 :\; m_1 \preceq m_2 \tag{2.6}$$
$$M_1 \curlyeqprec M_2 \quad :\Longleftrightarrow \quad \forall m_1 \in M_1\, \exists m_2 \in M_2 :\; m_1 \preceq m_2 \tag{2.7}$$

for $M_1, M_2 \in \mathcal{P}(W)$. If $M_2 \subseteq M_1$, then $M_1 \preccurlyeq M_2$ and $M_2 \curlyeqprec M_1$ by reflexivity of $\preceq$.

Observe that $W \preccurlyeq M$ and $M \curlyeqprec W$ for each $M \in \mathcal{P}(W)$, i.e. W is the smallest element for $\preccurlyeq$ and the largest for $\curlyeqprec$. Setting $M_1 = M$, $M_2 = \emptyset$ in (2.6) and $M_1 = \emptyset$, $M_2 = M$ in (2.7) we may find

$$\forall M \in \widehat{\mathcal{P}}(W) :\; M \preccurlyeq \emptyset, \quad \emptyset \curlyeqprec M. \tag{2.8}$$

This means, $\emptyset$ is the largest element for $\preccurlyeq$ and the smallest for $\curlyeqprec$. Note that for $M_i = \{w_i\}$, $w_i \in W$ for $i = 1, 2$, we have

$$M_1 \preccurlyeq M_2 \quad \Leftrightarrow \quad M_1 \curlyeqprec M_2 \quad \Leftrightarrow \quad w_1 \preceq w_2,$$

i.e., the ordering relations $\preccurlyeq$ and $\curlyeqprec$ can be understood to be extensions of $\preceq$ to $\widehat{\mathcal{P}}(W)$. In fact, they are quasiorders.

Proposition 22 *Let $(W, \preceq)$ be a quasiordered set. Then $(\mathcal{P}(W), \preccurlyeq)$, $(\mathcal{P}(W), \curlyeqprec)$, $\left(\widehat{\mathcal{P}}(W), \preccurlyeq\right)$ and $\left(\widehat{\mathcal{P}}(W), \curlyeqprec\right)$ are quasiordered as well.*

Proof. Reflexivity and transitivity of $\preccurlyeq$ and $\curlyeqprec$ on $\widehat{\mathcal{P}}(W)$ follow immediately from (2.6), (2.7) and (2.8). ■

Note that neither $\preccurlyeq$ nor $\curlyeqprec$ are partial orders in general, not even if $\preceq$ is antisymmetric. One can easy construct counterexamples for $\left(W = \mathrm{I\!R}^1, \leq\right)$. However, if we start with $(W, =)$, we obtain $(\mathcal{P}(W), \supseteq)$ and $(\mathcal{P}(W), \subseteq)$ being partial orders.

The next result contains formulas for infima and suprema in $\widehat{\mathcal{P}}(W)$ with respect to $\preccurlyeq$ and $\curlyeqprec$.

Theorem 6 *Let $(W, \preceq)$ be quasiordered. Then:*
(i) $(\mathcal{P}(W), \preccurlyeq)$ is Dedekind complete. If $\mathcal{M} \subseteq \mathcal{P}(W)$ is nonempty, then it is bounded below and

$$I^* := \bigcup_{M \in \mathcal{M}} \bigcup_{m \in M} \{w \in W :\; m \preceq w\} \tag{2.9}$$

is an infimum of $\mathcal{M}$ with respect to $\preccurlyeq$. If $\mathcal{M} \subseteq \mathcal{P}(W)$ is nonempty and bounded above, then the set

$$S^* := \bigcap_{M \in \mathcal{M}} \bigcup_{m \in M} \{w \in W :\; m \preceq w\} \tag{2.10}$$

is a supremum of $\mathcal{M}$.
(ii) $(\mathcal{P}(W), \preccurlyeq)$ *is Dedekind complete. If* $\mathcal{M} \subseteq \mathcal{P}(W)$ *is nonempty and bounded below, then the set*

$$I^\diamond := \bigcap_{M \in \mathcal{M}} \bigcup_{m \in M} \{w \in W : \, w \preceq m\} \tag{2.11}$$

is an infimum of $\mathcal{M}$. *If* $\mathcal{M} \subseteq \mathcal{P}(W)$ *is nonempty, then* $\mathcal{M}$ *is bounded above and the set*

$$S^\diamond := \bigcup_{M \in \mathcal{M}} \bigcup_{m \in M} \{w \in W : \, w \preceq m\} \tag{2.12}$$

is a supremum of $\mathcal{M}$.
(iii) $\left(\widehat{\mathcal{P}}(W), \curlyeqprec\right)$ *is order complete. If* $\mathcal{M} \subseteq \widehat{\mathcal{P}}(W)$ *is nonempty, then* I^* *from (2.9) is an infimum of* $\mathcal{M}$ *and* S^* *from (2.10) is a supremum of* $\mathcal{M}$. *If* $\emptyset$ *is the only upper bound of* $\mathcal{M}$, *then* $S^* = \emptyset$.
(iv) $\left(\widehat{\mathcal{P}}(W), \preccurlyeq\right)$ *is order complete. If* $\mathcal{M} \subseteq \widehat{\mathcal{P}}(W)$ *is nonempty, then* $I^\diamond$ *from (2.11) is an infimum of* $\mathcal{M}$ *and* $S^\diamond$ *from (2.12) is a supremum of* $\mathcal{M}$. *If* $\emptyset$ *is the only lower bound of* $\mathcal{M}$, *then* $I^\diamond = \emptyset$.

Proof. (i) Let $\mathcal{M} \subseteq \mathcal{P}(W)$ be nonempty. Then $\mathcal{M}$ is bounded below by $\emptyset \neq \overline{M} := \bigcup_{M \in \mathcal{M}} M$ since for each $M \in \mathcal{M}$ we have $M \subseteq \overline{M}$ and this implies $\overline{M} \curlyeqprec M$. Moreover, we have $\overline{M} \subseteq I^*$ implying that I^* is a lower bound of $\mathcal{M}$. It remains to show that $N \curlyeqprec I^*$ for any other lower bound N of $\mathcal{M}$. To see this, take $w \in I^*$. By definition of I^*, there is $m \in \overline{M}$ such that $m \preceq w$. Since N is a lower bound of $\mathcal{M}$, there is $n \in N$ such that $n \preceq m \preceq w$. Hence, for each $w \in I^*$ there is $n \in N$ such that $n \preceq w$, i.e. $N \curlyeqprec I^*$.

Now, let $\mathcal{M} \subseteq \mathcal{P}(W)$ be nonempty and bounded above with respect to $\curlyeqprec$ by $N \in \mathcal{P}(W)$. Since $M \curlyeqprec N$ for all $M \in \mathcal{M}$ we have

$$\forall M \in \mathcal{M} : \, \forall n \in N \, \exists m \in M : m \preceq n,$$

hence $N \subseteq S^*$. Hence S^* is nonempty and $S^* \curlyeqprec N$. On the other hand, for $M \in \mathcal{M}$ the definition of S^* implies

$$\forall w \in S^* \, \exists m \in M : \, m \preceq w,$$

hence $M \curlyeqprec S^*$ for all $M \in \mathcal{M}$. This proves that S^* is a supremum of $\mathcal{M}$.

(ii) By similar arguments as used for the proof of (i).

(iii) According to Remark 4, $\left(\widehat{\mathcal{P}}(W), \curlyeqprec\right)$ is order complete if and only if it contains a largest as well as a smallest element. This is true since

$$\forall M \in \widehat{\mathcal{P}}(W) : \, W \curlyeqprec M \curlyeqprec \emptyset.$$

Formulas (2.9) and (2.10) remain true: (2.9) yields $I^* = \emptyset$ if $\emptyset$ is the only member of $\mathcal{M}$ and (2.10) yields $S^* = \emptyset$ if $\emptyset \in \mathcal{M}$.

Finally, let $\emptyset$ be the only upper bound of $\mathcal{M}$. Assume $w \in S^*$ for some $w \in W$. The definition of S^* gives that $N = \{w\}$ is an upper bound of $\mathcal{M}$ with respect to $\curlyeqprec$. This is a contradiction, hence S^* must be empty.

(iv) The proof runs analogous to that of (iii). ∎

Remark 5 *Let $\mathcal{M} \subseteq \widehat{\mathcal{P}}(W)$ be given and define*

$$\overline{M} := \bigcup_{M \in \mathcal{M}} M.$$

(i) The set $\overline{M}$ is an infimum of $\mathcal{M}$ with respect to $\preccurlyeq$, i.e. $I^* \preccurlyeq \overline{M} \preccurlyeq I^*$ holds true. Since I^* is an infimum and $\overline{M}$ a lower bound of $\mathcal{M}$, certainly $\overline{M} \preccurlyeq I^*$ holds. On the other hand, $\overline{M} \subseteq I^*$, hence $I^* \preccurlyeq \overline{M}$.

(ii) The set $\overline{M}$ is a supremum of $\mathcal{M}$ with respect to $\curlyeqprec$, i.e. $S^\diamond \curlyeqprec \overline{M} \curlyeqprec S^\diamond$ holds true. Since $S^\diamond$ is a supremum and $\overline{M}$ an upper bound of $\mathcal{M}$, certainly $S^\diamond \curlyeqprec \overline{M}$ holds. On the other hand, $\overline{M} \subseteq S^\diamond$, hence $\overline{M} \curlyeqprec S^\diamond$.

Proposition 23 *Let $(W, \preceq)$ be quasiordered and $\mathcal{M} \subseteq \widehat{\mathcal{P}}(W)$ be given.*

If $I \in \widehat{\mathcal{P}}(W)$ is an infimum of $\mathcal{M}$ with respect to $\preccurlyeq$ (with respect to $\curlyeqprec$), then $I \subseteq I^*$ ($I \subseteq I^\diamond$) holds true.

If $S \in \widehat{\mathcal{P}}(W)$ is a supremum of $\mathcal{M}$ with respect to $\preccurlyeq$ (with respect to $\curlyeqprec$), then $S \subseteq S^*$ ($S \subseteq S^\diamond$) holds true.

Proof. Let I be an infimum of $\mathcal{M}$ with respect to $\preccurlyeq$. Take $w' \in I$. Since $I^* \preccurlyeq I$, there is $w \in I^*$ such that $w \preceq w'$. The definition of I^* implies

$$\exists M \in \mathcal{M}\ \exists m \in M : \ m \preceq w.$$

The transitivity of $\preceq$ implies $m \preceq w'$ for all these m's, hence $w' \in I^*$.

Let I be an infimum of $\mathcal{M}$ with respect to $\curlyeqprec$. Take $w' \in I$. Since $I \curlyeqprec I^\diamond$, there is $w \in I^\diamond$ such that $w' \preceq w$. The definition of $I^\diamond$ implies

$$\forall M \in \mathcal{M}\ \exists m \in M : \ w \preceq m.$$

The transitivity of $\preceq$ implies $w' \preceq m$ for all these m's, hence $w' \in I^\diamond$.

The proofs for the suprema run analogously. ■

The preceding result shows that the infima and suprema from Theorem 6 are the largest ones in the sense of set inclusion. The question arises how one can shrink these sets as much as possible. It turns out that the sets of minimal and maximal points, respectively, of the largest infima and suprema are good candidates.

In the following two theorems some relationships are established between the infimum (supremum) of a subset $\mathcal{M} \subseteq \widehat{\mathcal{P}}(W)$ with respect to $\preccurlyeq$ and $\curlyeqprec$ on one hand and the set of minimal points of I^* (S^*) and maximal points of $I^\diamond$ ($S^\diamond$) with respect to $\preceq$ on the other hand, respectively.

To state the result, we recall the so called domination condition. This concept plays an important role in vector optimization. Compare the book of Luc [85], [44] and the references therein.

Definition 15 *Let $(W, \preceq)$ be quasiordered. A subset $M \subseteq W$ is said to satisfy the* **lower domination condition** *iff*

$$\forall m \in M \; \exists n \in \min(M) : \; n \preceq m.$$

A subset $M \subseteq W$ is said to satisfy the **upper domination condition** *iff*

$$\forall m \in M \; \exists n \in \max(M) : \; m \preceq n.$$

For the sake of simplicity, we state the result for partial orders.

Theorem 7 *Let $(W, \preceq)$ be partially ordered and $\mathcal{M} \subseteq \widehat{\mathcal{P}}(W)$ be given.*

(i) Let $\mathcal{I}^* \subseteq \widehat{\mathcal{P}}(W)$ be the set of all infima of $\mathcal{M}$ with respect to $\preccurlyeq$. Then

$$\min(I^*) = \bigcap_{I \in \mathcal{I}^*} I.$$

The set I^* satisfies the lower domination condition if and only if $\min(I^*) \in \mathcal{I}^*$. In this case, $\min(I^*)$ is the smallest set being an infimum of $\mathcal{M}$ with respect to $\preccurlyeq$.

(ii) Let $\mathcal{S}^* \subseteq \widehat{\mathcal{P}}(W)$ be the set of all suprema of $\mathcal{M}$ with respect to $\preccurlyeq$. Then

$$\min(S^*) = \bigcap_{S \in \mathcal{S}^*} S.$$

The set S^* satisfies the lower domination condition if and only if $\min(S^*) \in \mathcal{S}^*$. In this case, $\min(S^*)$ is the smallest set being a supremum of $\mathcal{M}$ with respect to $\preccurlyeq$.

Proof. (i) Recall that $I^* = \bigcup_{M \in \mathcal{M}} \bigcup_{m \in M} \{w \in W : \; m \preceq w\}$, compare (2.9).

First, we show that $\min(I^*) \subseteq I$ for each $I \in \mathcal{I}^*$. Take $m \in \min(I^*) \subseteq I^*$. Since $I \preccurlyeq I^*$, there is $w \in I$ such that $w \preceq m$. Since $I^* \preccurlyeq I$, there is $m' \in I^*$ such that $m' \preceq w$. Since $\preceq$ is transitive, we get $m' \preceq m$ and since m is minimal in I^* and $\preceq$ is antisymmetric, this implies $m' = w = m$. Hence $m \in I$. Thus, we have proved that $\min(I^*) \subseteq \bigcap_{I \in \mathcal{I}^*} I$.

To show the converse inclusion, take $w \in \bigcap_{I \in \mathcal{I}^*} I$ and assume $w \notin \min(I^*)$. Then there must exist a $\bar{w} \in I^*$ such that $\bar{w} \preceq w$ and $\bar{w} \neq w$. For $I \in \mathcal{I}^*$ consider the set

$$I' := I \backslash \{w\} \cup \{\bar{w}\}.$$

Then

$$\forall w \in I \; \exists w' \in I' : \; w' \preceq w,$$

hence $I' \preccurlyeq I \preccurlyeq I^*$. On the other hand, since $I' \subseteq I^*$, we have $I^* \preccurlyeq I'$. Hence $I^* \preccurlyeq I' \preccurlyeq I^*$, i.e. $I' \in \mathcal{I}^*$. But $w \notin I'$, a contradiction.

Since $\min(I^*) \subseteq I^*$, we have $I^* \preccurlyeq \min(I^*)$. The lower domination conditions is equivalent to $\min(I^*) \preccurlyeq I^*$, hence $\min(I^*)$ is an infimum of $\mathcal{M}$ with respect to $\preccurlyeq$.

(ii) By similar arguments. ■

Note that $\min(I^*) = \min(I)$ for every $I \in \mathcal{I}^*$. Since $\overline{M} := \bigcup_{M \in \mathcal{M}} M \in \mathcal{I}^*$, it might be a good idea to look for minimal points of the union $\overline{M}$. This is the underlying idea of set valued optimization in the sense of Corley [20], Jahn [63] and many others since the middle of the 80ies. Theorem 7 tells us, among other things, that looking for minimal points of $\overline{M}$ yields a subset of an infimum with respect to $\preccurlyeq$.

On the other hand, the set $\min(S^*)$ is not contained in the union $\overline{M}$ in general. There are easy to construct examples in $\mathbb{R}^2$ with even $\min(S^*) \cap \overline{M} = \emptyset$.

The following corollary pays special attention to the case when the set $\mathcal{M}$ consists only of singletons. We obtain relationships between the set of minimal elements of a subset $M \subseteq W$ and the infimum with respect to $\preccurlyeq$ on one hand and the supremum with respect to $\preceq$ and the supremum with respect to $\preccurlyeq$ on the other hand.

We denote by $\sup(M)$ the set of suprema of M in W with respect to $\preceq$ whereas $\min(M) \subseteq M$ is the set of minimal points of M, compare Section 2.2.1. Assuming $(W, \preceq)$ to be a partially ordered set, $\sup(M)$ is empty or consists of a single point.

Corollary 5 *Let $(W, \preceq)$ be partially ordered and $\emptyset \neq M \subseteq W$. Considering*

$$\mathcal{M} := \{\{m\} : \; m \in M\} \subseteq \mathcal{P}(W)$$

the following assertions hold true:
(i) The set

$$I^* := \bigcup_{m \in M} \{w \in W : \; m \preceq w\}$$

is an infimum of $\mathcal{M}$ with respect to $\preccurlyeq$. The set $\min(I^*) = \min(M)$ is contained in every infimum of $\mathcal{M}$ with respect to $\preccurlyeq$ and is itself an infimum if and only if M satisfies the lower domination condition.
(ii) The set

$$S^* := \bigcap_{m \in M} \{w \in W : \; m \preceq w\}$$

is a supremum of $\mathcal{M}$ with respect to $\preccurlyeq$. If $\sup(M) \in W$ exists, then $S^* = \{w \in W : \; \sup(M) \preceq w\}$ and $\min S^* = \{\sup(M)\}$.

Proof. (i) I^* is an infimum of $\mathcal{M}$ with respect to $\preccurlyeq$ by Theorem 6, (i). The remaining part follows from Theorem 7, (i).

(ii) S^* is a supremum of $\mathcal{M}$ with respect to $\preccurlyeq$ by Theorem 6, (i). Moreover, $S^* = \{w \in W : \; \sup(M) \preceq w\}$, since S^* contains by definition all upper bounds of M with respect to $\preceq$ and $\sup(M)$ is the smallest upper bound by definition. ∎

Theorem 8 *Let $(W, \preceq)$ be partially ordered and $\mathcal{M} \subseteq \widehat{\mathcal{P}}(W)$ be given.*

(i) Let $\mathcal{I}^\diamond \subseteq \widehat{\mathcal{P}}(W)$ be the set of all infima of $\mathcal{M}$ with respect to $\preccurlyeq$. Then

$$\max(\mathcal{I}^\diamond) = \bigcap_{I \in \mathcal{I}^\diamond} I.$$

The set $I^\diamond$ *satisfies the upper domination condition if and only if* $\max(I^\diamond) \in \mathcal{I}^\diamond$*. In this case, it is the smallest set being an infimum of* $\mathcal{M}$ *with respect to* $\preccurlyeq$*.*

(ii) Let $\mathcal{S}^\diamond \subseteq \widehat{\mathcal{P}}(W)$ *be the set of all suprema of* $\mathcal{M}$ *with respect to* $\preccurlyeq$*. Then*

$$\max(S^\diamond) = \bigcap_{S \in \mathcal{S}^\diamond} S.$$

The set $S^\diamond$ *satisfies the upper domination condition if and only if* $\max(S^\diamond) \in \mathcal{S}^\diamond$*. In this case, it is the smallest set being a supremum of* $\mathcal{M}$ *with respect to* $\preccurlyeq$*.*

Proof. (i) Recall that $I^\diamond = \bigcap_{M \in \mathcal{M}} \bigcup_{m \in M} \{w \in W : \ w \preceq m\}$, compare (2.11).

First, we show that $\max(I^\diamond) \subseteq I$ for each $I \in \mathcal{I}^\diamond$. Take $m \in \max(I^\diamond) \subseteq I^\diamond$. Since $I^\diamond \preccurlyeq I$, there is $w \in I$ such that $m \preceq w$. Since $I \preccurlyeq I^\diamond$, there is $m' \in I^\diamond$ such that $w \preceq m'$. Hence $m \preceq m'$ and therefore $m = w = m'$ since m is maximal in $I^\diamond$ and $\preceq$ is transitive and antisymmetric. Hence $m \in I$ as desired.

To show the converse inclusion, take $w \in \bigcap_{I \in \mathcal{I}^\diamond} I$ and assume $w \notin \max(I^\diamond)$. Then there must exist a $\bar{w} \in I^\diamond$ such that $w \preceq \bar{w}$ and $w \neq \bar{w}$. For $I \in \mathcal{I}^\diamond$ consider the set

$$I' := I \backslash \{w\} \cup \{\bar{w}\}.$$

Then $I^\diamond \preccurlyeq I \preccurlyeq I'$ by construction of I'. On the other hand, $I' \preccurlyeq I^\diamond$ since $I' \subseteq I^\diamond$. Hence $I' \in \mathcal{I}^\diamond$ and $w \notin I'$, a contradiction.

Since $\max(I^\diamond) \subseteq I^\diamond$, we have $\max(I^\diamond) \preccurlyeq I^\diamond$. The upper domination conditions requires $I^\diamond \preccurlyeq \max(I^\diamond)$, hence $\max(I^\diamond)$ is an infimum of $\mathcal{M}$ with respect to $\preccurlyeq$.

(ii) By similar arguments. ■

The notes after the proof of Theorem 7 apply analogously with reversed roles of infimum and supremum.

Parallel to Corollary 5 we have the following result. Here, we denote by $\inf(M)$ the set of infima of M in W with respect to $\preceq$ whereas $\max(M) \subseteq M$ is the set of maximal points of M, compare Section 2.2.1. Assuming $(W, \preceq)$ to be a partially ordered set, $\inf(M)$ is empty or consists of a single point.

Corollary 6 *Let* $(W, \preceq)$ *be partially ordered and* $\emptyset \neq M \subseteq W$*. Considering*

$$\mathcal{M} := \{\{m\} : \ m \in M\} \subseteq \mathcal{P}(W)$$

the following assertions hold true:
(i) The set

$$I^\diamond := \bigcap_{m \in M} \{w \in W : \ w \preceq m\}$$

is an infimum of $\mathcal{M}$ *with respect to* $\preccurlyeq$*. If* $\inf(M) \in W$ *exists, then* $I^\diamond = \{w \in W : \ w \preceq \inf(M)\}$ *and* $\max I^\diamond = \{\inf(M)\}$*.*
(ii) The set

$$S^\diamond := \bigcup_{m \in M} \{w \in W : \ w \preceq m\}$$

is a supremum of $\mathcal{M}$ with respect to $\curlyeqprec$. The set $\max(S^\diamond) = \max(M)$ *is contained in every supemum of $\mathcal{M}$ with respect to $\curlyeqprec$ and is itself an supremum if and only if M satisfies the upper domination condition.*

Proof. Similar to the proof of Corollary 5. ∎

Example 9 *Consider $\mathcal{M} = \{M_1, M_2\} \subset \mathcal{P}(\mathbb{R})$ with $M_1 = [0,1]$ and $M_2 = [-1, 1000]$. Then $I^* = [-1, +\infty)$ and $I^\diamond = (-\infty, 1]$. If we interpret the numbers contained in M_1, M_2 as the possible financial loss we have to expect choosing M_1 and M_2, respectively, one might see that it is sometimes better to prefer M_2 to M_1, i.e. to deal with $\curlyeqprec$ instead of $\preccurlyeq$ or instead of simply to look for minimal points in the union.*

Since $\preccurlyeq$ and $\curlyeqprec$ are only quasiorders in general, one might ask for the structure of the corresponding partially ordered sets of equivalence classes on $\widehat{\mathcal{P}}(W)$. They are introduced as follows (compare Section 2.2.1):

$$M_1 \overset{*}{\sim} M_2 \quad :\Longleftrightarrow \quad M_1 \preccurlyeq M_2 \preccurlyeq M_1; \tag{2.13}$$

$$M_1 \overset{\diamond}{\sim} M_2 \quad :\Longleftrightarrow \quad M_1 \curlyeqprec M_2 \curlyeqprec M_1 \tag{2.14}$$

for $M_1, M_2 \in \widehat{\mathcal{P}}(W)$. The empty set $\emptyset$ is equivalent only to itself. For $M \in \widehat{\mathcal{P}}(W)$ we set

$$[M]^* := \left\{M' \in \widehat{\mathcal{P}}(W) : \ M' \overset{*}{\sim} M\right\}, \qquad \overline{M}^* := \bigcup_{M' \in [M]^*} M', \tag{2.15}$$

$$[M]^\diamond := \left\{M' \in \widehat{\mathcal{P}}(W) : \ M' \overset{\diamond}{\sim} M\right\}, \qquad \overline{M}^\diamond := \bigcup_{M' \in [M]^\diamond} M'. \tag{2.16}$$

The order relations on the set of equivalence classes have to be defined by

$$[M_1]^* \preccurlyeq [M_2]^* \quad :\Longleftrightarrow \quad \forall M_1' \in [M_1]^*, M_2' \in [M_2]^* : \ M_1' \preccurlyeq M_2'; \tag{2.17}$$

$$[M_1]^\diamond \curlyeqprec [M_2]^\diamond \quad :\Longleftrightarrow \quad \forall M_1' \in [M_1]^\diamond, M_2' \in [M_2]^\diamond : \ M_1' \curlyeqprec M_2'. \tag{2.18}$$

This leads to the following relationships containing $\emptyset$:

$$\forall M \in \widehat{\mathcal{P}}(W) : \ [M] \preccurlyeq [\emptyset], \quad [\emptyset] \curlyeqprec [M].$$

The following theorem tells us that the resulting partial order on the set of equivalence classes can be identified in a sense with $\supseteq$ and $\subseteq$, respectively.

Theorem 9 *Let $(W, \preceq)$ be quasiordered. Then:*
(i) $\overline{M}^* = \{w \in W : \ \exists m \in M : \ m \preceq w\}$. Moreover, $M' \overset{*}{\sim} \overline{M}^*$ for each $M' \in [M]^*$ and

$$[M_1]^* \preccurlyeq [M_2]^* \quad \Longleftrightarrow \quad \overline{M_1}^* \supseteq \overline{M_2}^*. \tag{2.19}$$

(ii) $\overline{M}^\diamond = \{w \in W : \ \exists m \in M : \ w \preceq m\}$. Moreover, $M' \overset{\diamond}{\sim} \overline{M}^\diamond$ for each $M' \in [M]^\diamond$ and

$$[M_1]^\diamond \curlyeqprec [M_2]^\diamond \quad \Longleftrightarrow \quad \overline{M_1}^\diamond \subseteq \overline{M_2}^\diamond. \tag{2.20}$$

PROOF. (i) Denote for the moment $\tilde{M}^* = \{w \in W : \exists m \in M : m \preceq w\}$. First, we show $\tilde{M}^* \subseteq \overline{M}^*$. Take $\tilde{w} \in \tilde{M}^*$ and set $M' := M \cup \{\tilde{w}\}$. Then $M' \overset{*}{\sim} M$, hence $\tilde{w} \in \overline{M}^*$. Conversely, we have $\overline{M}^* \subseteq \tilde{M}^*$, since $\bar{w} \in \overline{M}^*$ implies the existence of $M' \in [M]^*$ such that $\bar{w} \in M'$. Since $M \preccurlyeq M'$, there exists $m \in M$ such that $m \preceq \bar{w}$. Hence $\bar{w} \in \tilde{M}^*$.

For the second assertion, it suffices to show $M \overset{*}{\sim} \overline{M}^*$. This is true since $M \subseteq \overline{M}^*$ and each $m \in \overline{M}^*$ belongs to some $M' \overset{*}{\sim} M$.

It remains to show (2.19). Let $[M_1]^* \preccurlyeq [M_2]^*$ and take $m_2 \in \overline{M_2}^*$. Since $\overline{M_1}^* \preccurlyeq \overline{M_2}^*$, there is $m_1 \in \overline{M_1}^*$ such that $m_1 \preceq m_2$. This implies $\overline{M_1}^* \preccurlyeq \overline{M_1}^* \cup \{m_2\}$. Since $\overline{M_1}^* \subseteq \overline{M_1}^* \cup \{m_2\}$, we also have $\overline{M_1}^* \cup \{m_2\} \preccurlyeq \overline{M_1}^*$. Hence $\overline{M_1}^* \cup \{m_2\} \in [M_1]^*$ implying $m_2 \in \overline{M_1}^*$.

Conversely, take $M_2' \in [M_2]^*$. Then $M_2' \subseteq \overline{M_2}^* \subseteq \overline{M_1}^*$. This implies $\overline{M_1}^* \preccurlyeq M_2'$, hence we have $M_1 \preccurlyeq M_2'$ for each $M_1 \in [M_1]^*$.

(ii) The proof for $\curlyeqprec$ goes analogously. ■

Remark 6 *Let $M \in \widehat{\mathcal{P}}(W)$ and consider $\mathcal{M}^* := [M]^*$. Then $(\mathcal{M}^*, \supseteq)$ is partially ordered and $\overline{M}^*$ from above is the unique infimum of $\mathcal{M}^*$ that belongs itself to $\mathcal{M}^*$. Hence it is the smallest element of $\mathcal{M}^*$ with respect to $\supseteq$. If the intersection $\bigcap_{M' \in [M]^*} M'$ belongs to $\mathcal{M}^*$, then it is the largest element of $\mathcal{M}^*$. However, this is not the case in general.*

Of course, every $M' \in \mathcal{M}^*$ is an infimum of $\mathcal{M}^*$ with respect to $\preccurlyeq$. Applying Theorem 7 to $\mathcal{M}^*$, we get $I^* = \overline{M}^*$ and the lower domination condition for $\overline{M}^*$ is necessary and sufficient for $\min\left(\overline{M}^*\right) = \bigcap_{M' \in [M]^*} M'$ being the largest element in $(\mathcal{M}^*, \supseteq)$.

Analogously, $\overline{M}^\diamond$ is the largest element of $\mathcal{M}^\diamond := [M]^\diamond$ in $(\mathcal{M}^\diamond, \subseteq)$ being partially ordered as well. The smallest element is $\max\left(\overline{M}^\diamond\right) = \bigcap_{M' \in [M]^\diamond} M'$ if and only if $\overline{M}^\diamond$ satisfies the upper domination property according to Theorem 8.

Let $(W, \preceq)$ be quasiordered. Set

$$\left[\widehat{\mathcal{W}}\right]^* := \left\{[M]^* : M \in \widehat{\mathcal{P}}(W)\right\}, \qquad \left[\widehat{\mathcal{W}}\right]^\diamond := \left\{[M]^\diamond : M \in \widehat{\mathcal{P}}(W)\right\}.$$

Proposition 24 *Let $(W, \preceq)$ a quasiordered. Then $\left(\left[\widehat{\mathcal{W}}\right]^*, \preccurlyeq\right)$ and $\left(\left[\widehat{\mathcal{W}}\right]^\diamond, \curlyeqprec\right)$ are partially ordered.*

PROOF. This follows from using the standard procedure as described in Section 2.2.1. ■

The next theorem gives formulas for the supremum and infimum in the set of equivalence classes $\left[\widehat{\mathcal{W}}\right]^*$ and $\left[\widehat{\mathcal{W}}\right]^\diamond$.

Theorem 10 *Let $(W, \preceq)$ be quasiordered. Then:*
(i) $\left(\left[\widehat{\mathcal{W}}\right]^*, \preccurlyeq\right)$ is order complete. Let $[\mathcal{M}]^* \subseteq \left[\widehat{\mathcal{W}}\right]^*$ be nonempty and define $\mathcal{M}^* := \left\{\overline{M}^* : [M]^* \in [\mathcal{M}^*]\right\} \subseteq \widehat{\mathcal{P}}(W)$. Then

$$I^* = \bigcup_{\overline{M}^* \in \mathcal{M}^*} \bigcup_{m \in \overline{M}^*} \{w \in W : m \preceq w\}$$

is an infimum of $\mathcal{M}^*$ *and* $[I^*]^*$ *is an infimum of* $[\mathcal{M}]^*$*. The set*

$$S^* = \bigcap_{\overline{M}^* \in \mathcal{M}^*} \bigcup_{m \in \overline{M}^*} \{w \in W : \ m \preceq w\}$$

is a supremum of $\mathcal{M}^*$ *and* $[S^*]^*$ *is a supremum of* $[\mathcal{M}]^*$*.*
(ii) $\left(\left[\widehat{\mathcal{W}}\right]^\diamond, \preccurlyeq\right)$ *is order complete. Let* $[\mathcal{M}]^\diamond \subseteq \left[\widehat{\mathcal{W}}\right]^\diamond$ *be nonempty and define* $\mathcal{M}^\diamond := \left\{\overline{M}^\diamond : \ [M]^\diamond \in [\mathcal{M}^\diamond]\right\} \subseteq \widehat{\mathcal{P}}(W)$*. Then*

$$I^\diamond = \bigcap_{\overline{M}^\diamond \in \mathcal{M}^\diamond} \bigcup_{m \in \overline{M}^\diamond} \{w \in W : \ w \preceq m\}$$

is an infimum of $\mathcal{M}^\diamond$ *and* $[I^\diamond]^\diamond$ *is an infimum of* $[\mathcal{M}]^\diamond$*. The set*

$$S^\diamond = \bigcup_{\overline{M}^\diamond \in \mathcal{M}^\diamond} \bigcup_{m \in \overline{M}^\diamond} \{w \in W : \ w \preceq m\}$$

is a supremum of $\mathcal{M}^\diamond$ *and* $[S^\diamond]^\diamond$ *is a supremum of* $[\mathcal{M}]^\diamond$*.*

Proof. (i) This part follows from Theorem 6, (iii) and Theorem 9, (i).
(ii) This part follows from Theorem 6, (iii) and Theorem 9, (ii). ■

2.2.4 Ordered monoids

In this section, we investigate monoids supplied with a reflexive and transitive relation. The main application will be the monoid being the power set of an ordered group together with one of the relations $\preccurlyeq$ and $\curlyeqprec$ defined in the last subsection.

Definition 16 *Let* $(Y, \circ)$ *be a commutative monoid and* $\preceq$ *be a quasiordering on* Y *satisfying the following condition:*
(Q) $y_1, y_2, y_3 \in Y$*,* $y_1 \preceq y_2$ *implies* $y_1 \circ y_3 \preceq y_2 \circ y_3$*.*
Then $(Y, \circ, \preceq)$ *is called a* **quasiordered monoid**. *If* $\preceq$ *is a partial order satisfying (Q), then* $(Y, \circ, \preceq)$ *is called an* **ordered monoid**.

Let us note that (Q) is equivalent to

$$y_1, y_2, y_3, y_4 \in Y, \ y_1 \preceq y_2, \ y_3 \preceq y_4 \qquad \Longrightarrow \qquad y_1 \circ y_3 \preceq y_2 \circ y_4. \tag{2.21}$$

since $\preceq$ is reflexive and transitive. It is a crucial observation that a quasiordered monoid can be made into an order complete quasiordered monoid by adding a smallest and a largest elements, if necessary.

Proposition 25 *Every Dedekind complete quasiordered (ordered) monoid can be extended to an order complete quasiordered (ordered) monoid by adding at most two elements.*

Proof. Let $(Y, \circ, \preceq)$ be a Dedekind complete quasiordered monoid. Add an element l (u, respectively) such that

$$\forall y \in Y: \quad l \preceq y \quad (y \preceq u)\,,$$
$$\forall y \in Y: \quad l \circ y = y \circ l = l \quad (u \circ y = y \circ u = u)\,,$$

i.e., l (u, respectively) is the smallest (largest) element and dominant in the ordered monoid $(Y_l := Y \cup \{l\}, \circ, \preceq)$ and $(Y_u := Y \cup \{u\}, \circ, \preceq)$, respectively.

Add an element u (l, respectively) to Y_l (Y_u) being the largest (smallest) and again dominant. The result is the order complete commutative monoid $(Y^{\vartriangle} := Y_l \cup \{u\}, \circ, \preceq)$ and $(Y^{\triangledown} := Y_u \cup \{l\}, \circ, \preceq)$, respectively. (Q) is easy to check. ■

Remark 7 *1. Proposition 25 tells us that order completion does not destroy the monoidal structure. Order completion of a group in this way yields an order complete monoid.*
2. Of course, Proposition 25 is a generalization of the order completion of the reals, widely used in optimization theory, compare Example 4. Especially, $\mathbb{R}^{\vartriangle}$ is a fundamental structure in Convex Analysis.

Proposition 26 *Let $(Y, \circ, \preceq)$ be an ordered monoid with a largest (smallest) element. Then it is an idempotent element.*

Proof. Let $\hat{y} \in Y$ be the largest element, i.e.

$$\forall y \in Y: \quad y \preceq \hat{y}.$$

Especially, $\hat{y} \circ \hat{y} \preceq \hat{y}$ and $\theta \preceq \hat{y}$. From the latter inequality we obtain by (Q) $\hat{y} \preceq \hat{y} \circ \hat{y}$. The antisymmetry of $\preceq$ implies $\hat{y} \circ \hat{y} = \hat{y}$. The proof for the smallest element is similar. ■

In view of Proposition 6 we see that an order complete monoid can not be embedded into a group.

The set of positive elements for an ordered monoid is defined to be

$$P := \left\{y \in Y : \ \forall y' \in Y : \ y' \preceq y \circ y'\right\} = \{y \in Y : \ \theta \preceq y\}\,.$$

Moreover, $(P, \circ, \preceq)$ is an ordered submonoid of Y.

Next, we discuss ordering relations in the power set of an ordered monoid using the $\preccurlyeq$– and $\curlyeqprec$–relation introduced in the last section.

Theorem 11 *Let $(Y, \circ, \preceq)$ be a quasiordered monoid. Then:*
(i) $(\mathcal{P}(Y), \odot, \preccurlyeq)$ and $(\mathcal{P}(Y), \odot, \curlyeqprec)$ are quasiordered, Dedekind complete monoids.
(ii) $\left(\widehat{\mathcal{P}}(Y), \odot, \preccurlyeq\right)$ and $\left(\widehat{\mathcal{P}}(Y), \odot, \curlyeqprec\right)$ are quasiordered, order complete monoids.

Proof. (i) By Proposition 22 and Theorem 6, $(\mathcal{P}(Y), \preccurlyeq)$ and $(\mathcal{P}(Y), \curlyeqprec)$ are quasiordered and Dedekind complete. It remains to show that (Q) holds true for $\preccurlyeq$, $\curlyeqprec$. Take $M_1, M_2, M_3 \in \mathcal{P}(Y)$ such that $M_1 \preccurlyeq M_2$. Then by definition of $\preccurlyeq$,

$$\forall y_2 \in M_2 \; \exists y_1 \in M_1 : \; y_1 \preceq y_2.$$

(Q) implies

$$\forall y_3 \in M_3 \; \forall y_2 \in M_2 \; \exists y_1 \in M_1 : \; y_1 \circ y_3 \preceq y_2 \circ y_3.$$

Hence

$$\forall y \in M_2 \odot M_3 \; \exists y' \in M_1 \odot M_3 : \; y' \preceq y$$

which is $M_1 \odot M_3 \preccurlyeq M_2 \odot M_3$. The proof for $\curlyeqprec$ is similar. Hence $(\mathcal{P}(Y), \odot, \preccurlyeq)$ and $(\mathcal{P}(Y), \odot, \curlyeqprec)$ are quasiordered, Dedekind complete monoids.

(ii) It is easy to check that (Q) is true for $\preccurlyeq$, $\curlyeqprec$ if one or more of M_1, M_2, M_3 are the empty set. ■

Considering equivalence classes one can obtain ordered monoids. We start with a quasiordered monoid $(Y, \circ, \preceq)$. Denote by

$$\left[\widehat{\mathcal{Y}}\right]^* := \left\{[M]^* : \; M \in \widehat{\mathcal{P}}(Y)\right\} \quad \text{and} \quad \left[\widehat{\mathcal{Y}}\right]^\diamond := \left\{[M]^\diamond : \; M \in \widehat{\mathcal{P}}(Y)\right\}$$

the set of equivalence classes over $\widehat{\mathcal{P}}(Y)$ with respect to $\preccurlyeq$ and $\curlyeqprec$, respectively. For $M_1, M_2 \in \widehat{\mathcal{P}}(Y)$ define operations by

$$[M_1]^* \odot [M_2]^* := [M_1 \odot M_2]^*, \quad [M_1]^\diamond \odot [M_2]^\diamond := [M_1 \odot M_2]^\diamond. \tag{2.22}$$

Recall the definitions of the equivalence classes in (2.13), (2.14) and the order relations $\preccurlyeq$, $\curlyeqprec$ for equivalence classes in (2.17), (2.18), respectively.

Proposition 27 *Let $(Y, \circ, \preceq)$ be a quasiordered monoid. Then $\left(\left[\widehat{\mathcal{Y}}\right]^*, \odot, \preccurlyeq\right)$ and $\left(\left[\widehat{\mathcal{Y}}\right]^\diamond, \odot, \curlyeqprec\right)$ are order complete ordered monoids.*

Proof. First, let us note that the operations in (2.22) are well-defined: Take $M_1, M_1', M_2, M_2' \in \widehat{\mathcal{P}}(Y)$ such that $M_1 \overset{*}{\sim} M_1'$ and $M_2 \overset{*}{\sim} M_2'$. From $M_1 \preccurlyeq M_1'$, $M_2 \preccurlyeq M_2'$ and (2.21) with a view to Theorem 11 we get $M_1 \odot M_2 \preccurlyeq M_1' \odot M_2'$. Similar, from $M_1' \preccurlyeq M_1$, $M_2' \preccurlyeq M_2$ follows $M_1' \odot M_2' \preccurlyeq M_1 \odot M_2$. Hence $M_1 \odot M_2 \overset{*}{\sim} M_1' \odot M_2'$. The same procedure applies for $\curlyeqprec$. The remaining part of the theorem follows from Theorems 10, 11 and Proposition 24. ■

Theorem 9 admits the consideration of elements of $\widehat{\mathcal{P}}(Y)$ instead of equivalence classes. Define two subsets of $\widehat{\mathcal{P}}(Y)$ by

$$\widehat{\mathcal{Y}}^* := \left\{\overline{M}^* : \; M \in \widehat{\mathcal{P}}(Y)\right\} \quad \text{and} \quad \widehat{\mathcal{Y}}^\diamond := \left\{\overline{M}^\diamond : \; M \in \widehat{\mathcal{P}}(Y)\right\}$$

we obtain the following proposition by applying Theorem 9.

Proposition 28 *Let $(Y, \circ, \preceq)$ be a quasiordered monoid. Then $\left(\widehat{\mathcal{Y}}^*, \odot, \supseteq\right)$ and $\left(\widehat{\mathcal{Y}}^\diamond, \odot, \subseteq\right)$ are order complete ordered monoids.*

Proof. Invoke Theorem 9 and Proposition 27. ■

In the following, we consider the case of a quasiordered group $(Y, \circ, \preceq)$. In this case, the relation $\preceq$ on Y as well as $\preccurlyeq$ and $\curlyeqprec$ on $\widehat{\mathcal{P}}(Y)$ can be expressed equivalently using the sets P of positive elements and P' of "negative" elements which is defined to be

$$P' := \left\{y' \in Y : \; \exists y \in P : \; y \circ y' = \theta\right\} = \left\{y' \in Y : \; y' \preceq \theta\right\}.$$

Moreover, there is a close relationship between $\preccurlyeq$ and $\curlyeqprec$.

Theorem 12 *Let $(Y, \circ, \preceq)$ be a quasiordered group with neutral element θ, $M_i \in \widehat{\mathcal{P}}(Y)$ and $M_i' := \{y_i' \in Y : \; \exists y_i \in M_i : \; y_i \circ y_i' = \theta\}$ for $i = 1, 2$. Then it holds*

$$\begin{aligned}
M_1 \preccurlyeq M_2 \quad &\Longleftrightarrow \quad M_2 \subseteq M_1 \odot P, \\
M_1 \curlyeqprec M_2 \quad &\Longleftrightarrow \quad M_1 \subseteq M_2 \odot P', \\
M_1 \overset{*}{\sim} M_2 \quad &\Longleftrightarrow \quad M_1 \odot P = M_2 \odot P, \\
M_1 \overset{\diamond}{\sim} M_2 \quad &\Longleftrightarrow \quad M_1 \odot P' = M_2 \odot P', \\
M_1 \preccurlyeq M_2 \quad &\Longleftrightarrow \quad M_2' \curlyeqprec M_1'.
\end{aligned}$$

Proof. The proof of the first four equivalences relies on the fact that in quasiordered groups we have

$$y_1 \preceq y_2 \quad \Longleftrightarrow \quad y_2 \in \{y_1\} \odot P \quad \Longleftrightarrow \quad y_1 \in \{y_2\} \odot P'.$$

The relations

$$\begin{aligned}
M_1 \preccurlyeq M_2 \quad &\Longleftrightarrow \quad \forall y_2 \in M_2 \; \exists y_1 \in M_1 : \; y_1 \preceq y_2 \\
&\overset{(Q)}{\Longleftrightarrow} \quad \forall y_2' \in M_2' \; \exists y_1 \in M_1 : \; y_1 \circ y_2' \preceq \theta \\
&\overset{(Q)}{\Longleftrightarrow} \quad \forall y_2' \in M_2' \; \exists y_1' \in M_1' : \; y_2' \preceq y_1' \\
&\Longleftrightarrow \quad M_2' \curlyeqprec M_1'.
\end{aligned}$$

yield the last assertion. ■

Example 10 *The preceding theorem especially applies if V is a real linear space with a quasiorder $\preceq_K$ generated by a convex cone $K \in \mathcal{P}(V) =: Y$. By $\preccurlyeq_K$ and $\curlyeqprec_K$ we denote the two canonical extensions of $\preceq_K$ to Y. Then*

$$M_1 \preccurlyeq_K M_2 \quad \Longleftrightarrow \quad (-1)\, M_2 \curlyeqprec_K (-1)\, M_1 \quad \Longleftrightarrow \quad (-1)\, M_2 \preccurlyeq_{(-1)K} (-1)\, M_1$$

where $(-1)\, M := \{-v : \; v \in M\}$ for $M \in Y$.

2.2.5 Ordered conlinear spaces

We introduce the concept of an ordered conlinear space close to that of an ordered linear space.

Definition 17 *Let $(Y, +)$ be a conlinear space and $\preceq$ a quasiordering on Y satisfying the following conditions:*
(Q1) $y_1, y_2, y_3 \in Y$, $y_1 \preceq y_2$ implies $y_1 + y_3 \preceq y_2 + y_3$;
(Q2) $y_1, y_2 \in Y$, $y_1 \preceq y_2$, $t \geq 0$ implies $ty_1 \preceq ty_2$.
Then $(Y, +, \preceq)$ is called a **quasiordered conlinear space***. If $\preceq$ is a partial order satisfying (Q1), (Q2), then $(Y, +, \preceq)$ is called an* **ordered conlinear space***.*

Note that (Q1) of this definition coincides with (Q) of Definition 16 if we consider $(Y, +)$ to be a commutative monoid. Again, order completiton does not destroy the conlinear structure.

Proposition 29 *Every Dedekind complete quasiordered (ordered) conlinear space can be extended to an order complete quasiordered (ordered) conlinear space by adding at most two elements.*

Proof. Let $(Y, +, \preceq)$ be a quasiordered conlinear space. Proposition 25 ensures that $(Y, +, \preceq)$ as a quasiordered monoid can be supplemented, if necessary, by two elements l and u being the largest and the smallest. We obtain two possibilities for quasiordered monoids: $Y^{\vartriangle}$ (l is dominant) and $Y^{\triangledown}$ (u is dominant). Defining for $t > 0$ $t \cdot l = l$, $t \cdot u = u$ and $0 \cdot l = 0 \cdot u = \theta$ in both cases, it is easy to check that (C2), (Q1) and (Q2) are satisfied. Hence $(Y^{\vartriangle}, +, \preceq)$ and $(Y^{\triangledown}, +, \preceq)$ are order complete quasiordered conlinear spaces. If $(Y, +, \preceq)$ is ordered, so are the resulting spaces. ∎

By Proposition 26, the largest and the smallest element of an ordered conlinear space, if they exist, are idempotent elements of the underlying monoid. Moreover, they are cones.

Proposition 30 *Let $(Y, +, \preceq)$ be an ordered conlinear space. Then the largest and the smallest element of an ordered conlinear space, if they exist, are cones.*

Proof. Let $\hat{y} \in Y$ be the largest element. Then $\frac{1}{t} y \preceq \hat{y}$ holds for all $t > 0$ and $y \in Y$. ¿From (Q2) we may conclude $y \preceq t\hat{y}$ for all $t > 0$ and $y \in Y$. Since the largest element is unique by antisymmetry, it must be a cone. The proof for the smallest element is the same. ∎

Theorem 13 *Let $(Y, +, \preceq)$ be a quasiordered conlinear space. Then:*
(i) $(\mathcal{P}(Y), \oplus, \preccurlyeq)$ and $(\mathcal{P}(Y), \oplus, \curlyeqprec)$ are quasiordered, Dedekind complete conlinear spaces.
(ii) $\left(\widehat{\mathcal{P}}(Y), \oplus, \preccurlyeq\right)$ and $\left(\widehat{\mathcal{P}}(Y), \oplus, \curlyeqprec\right)$ are quasiordered, order complete conlinear spaces.

Proof. (i) By Proposition 22 and (i), (ii) of Theorem 6, $(\mathcal{P}(Y), \preccurlyeq)$ and $(\mathcal{P}(Y), \curlyeqprec)$ are quasiordered and Dedekind complete. On the other hand, by Proposition 10, $(\mathcal{P}(Y), \oplus)$ is a conlinear space. It remains to show that (Q1) and (Q2) hold true. Let us consider the case $\preccurlyeq$. Take $M_1, M_2, M_3 \in \mathcal{P}(Y)$ such that $M_1 \preccurlyeq M_2$. We have to show $M_1 \oplus M_3 \preccurlyeq M_2 \oplus M_3$, i.e.

$$\forall m \in M_2 \oplus M_3,\ \exists m' \in M_1 \oplus M_3 :\ m' \preceq m.$$

Take $m \in M_2 \oplus M_3$, i.e. $m = m_2 + m_3$, $m_2 \in M_2$, $m_3 \in M_3$. Since $M_1 \preccurlyeq M_2$, there is $m_1 \in M_1$ such that $m_1 \preceq m_2$. Applying (Q1) of Definition 17 in $(Y, +, \preceq)$ we obtain $m' := m_1 + m_3 \preceq m_2 + m_3 = m$ with $m' \in M_1 \oplus M_3$, i.e. (Q1) is valid in $(\mathcal{P}(Y), \oplus, \preccurlyeq)$. (Q2) is immediate. A similar procedure can be applied for $\curlyeqprec$.

(ii) The extension to $\widehat{\mathcal{P}}(Y)$ is straightforward. ■

We denote by $\left[\widehat{\mathcal{Y}}\right]^*$ and $\left[\widehat{\mathcal{Y}}\right]^\diamond$ the set of equivalence classes of $\widehat{\mathcal{P}}(Y)$ with respect to $\preccurlyeq$ and $\curlyeqprec$ as defined in (2.15) and (2.16), respectively.

The relations $\preccurlyeq$ and $\curlyeqprec$ for equivalence classes are defined as in (2.17) and (2.18), respectively. The algebraic operation on $\left[\widehat{\mathcal{Y}}\right]^*$ and $\left[\widehat{\mathcal{Y}}\right]^\diamond$ are defined by

$$[Y_1]^* \oplus [Y_2]^* := [Y_1 \oplus Y_2]^*, \quad [Y_1]^\diamond \oplus [Y_2]^\diamond := [Y_1 \oplus Y_2]^\diamond,$$

$$t \cdot [Y_1]^* := [tY_1]^*, \quad t \cdot [Y_1]^\diamond := [tY_1]^\diamond$$

for $Y_1, Y_2 \in \widehat{\mathcal{P}}(Y)$ and $t \geq 0$. As in (2.15) and (2.16) we set $\widehat{\mathcal{Y}}^* = \left\{\overline{M}^* :\ M \in \widehat{\mathcal{P}}(Y)\right\}$ and $\widehat{\mathcal{Y}}^\diamond = \left\{\overline{M}^\diamond :\ M \in \widehat{\mathcal{P}}(Y)\right\}$ where $\overline{M}^* = \bigcup_{M' \in [M]^*} M'$ and $\overline{M}^\diamond = \bigcup_{M' \in [M]^\diamond} M'$, respectively.

Theorem 14 *Let $(Y, +, \preceq)$ be a quasiordered conlinear space. Then:*
(i) $\left(\left[\widehat{\mathcal{Y}}\right]^*, \oplus, \preccurlyeq\right)$ and $\left(\left[\widehat{\mathcal{Y}}\right]^\diamond, \oplus, \curlyeqprec\right)$ are order complete, ordered conlinear spaces.
(ii) $\left(\widehat{\mathcal{Y}}^*, \oplus, \supseteq\right)$ and $\left(\widehat{\mathcal{Y}}^\diamond, \oplus, \subseteq\right)$ are order complete, ordered conlinear spaces.

Proof. (i) $\left(\left[\widehat{\mathcal{Y}}\right]^*, \oplus, \preccurlyeq\right)$ and $\left(\left[\widehat{\mathcal{Y}}\right]^\diamond, \oplus, \curlyeqprec\right)$ are order complete, ordered monoids by Proposition 27. The conditions of (C2) of Definition 5 may be checked straightforward.

(ii) Similar to (i) invoking Proposition 28. ■

Let $(Y, +, \preceq)$ be a quasiordered conlinear space. We define the set K of positive elements by

$$K := \{y \in Y :\ \theta \preceq y\}.$$

It can easily be seen that K is a convex subset of Y by (Q1), (Q2) and transitivity and that it is a cone in $\mathcal{P}(Y)$ by (Q2) containing $\theta \in Y$ by reflexivity. Therefore, from Proposition 17, we know that K is a convex element of $\mathcal{P}(Y)$.

Since $(Y, +)$ is not a group, it is not possible to get back the relation $\preceq$ from K in general by defining

$$y \preceq_K y' \qquad \Longleftrightarrow \qquad y' \in \{y\} \oplus K,$$

i.e. the relations $\preceq_K$ and $\preceq$ do not coincide. More precisely, we have $y_1 \preceq_K y_2$ implies $y_1 \preceq y_2$, but not conversely in general.

It is beyond the scope of this thesis to develop a theory of (quasi)ordered conlinear spaces although this seems to be worth doing. We shall give only one more example for a difference to the linear case.

Let $(Y, +, \mathbb{R}, \preceq)$ be an ordered conlinear space and $y_1, y_2 \in Y$. We call the set $[y_1, y_2] := \{y \in Y : y_1 \preceq y \preceq y_2\}$ the **order intervall** between y_1 and y_2. In ordered linear spaces, the convex hull of $\{y_1, y_2\}$ is always contained in the $[y_1, y_2]$. This is no longer true in conlinear spaces as the following example shows.

Example 11 *Take* $Y = \mathcal{P}\left(\mathbb{R}^2\right)$, $K = \mathbb{R}_+ \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *and set for* $M_1, M_2 \in Y$

$$M_1 \preceq_K M_2 \quad \Longleftrightarrow \quad M_2 \subseteq M_1 \oplus K.$$

Take $M_1 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$, $M_1 = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix} \right\}$. *Then* $M_1 \preceq_K M_2$ *but we have neither* $M_1 \preceq_K tM_1 + (1-t) M_2$ *nor* $tM_1 + (1-t) M_2 \preceq_K M_2$ *for* $t \in (0,1)$.

2.2.6 Ordered semilinear spaces

The order relations $\preccurlyeq$ and $\curlyeqprec$ for the power set of a linear space are the main motivation for the considerations of this section. Therefore, we extend the definitions to the case where a multipication with negative real numbers is available but the group property still does not hold. Recall Definition 10.

Definition 18 *Let* $(Y, +)$ *be a semilinear space and* $\preceq$ *be a quasiorder on* Y *satisfying (Q1) and (Q2) of Definition 17. Then* $(Y, +, \preceq)$ *is called a* **quasiordered semilinear space**. *If* $\preceq$ *is additionally antisymmetric, i.e. a partial order, then* $(Y, +, \preceq)$ *is called an* **ordered semilinear space**.

Since a semilinear space is especially conlinear, the results concerning order completion and the extension of the order to the power set remain true for semilinear spaces. We shall present the results indicating in the proofs only the main differences to the conlinear case.

Proposition 31 *Every Dedekind complete quasiordered (ordered) semilinear space can be extended to an order complete quasiordered (ordered) semilinear space by adding at most two elements.*

PROOF. Proceed as in Proposition 29: To maintain the semilinear structure of the extensions, one has to define $t \cdot l = l$ and $t \cdot u = u$ for the largest element l and the smallest u and all $t \in \mathbb{R} \backslash \{0\}$ as well as $0 \cdot l = 0 \cdot u = \theta$. Depending on the dominance property, two cases are possible: $l + u = l$ or $l + u = u$. The conditions of Definition 10 are easy to check. ■

In the following theorm, we use the definitions of $\left[\widehat{\mathcal{Y}}\right]^*$, $\left[\widehat{\mathcal{Y}}\right]^\diamond$, $\widehat{\mathcal{Y}}^*$, $\widehat{\mathcal{Y}}^\diamond$ and the algebraic operations and the order relations as in the preceding subsection on conlinear spaces. Naturally, the multiplication with real numbers is defined as follows:

$$t \cdot M := \{t \cdot m : \ m \in M\}, \quad t \cdot [M] := [t \cdot M]$$

for $t \in \mathbb{R}$, $M \in \widehat{\mathcal{P}}(Y)$, $[M] \in \{[M]^*, [M]^\diamond\}$.

Theorem 15 *Let $(Y, +, \preceq)$ be a quasiordered semilinear space. Then:*
(i) $\left(\widehat{\mathcal{P}}(Y), \oplus, \preccurlyeq\right)$ and $\left(\widehat{\mathcal{P}}(Y), \oplus, \curlyeqprec\right)$ are quasiordered, order complete semilinear spaces.
(ii) $\left(\left[\widehat{\mathcal{Y}}\right]^*, \oplus, \preccurlyeq\right)$ and $\left(\left[\widehat{\mathcal{Y}}\right]^\diamond, \oplus, \curlyeqprec\right)$ are order complete, ordered semilinear spaces.
(iii) $\left(\widehat{\mathcal{Y}}^*, \oplus, \supseteq\right)$ and $\left(\widehat{\mathcal{Y}}^\diamond, \oplus, \subseteq\right)$ are order complete, ordered semilinear spaces.

Proof. (i) This is true since $\left(\widehat{\mathcal{P}}(Y), \oplus, \preccurlyeq\right)$ and $\left(\widehat{\mathcal{P}}(Y), \oplus, \curlyeqprec\right)$ are quasiordered, order complete conlinear spaces by Theorem 13 and $\left(\widehat{\mathcal{P}}(Y), \oplus\right)$ is a semilinear space by Proposition 19.

(ii) The extension of the semilinear structure follows essentially from

$$t \cdot ([M_1] \oplus [M_2]) = t \cdot ([M_1 \oplus M_2]) = [t \cdot (M_1 \oplus M_2)] = [t \cdot M_1] \oplus [t \cdot M_2]$$

for $M_1, M_2 \in \widehat{\mathcal{P}}(Y)$, $[\cdot] \in \{[\cdot]^*, [\cdot]^\diamond\}$.

(iii) This is a consequence of (ii) and Theorem 14. ■

2.2.7 Historical comments

In the preceding section, basic order theoretic notation has been presented with special emphasis on the two canonical extensions of an order relation from a set W to its power set $\widehat{\mathcal{P}}(W)$. We refer the reader to the comprehensive 1993 survey [7] of a more algebraic motivated approach to the topic of power structures. In this paper, the relations $\curlyeqprec$ and $\preccurlyeq$ are denoted by R_0^+ and R_1^+, respectively. These and similar structures mostly defined on finite or countable sets are widely used in theoretical infomation sciences, compare for example the reference list of [7].

However, the question how algebraic and order structures have to be extended from a given set to its power set has been investigated from several, quite different viewpoints. Without intending to give a complete list we mention a few authors being of influence for the present work.

The paper [129] by R. C. Young already contains the definitions of $\preccurlyeq$ and $\curlyeqprec$ implicitly and presents applications to the analysis of upper and lower limits of sequences of numbers. Nishianidze [95] also used the relations $\preccurlyeq$ and $\curlyeqprec$. Construction mainly motivated by applications in economical and social choice theory can be found e.g. in [88]. Compare also the references therein, especially [68].

In [78] one can find a systematic investigation of six extensions of a quasiorder $\leq_K$ on a topological linear space with convex ordering cone K with nonempty interior to its power set; the relations $\preccurlyeq_K$ and $\curlyeqprec_K$ are proven to be the only relations being reflexive and transitive and definitions for in some sense convex setvalued maps are given. Several subsequent papers of the three authors of [78] contain applications, see for example [77], [76], [117] within the field of optimization with a setvalued objective function. For this topic, compare also the book [63], especially Chapter V.

Finally, in [116] an algebraic approach to vector optimization has been presented including some results on hull structures being in some sense related to power structures as used in this section.

The formulas for infimum and supremum with repect to $\preccurlyeq$ and $\curlyeqprec$ in this section and the relationships between extrema for these relations on one hand and infimal/minimal points for $\preceq$ on the other hand seems to be new.

2.3 Topological and uniform structures

For the convenience of the reader, we recall in this section definitions, facts and references concerning basic uniform and topological structures that are used in the subsequent chapters. Moreover, some results are collected not being very much standard such as the equivalent characterization of a uniformity by means of quasimetrics or an order metric.

Our standard references for this section are [75] and [72], for uniform structures also [64] and [16].

2.3.1 Topological spaces

There are several possibilities to introduce the concept of a topology. In the following definition, the neighborhoods of a point are used as the starting point since this is the most convenient method for the proofs of the next subsection on uniform topologies.

Definition 19 *Let Z be a nonempty set and for each $z \in Z$ let there be a nonempty set $\mathcal{N}(z) \subseteq \mathcal{P}(Z)$ satisfying*
(T1) If $N \in \mathcal{N}(z)$, then $z \in N$;
(T2) If $N \in \mathcal{N}(z)$ and $N \subseteq N' \in \mathcal{P}(Z)$, then $N' \in \mathcal{N}(z)$;
(T3) If $N_1, N_2 \in \mathcal{N}(z)$, then $N_1 \cap N_2 \in \mathcal{N}(z)$;
(T4) If $N \in \mathcal{N}(z)$, then there is $N' \in \mathcal{N}(z)$ such that $N \in \mathcal{N}(z')$ for each $z' \in N'$.
An element $N \in \mathcal{N}(z)$ is called a **neighborhood** *of $z \in Z$. The entity $\mathcal{N}(z)$ is called a* **system of neighborhoods** *of $z \in Z$. A subset $T \subseteq Z$ is called* **open set** *iff $T \in \mathcal{N}(z)$ whenever $z \in T$. The set $\mathcal{T}$ of all open sets is called a* **topology** *on Z and the pair $(Z, \mathcal{T})$ is called a* **topological space**.

The axioms (T2) and (T3) imply that $\mathcal{N}(z)$ is a filter (see Definiton 0.1, 0.3 on p. 5f of [64]). A filter base for $\mathcal{N}(z)$ is called a **neighborhood base** for $z \in Z$. A subset $\mathcal{B}(z) \subseteq \mathcal{P}(Z)$ is a neighborhood base for $z \in Z$ if it satisfies (T1), (T4) with $\mathcal{N}$ replaced by $\mathcal{B}$ and

(T3') If $B_1, B_2 \in \mathcal{B}(z)$, then there is $B \in \mathcal{B}(z)$ such that $B \subseteq B_1 \cap B_2$.

In this case, the subset of $\mathcal{P}(Z)$ that contains the supersets of members of $\mathcal{B}(z)$ satisfies (T1) – (T4), i.e., it is a neighborhood system $\mathcal{N}(z)$ for $z \in Z$. If a neighborhood base for each $z \in Z$ is given, the corresponding topology is uniquely defined.

The complement $Z \backslash T$ for $T \in \mathcal{T}$ is said to be a **closed** set. Let $M \subseteq Z$ be a subset of Z. An element $z \in M$ is called **interior point** of M iff there is $N \in \mathcal{N}(z)$ such that $N \subseteq M$. The set of interior points of M is denoted by $\operatorname{int} M$. The set $\operatorname{cl} M := Z \backslash \operatorname{int}(Z \backslash M)$ is called the **closure** of M.

A topological space is called **separated** or **Hausdorff** iff for any two distinct points $z_1, z_2 \in Z$ there are disjoint open sets $T_1, T_2 \in \mathcal{T}$ such that $z_1 \in T_1$ and $z_2 \in T_2$. This is equivalent to the property that the intersection of the closed neighborhoods of a point $z \in Z$ contains only z itself. In a Hausdorff topological space, each set $\{z\}$ for $z \in Z$ is closed.

Prominent examples of topological spaces are topological (Abelian) groups. We give the definition, compare for example [72], [54] or [64]. In order to do this one needs the concepts of a continuous function and of a product topology.

Let $(Z_1, \mathcal{T}_1)$, $(Z_2, \mathcal{T}_2)$ be two topological spaces. A function $f : Z_1 \to Z_2$ is said to be **continuous** iff the inverse image of a member of $\mathcal{T}_2$ is a member of $\mathcal{T}_1$. The collection of all Cartesian products $T_1 \times T_2$ for $T_1 \in \mathcal{T}_1$, $T_2 \in \mathcal{T}_2$ form the base for a uniquely determined topology on the Cartesian product $Z_1 \times Z_2$ called the **product topology**.

Definition 20 *Let $(Y, \circ)$ be a group supplied with a topology $\mathcal{T}$. Suppose further that (i) the mapping $(y_1, y_2) \to y_1 \circ y_2$ is a continuous function of the Cartesian product $Y \times Y$ onto Y and (ii) the mapping $y \to y^{-1}$ is a continuous function of Y onto Y (y^{-1} being the inverse element of y with respect to $\circ$). Then $(Y, \circ, \mathcal{T})$ is called a* **topological group**. *If the group is additionally commutative, it is called* **topological Abelian group**.

A neighborhood N of θ in a topological group $(Y, \circ, \mathcal{T})$ is called **symmetric** if $y \in N$ implies $y^{-1} \in N$. The symmetric neighborhoods of the neutral element $\theta \in Y$ form a neighborhood base of $\theta \in Y$. The topology of a topological Abelian group is uniquely defined by a base of symmetric neighborhoods of the neutral element since $\{y\} \odot N$ is a neighborhood of $y \in Y$ if and only if N is a neighborhood of $\theta \in Y$. If $\mathcal{B}(\theta)$ is a neighborhood base of $\theta \in Y$ and $\mathcal{T}$ the topology generated by $\mathcal{B}(\theta)$, then $(Y, \mathcal{T})$ is separated if and only if $\bigcap_{B \in \mathcal{B}(\theta)} B = \{\theta\}$.

Let $(Y, \circ, \mathcal{T})$ be a topological Abelian group with neutral element $\theta \in Y$. Further, let $\leq$ be a quasiorder on Y such that $(Y, \circ, \leq)$ is a quasiordered monoid in the sense of Definition 16. Then $(Y, \circ, \leq, \mathcal{T})$ is called a **quasiordered topological Abelian group**. We study two properties linking the order structure with the topological structure.

(A) There is a neighborhood base $\mathcal{B}(\theta)$ of $\theta \in Y$ such that

$$\forall B \in \mathcal{B}(\theta): \ \left(\theta \leq y \leq y', \, y' \in B\right) \implies y \in B.$$

(B) There exists a neighborhood base $\mathcal{N}(\theta)$ of $\theta \in Y$ such that

$$\forall N \in \mathcal{N}(\theta): \ (y_1 \leq y \leq y_2, \ y_1, y_2 \in N) \implies y \in N.$$

Lemma 1 *Let $(Y, \circ, \leq, \mathcal{T})$ be a quasiordered topological Abelian group with neutral element $\theta \in Y$. Then, the properties (A) and (B) are equivalent.*

Proof. First, we show that (A) implies (B). Let $M \in \mathcal{N}(\theta)$. We shall show that there is $N \in \mathcal{N}(\theta)$, $N \subseteq M$ satisfying $N = \{y \in Y : \ y_1 \leq y \leq y_2, \ y_1, y_2 \in N\}$. Take $B \in \mathcal{B}(\theta)$ such that $B \odot B \subseteq M$ and a symmetric $N' \in \mathcal{N}(\theta)$ such that $N' \odot N' \subseteq B$. Let $y \in Y$ such that $y_1 \leq y \leq y_2$ for $y_1, y_2 \in N'$. Then

$$\theta \leq y \circ y_1^{-1} \leq y_2 \circ y_1^{-1} \in N' \odot N' \subseteq B.$$

Hence $y \circ y_1^{-1} \in B$ implying $y \in \{y_1\} \odot B \subseteq N' \odot B \subseteq B \odot B \subseteq M$. Hence $N := \{y \in Y : \ y_1 \leq y \leq y_2, \ y_1, y_2 \in N'\}$ as desired.

Conversely, if $N \in \mathcal{N}(\theta)$, $\mathcal{N}(\theta)$ satisfies (B) and $\theta \leq y \leq y' \in N$, then $y \in N$ since $\theta \in N$. ■

Lemma 2 *A quasiorder on a separated topological Abelian group satisfying (A) (and (B)) is antisymmetric, i.e., a partial order.*

Proof. If $\theta \leq y \leq \theta$ for some $y \in Y$, then $y \in B$ for each $B \in \mathcal{B}(\theta)$ satisfying (A). Since $\mathcal{B}$ is separated, the result follows. ■

Definition 21 *A partial order $\leq$ on a topological Abelian group $(Y, \circ, \leq, \mathcal{T})$ satisfying (A) (and (B), too) is called* **normal**. *In this case, $(Y, \circ, \leq, \mathcal{T})$ is called* **normally ordered**.

A set $M \subseteq Y$ satisfying $M = \{y \in Y : \ y_1 \leq y \leq y_2, \ y_1, y_2 \in M\}$ is called **full** or **saturated**. Thereby, condition (B) can be rewritten as: There is a neighborhood base of the neutral element consisting of full (saturated) sets.

Let $(Y, \circ, \leq, \mathcal{T})$ be a quasiordered topological Abelian group. Defining

$$P := \{y \in Y : \ \theta \leq y\}, \qquad P^{-1} := \left\{y' \in Y : \ y' \leq \theta\right\}$$

we may see that

$$y_1 \leq y_2 \quad \iff \quad y_2 \circ y_1^{-1} \in P \quad \iff \quad y_1 \circ y_2^{-1} \in P^{-1}.$$

Hence, a set $M \subseteq Y$ is full (saturated) if and only if $M = (M \odot P) \bigcap \left(M \odot P^{-1}\right)$.

If a quasiorder with the set P of "positive" elements is given, another quasiorder is generated by $\operatorname{cl} P$ (note that $(\operatorname{cl} P)^{-1} = \operatorname{cl} P^{-1}$):

$$y_1 \leq_{\operatorname{cl} P} y_2 \quad :\iff \quad y_2 \circ y_1^{-1} \in \operatorname{cl} P \quad \iff \quad y_1 \circ y_2^{-1} \in \operatorname{cl} P^{-1}.$$

The following relationship holds true concerning normality.

Lemma 3 *Let $(Y, \circ, \mathcal{T})$ be a topological Abelian group and $\leq_P$ a partial order with $P = \{y \in Y : \theta \leq y\}$. Then, $\leq_P$ is normal if and only if $\leq_{\operatorname{cl} P}$ is normal.*

PROOF. Since $y_1 \leq_P y_2$ implies $y_1 \leq_{\operatorname{cl} P} y_2$, it is clear that the normality of $\leq_{\operatorname{cl} P}$ implies the normality of $\leq_P$.

Conversely, assume the normality of $\leq_P$, i.e. there is a neighborhood base $\mathcal{B}(\theta)$ of $\theta \in Y$ such that $B = (P \odot B) \bigcap \left(P^{-1} \odot B\right)$ for each $B \in \mathcal{B}(\theta)$. Take $B \in \mathcal{B}(\theta)$ and $y_1, y_2 \in Y$ such that $\theta \leq_{\operatorname{cl} P} y_1 \leq_{\operatorname{cl} P} y_2 \in B$. Since Y is a topological Abelian group, there is $B' \in \mathcal{B}(\theta)$ such that $B' \odot B' \subseteq B$ (see e.g. [64], p. 37f). From $\theta \leq_{\operatorname{cl} P} y_1$ we obtain $y_1 \in \operatorname{cl} P \subseteq P \odot B' \subseteq P \odot B$. From $\theta \leq_{\operatorname{cl} P} y_2$ we get $y_2^{-1} \in \operatorname{cl} P^{-1} \subseteq P^{-1} \odot B'$ and using this from $y_1 \leq_{\operatorname{cl} P} y_2$ it follows

$$y_1 \in \left\{y_2^{-1}\right\} \odot \operatorname{cl} P^{-1} \subseteq \left(P^{-1} \odot B'\right) \odot \left(P^{-1} \odot B'\right) \subseteq P^{-1} \odot B.$$

Therefore, $y_1 \in (P \odot B) \bigcap \left(P^{-1} \odot B'\right) = B$ as desired. ■

The notion of a normal partial order and the corresponding normal cone (in the classical sense) is a central concept in the theory of ordered topological linear spaces. Details, especially some more equivalent characterizations can be found e.g. in [99], Chapter 2, or [107], Chapter V, §3. Compare also [44], p. 24ff, and [58], Chapter 1.2. An interesting result using the normality of orders on topological Abelian groups is Proposition 12 of [87], p. 76. Note that condition (0.5) of [108], p. 1 is also a normality condition that is used for the definition of quasimetric spaces.

2.3.2 Uniform spaces

Let X be a nonempty set. We consider a collection $\mathcal{U}$ of subsets E of $X \times X := \{(x_1, x_2) : x_1, x_2 \in X\}$. The set $\Delta := \{(x, x) \in X \times X : x \in X\}$ is called the **diagonal**. For $E \subseteq X \times X$ we denote $E^{-1} := \{(x_2, x_1) : (x_1, x_2) \in E\}$ and $E \circ E := \{(x_1, x_2) \in X \times X : \exists x \in X : (x_1, x), (x, x_2) \in E\}$.

Definition 22 *Let X be a nonempty set. A set $\mathcal{U} \subseteq \mathcal{P}(X \times X)$ is said to be a* **uniformity** *on X iff*
(U1) $E \in \mathcal{U}$, $E \subseteq E'$ implies $E' \in \mathcal{U}$ and $E_1, E_2 \in \mathcal{U}$ implies $E_1 \cap E_2 \in \mathcal{U}$;
(U2) If $E \in \mathcal{U}$, then $\Delta \subseteq E$;
(U3) If $E \in \mathcal{U}$, then there is $E' \in \mathcal{U}$ such that $E' \subseteq E^{-1}$;
(U4) For all $E \in \mathcal{U}$ there is $E' \in \mathcal{U}$ such that $E' \circ E' \subseteq E$.
The pair $(X, \mathcal{U})$ is called a uniform space. The elements of $\mathcal{U}$ are called **entourages** *or* **surroundings**. *The uniformity $\mathcal{U}$ is called* **separated** *iff*
(U5) $\bigcap_{E \in \mathcal{U}} E = \Delta$.

A uniformity $\mathcal{U}$ on X is a filter by (U1), (U4). A filter base $\mathcal{U}_B$ for the filter $\mathcal{U}$ is called a **base of the uniformity $\mathcal{U}$**.

A set $\mathcal{U}_B \subseteq \mathcal{P}(X \times X)$ is a base of some uniformity $\mathcal{U}$ if it satisfies (U2), (U3), (U4) with $\mathcal{U}$ replaced by $\mathcal{U}_B$ and, additionally,

(U1') If $E_1, E_2 \in \mathcal{U}_B$, then there is $E \in \mathcal{U}_B$ with $E \subseteq E_1 \cap E_2$.

In this case, a uniformity is obtained by taking supersets in $\mathcal{P}(X \times X)$ of the members of $\mathcal{U}_B$.

Let $(X,\mathcal{U})$ be a uniform space. The family of sets $U(x) := \{U_E(x) : E \in \mathcal{U}\}$ where $U_E(x) := \{x' \in X : (x, x') \in E\}$ is a neighborhood system for $x \in X$, i.e., it satisfies (T1) – (T4) of Definiton 19. Similarly, a base of the uniformity generates a neighborhood base for each $x \in X$. In this way, a uniquely defined topology, called the **uniform topology** on X can be generated. If the uniformity is separated, then so is the uniform topology.

The class of separated uniform spaces coincides with the class of completely regular (Tychonoff) spaces. This result is well–known, [72], Corollary 17, p. 188 or [75], p. 48-50.

Metric spaces, topological groups and hence topological linear spaces can be supplied with a uniform structure such that the given topology is the corresponding uniform topology. For example, let $(Y, \circ, \mathcal{T})$ be a topological Abelian group and $\mathcal{B}(\theta)$ a neighborhood base of the neutral element $\theta \in Y$. Take $B \in \mathcal{B}(\theta)$ and define a subset of $Y \times Y$ by

$$E_B := \left\{(y_1, y_2) \in Y \times Y : \, y_1^{-1} \circ y_2 \in B\right\}.$$

The set $\{E_B : B \in \mathcal{B}(\theta)\}$ is a base for a uniform structure on Y. Of course, if Y is not commutative one has to distinguish between a "left" and a "right" uniformity. See the book of James [64] for this and further related results.

There are other possibilities to generate a uniform structure: via a family of realvalued pseudometrics or quasimetrics and via an order metric. We shall give the definitions since they will admit a greater degree of freedom in defining an order relation on X that is an essential ingredient of minimal element theorems.

The first equivalent description of a uniformity is via a family of pseudometrics. This notion is standard in textbooks on uniform spaces, compare [72], [75], [21], [64], [16].

Definition 23 *Let X be a nonempty set. A function $p : X \times X \to \mathbb{R}$ is called a* **pseudometric** *on X iff for all $x, x_1, x_2 \in X$ the following conditions are satisfied:*
(UP1) $p(x_1, x_2) \geq 0$ and $p(x, x) = 0$;
(UP2) $p(x_1, x_2) = p(x_2, x_1)$;
(UP3) $p(x_1, x_2) \leq p(x_1, x) + p(x, x_2)$.
Let $(\Lambda, \prec)$ be a directed set. A set $\{p_\lambda\}_{\lambda\in\Lambda}$ of pseudometrics $p_\lambda : X \times X \to \mathbb{R}$ satisfying
(UP4) $\lambda, \mu \in \Lambda$, $\lambda \prec \mu$ implies $p_\lambda(x_1, x_2) \leq p_\mu(x_1, x_2)$ for all $x_1, x_2 \in X$
is called a **family of pseudometrics**. *If, additionally, the condition*
(UP5) If $p_\lambda(x_1, x_2) = 0$ for all $\lambda \in \Lambda$, then $x_1 = x_2$.
is satisfied, then the family of pseudometrics is called **separating**.

The following proposition gives the relationsship between uniformities and pseudometric spaces. It is a fundamental result on uniform spaces.

Proposition 32 *(i) Let* $(X,\mathcal{U})$ *be a (separated) uniform space. Then there is a (separating) family* $\{p_\lambda\}_{\lambda\in\Lambda}$ *of pseudometrics on* X *such that the entity of sets*

$$E_{\lambda,r} := \{(x_1,x_2)\in X\times X : \ p_\lambda(x_1,x_2) < r\}, \quad \lambda\in\Lambda,\ r>0, \tag{2.23}$$

is a base for the uniformity $\mathcal{U}$.

(ii) Let X *be a nonempty set and* $\{p_\lambda\}_{\lambda\in\Lambda}$ *a (separating) family of pseudometrics on* X*. Then the entity of sets given by (2.23) is a base for a (separated) uniformity* $\mathcal{U}$ *on* X.

Proof. The proof is standard in text books on uniform spaces. Compare [72], Metrization Lemma, Theorems 15, 16. The proof of (i) is essentially based on a metrization procedure, (ii) can be verified directly checking the properties for bases of uniformities. ■

A family of pseudometrics generates a uniformity and hence a corresponding uniform topology. More precisely, we have the following result.

Corollary 7 *A topological space* $(X,\mathcal{T})$ *is a (separated) uniform space if and only if its topology* $\mathcal{T}$ *can be generated by a (separating) family of pseudometrics.*

Proof. If $\{p_\lambda\}_{\lambda\in\Lambda}$ is a family of pseudometrics on X, then the entity of sets

$$N_{\lambda,r}(x) := \left\{x'\in X : \ p_\lambda\left(x',x\right) < r\right\}, \quad \lambda\in\Lambda,\ r>0,\ x\in X \tag{2.24}$$

is a neighborhood base for some topology on X that coincides with the uniform topology generated by $\{p_\lambda\}_{\lambda\in\Lambda}$. Conversely, if $\mathcal{T}$ is a topology on X generated by some uniformity $\mathcal{U}$ on X, then there is a family of pseudometrics yielding a base for $\mathcal{U}$ via (2.23) and also a base for the uniform topology via (2.24). ■

Fang [33] introduced so called F–type topological spaces using families of quasimetrics. It has been observed in [51] that this is just another way to generate the topology via a uniformity.

Definition 24 *Let* X *be a nonempty set and* $(\Lambda,\prec)$ *be a directed set. A set* $\{q_\lambda\}_{\lambda\in\Lambda}$ *of functions* $q_\lambda : X\times X\to \mathbb{R}$ *is called a* **family of quasimetrics** *on* X *iff the following conditions are satisfied:*
(UQ1) $q_\lambda(x_1,x_2)\geq 0$ *and* $q_\lambda(x,x)=0$ *for all* $\lambda\in\Lambda$ *and all* $x,x_1,x_2\in X$;
(UQ2) $q_\lambda(x_1,x_2)=q_\lambda(x_2,x_1)$ *for all* $\lambda\in\Lambda$ *and all* $x_1,x_2\in X$;
(UQ3) For all $\lambda\in\Lambda$ *there is* $\mu\in\Lambda$ *such that* $\lambda\prec\mu$ *and* $q_\lambda(x_1,x_2)\leq q_\mu(x_1,x)+q_\mu(x,x_2)$ *for all* $x,x_1,x_2\in X$;
(UQ4) $\lambda,\mu\in\Lambda$, $\lambda\prec\mu$ *implies* $q_\lambda(x_1,x_2)\leq q_\mu(x_1,x_2)$ *for all* $x_1,x_2\in X$.
If, additionally, the condition
(UQ5) if $q_\lambda(x_1,x_2)=0$ *for all* $\lambda\in\Lambda$, *then* $x_1=x_2$;
is satisfied, then the family of quasimetrics is called **separating**.

Proposition 33 *(i) Let $(X,\mathcal{U})$ be a (separated) uniform space. Then there is a (separating) family $\{q_\lambda\}_{\lambda\in\Lambda}$ of quasimetrics on X such that the entity of sets*

$$E_{\lambda,r} := \{(x_1,x_2)\in X\times X:\ q_\lambda(x_1,x_2)<r\},\quad \lambda\in\Lambda,\ r>0, \tag{2.25}$$

is a base for the uniformity $\mathcal{U}$.

(ii) Let X be a nonempty set and $\{q_\lambda\}_{\lambda\in\Lambda}$ a (separating) family of quasimetrics on X. Then the entity of sets given by (2.25) is a base for a (separated) uniformity $\mathcal{U}$ on X.

Proof. (i) Apply Proposition 32, (i) to get a family of pseudometric generating the topology. Since every pseudometric is all the more a quasimetric, the result follows. (ii) Check the properties for a base of a uniformity for the entity of sets $E_{\lambda,r}$ in (2.25). ■

Corollary 8 *A topological space $(X,\mathcal{T})$ is a (separated) uniform space if and only if its topology $\mathcal{T}$ can be generated by a (separating) family of quasimetrics.*

Proof. This can been seen by the same arguments as in the proof of Corollary 7. See also [51], p. 579, Theorem 2.4. ■

The next possibility to introduce a uniformity uses functions on $X\times X$ to some ordered topological group Y with properties very close to the properties of a metric in the usual sense.

Definition 25 *Let X be a nonempty set, $(Y,\circ,\leq,\mathcal{T})$ a normally ordered topological Abelian group with neutral element $\theta\in Y$. A function $D: X\times X\to Y$ is called an* **order pseudometric** *iff for all $x,x_1,x_2\in X$ the following conditions are satisfied:*
(UM1) $\theta\leq D(x_1,x_2)$ and $D(x,x)=\theta$;
(UM2) $D(x_1,x_2)=D(x_2,x_1)$;
(UM3) $D(x_1,x_2)\leq D(x_1,x)\circ D(x,x_2)$.
If, additionally, the condition
(UM4) $D(x_1,x_2)=\theta$ implies $x_1=x_2$;
is satisfied, then D is called an **order metric**[1]*. The pair (X,D) is called an* **order (pseudo)metric space***.*

If $Y=\mathbb{R}$, i.e. the set of real numbers together with the usual addition, order relation and topology, the widely used definition of a **metric space** is obtained.

The above definition can be generalized in different directions. For example, Y can be assumed to be not a group but an ordered monoid. Compare e. g. [108] for details and the practical importance of more general structures. Also, some of the axioms can be relaxed. In the following chapter, a generalization is given in order to obtain as much freedom as possible to define order relations on uniform space.

[1] We would prefer simply to speak of (pseudo)metrics. For historical reasons, we keep on using the term "(pseudo)metric" only in case $Y=\mathbb{R}$.

Proposition 34 *Let X be a nonempty set, $(Y,\circ,\leq,\mathcal{T})$ a normally ordered topological Abelian group with neutral element $\theta\in Y$ and $D: X\times X\to Y$ an order pseudometric. Let $\mathcal{B}(\theta)$ be a neighborhood base of $\theta\in Y$ consisting of symmetric neighborhoods. Then, the entity of sets*

$$E_B := D^{-1}(B) = \{(x_1,x_2)\in X\times X:\ D(x_1,x_2)\in B\}, \quad B\in\mathcal{B}(\theta) \tag{2.26}$$

is a base of a uniform structure on X. The corresponding uniform topology coincides with the topology generated by D. If D is an order metric, the uniformity is separated.

Proof. Denote $\mathcal{U}_B = \{E_B:\ B\in\mathcal{B}(\theta)\}$. First, take $E_1,E_2\in\mathcal{U}_B$. Then there are $B_1,B_2\in\mathcal{B}(\theta)$ such that $E_1=D^{-1}(B_1)$, $E_2=D^{-1}(B_2)$. Since $\mathcal{B}(\theta)$ is a neighborhood base of $\theta\in Y$, there is $B\in\mathcal{B}(\theta)$ such that $B\subseteq B_1\cap B_2$. It is easy to see $E_B\in\mathcal{U}_B$ and $E_B\subseteq E_1\cap E_2$.

It remains to show that $\mathcal{U}_B$ satisfies (U2), (U3) and (U4). (U2) follows from (UM1) whereas (U3) is a consequence of (UM2) since the elements $B\in\mathcal{B}(\theta)$ are symmetric.

To show (U4), take $E_B\in\mathcal{U}_B$. Since $(Y,\circ,\leq,\mathcal{T})$ is a topological Abelian group, there is $B'\in\mathcal{B}(\theta)$ such that $B'\odot B'\subseteq B$. Set $E'=D^{-1}(B')$ and take $(x_1,x_2)\in E'\circ E'$. Then there is $x\in X$ such that $(x_1,x),(x,x_2)\in E'$. From (UM3) we obtain

$$D(x_1,x_2)\leq D(x_1,x)\circ D(x,x_2)\in B'\odot B'\subseteq B.$$

Therefore $D(x_1,x_2)\in B$ by normality of $\leq$. Hence $(x_1,x_2)\in E_B$ and $E'\circ E'\subseteq E_B$ as desired.

The last assertions are obvious. ■

Proposition 35 *Let $(X,\mathcal{U})$ be a (separated) uniform space. Then, there is a topological Abelian group and an order pseudometric (metric) such that D generates the uniform structure $\mathcal{U}$ on X via (2.26).*

Proof. Since $(X,\mathcal{U})$ is uniform, there exists a family of pseudometrics $\{p_\lambda\}_{\lambda\in\Lambda}$ such that the entity of sets

$$E_{\lambda,r} := \{(x_1,x_2)\in X\times X:\ p_\lambda(x_1,x_2)\leq r\}, \quad \lambda\in\Lambda,\ r>0$$

is a base for the uniformity $\mathcal{U}$.

Define $y:=(r_\lambda)_{\lambda\in\Lambda}\in\mathbb{R}^\Lambda =: Y$, $r_\lambda\in\mathbb{R}$. With the usual componentwise addition and $y'\leq y$ iff $r_\lambda - r'_\lambda\geq 0$ for all $\lambda\in\Lambda$, we obtain a partially ordered Abelian group $(Y,+,\leq)$ with neutral element $\theta:=(0_\lambda)_{\lambda\in\Lambda}$. It is not hard to see that the sets

$$B_{t,\lambda(n)} := \{y\in Y:\ |r_{\lambda_1}|<t,\ldots,|r_{\lambda_n}|<t\}$$

for $t>0$, $n\in\mathbb{N}\setminus\{0\}$, $\lambda(n)=\{\lambda_1,\ldots,\lambda_n\}\subseteq\Lambda$ form a neighborhood base of a topology $\mathcal{T}$ such that $(Y,+,\leq,\mathcal{T})$ is a normally ordered, topological Abelian group. Defining $D: X\times X\to Y$ by

$$D(x_1,x_2) := (p_\lambda(x_1,x_2))_{\lambda\in\Lambda}$$

we may easily verify the axioms (UM1)–(UM4) from (UP1)–(UP4). Moreover, the definition of D ensures that the uniformity generated by D coincides with $\mathcal{U}$. This completes the proof. ∎

Remark 8 *Propositions 34 and 35 show that a space admits a uniformity if and only if it admits a metric in the sense of Definition 25. This observation is due to [66], but it is much less standard than the formulation via families of pseudometrics.*

Proposition 36 *Let X be a nonempty set, $(Y, \circ, \leq, \mathcal{T})$ a normally ordered topological Abelian group with neutral element $\theta \in Y$ and $D : X \times X \to Y$ an order pseudometric. The sets*

$$N_B(x) := \left\{x' \in X : \; D\left(x, x'\right) \in B\right\}, \quad B \in \mathcal{B}(\theta)$$

form a neighborhood base of $x \in X$. The collection

$$\mathcal{N} := \{\{N_B(x) : \; B \in \mathcal{B}(\theta)\} : \; x \in X\}$$

is a neighborhood base on X generating a topology $\mathcal{T}$. If D is an order metric, the $(X, \mathcal{T})$ is a separated topological space.

Proof. It suffices to show that $\mathcal{N}(x) := \{N_B(x) : \; B \in \mathcal{B}(\theta)\}$ satisfies (T1), (T3) and (T4) of Definition 19. (T1) is clear from (UM1).

To prove (T3) take $B_1, B_2 \in \mathcal{B}(\theta)$. Since $\mathcal{B}(\theta)$ is a neighborhood base of $\theta \in Y$, there is $B \in \mathcal{B}(\theta)$ such that $B \subseteq B_1 \bigcap B_2$. Since for $x' \in N_B(x)$ we have $D(x, x') \in B \subseteq B_1 \bigcap B_2$ we may conclude $N_B(x) \subseteq N_{B_1}(x) \bigcap N_{B_2}(x)$, hence $\mathcal{N}(x)$ satisfies (T3).

To show (T4) take $B \in \mathcal{B}(\theta)$ and $x' \in N_B(x)$. Since $\mathcal{B}(\theta)$ is a neighborhood base of $\theta \in Y$ and $y := D(x, x') \in B$ there is a neighborhood $N(y) \subseteq Y$ such that $N(y) \subseteq B$. Since $\left\{y^{-1}\right\} \odot N(y)$ is a neighborhood of $\theta \in Y$, there is $B' \in \mathcal{B}(\theta)$ such that $B' \subseteq \left\{y^{-1}\right\} \odot N(y)$. This implies $\{y\} \odot B' \subseteq N(y) \subseteq B$. Take $u \in N_{B'}(x')$. From (UM3) we get

$$D(x, u) \leq D\left(x, x'\right) \circ D\left(x', u\right) \in \{y\} \odot B' \subseteq B.$$

The normality of $\leq$ implies $D(x, u) \in B$, hence $u \in N_B(x)$ and therefore $N_{B'}(x') \subseteq N_B(x)$. This completes the proof. ∎

Note that the group Y in Proposition 34 – 36 can be replaced by a locally convex space with a normal cone, see [91]. A definition of locally convex spaces is given below. To the opinion of the author, the choice of a normally ordered topological Abelian group admits a lucid formulation without to much non–standard technicalities. We refer the reader also to [27] and the various papers of A. B. Nemeth about so called cone valued metrics.

Thus, if a uniform space is given (say, by a base for the uniformity, i.e. for the system of entourages), there are at least three further possibilities to generate its uniform structure as well as the corresponding uniform topology.

2.3.3 Completeness in uniform spaces

Many topological concepts carry over from the metric to the uniform case. The concepts of a Cauchy sequence (net) and completeness are of importance for the formulation of variational principles. Therefore, we shall give the definitions.

Let $(X,\mathcal{U})$ be a uniform space. Let $(A,\prec)$ be a directed set. A net $\{x_\alpha\}_{\alpha\in A}\subseteq X$ is called a **Cauchy net** iff

$$\forall E\in\mathcal{U}\ \exists\alpha_E\in A:\ (\alpha_1,\alpha_2\in A,\ \alpha_E\prec\alpha_1\prec\alpha_2 \implies (x_{\alpha_1},x_{\alpha_2})\in E)\,.$$

The net $\{x_\alpha\}_{\alpha\in A}\subseteq X$ is called **convergent** to $x\in X$ iff

$$\forall E\in\mathcal{U}\ \exists\alpha_E\in A:\ (\alpha,\in A,\ \alpha_E\prec\alpha \implies (x_\alpha,x)\in E)\,.$$

The uniform space $(X,\mathcal{U})$ is called **complete** iff every Cauchy net converges to some element $x\in X$. The uniform space $(X,\mathcal{U})$ is called **sequentially complete** iff every Cauchy sequence ($(A,\prec)=(\mathbb{N},\leq)$) converges to some element $x\in X$.

In the above definitions, it suffices to involve elements $E\in\mathcal{U}_B$ for a base $\mathcal{U}_B$ of the uniformity $\mathcal{U}$. A base may be generated by a family of pseudometric, of quasimetrics or by an order pseudometric. Therefore, there are several possibilities to characterize Cauchy (convergent) nets. For example, if an order pseudometric $D:X\times X\to Y$ is used, $(Y,\circ,\leq,\mathcal{T})$ being a normally ordered topological Abelian group, the property of being a Cauchy net can be expressed as follows:

$$\forall B\in\mathcal{B}(\theta)\ \exists\alpha_B\in A:\ (\alpha_1,\alpha_2\in A,\alpha_B\prec\alpha_1\prec\alpha_2 \implies D(x_{\alpha_1},x_{\alpha_2})\in B)\,,$$

where $\mathcal{B}(\theta)$ is a neighborhood base of $\theta\in Y$.

2.3.4 The linear case

Linear spaces are special cases of conlinear spaces defined in Section 2.1.2.

Definition 26 *Let $(X,+)$ be a linear space supplied with a topology $\mathcal{T}$. Suppose further that (i) the mapping $(x_1,x_2)\to x_1+x_2$ is a continuous mapping of the Cartesian product $X\times X$ onto X and (ii) the mapping $(t,x)\to tx$ is a continuous mapping of $\mathbb{R}\times X$ onto X. Then $(X,+,\mathcal{T})$ is called a (real)* **topological linear space***.*

A standard reference is the book of Köthe [75]. Since a topological linear space is especially a topological Abelian group with respect to addition, it is clear that it can be provided with a unifom structure in the same way, i.e., the topology of a topological linear space is uniformizable ([75], §15, (3), p. 150). Hence, Propositions 32 and 7 as well as 34 and 36 remain in force, i.e., the uniform structure as well as the topology of a topological linear space can be generated by an order pseudometric or by a family of realvalued pseudometrics.

An important subclass of the class of topological linear spaces is the class of locally convex spaces. A definition is as follows.

Definition 27 *A topological linear space* $(X, +, \mathcal{T})$ *is called* **locally convex topological linear space** *(for short: locally convex space) iff there is a neighborhood base of* $\theta \in X$ *consisting of convex sets.*

Hyers [56], [57] and LaSalle [79] observed that every topological linear space admits a system of quasinorms generating the topology. The definition is as follows.

Definition 28 *Let* $(X, +)$ *be a real linear space and* $(\Lambda, \prec)$ *be a directed set. A system* $\{\|\cdot\|_\lambda\}_{\lambda\in\Lambda}$ *of functions* $\|\cdot\|_\lambda : X \to \mathbb{R}$ *is called a* **family of quasinorms** *on* X *iff for all* $x, x_1, x_2 \in X$ *the following conditions are satisfied:*
(NQ1) $\|x\|_\lambda \geq 0$ *for all* $\lambda \in \Lambda$*;*
(NQ2) $\|tx\|_\lambda = |t| \|x\|_\lambda$ *for all* $t \in \mathbb{R}$ *and* $\lambda \in \Lambda$*;*
(NQ3) For all $\lambda \in \Lambda$ *there is* $\mu \in \Lambda$ *such that* $\lambda \prec \mu$ *and* $\|x_1 + x_2\|_\lambda \leq \|x_1\|_\mu + \|x_2\|_\mu$*.*
(NQ4) $\lambda, \mu \in \Lambda$*,* $\lambda \prec \mu$ *implies* $\|x\|_\lambda \leq \|x\|_\mu$*.*
If, additionally, the condition
(NQ5) If $\|x\|_\lambda = 0$ *for all* $\lambda \in \Lambda$*, then* $x = \theta$*;*
is satisfied, then the family of quasimetrics is called **separating**.

Given a family of quasinorms $\{\|\cdot\|_\lambda\}_{\lambda\in\Lambda}$ on the linear space X, the expressions

$$q_\lambda(x_1, x_2) := \|x_1 - x_2\|_\lambda\,,\ \lambda \in \Lambda,\ x_1, x_2 \in X$$

define a family of quasimetrics on X generating a uniform structure. The corresponding uniform topology $\mathcal{T}$ on X can be generated by the sets

$$B_{\lambda,r} := \{x \in X :\ \|x\|_\lambda < r\}\,,\ \lambda \in \Lambda,\ r > 0$$

forming a neighborhood base of $\theta \in X$ on X. The couple $(X, \mathcal{T})$ is a topological linear space. The following result is due to Hyers and LaSalle. A concise proof and some more details can be found in [81].

Proposition 37 *A linear space* $(X, +)$ *is a (separated) topological linear space with respect to a topology* $\mathcal{T}$ *on* X *if and only if the topology can be generated by a (separating) family of quasinorms on* X*.*

Proof. See [56], [79] or [81], Theorem 1.6. ∎

Remark 9 *If the choice* $\mu = \lambda$ *is always possible in (NQ3) of Definition 28, the function* $\|\cdot\|_\lambda$ *is called a (realvalued)* **seminorm**[2]*. If the topology of the linear space* X *can be generated by a family of seminorms, the resulting topological linear space is a locally convex space. In this case, the sets* $B_{\lambda,r}$ *are convex, hence there is a neighborhood base of* $\theta \in X$ *consisting of convex sets.*

[2]This term is not consistent with Definition 23. It seems to be preferable to replace the term "seminorm" by "pseudonorm" since every seminorm generates a pseudometric in an obvious way. However, we keep on using "seminorm" for historical reasons.

As it is the case for uniform spaces, the family of quasinorms can be replaced by a single norm with values in a set with less structure than the set of nonnegative real numbers.

Definition 29 *Let $(X,+)$ be a linear space and $(V,+,\leq_K)$ be a quasiordered linear spaces where $K \subseteq V$ is the convex ordering cone. A function $N : X \times X \to V$ is called an* **order quasinorm** *iff there is a linear mapping $T : V \to V$ such that $T(K) \subseteq K$ and for all $x, x_1, x_2 \in X$ the following conditions are satisfied:*
(N1) $\theta \leq_K N(x)$ where $\theta \in V$ is the neutral element of $(V,+)$;
(N2) $N(tx) = |t| N(x)$ for all $t \in \mathbb{R}$;
(N3) $N(x_1 + x_2) \leq_K T(N(x_1) + N(x_2))$.
If, additionally, the condition
(N4) $N(x) = \theta \in V$ implies $x = \theta \in X$;
is satisfied, then the order quasinorm N is called **separating***.*
If T is the identity, the function N satisfying (N1) – (N3) is called **order seminorm***. A separating order seminorm is called* **order norm***.*

The concept of an order norm has been introduced by Kantorovich [69]. It is used in vector optimization and approximation theory, compare the books of Jahn [62], [63] as well as [44], for example. Order norms are sometimes called vector-valued or cone-valued norms.

Order quasinorms with values in topological semifields appeared in [2]. The following two propositions contain a complete characterization of topological linear spaces using order quasinorms. For this purpose we use a procedure close to that of [70], Theorem 3 and 4, but avoiding the use of topological semifields explicitly.

Proposition 38 *Let $(X,+)$ be a linear space and $(V,+,\mathcal{S},\leq_K)$ be a normally ordered, locally convex topological linear space with convex ordering cone K. Let $\mathcal{S}(\theta)$ be a neighborhood base of $\theta \in V$ consisting of convex full sets. If $N : X \to V$ is an order quasinorm on X, then the following assertions hold true:*
(i) The entity of sets

$$B_S := \{x \in X : \ N(x) \in S\}, \ S \in \mathcal{S}(\theta)$$

form a neighborhood base of $\theta \in X$ for some topology $\mathcal{T}$ on X; it is the coarsest topology on X such that $(X,+,\mathcal{T})$ is a topological linear space and N is continuous at $\theta \in X$;
(ii) The entity of sets

$$E_S := \{(x_1,x_2) \in X \times X : \ N(x_1 - x_2) \in S\}, \ S \in \mathcal{S}(\theta)$$

form a base of a uniformity $\mathcal{U}$ on X such that $\mathcal{T}$ is the uniform topology generated by $\mathcal{U}$.
(iii) If N is separating and $(V,\mathcal{S})$ is separated, then the topology $\mathcal{T}$ and the uniformity $\mathcal{U}$ are separated as well.

Proof. (i) Since V is normally ordered, there is a neighborhood base $\mathcal{S}(\theta)$ of $\theta \in V$ satisfying (B), i.e.

$$\forall S \in \mathcal{S}(\theta): \ v_1, v_2 \in S, \ v_1 \leq_K v \leq_K v_2 \in S \ \Rightarrow \ v \in S. \tag{2.27}$$

Define $\mathcal{B} := \{B_S : \ S \in \mathcal{S}(\theta)\}$. Let us show that $\mathcal{B}$ is a neighborhood base of $\theta \in X$ for some topology $\mathcal{T}$.

First, let $B_1, B_2 \in \mathcal{B}$, i.e. there are $S_1, S_2 \in \mathcal{S}(\theta)$ such that $B_i = \{x \in X : \ N(x) \in S_i\}$, $i = 1, 2$. Since $\mathcal{S}(\theta)$ is a neighborhood base of $\theta \in V$, there is $S \in \mathcal{S}(\theta)$ such that $S \subseteq S_1 \bigcap S_2$. It follows that $B_S \subseteq B_1 \bigcap B_2$.

Next, take $B \in \mathcal{B}$, i.e. there is $S \in \mathcal{S}(\theta)$ such that $B = \{x \in X : \ N(x) \in S\}$. We are going to show that there is $B'' \in \mathcal{B}$ such that $B'' \oplus B'' \subseteq B$. Since $\mathcal{S}(\theta)$ is a neighborhood base of $\theta \in V$, there is $S' \in \mathcal{S}(\theta)$ such that $S' \oplus S' \subseteq S$. Since T is continuous, there is $S'' \in \mathcal{S}(\theta)$ such that $T(S'') \subseteq S'$, hence $T(S'') \oplus T(S'') \subseteq S$. Take $x_1, x_2 \in B'' := \{x \in X : \ N(x) \in S''\}$. By (N3) of Definition 29 we obtain

$$N(x_1 + x_2) \leq_K T(N(x_1) + N(x_2)) = T(N(x_1)) + T(N(x_2)) \in T\left(S''\right) \oplus T\left(S''\right) \subseteq S.$$

(2.27) implies $N(x_1 + x_2) \in S$, i.e. $x_1 + x_2 \in B$. Therefore, $B'' \oplus B'' \subseteq B$ as desired.

Take $B \in \mathcal{B}$, i.e. $B = B_S$ for some $S \in \mathcal{S}(\theta)$. We show that there is $S' \in \mathcal{S}(\theta)$ such that $tB_{S'} \subseteq B$ whenever $|t| \leq 1$. Indeed, since $\mathcal{S}(\theta)$ is a neighborhood base of $\theta \in V$, there is $S' \in \mathcal{S}(\theta)$ such that $tS' \subseteq S$ whenever $|t| \leq 1$. Take $x \in B_{S'}$, i.e. $N(x) \in S'$. Then $tx \in tB_{S'}$ and by (N2) of Definition 29 it follows

$$N(tx) = |t| N(x) \in |t| S' \subseteq S,$$

therefore $tx \in B$ and consequently $tB_{S'} \subseteq B$.

Finally, we shall show that the sets B_S are absorbing. Take $S \in \mathcal{S}(\theta)$ and $x \in X$. Since S is absorbing, there is $t > 0$ such that $N(x) \in tS$. (N2) of Definition 29 and the definition of B_S imply $x \in tB_S$ as desired.

Concludingly, there is a topology $\mathcal{T}$ such that the couple $(X, \mathcal{T})$ is a topological linear space such that $\mathcal{B}$ is a neighborhood base of $\theta \in X$. Of course, this topology is the coarsest one making N continuous at $\theta \in X$.

(ii) Is clear since the topology of a topological linear space is uniformizable.

(iii) Is obvious from (N5) of Definition 29, since $\bar{x} \in \{x \in X : \ N(x) \in S\}$ for all $S \in \mathcal{S}(\theta)$ implies $N(\bar{x}) \in \bigcap_{S \in \mathcal{S}(\theta)} S$. Hence $N(\bar{x}) = \theta$ since $(V, \mathcal{S})$ is separated. ■

Remark 10 *If the order quasinorm is such that T can be chosen to be the identity, the resulting topology on X is locally convex since the sets $B_S = \{x \in X : \ N(x) \in S\}$, $S \in \mathcal{S}(\theta)$ are convex: Take $x_1, x_2 \in B_S$, $t \in (0, 1)$. Then $N(x_1), N(x_2) \in S$. (N3) and (N2) of Definiton 29 and the convexity of S imply*

$$N(tx_1 + (1-t)x_2) \leq_K tN(x_1) + (1-t)N(x_2) \in S.$$

The next proposition shows that every topological linear space can be supplied with an order quasinorm generating the topology.

Proposition 39 *Let $(X, +, \mathcal{T})$ be a topological linear space. Then there are a normally ordered, locally convex space $(V, +, \mathcal{S}, \leq_K)$ with convex ordering cone K, an order quasinorm $N : X \to V$ and a continuous linear operator $T : V \to V$ such that $T(K) \subseteq K$ and*
(i) For every neighborhood B of $\theta \in X$ there is a $S \in \mathcal{S}(\theta)$, $\mathcal{S}(\theta)$ being a neighborhood base of $\theta \in V$ such that

$$\{x \in X : \ N(x) \in S\} \subseteq B;$$

(ii) For every neighborhood S of $\theta \in V$, there is a neighborhood B of $\theta \in X$ such that

$$B \subseteq \{x \in X : \ N(x) \in S\}.$$

Proof. Proposition 37 tells us that there is a family $\{\|\cdot\|_\lambda\}_{\lambda \in \Lambda}$ of quasinorms generating the topology, i.e. the family of sets

$$\{x \in X : \ \|x\|_\lambda \leq r\}, \ r > 0, \ \lambda \in \Lambda$$

is a neighborhood base of $\theta \in X$. Consider the locally convex space $\mathbb{R}^\Lambda$ being normally ordered by $K = \mathbb{R}^\Lambda_+$. Define

$$N(x) := (\|x\|_\lambda)_{\lambda \in \Lambda}.$$

The space $\mathbb{R}^\Lambda$ can be identified with the set of all functions mapping Λ into $\mathbb{R}$. In this sense,

$$N(x)(\lambda) = \|x\|_\lambda, \ \lambda \in \Lambda.$$

By (N3) of Definition 28, there is a mapping $\phi : \Lambda \to \Lambda$ such that for each $\lambda \in \Lambda$

$$\|x_1 + x_2\|_\lambda \leq \|x_1\|_{\phi(\lambda)} + \|x_2\|_{\phi(\lambda)}.$$

Denoting an element of $\mathbb{R}^\Lambda$ by $v = (v_\lambda)_{\lambda \in \Lambda}$, we define a mapping $T : \mathbb{R}^\Lambda \to \mathbb{R}^\Lambda$ by

$$T\left((v_\lambda)_{\lambda \in \Lambda}\right) := \left(v_{\phi(\lambda)}\right)_{\lambda \in \Lambda}.$$

We claim that T is linear, positive and continuous at $\theta \in \mathbb{R}^\Lambda$. Linearity and positivity are obvious. To show the continuity, take a neighborhood of $\theta \in \mathbb{R}^\Lambda$, i.e. choose $r_1, \ldots, r_n > 0$, $\lambda_1, \ldots, \lambda_n \in \Lambda$ and consider

$$S := \left\{v \in \mathbb{R}^\Lambda : \ |v_{\lambda_i}| < r_i, \ i = 1, \ldots, n\right\}.$$

The set

$$S' := \left\{w \in \mathbb{R}^\Lambda : \ \left|w_{\phi(\lambda_i)}\right| < r_i, \ i = 1, \ldots, n\right\}$$

is also a neighborhood of $\theta \in V$. Moreover, $\left\{v \in \mathbb{R}^\Lambda : \ v = Tw, \ w \in S'\right\} = T(S') \subseteq S$, since $v \in T(S')$ means $v = Tw$ for some $w \in S'$, hence $|v_{\lambda_i}| = \left|w_{\phi(\lambda_i)}\right| < r_i$ for $i = 1, \ldots, n$. This proves the claim.

The mapping N defined above is an order quasinorm. The conditions (N1), (N2) and (N3) of Definition 29 are easy to check.

Let us prove (i). It suffices to show that for each

$$B := \{x \in X : \|x\|_\lambda \le r\}, \; r > 0, \; \lambda \in \Lambda$$

there is $S \in \mathcal{S}(\theta)$, $S \subseteq \mathrm{I\!R}^\Lambda$ such that $B_S := \{x \in X : N(x) \in S\} \subseteq B$. This is obvious for $S = \left\{v \in \mathrm{I\!R}^\Lambda : |v_\lambda| < r\right\}$.

Finally, we show (ii). Take $S \in \mathcal{S}(\theta)$, $S \subseteq \mathrm{I\!R}^\Lambda$, i.e.

$$S = \left\{v \in \mathrm{I\!R}^\Lambda : |v_{\lambda_i}| < r_i, \; i = 1, \ldots, n\right\}.$$

Then

$$\{x \in X : N(x) \in S\} = \bigcap_{i=1}^{n} \left\{x \in X : \|x\|_{\lambda_i} \le r_i\right\}.$$

The sets $\left\{x \in X : \|x\|_{\lambda_i} \le r_i\right\}$, $i = 1, \ldots, n$, are neighborhoods of $\theta \in X$. Hence there is a neighborhood $B \subseteq \{x \in X : N(x) \in S\}$ of $\theta \in X$. This completes the proof of the proposition. ■

Remark 11 *If $(X, +, \mathcal{T})$ is a locally convex space, the mapping T can be chosen to be the identity. This is due to the fact that in the proof of Proposition 39 the family of quasinorms can be replaced by a family of seminorms.*

Taking Remark 10 into account, the class of separated locally convex spaces coincides with that of order normed spaces.

2.3.5 Conlinear spaces via topological constructions

Starting with a topological linear space $(V, +, \mathcal{T})$ we may construct conlinear subspaces of $\left(\widehat{\mathcal{P}}(V), \oplus\right)$ with the help of topological properties. We refer the reader to the thesis [83] for a far reaching application for the case $V = \mathrm{I\!R}^n$.

Example 12 *Let $(V, +, \mathcal{T})$ be a topological linear space and denote by $\mathcal{F}(V)$ the set of all closed subsets of V. The Minkowski sum of two closed sets is not closed in general. Therefore, we define for $W_1, W_2 \in \mathcal{F}(V)$*

$$W_1 \widehat{\oplus} W_2 := \mathrm{cl}\,(W_1 \oplus W_2).$$

Then $\left(\mathcal{F}(V), \widehat{\oplus}\right)$ is a semilinear space, hence also conlinear.

Using the conventions of Proposition 19 we only have to show the law of assoziativity for $\widehat{\oplus}$, see (S1) of Definition 10. We show that $\mathrm{cl}\,(W_1 \oplus W_2) \oplus W_3 \subseteq \mathrm{cl}\,(W_1 \oplus W_2 \oplus W_3)$. Indeed, if $w \in \mathrm{cl}\,(W_1 \oplus W_2) \oplus W_3$, then $w \in (W_1 \oplus W_2 \oplus B) \oplus W_3$ for each member B of a neighborhood base of $\theta \in V$. Since $\oplus$ is associative, this implies $w \in (W_1 \oplus W_2 \oplus W_3) \oplus B$, hence $w \in \mathrm{cl}\,(W_1 \oplus W_2 \oplus W_3)$. The opposite inclusion follows by symmetry.

In a similiar way, the set of all closed convex subsets of a topological linear space can be provided with a semilinear structure.

Chapter 3

Order Premetrics and their Regularity

A basic ingredients for results in the spirit of Ekeland's variational principle is a metric space and an order relation on the space defined in terms of the metric itself. If a function is involved that maps not into the reals but into a more general set, for example a linear space, the metric has to be replaced by an expression mapping into the same set. On the other hand, not all properties of a metric are really essential for a proof of a variational principle. This has been already realized in [8]. Therefore, we extend the concept of a (realvalued) metric to functions into ordered monoids maintaining only a few but not all properties of a metric. We do not focus on topological structures which may be generated by such extensions of a metric as it has been done in Section 2.3 with the concept of order (pseudo)metrics. Of course, an order (pseudo)metric is an example of an order premetric that is introduced in the next definition.

Definition 30 *Let X be a nonempty set and $(Y, \circ, \leq)$ a quasiordered monoid with neutral element $\theta \in Y$. A function $\Phi : X \times X \to Y$ is called an* **order premetric** *iff the following conditions are satisfied:*
(P1) $\forall x \in X$*:* $\theta = \Phi(x, x)$*;*
(P2) $\forall x_1, x_2 \in X$*:* $\theta \leq \Phi(x_1, x_2)$*;*
(P3) $\forall x_1, x_2, x_3 \in X$*:* $\Phi(x_1, x_3) \leq \Phi(x_1, x_2) \circ \Phi(x_2, x_3)$.

The condition (P1) is not a true restriction as the following lemma shows.

Lemma 4 *Let X, Y be as in Definition 30 and $\Psi : X \times X \to Y$ be a function satisfying (P2) and (P3). Then the function $\Phi : X \times X \to Y$ defined by*

$$\Phi(x_1, x_2) := \begin{cases} \Psi(x_1, x_2) & : \quad x_1 \neq x_2 \\ \theta & : \quad x_1 = x_2 \end{cases} \tag{3.1}$$

is an order premetric.

PROOF. For Φ, the conditions (P1), (P2), (P3) may be checked straightforward. ■

Definition 31 *Let* $(X,\mathcal{U})$ *be a uniform space and* $(Y,\circ,\leq)$ *a quasiordered monoid with neutral element* $\theta\in Y$. *A function* $\Phi : X\times X\to Y$ *satisfying (P2), (P3) of Definition 30 is called (sequentially)* **regular** *with respect to* $y_1,y_2\in Y$ *iff it satisfies:*
(P4) If $\{x_n\}_{n\in\mathbb{N}}\subseteq X$ *and*

$$\forall n\in\mathbb{N}:\ y_1\circ\sum_{k=0}^{n}\Phi\left(x_{k+1},x_k\right)\leq y_2,$$

then $\{x_n\}_{n\in\mathbb{N}}$ *is asymptotic, i.e.*

$$\forall E\in\mathcal{U}\ \exists n_E\in\mathbb{N}\ \forall n\geq n_E:\ \left(x_{n+1},x_n\right)\in E.$$

Note that, if (X,d) is a metric space, a sequence $\{x_n\}_{n\in\mathbb{N}}$ is asymptotic if and only if $\lim_{n\to\infty}d\left(x_{n+1},x_n\right)=0$. The definition above applies also to this case.

Let Y be not only a monoid but also a group. Then $y_1\circ\sum_{k=0}^{n}\Phi\left(x_{k+1},x_k\right)\leq y_2$ for all $n\in\mathbb{N}$ if and only if $\theta\leq\sum_{k=0}^{n}\Phi\left(x_{k+1},x_k\right)\leq y_2\circ y_1^{-1}$ for all $n\in\mathbb{N}$. Hence, it is enough to assume that the boundedness of above of $\{\sum_{k=0}^{n}\Phi\left(x_{k+1},x_k\right):\ n\in\mathbb{N}\}$ implies that $\{x_n\}_{n\in\mathbb{N}}$ is asymptotic.

Lemma 5 *Let* $(X,\mathcal{U})$ *be a uniform space,* $(Y,\circ,\leq)$ *a quasiordered monoid and* $\Psi : X\times X\to Y$ *be a function satisfying (P2) and (P3). Then, the order premetric* Φ*, defined via (3.1) is regular if and only if* Ψ *is regular.*

PROOF. Clearly, the regularity of Φ implies the regularity of Ψ. To show the converse, assume the regularity of Ψ and take a sequence $\{x_n\}_{n\in\mathbb{N}}\subseteq X$ such that

$$\forall n\in\mathbb{N}:\ y_1\circ\sum_{k=0}^{n}\Phi\left(x_{k+1},x_k\right)\leq y_2.$$

If $x_{n+1}=x_n$ for some $n\in\mathbb{N}$, we may delete x_{n+1} from the sequence since $\left(x_{n+1},x_n\right)\in E$ for each $E\in\mathcal{U}$ in this case. Doing this as long as possible, we either obtain only finitly many elements of the original sequence or a subsequence $\{x_{n_l}\}_{l\in\mathbb{N}}\subseteq X$ such that $x_{n_l+1}\neq x_{n_l}$. In the first case, the original sequence is constant up to finitely many elements and hence asymptotic. In the second case, we have $\Phi\left(x_{n_l+1},x_{n_l}\right)=\Psi\left(x_{n_l+1},x_{n_l}\right)$ for all $l\in\mathbb{N}$. This implies

$$\forall l\in\mathbb{N}:\ y_1\circ\sum_{k=0}^{l}\Psi\left(x_{n_k+1},x_{n_k}\right)\leq y_2.$$

¿From the regularity of Ψ we may deduce that for $E\in\mathcal{U}$ and $n_l\in\mathbb{N}$ sufficiently large, we have $\left(x_{n_l+1},x_{n_l}\right)\in E$. This completes the proof. ■

Example 13 *In [65], Kada et al. introduced the concept of a w-distance as follows: Let* (X,d) *be a metric space and* $w : X \times X \to \mathbb{R}_+$ *be a function satisfying (i)* $w(x_1,x_3) \leq w(x_1,x_2) + w(x_2,x_3)$ *for all* $x_1,x_2,x_3 \in X$*; (ii) For each* $x_0 \in X$*, the function* $x \to w(x_0,x)$ *is lower semicontinuous; (iii) For each* $\varepsilon > 0$*, there is* $\delta > 0$ *such that* $w(x,x_1) < \delta$*,* $w(x,x_2) < \delta$ *imply* $d(x_1,x_2) < \varepsilon$*.*

We show that a w-distance is a regular premetric with $(Y,\circ) = (\mathbb{R}_+,+)$ *and the usual* $\leq$*-relation for real numbers for* $y_1 = 0$ *and each* $y_2 = r \in \mathbb{R}_+$*. To this purpose, take a sequence* $\{x_n\}_{n\in\mathbb{N}} \subset X$ *such that* $\sum_{k=0}^{n} w(x_{k+1},x_k) \leq r$ *is true for all* $n \in \mathbb{N}$*. Since* $0 \leq w(x_{n+1},x_n)$ *for all* $n \in \mathbb{N}$*, this implies* $0 = \lim_{n\to\infty} w(x_{n+1},x_n)$*. Fix* $\varepsilon > 0$*. Then there is* $n_\varepsilon \in \mathbb{N}$ *such that*

$$\forall n \geq n_\varepsilon : \; w(x_n,x_{n+1}) < \frac{\delta}{2} < \delta, \qquad w(x_{n+1},x_{n+2}) < \frac{\delta}{2}$$

with $\delta > 0$ *from (iii). By (i), we obtain*

$$w(x_n,x_{n+2}) \leq w(x_n,x_{n+1}) + w(x_{n+1},x_{n+2}) < \delta.$$

and therefore from (iii) $d(x_{n+1},x_{n+2}) < \varepsilon$ *for all* $n \geq n_\varepsilon$*. Hence the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *is asymptotic.*

In [65], a list of examples can be found showing that the set of w-distances contains the metric d but much more elements. Note that already in Brønstedts paper [8] similar functions has been used on uniform spaces.

Example 14 *A simple example of an ordered monoid is* $(Y := \mathbb{R}_+ \cup \{+\infty\}, +, \leq)$*. Let* (X,d) *be a metric space. Then d is a regular order premetric with respect to* $y_1 = 0$*,* $y_2 = r \in \mathbb{R}_+ \subsetneqq Y$*, but not with respect to* $y_2 = +\infty \in Y$*, of course.*

Example 15 *Let* (X,d) *be a metric space and let* $(Y,+,\mathcal{T},\leq_K)$ *be a normally ordered separated locally convex space with ordering cone* $K \subseteq Y$*. Take* $k \in K\backslash\{0\}$*. Then* $\Phi(x_1,x_2) := kd(x_1,x_2)$ *is a regular order premetric in the sense of Definition 30. This result is presented by Isac, compare Proposition 1 and the proof of Theorem 3 of [59].*

Example 16 *Let* $(X,\mathcal{U})$ *be a uniform space and let* $(Y,\circ,\leq,\mathcal{T})$ *be a normally ordered topological Abelian group. Then, every order pseudometric* $D : X \times X \to Y$ *in the sense of Definition 25 is an order premetric.*

Chapter 4

Variational Principles on Metric Spaces

In this chapter, we are dealt with ordered metric spaces. We ask for circumstances ensuring the existence of minimal elements with respect to the given order relation. The completeness of the space in connection with a certain regularity assumption of the order turns out to be the crucial point. We state three equivalent formulations of the main result and draw a series of corollaries including new results as well as almost all known theorems that are equivalent to or generalizations of Ekeland's variational principle on metric spaces.

4.1 The basic theorem on metric spaces

4.1.1 Preliminaries

Let (X, d) be a metric space provided with a quasiorder $\preceq$, i.e. a reflexive and transitive relation. In the following, we simply denote the lower sections $S_l(x) = \{x' \in X : x' \preceq x\}$ by $S(x)$ for $x \in X$, compare Definiton 12.

A sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ is said to be **decreasing** with respect to $\preceq$ iff

$$\forall n \in \mathbb{N} : \; x_{n+1} \preceq x_n.$$

The metric space X is said to be **$\preceq$–complete** iff every decreasing Cauchy sequence in X converges to some element of X. Of course, every complete metric space is $\preceq$–complete for every quasiordering $\preceq$ while the converse is not true: Take $X = [0, 1)$ and the usual $\leq$–relation. A quasiorder $\preceq$ is called **lower closed** iff for any decreasing sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ converging to some $x \in X$

$$\forall n \in \mathbb{N} : \; x \preceq x_n$$

holds true. A quasiorder is lower closed if and only if the sections $S(x)$ are closed with respect to decreasing sequences, i.e. if $\{x_n\}_{n \in \mathbb{N}} \subseteq S(x)$ and $\lim_{n \to \infty} x_n = \hat{x}$, then $\hat{x} \in$

$S(x)$. A quasiorder $\preceq$ on a metric space X is called **regular** iff every decreasing sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is asymptotic, i.e.,

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

Regularity improves the properties of the order as the following proposition shows.

Proposition 40 *A regular quasiorder $\preceq$ on a metric space X is antisymmetric.*

PROOF. Take $x, x', \in X$ such that $x' \preceq x$ as well as $x \preceq x'$. Define

$$\{x_n\} = \left\{x, x', x, x', x, \ldots\right\}$$

being a $\preceq$–decreasing sequence. The regularity of $\preceq$ implies $x = x'$. ∎

If $\preceq$ is a regular quasiorder on X, Proposition 40 admits to say that $\bar{x} \in X$ is a minimal point with respect to $\preceq$ iff $\{\bar{x}\} = S(\bar{x})$.

4.1.2 The basic theorem

The stage is set for the basic minimal element theorem on metric spaces.

Theorem 16 *Let the following assumptions be satisfied:*
(M1) (X, d) is a metric space;
(M2) $\preceq$ is a reflexive and transitive relation on X such that X is $\preceq$–complete;
(M3) $\preceq$ is regular;
(M4) $\preceq$ is lower closed.
Then, for each $x_0 \in X$, there exists $\bar{x} \in X$ such that

$$\bar{x} \in S(x_0) \qquad \textit{and} \qquad \{\bar{x}\} = S(\bar{x}).$$

PROOF. Starting with $x_0 \in X$, we define a sequence by choosing $x_{n+1} \in S(x_n)$ such that

$$d(x_{n+1}, x_n) \geq \sup_{x\in S(x_n)} d(x, x_n) - \frac{1}{n} \qquad \text{if} \qquad \sup_{x\in S(x_n)} d(x, x_n) < +\infty$$

or $d(x_{n+1}, x_n) \geq 1$ if $\sup_{x\in S(x_n)} d(x, x_n) = +\infty$. There is $n_0 \in \mathbb{N}$ such that the latter case can not occur for each $n \geq n_0$. Otherwise, we may obtain a decreasing sequence with $d(x_{n+1}, x_n) \geq 1$ for all $n \in \mathbb{N}$ contradicting the regularity.

The transitivity of $\preceq$ implies $x_m \in S(x_n)$ for $m \geq n$. From this, we get

$$d(x_m, x_n) \leq \sup_{x\in S(x_n)} d(x, x_n) \leq d(x_{n+1}, x_n) + \frac{1}{n}$$

for all $n \in \mathbb{N}$, $n \geq n_0$. Assumption (M3) implies that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, hence convergent to some $\bar{x} \in X$. By (M4), we have $\bar{x} \in S(x_n)$ for all $n \in \mathbb{N}$. Especially, $\bar{x} \in S(x_0)$.

Take $x' \in S(\bar{x})$. Then by transitivity, $x' \preceq \bar{x} \preceq x_n$, hence $x' \in S(x_n)$ for all $n \in \mathbb{N}$. This implies

$$d\left(x', x_n\right) \leq \sup_{x \in S(x_n)} d(x, x_n) \leq d(x_{n+1}, x_n) + \frac{1}{n}.$$

Concludingly, $x_n \to x'$, hence $x' = \bar{x}$ and therefore, $\{\bar{x}\} = S(\bar{x})$. This completes the proof. ■

Remark 12 *Starting with a relation $\preceq'$ being only transitive, we can obtain a quasiordering $\preceq$ by defining*

$$x' \preceq x \quad :\Longleftrightarrow \quad x' \preceq' x \quad \text{or} \quad x' = x.$$

If (M3) and (M4) are satisfied for $\preceq'$, then for $\preceq$, too. Therefore, the restriction to quasiorders is not essential. Hence, taking Proposition 40 into account, we may assume that $\preceq$ in Theorem 16 is a partial order without loss of generality.

4.1.3 Equivalent formulations of the basic theorem

At an early stage, it has been observed that Ekeland's variational principle has a number of equivalent formulations. The papers [24] and [98] by Daneš and Penot, respectively, are the first systematic surveys about this topic. Further results in this direction can be found e.g. in [38], [3], [96] and [97].

In [118] (Theorem 3.2) and in [67] (Corollary) a fixed point result has been established as a corollary of the the main result.

In the 1976 paper [9], Caristi observed that his fixed point theorem is an equivalent formulation of Ekeland's variational principle. We establish a generalized version and show the equivalence to Theorem 16. The following result is close to Theorem 3.1 in [22].

We consider a set-valued mapping $T : X \to \widehat{\mathcal{P}}(X)$. A point $\bar{x} \in X$ is said to be a **fixed point** of T iff $\bar{x} \in T(\bar{x})$. A point $\bar{x} \in X$ is said to be an **invariant point** of T iff $\{\bar{x}\} = T(\bar{x})$.

Theorem 17 *Let (M1) through (M4) of Theorem 16 be in force and, additionally, $T : X \to \widehat{\mathcal{P}}(X)$ be a set-valued mapping. If T satisfies*

$$\forall x \in X,\ \exists x' \in T(x) :\ x' \preceq x, \tag{WC}$$

then there is $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$, i.e. $\bar{x}$ is a fixed point of T. If T satisfies

$$\forall x \in X,\ \forall x' \in T(x) :\ x' \preceq x, \tag{SC}$$

then there is $\bar{x} \in X$ such that $\{\bar{x}\} = T(\bar{x})$, i.e. $\bar{x}$ is an invariant point of T.

Proof. Each point $\bar{x}$ satisfying the conclusions of Theorem 16 does the job. ■

Conversely, Theorem 16 can be proven using Theorem 17. To see this, replace X in

Theorem 17 by $S(x_0)$ of Theorem 16 and consider the map $T(x) := S(x)$ that satisfies (SC).

In their 1993 paper [96], Oettli and Théra proved an equivalent formulation of Ekeland's principle. See also [97]. This theorem can be generalized in order to produce a reformulation of Theorem 16.

Theorem 18 *Let (M1) through (M4) of Theorem 16 be in force and, additionally:*
(M5) The set $M \subseteq X$ satisfies

$$\forall x \in S(x_0) \backslash M \quad \exists x' \in S(x) \setminus \{x\}.$$

Then, there exists $\bar{x} \in S(x_0) \cap M$.

Proof. By Theorem 16, there exists $\bar{x} \in S(x_0)$ such that $\{\bar{x}\} = S(\bar{x})$. By assumption (M5), $\bar{x} \in M$, hence $\bar{x} \in M \cap S(x_0)$. ∎

Conversely, Theorem 16 can be proven using Theorem 18. To see this, let (M1) through (M4) be in force. Define $M := \{x \in X : \{x\} = S(x)\}$. If $x \notin M$, then there exists $x' \in X$ such that $x' \neq x$, $x' \preceq x$, hence (M5) is satisfied. By Theorem 18, there exists $\bar{x} \in S(x_0) \cap M$, hence $\{\bar{x}\} = S(\bar{x})$.

Let us note that the equivalence of the Theorems 16, 17 and 18 is understood in the sense that each of it can be proven using each of the others without any reference to the induction process that appears in the proof of Theorem 16.

4.1.4 The regularity assumptions

The next proposition shows that under the regularity assumption decreasing sequences are even Cauchy.

Proposition 41 *Let (X, d) be a metric space, quasiordered by $\preceq$. Then $\preceq$ is regular if and only if every decreasing sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is a Cauchy sequence, i.e.*

$$\forall \varepsilon > 0, \exists p \in \mathbb{N}, \forall n, m \geq p: \ d(x_m, x_n) \leq \varepsilon.$$

Proof. Of course, every Cauchy sequence is asymptotic. Conversely, let $\preceq$ be regular and $\{x_n\}_{n\in\mathbb{N}} \subset X$ a decreasing, hence asymptotic sequence. Assume $\{x_n\}_{n\in\mathbb{N}}$ is not Cauchy. Then there is $\varepsilon > 0$ such that

$$\forall p \in \mathbb{N}, \exists m > n \geq p: \ d(x_m, x_n) \geq \varepsilon.$$

Set $p = 1$. Then there are $m_1 > n_1 \geq p$ such that $d(x_{m_1}, x_{n_1}) \geq \varepsilon$. Set $x'_1 = x_{n_1}$ and $x'_2 = x_{m_1}$. Set $p = m_1$. Then there are $m_2 > n_2 \geq p$ such that $d(x_{m_2}, x_{n_2}) \geq \varepsilon$. Set $x'_3 = x_{n_2}$ and $x'_4 = x_{m_2}$. Continue this procedure to obtain a subsequence $\{x'_k\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ with $d\left(x'_{2k+2}, x'_{2k+1}\right) \geq \varepsilon$ for $k \in \mathbb{N}$. This sequence is decreasing, but not asymptotic contradicting the assumption. ∎

Proposition 41 yields: If $\preceq$ is regular, then X is $\preceq$–complete if and only if every decreasing sequence in X converges to some element of X.

Remark 13 *The regularity assumption (M3) has been used by Turinici [118] as well as Dancs, Hegedüs and Medvegyev [22]. Theorem 3.1 in [118] has been proved using Zorn's Lemma whereas Theorem 3.2 in [22] deals with complete metric spaces.*

Several attempts have been made to replace the regularity assumption (M3) by a more general one. We mention the following conditions:

$$\forall \varepsilon > 0, \forall x \in X, \exists u \in S(x) : \ diam\, S(u) \leq \varepsilon; \tag{M3-1}$$

$$\forall \varepsilon > 0, \forall x \in X, \exists u \in S(x) : \sup_{x' \in S(u)} d\left(x', u\right) \leq \varepsilon; \tag{M3-2}$$

$$\forall \varepsilon > 0, \forall x \in X, \exists u \in S(x) : \ \left(x'' \preceq x' \preceq u \implies d\left(x', x''\right) < \varepsilon\right). \tag{M3-3}$$

The following relationships can be established.

Lemma 6 *Let (X, d) be a metric space. Then, the conditions (M3-1), (M3-2) and (M3-3) are mutually equivalent. Moreover, (M3) implies each of (M3-1), (M3-2) and (M3-3).*

Proof. The implications (M3-1) $\Rightarrow$ (M3-3) $\Rightarrow$ (M3-2) are immediate. Let (M3-2) be in force. Fix $\varepsilon > 0$ and $x \in X$. Take $u \in S(x)$ such that $\sup_{x' \in S(u)} d(x', u) \leq \frac{\varepsilon}{2}$. For $x', x'' \in S(u)$ we obtain

$$d\left(x', x''\right) \leq d\left(x', u\right) + d\left(u, x''\right) \leq \varepsilon,$$

hence (M3-1) is satisfied.

Finally, we show that regularity implies (M3-2). Assume the contrary. Then there are $\varepsilon > 0$ and $x \in X$ such that

$$\forall u \in S(x) : \sup_{x' \in S(u)} d\left(x', u\right) > \varepsilon.$$

Starting with $x_0 = x$ one may find a sequence such that $x_n \in S(x)$ and $d(x_{n+1}, x_n) \geq \varepsilon$ for all $n \in \mathbb{N}$. This contradicts regularity. ■

By a simple modification of the proof of Theorem 16 one obtains the same conclusions assuming (MP-i), $i = 1, 2, 3$, instead of (M3). However, it is not sure if this is a true generalization.

Theorem 19 *Let the assumptions (M1), (M2) and (M4) of Theorem 16 be in force and additionally either of (M3-1), (M3-2) and (M3-3). Then, for each $x_0 \in X$, there exists $\bar{x} \in X$ such that*

$$\bar{x} \in S(x_0) \qquad \text{and} \qquad \{\bar{x}\} = S(\bar{x}).$$

Proof. We use (M3-1) for the proof. Starting with $x_0 \in X$, we define a sequence by choosing $x_{n+1} \in S(x_n)$ such that $diam\, S(x_n) \leq \frac{1}{n}$. The transitivity of $\preceq$ implies $x_m \in S(x_n)$ for $m \geq n$. From this, we get

$$d(x_m, x_n) \leq diam\, S(x_n) \leq \frac{1}{n}$$

for all $n \in \mathbb{N}$. The remaining part of the proof is the same as that for Theorem 16. ■

4.1.5 Completeness

By several authors, it has been observed that Ekeland's principle or its reformulations characterize the completeness of the metric space X. Compare [74], [126], [111], [22] and [112]. The same is true for Theorem 16. We state the result in the following form.

Theorem 20 *Let (X, d) be a metric space. If for all reflexive and transitive relations being regular and lower sequentially closed the assertions of Theorem 16 hold true, then X is complete.*

Proof. Assume that X is not complete. Then there is a sequence $X = M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$ of nonempty closed subsets of X such that $diam\, M_n \to 0$ as $n \to \infty$, but $\cap_{n=1}^{\infty} M_n = \emptyset$. Define an ordering relation by

$$S(x) := M_{n+1} \cup \{x\} \quad \text{if} \quad x \in M_n \quad \text{and} \quad x \notin M_{n+1}$$

and $x' \preceq x$ iff $x' \in S(x)$. One may check that $\preceq$ is reflexive, transitive (lower sequentially) closed and regular by assumption. A minimal point $\bar{x}$ of $\preceq$ would satisfy $\{\bar{x}\} = S(\bar{x}) = M_{\bar{n}} \cup \{\bar{x}\}$ for some $\bar{n} \in \mathbb{N}$. This would imply $M_{\bar{n}} = \emptyset$, a contradiction. ■

In fact, it is enough that the assumption of Theorem 20 holds true for all ordering relations generated by uniformly continuous functions $f : X \to \mathbb{R}$, being bounded below, in the following way:

$$x' \preceq x \quad \Longleftrightarrow \quad f(x') + d(x', x) \le f(x).$$

This shows the proof of the theorem of Weston in [126].

4.1.6 Set relation formulation

The variational principle of Ekeland is bound up with so–called minimal point theorems in procduct spaces. This idea probably goes back to Phelps (compare [102]) and has been put explicitely in [3].

At this early stage of our development, we shall establish a minimal point theorem involving order relations on product sets $X \times Y$. The crucial point of the proof is to generate a suitable order relation defined on X only in order to apply Theorem 16. This is possible using the set relations introduced in Section 2.2.1.

Note that the set Y is merely assumed to be nonempty. Neither algebraic nor topological structure concerning Y appears.

We need some notation to formulate the results. Let (X, d) be a metric space and Y as well as $M \subseteq X \times Y$ be nonempty sets. For $x \in X$, let us define $M(x) := \{(x', y) \in X \times Y : \; x' = x, \, (x', y) \in M\} \in \widehat{\mathcal{P}}(X \times Y)$ and $M_Y(x) := \{y \in Y : \; (x, y) \in M\} \in \widehat{\mathcal{P}}(Y)$. Let $\preceq$ be a quasiorder on M. Then, $(\{M(x) : \; x \in X\}, \preccurlyeq)$ as well as $(\{M(x) : \; x \in X\}, \curlyeqprec)$ is quasiordered. Here, the relations $\preccurlyeq$ and $\curlyeqprec$ are the extensions of $\preceq$ to subsets of $\widehat{\mathcal{P}}(X \times Y)$, compare (2.6), (2.7). Note that $M(x') \preccurlyeq M(x)$ if and only if

$$\forall y \in M_Y(x), \; \exists y' \in M_Y(x') : \; (x', y') \preceq (x, y) \tag{4.1}$$

and $M(x') \preccurlyeq M(x)$ if and only if

$$\forall y' \in M_Y\left(x'\right), \ \exists y \in M_Y(x): \ \left(x', y'\right) \preceq (x, y). \tag{4.2}$$

Theorem 21 *Let the following assumptions be satisfied:*
(M1') (X,d) is a metric space and Y as well as $M \subseteq X \times Y$ are nonempty sets;
(M2') $\preceq$ is a quasiorder, i.e., a reflexive and transitive relation on $X \times Y$;
(M3') If $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq M$ is a decreasing sequence, i.e.

$$\forall n \in \mathbb{N}: \ (x_{n+1}, y_{n+1}) \preceq (x_n, y_n)$$

and $\{x_n\}_{n\in\mathbb{N}}$ converges to $x \in X$, then there exists $y \in Y$ such that $(x,y) \in M$ and

$$\forall n \in \mathbb{N}: \ (x, y) \preceq (x_n, y_n);$$

(M4') If $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq M$ is a decreasing sequence, then $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.
Then, for each $x_0 \in X$ with $M_Y(x_0) \neq \emptyset$, there exists $\bar{x} \in X$ such that $M_Y(\bar{x}) \neq \emptyset$ and

$$\begin{array}{lll} (i) & M(\bar{x}) \preccurlyeq M(x_0) & \\ (ii) & M(x) \preccurlyeq M(\bar{x}) & \Longrightarrow \quad x = \bar{x}. \end{array}$$

Proof. We define a binary relation on X be setting

$$x' \preceq_X x \quad \Longleftrightarrow \quad M\left(x'\right) \preccurlyeq M(x).$$

in order to apply Theorem 16. Of course, $\preceq_X$ is reflexive and transitive. Let us check the regularity assumption (M3) of Theorem 16. Take a sequence $\{x_n\}_{n\in\mathbb{N}}$ decreasing with respect to $\preceq_X$, i. e.

$$\forall y_n \in M_Y(x_n) \ \exists y_{n+1} \in M_Y(x_{n+1}): \ (x_{n+1}, y_{n+1}) \preceq (x_n, y_n). \tag{4.3}$$

Take $y_0 \in M_Y(x_0)$. Find $y_1 \in M_Y(x_1)$ via (4.3) such that $(x_1, y_1) \preceq (x_0, y_0)$. Find $y_2 \in M_Y(x_2)$ via (4.3) such that $(x_2, y_2) \preceq (x_1, y_1)$. Continuing this procedure, one gets a sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ being decreasing with respect to $\preceq$. Since $\preceq$ is regular by (M4'), we obtain $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$ as desired.

Finally, we show that $\preceq_X$ is lower closed. Take a sequence $\{x_n\}_{n\in\mathbb{N}}$ decreasing with respect to $\preceq_X$ and converging to $x \in X$. We have to show that $x \preceq_X x_n$ for each $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. By (4.3), one can find $y_{n+1} \in M_Y(x_{n+1})$ such that $(x_{n+1}, y_{n+1}) \preceq (x_n, y_n)$ and, as before, gets a sequence $\{(x_{n+m}, y_{n+m})\}_{m\in\mathbb{N}}$ being decreasing with respect to $\preceq$. Of course, we still have $\lim_{m\to\infty} x_{n+m} = x$. Assumption (M3') implies the existence of $y \in M_Y(x)$ such that for each $m \in \mathbb{N}$

$$(x, y) \preceq (x_{n+m}, y_{n+m}) \preceq (x_n, y_n).$$

This procedure is applicable for every $n \in \mathbb{N}$ (the corresponding $y \in M_Y(x)$ may depend on n). The lower closedness of $\preceq_X$ is proven.

The final step of the proof is an application of Theorem 16 to the metric space (X, d) and the relation $\preceq_X$ in order to obtain (i) and (ii). This is straightforward. ■

Analyzing the proof above, one may see that it is not possible to show the regularity and lower closedness of the order $\preceq_X$ if simply $\preccurlyeq$ is replaced by $\curlyeqprec$. The corresponding result for $\curlyeqprec$ reads as follows.

Theorem 22 *Let the following assumptions be satisfied:*
(M1') (X, d) is a metric space and Y as well as $M \subseteq X \times Y$ are nonempty sets;
(M2') $\preceq$ is a quasiorder, i.e., a reflexive and transitive relation on $X \times Y$;
(M3') If $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq M$ is a increasing sequence, i.e.

$$\forall n \in \mathbb{N}: \ (x_n, y_n) \preceq (x_{n+1}, y_{n+1})$$

and $\{x_n\}_{n\in\mathbb{N}}$ converges to $x \in X$, then there exists $y \in Y$ such that $(x, y) \in M$ and

$$\forall n \in \mathbb{N}: \ (x_n, y_n) \preceq (x, y)\,;$$

(M4') If $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq M$ is a increasing sequence, then $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.
Then, for each $x_0 \in X$ with $M_Y(x_0) \neq \emptyset$, there exists $\bar{x} \in X$ such that $M_Y(\bar{x}) \neq \emptyset$ and

$$\begin{array}{ll} (i) & M(x_0) \curlyeqprec M(\bar{x}) \\ (ii) & M(\bar{x}) \curlyeqprec M(x) \quad \Longrightarrow \quad x = \bar{x}. \end{array}$$

Proof. There are at least two proofs possible. The first idea is to reformulate Theorem 16 as a maximal element theorem and proceed as in the proof of Theorem 21. Another plan is to apply Theorem 21 to a suitable order relation. We shall do the latter. Define a binary relation $\preceq'$ on $X \times Y$ by

$$\left(x', y'\right) \preceq' (x, y) \quad \Longleftrightarrow \quad (x, y) \preceq \left(x', y'\right).$$

Of course, $\preceq'$ is a quasiorder. Moreover, a sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq X \times Y$ is decreasing with respect to $\preceq'$ if and only if it is increasing with respect to $\preceq$. Hence (M3') and (M4') of Theorem 21 are satisfied for $\preceq'$ if and only if (M3') and (M4') of Theorem 22 are satisfied for $\preceq$, respectively. Denote by $\preccurlyeq'$ the relation defined by (4.1) replacing $\preceq$ by $\preceq'$. We can apply Theorem 21 to get an $\bar{x} \in X$ such that $M_Y(\bar{x}) \neq \emptyset$ and

$$\begin{array}{ll} (i') & M(\bar{x}) \preccurlyeq' M(x_0) \\ (ii') & M(x) \preccurlyeq' M(\bar{x}) \quad \Longrightarrow \quad x = \bar{x}. \end{array}$$

Observing that $M(x') \preccurlyeq' M(x)$ if and only if $M(x) \curlyeqprec M(x')$ we see that (i') and (ii') are equivalent to (i) and (ii) of Theorem 22, respectively. This completes the proof. ■

Remark 14 *A special case of Theorem 21 (as well as of Theorem 22) is the case if Y consists of a single element only. In this case, Theorem 21 reduces to Theorem 16 (as well as Theorem 22 to a maximal element reformulation of Theorem 16). On the other hand, Theorem 21 (as well as Theorem 22) are proven using Theorem 16 without any reference to the induction process in the proof of Theorem 16. In this sense, the theorems are equivalent.*

Remark 15 *Another special case is $M(x) = \{(x, y)\}$, i.e., $M(x)$ is a singleton. In this case, the set $M \subseteq X \times Y$ defines a function $f : X \to Y$. The relation $\preceq$ coincides with $\preccurlyeq$ and $\preceq_X$, and they compare arguments and values of f at the same time:*

$$x' \preceq_X x \quad \Longleftrightarrow \quad \left(x', f\left(x'\right)\right) \preceq \left(x, f\left(x\right)\right).$$

Remark 16 *Assumption (M3') of the Theorems 21 and 22 coincides with assumption (H1) in [47] if Y is assumed to be a topological linear space. Thanks to this assumption, we can get rid of assumptions concerning topological and algebraic properties of Y or the concrete form of $\preceq$. On the other hand, it is by no means a trivial task to verify assumption (M3') in special cases. Compare the discussion in [47] and [44], Section 3.10. Finally, note that assumption (2) of the famous Theorem 1 due to Brézis and Browder has a similar structure, but does not involve product sets.*

4.2 Results with functions into ordered monoids

The Theorems 16, 17 and 18 are formulated in such a way that the order relation may or may not depend on the metric d. The charming character of Ekeland's variational principle and related theorems relies on its recursive structure: the order relation is defined in terms involving the metric itself. Actually, this ensures the topological requirements such as (M3) and (M4) of Theorem 16. It has already been observed by Brønsted [8] that not all properties of a metric are necessary for defining order relations in order to produce Ekeland type theorems.

To the authors opinion, it should be possible to prove (almost) all Ekeland type theorems and its equivalent reformulations on metric spaces by defining a suitable order relation and applying Theorem 16. To carry out this program, is the main goal of the remaining part of this chapter.

Results in the spirit of Ekeland's principle usually involve a function f mapping the metric space X to a set Y provided with some algebraic, order and topological structure.

The next sections are devoted to such results supplying Y with more and more structure from step to step. Each of the Theorems 16, 17 and 18 (or even Theorems 21 and 22) may be chosen as starting points for the these developments. We prefer to use Theorem 16.

The first corollary seems to require less algebraic structure of the image space than any other result in this direction up to now. It involves functions from a metric space (X, d) to an ordered monoid. Most of the so called setvalued or vectorvalued variants of Ekeland's principle on metric spaces are special cases of the theorems of the next subsection.

4.2.1 Ekeland's variational principle

The following result is parallel to Ekeland's variational principle from [30], but for functions with values in ordered monoids.

Corollary 9 *Let the following assumptions be satisfied:*
(A1) (X,d) *is a metric space and* $(Y,\circ,\leq)$ *a quasiordered monoid;*
(A2) $\Phi : X \times X \to Y$ *is an order premetric;*
(A3) The function $f : X \to Y$ *and* $\tilde{y} \in Y$ *are such that*
(i) $\tilde{y} \leq f(x)$ *for all* $x \in X$*;*
(ii) Φ *is regular with respect to* $\tilde{y}, f(x_0) \in Y$ *for* $x_0 \in X$*;*
(iii) if $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *is a Cauchy sequence with*

$$\forall n \in \mathbb{N}: \; f(x_{n+1}) \circ \Phi(x_{n+1}, x_n) \leq f(x_n) \tag{4.4}$$

then it converges to some $x \in X$*;*
(A4) If $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *converges to* $x \in X$ *and satisfies (4.4), then* $f(x)\circ\Phi(x_n,x) \leq f(x_n)$ *for all* $n \in \mathbb{N}$.
Then, there exists $\bar{x} \in X$ *such that*

$$\begin{array}{llcll} (i) & \quad f(\bar{x}) \circ \Phi(\bar{x}, x_0) & \leq & f(x_0) & \\ (ii) & x \in X, \; f(x) \circ \Phi(x, \bar{x}) & \leq & f(\bar{x}) & \Longrightarrow \; x = \bar{x}. \end{array}$$

Proof. The proof is by checking the assumptions of Theorem 16 for the relation

$$x' \preceq x \qquad :\Longleftrightarrow \qquad f\left(x'\right) \circ \Phi\left(x', x\right) \leq f(x).$$

The relation $\preceq$ is reflexive since $\leq$ is reflexive and Φ satisfies (P1) of Definition 30. It is transitive by (P2) of Definition 30 and the transitivity of $\leq$. The $\preceq$–completeness of X follows from (A3, (iii)). (M4) follows directly from assumption (A4). It remains to check (M3). Let $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ be such that $x_{n+1} \preceq x_n$ for all $n \in \mathbb{N}$, i.e.

$$f(x_{n+1}) \circ \Phi(x_{n+1}, x_n) \leq f(x_n).$$

The transitivity of $\leq$ implies

$$f(x_{n+1}) \circ \Phi(x_{n+1}, x_n) \circ \Phi(x_n, x_{n-1}) \leq f(x_n) \circ \Phi(x_n, x_{n-1}) \leq f(x_{n-1}).$$

Continuing this process, we obtain for each $n \in \mathbb{N}$

$$f(x_{n+1}) \circ \sum_{k=0}^{n} \Phi(x_{k+1}, x_k) \leq f(x_0),$$

where the "sum" ($\sum$) is understood with respect to the operation $\circ$. Since $\tilde{y} \leq f(x_m)$ for each $m \in \mathbb{N}$ by (A3, (i)), it follows

$$\tilde{y} \circ \sum_{k=0}^{n} \Phi(x_{k+1}, x_k) \leq f(x_0).$$

Since by (A3, (ii)) Φ is regular with respect to $\tilde{y}, f(x_0) \in Y$ (see Definition 31), this implies $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$. An application of Theorem 16 yields the desired result. ■

Remark 17 *Let $\Psi : X \times X \to Y$ be a function satisfying (P2) and (P3) of Definition 30. Then Φ according to Lemma 4 is a regular order premetric. Getting $\bar{x}$ from Corollary 9, we have either $\bar{x} = x_0$ or $f(\bar{x}) \circ \Psi(\bar{x}, x_0) \leq f(x_0)$. Relationship (ii) of the corollary remains in force substituting Φ by Ψ. In the following, we do not mention this possibility, but work with regular order premetrics.*

Remark 18 *We do not need topological structure in Y. Note further, that it is not necessary to have Y being a group. Thus, Corollary 9 generalizes the result of [93] with respect to the image space Y. In fact, this generalization makes it possible for dealing with setvalued maps: Let $(Y, \circ, \leq)$ be a quasiordered monoid with neutral element $\theta \in Y$. Then, $\left(\widehat{\mathcal{P}}(Y), \odot, \preccurlyeq\right)$ and $\left(\widehat{\mathcal{P}}(Y), \odot, \curlyeqprec\right)$ are quasiordered monoids as well. Hence Corollary 9 can be applied to functions $f : X \to \widehat{\mathcal{P}}(Y)$. We shall discuss this situation e.g. in Section 4.3.1 obtaining Ekeland type theorems for setvalued maps.*

We shall indicate a sufficient condition for (A4) of Corollary 9. A function $f : X \to Y$ is called **lower monotone** iff for each sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converging to some $x \in X$ and satisfying $f(x_{n+1}) \leq f(x_n)$ the inequality $f(x) \leq f(x_n)$ holds true for all $n \in \mathbb{N}$. Compare [93] for this kind of condition.

An order premetric $\Phi : X \times X \to Y$ is called **lower monotone** with respect to the first variable iff for each $x' \in X$ and each sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converging to $x \in X$ and $y_1, y_2 \in Y$ the condition

$$\forall n \in \mathbb{N} : \; y_1 \circ \Phi\left(x_n, x'\right) \leq y_2$$

implies $y_1 \circ \Phi(x, x') \leq y_2$.

Lemma 7 *Let (X, d) be a metric space and $(Y, \circ, \leq)$ be an ordered monoid. Let the function $f : X \to Y$ be lower monotone and the order premetric $\Phi : X \times X \to Y$ lower monotone with respect to the first variable. Then, (A4) of Corollary 9 is satisfied.*

PROOF. Take a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converging to $x \in X$ such that

$$\forall n \in \mathbb{N} : \; f(x_{n+1}) \circ \Phi(x_{n+1}, x_n) \leq f(x_n).$$

This implies $f(x_m) \circ \Phi(x_m, x_n) \leq f(x_n)$ for $m \geq n$ on the one hand and, since $\theta \leq \Phi(x_{n+1}, x_n)$,

$$f(x_{n+1}) \leq f(x_{n+1}) \circ \Phi(x_{n+1}, x_n) \leq f(x_n)$$

on the other hand. Therefore $f(x_{n+1}) \leq f(x_n)$ for all $n \in \mathbb{N}$ since $\leq$ is transitive. The lower monotonicity of f implies $f(x) \leq f(x_n)$ for all $n \in \mathbb{N}$. For $m \in \mathbb{N}$, $m \geq n$ we obtain

$$f(x) \circ \Phi(x_m, x_n) \leq f(x_m) \circ \Phi(x_m, x_n) \leq f(x_n).$$

The lower monotonicity property of Φ yields the result. ■

4.2.2 Kirk-Caristi fixed point theorem

The next result is a fixed point theorem. The original variant goes back to Caristi and Kirk, see [9], [74], [127], [26] and [25]. A concise proof, being constructive in some sense, can be found in [110]. See also [4] for a thorough discussion of the proof as well as several applications. There are many generalizations and variants, see for example [96] for an equilibrium version, [73] and [115] for vector valued variants. Most of them are special cases of the following corollary of Theorem 16.

We consider a set valued mapping $T : X \to \widehat{\mathcal{P}}(X)$. Recall the definitons of a fixed point and an invariant point of T given in Section 1.1.3.

Corollary 10 *Let the following assumptions be satisfied:*
(A1) (X, d) *is a metric space and* $(Y, \circ, \leq)$ *a quasiordered monoid;*
(A2) $\Phi : X \times X \to Y$ *is an order premetric;*
(A3) The function $f : X \to Y$ *and* $\tilde{y} \in Y$ *are such that*
(i) $\tilde{y} \leq f(x)$ *for all* $x \in X$*;*
(ii) Φ *is regular with respect to* $\tilde{y}, f(x_0) \in Y$ *for* $x_0 \in X$*;*
(iii) if $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *is a Cauchy sequence with*

$$\forall n \in \mathbb{N} :\ f(x_{n+1}) \circ \Phi(x_{n+1}, x_n) \leq f(x_n) \tag{4.5}$$

then it converges to some $x \in X$*;*
(A4) If $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *converges to* $x \in X$ *and satisfies (4.5), then* $f(x) \circ \Phi(x_n, x) \leq f(x_n)$ *for all* $n \in \mathbb{N}$.
If, additionally, the mapping $T : X \to \widehat{\mathcal{P}}(X)$ *satisfies the weak contraction condition*

$$\forall x \in X,\ \exists x' \in T(x) :\ f\left(x'\right) \circ \Phi\left(x', x\right) \leq f(x), \tag{WC}$$

then T *has a fixed point.*
If the mapping $T : X \to \mathcal{P}(X)$ *satisfies the strong contraction condition*

$$\forall x \in X,\ \forall x' \in T(x) :\ f\left(x'\right) \circ \Phi\left(x', x\right) \leq f(x), \tag{SC}$$

then T *has an invariant point.*

Proof. By contradiction: Assume there is no fixed point and no stationary point, respectively. By Corollary 9, there is $\bar{x} \in X$ such that

$$f(x) \circ \Phi(x, \bar{x}) \leq f(\bar{x}) \quad \Longrightarrow \quad x = \bar{x}.$$

Hence, $\bar{x}$ is the only point that can satisfy (WC) and (SC), respectively. This proves the corollary. ■

Conversely, Corollary 9 can be proven using the fixed point result above. Indeed, assume that (ii) of Corollary 9 does not hold, i.e.

$$\forall x \in X,\ \exists x' \neq x :\ f\left(x'\right) \circ \Phi\left(x', x\right) \leq f(x).$$

Then, the mapping $T : X \to \widehat{\mathcal{P}}(X)$ satisfies (SC) and has no invariant point, i.e., the assertions of Corollary 10 can not hold. In this sense, the two corollaries are equivalent.

Of course, Corollary 10 is also a direct consequence of Theorem 17.

4.2.3 Takahashi's existence principle

The following existence principle is, for the real valued case, due to Takahashi [113]. It's equivalence to Ekeland's principle has been observed in [96] and [48]. Compare e.g. [97] and [128] for similiar results.

Corollary 11 *Let the following assumptions be satisfied:*
(A1) (X, d) *is a metric space and* $(Y, \circ, \leq)$ *a quasiordered monoid;*
(A2) $\Phi : X \times X \to Y$ *is an order premetric;*
(A3) The function $f : X \to Y$ *and* $\tilde{y} \in Y$ *are such that*
(i) $\tilde{y} \leq f(x)$ *for all* $x \in X$*;*
(ii) Φ *is regular with respect to* $\tilde{y}, f(x_0) \in Y$ *for* $x_0 \in X$*;*
(iii) if $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *is a Cauchy sequence with*

$$\forall n \in \mathbb{N} : \; f(x_{n+1}) \circ \Phi(x_{n+1}, x_n) \leq f(x_n) \tag{4.6}$$

then it converges to some $x \in X$*;*
(A4) If $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *converges to* $x \in X$ *and satisfies (4.6), then* $f(x) \circ \Phi(x_n, x) \leq f(x_n)$ *for all* $n \in \mathbb{N}$*.*
Assume, additionally, that

$$x_1, x_2 \in X, \; f(x_1) \leq f(x_2), \; f(x_2) \not\leq f(x_1)$$

implies

$$\exists x_3 \in X : \; x_3 \neq x_1, \; f(x_3) \circ \Phi(x_3, x_1) \leq f(x_1).$$

Then, there exists $\bar{x} \in X$ *such that* $f(\bar{x}) \in \min f(X)$ *where* $f(X) = \bigcup_{x\in X} \{f(x)\}$*.*

Proof. By contradiction: By Corollary 9, there is $\bar{x} \in X$ such that

$$f(x) \circ \Phi(x, \bar{x}) \leq f(\bar{x}) \quad \Longrightarrow \quad x = \bar{x}. \tag{4.7}$$

Let $f(\bar{x}) \notin \min f(X)$, i.e.

$$\exists u \in X : \; f(u) \leq f(\bar{x}), f(\bar{x}) \not\leq f(u).$$

By assumption, there is $x' \neq \bar{x}$ such that

$$f(x') \circ \Phi(x', \bar{x}) \leq f(\bar{x}),$$

contradicting (4.7). ■

Assume additionally that Φ is symmetric, i.e.,

$$\forall x', x \in X: \ \Phi\left(x', x\right) = \Phi\left(x, x'\right).$$

Then, Corollary 9 can be proven using Corollary 11. Indeed, assume that (ii) of Corollary 9 does not hold, i.e.

$$\forall x \in X, \ \exists x' \neq x: \ f\left(x'\right) \circ \Phi\left(x', x\right) \leq f(x).$$

Take $\bar{x} \in X$ such that $f(\bar{x}) \in \min f(X)$, i.e. $x \in X, \ f(x) \leq f(\bar{x})$ imply $f(\bar{x}) \leq f(x)$. Such a point does exist by Corollary 11. By assumption, there is also $\bar{x}' \in X$ such that

$$f\left(\bar{x}'\right) \circ \Phi\left(\bar{x}', \bar{x}\right) \leq f(\bar{x}).$$

¿From $\theta \leq \Phi(\bar{x}', \bar{x})$ we obtain

$$f\left(\bar{x}'\right) \leq f\left(\bar{x}'\right) \circ \Phi\left(\bar{x}', \bar{x}\right).$$

The transitivity of $\leq$ implies $f(\bar{x}') \leq f(\bar{x})$ and therefore the minimality of $f(\bar{x})$ gives $f(\bar{x}) \leq f(\bar{x}')$. Using this and the symmetry of Φ, one may conclude

$$f(\bar{x}) \circ \Phi\left(\bar{x}', \bar{x}\right) \leq f\left(\bar{x}'\right) \circ \Phi\left(\bar{x}', \bar{x}\right) \leq f(\bar{x}) \leq f\left(\bar{x}'\right).$$

Since $\preceq$ is antisymmetric (this is due to the regularity of Φ, compare Remark 40), $\bar{x}' = \bar{x}$, a contradiction.

4.2.4 The flower petal theorem

In [98], Penot proved a geometric theorem being equivalent to Ekeland's principle for extended realvalued functions. It is called the flower petal theorem because of the shape of certain sets in case $X = \mathbb{R}^2$. We need the following constructions to formulate a similar theorem in the present general framework.

Let X, Y be as in the last corollaries. Let $\Phi : X \times X \to Y$ be an order premetric and $\Psi : X \times X \to Y$ a function satisfying (P1) and (P2) of Definition 30. We call the set

$$P_\Phi(u, v) := \{x \in X : \ \Psi(x, v) \circ \Phi(x, u) \leq \Psi(u, v)\}$$

the **flower petal** generated by $u, v \in X$. It is always nonempty since $u \in P_\Phi(u, v)$ for each $u \in X$ because $\Phi(u, u) = \theta$ and $\leq$ is reflexive. The metric space (X, d) is called **Φ–complete** iff every Cauchy sequence $\{x_n\}_{n \in \mathbb{N}}$ satisfying $x_{n+1} \in P_\Phi(x_n, v)$ converges to some $x \in X$.

Corollary 12 *Let the following assumptions be satisfied:*
(A1) (X, d) is a metric space and $(Y, \circ, \leq)$ a quasiordered monoid; $M \subseteq X$, $x_0 \in M$ and $v \in X \backslash M$;
(A2) The function $\Psi : X \times X \to Y$ satisfies (P1) and (P2) of Definition 30;

(A3) $\Phi : X \times X \to Y$ *is a regular order premetric with respect to* $\theta, \Psi(x_0, v) \in Y$ *for* $x_0 \in X$ *such that* X *is* Φ*–complete;*
(A4) If $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *converges to* $x \in X$ *and*

$$\forall n \in \mathbb{N} : \ x_{n+1} \in P_\Phi(x_n, v) \cap M,$$

then $x \in P_\Phi(x_n, v) \cap M$ *for all* $n \in \mathbb{N}$.
Then, there exists $\bar{x} \in M$ *such that*

$$\bar{x} \in P_\Phi(x_0, v) \cap M \qquad \textit{and} \qquad \{\bar{x}\} = P_\Phi(\bar{x}, v) \cap M.$$

Proof. We check the assumptions of Theorem 16 for the relation

$$x' \preceq x \qquad :\Longleftrightarrow \qquad x' \in P_\Phi(x, v) \cap M.$$

on $X' := P_\Phi(x_0, v) \cap M$. Of course, (X', d) is a metric space. The relation $\preceq$ is reflexive and transitive since Φ is an order premetric. X' is $\preceq$–complete since X is Φ–complete. (M4) of Theorem 16 follows directly from (A4). It remains to show the regularity assumption (M3). Since $x_{n+1} \preceq x_n$ if and only if $\Psi(x_{n+1}, v) \circ \Phi(x_{n+1}, x_n) \le \Psi(x_n, v)$ we obtain for each $n \in \mathbb{N}$

$$\Psi(x_n, v) \circ \sum_{k=0}^{n-1} \Phi(x_{k+1}, x_k) \le \Psi(x_0, v).$$

The regularity of Φ implies that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is asymptotic, i.e., (M3) is satisfied.

Hence there exists $\bar{x} \in S(x_0) = P_\Psi(x_0, v) \cap M$ such that $\{\bar{x}\} = S(\bar{x}) = P_\Psi(\bar{x}, v)$. This completes the proof. ■

The flower petal theorem is also a consequence of Corollary 9. To see this, simply take $f(x) := \Psi(x, v)$.

4.2.5 An equilibrium formulation of Ekeland's principle

The next result deals with a function $F : X \times X \to Y$ instead of $f : X \to Y$. As far as the author is aware, for realvalued functions this idea is due to Oettli and Théra [96] causing several subsequent similiar considerations, see for example [59], [97].

Corollary 13 *Let the following assumptions be satisfied:*
(A1) (X, d) *is a metric space and* $(Y, \circ, \le)$ *a quasiordered monoid;*
(A2) $F : X \times X \to Y$ *is a function and* $x_0 \in X$, $\tilde{y} \in Y$ *such that*
(i) $F(x_1, x_3) \le F(x_1, x_2) \circ F(x_2, x_3)$ *for all* $x_1, x_2, x_3 \in X$*;*
(ii) $\tilde{y} \le F(x_0, x)$ *for all* $x \in X$*;*
(A3) $\Phi : X \times X \to Y$ *is a regular order premetric with respect to* $\tilde{y}, \theta \in Y$*;*
(A4) If $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *is a Cauchy sequence such that*

$$F(x_n, x_{n+1}) \circ \Phi(x_{n+1}, x_n) \le \theta \quad \textit{whenever} \quad x_{n+1} \ne x_n, \tag{4.8}$$

then it converges to some $x \in X$;
(A5) If $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *is a sequence satisfying (4.8) and converging to* $x \in X$*, then* $F(x_n, x) \circ \Phi(x, x_n) \leq \theta$ *for all* $n \in \mathbb{N}$ *with* $x \neq x_n$.

Then, there exists $\bar{x} \in X$ *such that*

$$\begin{array}{lrcl} (i) & F(x_0, \bar{x}) \circ \Phi(\bar{x}, x_0) & \leq & \theta \\ (ii)\ x \in X, & F(\bar{x}, x) \circ \Phi(x, \bar{x}) & \leq & \theta \implies x = \bar{x}. \end{array}$$

Proof. We check the assumptions of Theorem 16 for the relation

$$x' \preceq x \quad :\Longleftrightarrow \quad x' = x \quad \text{or} \quad F\left(x, x'\right) \circ \Phi\left(x', x\right) \leq \theta.$$

being reflexive and transitive by the properties of Φ, F and $\leq$. (M4) of Theorem 16 follows directly from (A5). It remains to check (M3). Let $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ be such that $x_{n+1} \preceq x_n$ for all $n \in \mathbb{N}$, i.e.,

$$x_{n+1} = x_n \quad \text{or} \quad F(x_n, x_{n+1}) \circ \Phi(x_{n+1}, x_n) \leq \theta$$

for all $n \in \mathbb{N}$. Deleting x_{n+1} from the sequence if $x_{n+1} = x_n$ we obtain a finite number of x_n's or a subsequence again denoted by $\{x_n\}_{n\in\mathbb{N}}$. In the first case, the original sequence is constant up to finitely many elements, hence asymptotic. In the second case, using (A2, (i)), the properties of Φ and the transitivity of $\leq$ we obtain for $n \in \mathbb{N}$:

$$F(x_0, x_n) \circ \sum_{k=0}^{n-1} \Phi(x_{k+1}, x_k) \leq \theta.$$

With the help of assumption (A2, (ii)) we may conclude

$$\tilde{y} \circ \sum_{k=0}^{n-1} \Phi(x_{k+1}, x_k) \leq \theta.$$

Therefore, the regularity of Φ implies $d(x_{n+1}, x_n) \to 0$. We can apply Theorem 16 in order to obtain the desired result. ■

Remark 19 *We are moving within the setting of Corollary 13. Define a function* $f : X \to Y$ *by*

$$f(x) := F(x_0, x), \quad x \in X$$

and an order relation on X *by*

$$x' \preceq_f x \quad :\Longleftrightarrow \quad x' = x \quad \text{or} \quad f\left(x'\right) \circ \Phi\left(x', x\right) \leq f(x).$$

being reflexive and transitive. Observe that $x' \preceq x$ *implies* $x' \preceq_f x$ *since by (A2, i)*

$$f\left(x'\right) \circ \Phi\left(x', x\right) = F\left(x_0, x'\right) \circ \Phi\left(x', x\right) \leq F(x_0, x) \circ F\left(x, x'\right) \circ \Phi\left(x', x\right) \leq F(x_0, x)$$

whenever $F(x, x') \circ \Phi(x', x) \leq \theta$.

Let the assumptions of Corollary 9 be satisfied for f as defined above. Then we get $\bar{x} \in X$ such that

$$\begin{array}{llcll} (i') & f(\bar{x}) \circ \Phi(\bar{x}, x_0) & \leq & f(x_0) & \\ (ii') & x \in X, \quad f(x) \circ \Phi(x, \bar{x}) & \leq & f(\bar{x}) & \Longrightarrow \quad x = \bar{x}. \end{array}$$

Assuming that $F(x_0, x_0) \leq \theta$ we obtain (i) of Corollary 13 from (i'). Moreover, if $\bar{x}$ satisfies (ii'), then it satisfies (ii) of Corollary 13 since $x' \preceq x$ implies $x' \preceq_f x$.

These considerations show that Corollary 13 is a consequence of Corollary 9 in case if $F(x_0, x_0) \leq \theta$ and the function $f(x) = F(x_0, x)$ satisfies (A5) of Corollary 9.

4.2.6 Ekeland's variational principle on groups

We shall consider the case Y being a group separately since it is interesting from a theoretical point of view. Especially, Corollary 13 will turn out to be an equivalent formulation to Corollary 9 in this situation. Besides, in many applications Y is even a linear space.

Nemeth [93] first investigated the case of an ordered topological Abelian group $(G, \circ)$. In the following corollary, G is an ordered group not order complete in general. We can adjoin a largest element y_l (as well as a smallest one if necessary) obtaining an ordered monoid. Note that no topological requirements concerning G do appear in contrast to Nemeth's results [93], [94].

Corollary 14 *Let the following assumptions be satisfied:*
(A1) (X, d) is a metric space and $(G, \circ, \leq)$ a quasiordered Abelian group;
(A2) $f : X \to G \cup \{y_l\}$ is a function and $\tilde{y} \in G$ such that and $\tilde{y} \leq f(x)$ for all $x \in X$;
(A3) $\Phi : X \times X \to Y$ is a regular order premetric with respect to $\tilde{y} \circ [f(x_0)]^{-1}$ for $x_0 \in G$;
(A4) If $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ converges to $x \in X$ and

$$\forall n \in \mathbb{N} : \ f(x_{n+1}) \circ \Phi(x_{n+1}, x_n) \leq f(x_n)$$

then $f(x) \circ \Phi(x_n, x) \leq f(x_n)$ for all $n \in \mathbb{N}$.

Then, there is $\bar{x} \in X$ such that

$$\begin{array}{llcll} (i) & f(\bar{x}) \circ \Phi(\bar{x}, x_0) & \leq & f(x_0) & \\ (ii) & x \in X, \quad f(x) \circ \Phi(x, \bar{x}) & \leq & f(\bar{x}) & \Longrightarrow \quad x = \bar{x}. \end{array}$$

Proof. Set $Y := G \cup \{y_l\}$ and extend $\circ$ to Y according to Proposition 25. Apply Corollary 9 with the ordered monoid $(Y, \circ, \leq)$. ■

Remark 20 *In the present case, one can define*

$$F(x, x') := f(x') \circ [f(x)]^{-1},$$

where $y^{-1} \in G$ denotes the inverse element of $y \in G$ with respect to $\circ$: $y \circ y^{-1} = y^{-1} \circ y = \theta \in G$. Then, $F(x, x') \circ \Phi(x', x) \leq \theta$ if and only if $f(x') \circ \Phi(x', x) \leq f(x)$. This means, the assumptions of Corollary 13 are satisfied if f satisfies the assumptions of Corollary 14. In this sense, Corollary 14 is a special case of Corollary 13.

On the other hand, it seem to be not possible to obtain Corollary 9 from Corollary 13 since Y is not a group.

4.3 Ekeland's principle for set valued maps

The first Ekeland type theorems for set valued maps seems to be the results of Chen and Huang in [11]. More can be found in [12], [50], [52], [117], [55]. In fact, the appearance of set valued variants of Ekeland's principle has been the main motivation to allow the image space a more general algebraic structure than a linear space or a group. Note that the results of Nemeth [92], [93], [94] do not cover Ekeland type theorems with set valued mappings since he assumed the image space of f to be a topological Abelian group.

4.3.1 Power set of ordered monoids

The following two corollaries involve the power set of an ordered monoid $(Y, \circ, \leq)$ supplied with the relation $\preccurlyeq$ and $\curlyeqprec$, respectively.

Corollary 15 *Let the following assumptions be satisfied:*
(A1) (X, d) *is a metric space,* $(Y, \circ, \leq)$ *an ordered monoid and* $(\mathcal{Y}, \odot, \preccurlyeq)$ *the quasiordered monoid generated by* $\mathcal{Y} := \widehat{\mathcal{P}}(Y)$*;*
(A2) $\Phi : X \times X \to \mathcal{Y}$ *is an order premetric;*
(A3) The function $f : X \to \mathcal{Y}$ *and* $M \in \mathcal{Y}$ *are such that*
(i) $M \preccurlyeq f(x)$ *for all* $x \in X$*;*
(ii) Φ *is regular with respect to* $M, f(x_0) \in \mathcal{Y}$ *for* $x_0 \in X$*;*
(iii) if $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *is a Cauchy sequence with*

$$\forall n \in \mathbb{N} : \ f(x_{n+1}) \odot \Phi(x_{n+1}, x_n) \preccurlyeq f(x_n) \tag{4.9}$$

then it converges to some $x \in X$*;*
(A4) If $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *is a sequence converging to* $x \in X$ *and satisfying (4.9), then*

$$\forall n \in \mathbb{N} : \ f(x) \odot \Phi(x, x_n) \preccurlyeq f(x_n).$$

Then, there exists $\bar{x} \in X$ *such that*

$$\begin{array}{lll} (i) & & f(\bar{x}) \odot \Phi(\bar{x}, x_0) \preccurlyeq f(x_0) \\ (ii) & x \in X, & f(x) \odot \Phi(x, \bar{x}) \preccurlyeq f(\bar{x}) \quad \Longrightarrow \quad x = \bar{x}. \end{array}$$

PROOF. By Theorem 11, $(\mathcal{Y}, \odot, \preccurlyeq)$ is an order complete quasiordered monoid. Defining the relation

$$x' \preceq x \qquad :\Longleftrightarrow \qquad f(x') \odot \Phi(x', x) \preccurlyeq f(x)$$

on X, the assumptions of Corollary 9 are easy to check. Its conclusions yield the desired result. ■

Corollary 16 *Let the following assumptions be satisfied:*
(A1) (X, d) *is a metric space,* $(Y, \circ, \leq)$ *an ordered monoid and* $(\mathcal{Y}, \odot, \curlyeqprec)$ *the quasiordered monoid generated by* $\mathcal{Y} := \widehat{\mathcal{P}}(Y)$*;*

(A2) $\Phi : X \times X \to \mathcal{Y}$ *is an order premetric;*
(A3) The function $f : X \to \mathcal{Y}$ *and* $M \in \mathcal{Y}$ *are such that*
(i) $M \curlyeqprec f(x)$ *for all* $x \in X$*;*
(ii) Φ *is regular with respect to* $M, f(x_0) \in \mathcal{Y}$ *for* $x_0 \in X$*;*
(iii) if $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *is a Cauchy sequence with*

$$\forall n \in \mathbb{N} : \ f(x_{n+1}) \odot \Phi(x_{n+1}, x_n) \curlyeqprec f(x_n) \tag{4.10}$$

then it converges to some $x \in X$*;*
(A4) If $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *is a sequence converging to* $x \in X$ *and satisfying (4.10), then*

$$\forall n \in \mathbb{N} : \ f(x) \odot \Phi(x, x_n) \curlyeqprec f(x_n).$$

Then, there exists $\bar{x} \in X$ *such that*

$$\begin{array}{lll} (i) & f(\bar{x}) \odot \Phi(\bar{x}, x_0) \curlyeqprec f(x_0) & \\ (ii) \quad x \in X, & f(x) \odot \Phi(x, \bar{x}) \curlyeqprec f(\bar{x}) & \Longrightarrow \quad x = \bar{x}. \end{array}$$

Proof. Replace $\preccurlyeq$ by $\curlyeqprec$ in the proof of Corollary 15. ■

The next results deals with an image space being the power set of a topological linear space $(V, +, \mathcal{T})$. Let $K \in \mathcal{P}(V)$ be a cone and a convex element at the same time, i.e., K is a convex cone in the classical sense containing $\theta \in V$. It generates the quasiorder $\leq_K$ by $v' \leq_K v$ iff $v \in \{v'\} \oplus K$. We denote by $\preccurlyeq_K$ and $\curlyeqprec_K$ the two quasiorders generated in $\mathcal{V} := \widehat{\mathcal{P}}(V)$ by $\leq_K$.

Corollary 17 *Let the following assumptions be satisfied:*
(A1) (X, d) *is a metric space and* $(V, +, \mathcal{T})$ *a topological linear space;*
(A2) $K \in \mathcal{V}$ *is a cone and a convex element in* $(\mathcal{V}, \oplus)$*,* $K^0 \subseteq K \backslash (-\operatorname{cl} K)$ *is a nonempty convex and sequentially compact set;*
(A3) The function $f : X \to \mathcal{V}$ *and the topological bounded set* $M \subseteq V$ *are such that* $M \preccurlyeq_K f(x)$ *for all* $x \in X$*;*
(A4) $\varphi : X \times X \to \mathbb{R}_+$ *is a regular premetric;*
(A5) If $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *is a Cauchy sequence with*

$$\forall n \in \mathbb{N} : \ f(x_{n+1}) \oplus \varphi(x_{n+1}, x_n) K^0 \preccurlyeq_K f(x_n) \tag{4.11}$$

then it converges to some $x \in X$*;*
(A6) If $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *is a sequence satisfying (4.11) such that* $\{x_n\}_{n\in\mathbb{N}}$ *converges to* $x \in X$*, then*

$$\forall n \in \mathbb{N} : \ f(x) \oplus \varphi(x, x_n) K^0 \preccurlyeq_K f(x_n).$$

Then, for each $x_0 \in X$ *with* $f(x_0) \neq \emptyset$*, there exists* $\bar{x} \in X$ *such that*

$$\begin{array}{lll} (i) & f(\bar{x}) \oplus \varphi(\bar{x}, x_0) K^0 \preccurlyeq_K f(x_0) & \\ (ii) \quad x \in X, & f(x) \oplus \varphi(x, \bar{x}) K^0 \preccurlyeq_K f(\bar{x}) & \Longrightarrow \quad x = \bar{x}. \end{array}$$

Proof. Consider $(\mathcal{V}, \oplus, \preccurlyeq_K)$ being a quasiordered monoid. We check the assumptions of Corollary 15 replacing Y by V, $\mathcal{Y}$ by $\mathcal{V}$ and setting $\Phi := \varphi K^0$. Most of the assumptions can be checked straightforward, only the regularity of Φ requires some extra words. In fact, Φ is regular with respect to $M, f(x_0) \in \mathcal{V}$. To see this, let

$$\forall n \in \mathbb{N}: \; M \oplus \left[\sum_{k=0}^{n} \Phi(x_{k+1}, x_k)\right] = M \oplus \left[\sum_{k=0}^{n} \varphi(x_{k+1}, x_k) K^0\right] \preccurlyeq_K f(x_0)$$

be satisfied. Since K^0 is a convex subset of a linear space, $(t+s) K^0 = sK^0 \oplus tK^0$ holds for all $s, t \geq 0$. Therefore,

$$\forall n \in \mathbb{N}: \; M \oplus \left[\sum_{k=0}^{n} \varphi(x_{k+1}, x_k) K^0\right] = M \oplus \left[\sum_{k=0}^{n} \varphi(x_{k+1}, x_k)\right] K^0 \preccurlyeq_K f(x_0).$$

Hence, for all $v \in f(x_0)$ and $n \in \mathbb{N}$ there exist $\tilde{v}_n \in M$, $k_n^0 \in K^0$ such that

$$\tilde{v}_n + \left[\sum_{k=0}^{n} \varphi(x_{k+1}, x_k)\right] k_n^0 \leq_K v.$$

Assume $\alpha_n := \sum_{k=0}^{n} \varphi(x_{k+1}, x_k) \to \infty$. Then

$$\frac{1}{\alpha_n}\tilde{v}_n + k_n^0 \leq_K \frac{1}{\alpha_n} v \quad \Leftrightarrow \quad \frac{1}{\alpha_n} v - \frac{1}{\alpha_n}\tilde{v}_n - k_n^0 \in K.$$

Since K^0 is sequentially compact, there exists a subsequence of $\{n\}_{n\in\mathbb{N}}$ such that $k_n^0 \to k^0 \in K^0$ along this subsequence. Since $\{v\}$ and M are bounded subsets of V, the above relationship implies $-k^0 \in \operatorname{cl} K$, a contradiction to assumption (A2). Therefore, the α_n's remain bounded, hence $\varphi(x_{n+1}, x_n) \to 0$. The regularity of φ gives $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.

The conclusions of Corollary 15 yield (i) and (ii) of Corollary 17. ■

Corollary 17 may be considered as a minimal element theorem in the product space $X \times \widehat{\mathcal{P}}(V)$ with respect to the order relation

$$(x', M') \preccurlyeq_{K^0} (x, M) \qquad \Longleftrightarrow \qquad M' \oplus \varphi(x', x) K^0 \preccurlyeq_K M.$$

Indeed, let a subset $A \subseteq X \times V$ be given and define a function (a set valued map) $f : X \to \widehat{\mathcal{P}}(V)$ by

$$f(x) := \{v \in V : \; (x, v) \in A\}.$$

If $f(x_0) \neq \emptyset$ and assumptions (A3), (A5) of Corollary 17 are satisfied for this f, we obtain a minimal point $(\bar{x}, f(\bar{x}))$ with respect to $\preccurlyeq_{K^0}$ of the set

$$\{(x, f(x)) : \; x \in X\} \subseteq X \times \mathcal{P}(V).$$

Hence, the question "authentic" or "not authentic" ([47], [44]) depends on the order relation. A theorem of the type of Corollary 17 has been called a *Minimal Set Theorem* in [50].

Note that more assumptions are necessary to ensure the existence of minimal elements of a subset $A \subseteq X \times V$ with respect to order relations in $X \times V$ such as

$$\left(x', v'\right) \leq_{k^0} (x, v) \qquad \Longleftrightarrow \qquad v' + \varphi\left(x', x\right) k^0 \leq_K v$$

where $k^0 \in K \backslash (-\operatorname{cl} K)$. We refer to Sections 4.7 and 7.2.

It follows the counterpart of the last corollary for the relation $\preccurlyeq_K$. One may notice the difference concerning the sets M and K^0: They have to be only nonempty. On the other hand, the set $f(x_0)$ must be topological bounded.

Corollary 18 *Let the following assumptions be satisfied:*
(A1) (X, d) is a metric space and $(V, +, \mathcal{T})$ a topological linear space;
(A2) $K \in \mathcal{V}$ is a cone and a convex element in $(\mathcal{V}, \oplus)$, $K^0 \subseteq K \backslash (-\operatorname{cl} K)$ is a nonempty convex set;
(A3) The function $f : X \to \mathcal{V}$ and the nonempty set $M \subseteq V$ are such that $M \preccurlyeq_K f(x)$ for all $x \in X$;
(A4) $\varphi : X \times X \to \mathbb{R}_+$ is a regular premetric;
(A5) If $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ is a Cauchy sequence with

$$\forall n \in \mathbb{N} : \; f(x_{n+1}) \oplus \varphi(x_{n+1}, x_n) K^0 \preccurlyeq_K f(x_n) \tag{4.12}$$

then it converges to some $x \in X$;
(A6) If $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ is a sequence satisfying (4.12) and $\{x_n\}_{n \in \mathbb{N}}$ converges to $x \in X$, then

$$\forall n \in \mathbb{N} : \; f(x) \oplus \varphi(x, x_n) K^0 \preccurlyeq_K f(x_n).$$

Then, for each x_0 with $f(x_0)$ topological bounded, there exists $\bar{x} \in X$ such that

$$\begin{array}{lll} (i) & & f(\bar{x}) \oplus \varphi(\bar{x}, x_0) K^0 \preccurlyeq_K f(x_0) \\ (ii) & x \in X, & f(x) \oplus \varphi(x, \bar{x}) K^0 \preccurlyeq_K f(\bar{x}) \quad \Longrightarrow \quad x = \bar{x}. \end{array}$$

Proof. We check the assumptions of Corollary 16 replacing $\left(\widehat{\mathcal{P}}(Y), \odot, \preccurlyeq\right)$ by $(\mathcal{V}, \oplus, \preccurlyeq_K)$ and setting $\Phi := \varphi K^0$. Again, the regularity of Φ is the crucial point to check. Let

$$\forall n \in \mathbb{N} : \; M \oplus \left[\sum_{k=0}^{n} \Phi(x_{k+1}, x_k)\right] = M \oplus \left[\sum_{k=0}^{n} \varphi(x_{k+1}, x_k) K^0\right] \preccurlyeq_K f(x_0)$$

be satisfied. As in the proof of Corollary 17, we may conclude

$$\forall n \in \mathbb{N} : \; M \oplus \left[\sum_{k=0}^{n} \varphi(x_{k+1}, x_k)\right] K^0 \preccurlyeq_K f(x_0).$$

According to the definition of $\preccurlyeq_K$, for all $v \in M$ and all $k^0 \in K^0$ there is $v_n \in f(x_0)$ such that

$$v + k^0 \alpha_n \leq_K v_n,$$

where again $\alpha_n := \sum_{k=0}^{n} \varphi(x_{k+1}, x_k)$ Assuming $\alpha_n \to \infty$, we arrive at the contradiction $k^0 \in -\mathrm{cl}\, K$ since $f(x_0)$ is bounded. This means, $\Phi = \varphi K^0$ is a regular order premetric with respect to $M, f(x_0)$. Applying Corollary 16 we obtain the desired result. ■

Of course, Corollary 18 can be considered as a minimal element theorem in the product space $X \times \widehat{\mathcal{P}}(V)$ in a similar way as Corollary 17.

4.4 Ekeland's principle for single valued Functions

It seems to the author that Nemeth [89], [91], [90] proved the first vector valued versions of Ekeland's theorem, even for functions on a space X more general than a complete metric space. Compare also [92]. Related results including a fixed point theorem of Kirk–Caristi type have been obtained by Khanh [73]

In [84], Proposition 4.2., a variant has been given for functions mapping a real Banach space into $\mathbb{R}^p$, $p \in \mathbb{N}$, $p > 1$: The proof is an elementary application of Ekeland's original result [30] to a scalarized problem.

Using nonlinear scalarization technique, Tammer [114] established an Ekeland type theorem for functions mapping a complete metric space into an topological linear space with an order relation not necessarily generated by a cone.

Related results can be found in [59], [58], [10], [14], [80], [132], [34], [35].

Another approach has been developed by Göpfert, Tammer and Zălinescu by proving a so–called minimal point theorem in the product space $X \times Y$ and deriving from this Ekeland type theorems. Compare [43], [46], [47] and the book [44].

The next result is a special case of Corollary 17 as well as of Corollary 18. In [47], [44] it is called a non–authentic minimal point theorem.

We consider an quasiordered linear space $(V, +, \leq_K)$ where the order relation $\leq_K$ is generated by a convex cone K. Again, a largest element v_l can be added to V obtaining a quasiordered monoid $(V \cup \{v_l\}, +)$.

Corollary 19 *Let the following assumptions be satisfied:*
(A1) (X, d) is a metric space and $(V, +, \mathcal{T})$ a topological linear space;
(A2) $K \in \mathcal{V}$ is a cone and a convex element in $(\mathcal{V}, \oplus)$ and $k^0 \in K \backslash (-\mathrm{cl}\, K)$;
(A3) The function $f : X \to V \cup \{v_l\}$ and the topological bounded set $M \subseteq V$ are such that

$$\forall x \in X : \ f(x) \in (M \oplus K) \cup \{v_l\};$$

(A4) $\varphi : X \times X \to \mathbb{R}_+$ is a regular premetric;
(A5) If $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is a Cauchy sequence with

$$\forall n \in \mathbb{N} : \ f(x_{n+1}) + \varphi(x_{n+1}, x_n)\, k^0 \leq_K f(x_n) \tag{4.13}$$

then it converges to some $x \in X$;
(A6) If $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is a sequence satisfying (4.13) and converging to $x \in X$, then

$$\forall n \in \mathbb{N} : \ f(x) + \varphi(x, x_n)\, k^0 \leq_K f(x_n).$$

Then, for each x_0 *with* $f(x_0)$, *there exists* $\bar{x} \in X$ *such that*

$$\begin{array}{llll} (i) & f(\bar{x}) + \varphi(\bar{x}, x_0)\, k^0 \leq_K f(x_0) & & \\ (ii) & x \in X, \quad f(x) + \varphi(x, \bar{x})\, k^0 \leq_K f(\bar{x}) & \Longrightarrow & x = \bar{x}. \end{array}$$

First Proof. Specialize Corollary 17. ■

Second Proof. Set $(Y, \circ, \leq) = (V \cup \{v_l\}, +, \leq_K)$ and apply Corollary 9. ■

4.5 Ekeland's principle for real valued functions

In this section, we prove a series of corollaries of Theorem 16 all being equivalent to Ekeland's principle from 1972 for extended real valued functions. This includes results with a more geometric nature such as the drop theorem, the flower petal theorem and Phelps' lemma. We do not focus on the equivalence proofs. They are well–known and can be found e.g. in [98], [96], [97].

In contrast, our proofs rely on Theorem 16. As in the last section, in each case, we shall construct an order relation and check the assumptions of Theorem 16. In order to simplify the exposition we assume (X, d) to be a complete metric space and only use the metric instead of a real valued premetric. The corresponding generalizations can be obtained easily parallel to the results in Section 4.2.

The first corollary is due to Dancs, Hegedüs, Medvegyev and can be found in [22]. Compare also [4], chapter 6. Therein, the set valued mapping f occuring in the corollary below is called a *dynamical system*.

Corollary 20 *Let the following assumptions be satisfied:*
(A1) (X, d) *is a complete metric space and* $f : X \to \mathcal{P}(X)$ *a set valued map;*
(A2) $x \in f(x)$ *for all* $x \in X$*;*
(A3) $x' \in f(x)$ *implies* $f(x') \subseteq f(x)$*;*
(A4) If $\{x_n\}_{n\in\mathbb{N}} \subset X$ *is a sequence such that* $x_{n+1} \in f(x_n)$ *for all* $n \in \mathbb{N}$*, then* $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$*;*
(A5) $f(x)$ *is closed for each* $x \in X$*.*
Then, for each $x_0 \in X$*, there is* $\bar{x} \in X$ *such that*

$$\bar{x} \in f(x_0) \qquad \text{and} \qquad \{\bar{x}\} = f(\bar{x}).$$

Proof. Define a relation $\preceq$ on X by

$$x' \preceq x \qquad :\Longleftrightarrow \qquad x' \in f(x)$$

being reflexive and transitive by (A2) and (A3), respectively. Then $S(x) = f(x)$. Assumptions (A4) and (A5) imply (M3) and (M4) of Theorem 16. Its conclusion yields the result. ■

Next, we prove a slightly generalized version of Ekeland's original theorem [30], [31], [98].

Corollary 21 *Let the following assumptions be satisfied:*
(A1) (X, d) *is a complete metric space;*
(A2) $f: X \to \mathrm{I\!R} \cup \{+\infty\}$ *is bounded from below;*
(A3) For each $x \in X$*, the set* $\{x' \in X : f(x') + d(x', x) \leq f(x)\}$ *is closed.*
Then, for each $x_0 \in X$ *with* $f(x_0) \in \mathrm{I\!R}$*, there is* $\bar{x} \in X$ *such that*

$$\begin{array}{ll} (i) & f(\bar{x}) + d(\bar{x}, x_0) \leq f(x_0) \\ (ii) & \forall x \in X \setminus \{\bar{x}\} : f(\bar{x}) < f(x) + d(x, \bar{x}) \end{array}$$

Proof. Define a partial order by

$$x' \preceq x \qquad :\Longleftrightarrow \qquad f(x') + d(x', x) \leq f(x).$$

By (A3), the set $S(x)$ is closed for each $x \in X$. We check (M4) of Theorem 16. Take a sequence $\{x_n\}_{n \in \mathrm{I\!N}} \subset X$ such that $x_{n+1} \preceq x_n$, i.e.

$$f(x_{n+1}) + d(x_{n+1}, x_n) \leq f(x_n)$$

for each $n \in \mathrm{I\!N}$. Since $f(x_0) \in \mathrm{I\!R}$ we have $\{f(x_n)\}_{n \in \mathrm{I\!N}} \subset \mathrm{I\!R}$. The sequence is nonincreasing and bounded from below by (A2), hence convergent. This implies $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. Applying Theorem 16, we obtain $\bar{x} \in X$ with properties (i) and (ii). ■

Remark 21 *1. Assumption (A3) is weaker than lower semiconituity of* f*. Consider the function* $f : \mathrm{I\!R} \to \mathrm{I\!R}$ *defined by* $f(x) = Exp(-|x|)$ *if* $x \neq 0$ *and* $f(0) = 2$ *not being lower semicontinuous at* $x = 0$*. This example is taken from [47]. The attempt to weaken the classical assumptions to* f *such as lower semicontinuity is due to [37].*
2. The above formulation of the conclusions (i), (ii) is probably due to Penot [98].
3. The original formulation of Ekeland is as follows. Start with $x_0 \in X$ *and* $\varepsilon > 0$, $\lambda > 0$ *such that* $f(x_0) \leq \inf_{x \in X} f(x) + \varepsilon$*. Replace* d *in Corollary 21 by* $\frac{\varepsilon}{\lambda} d$*. Then (i) implies that* $f(\bar{x}) \leq f(x_0)$ *and*

$$\frac{1}{\lambda} d(\bar{x}, x_0) \leq \frac{1}{\varepsilon}\left[f(x_0) - f(\bar{x})\right] \leq \frac{1}{\varepsilon}\left[\inf_{x \in X} f(x) + \varepsilon - f(\bar{x})\right] \leq 1,$$

hence $d(\bar{x}, x_0) \leq \lambda$*. Choosing* $\lambda = \sqrt{\varepsilon}$ *one can ensure that the difference* $f(\bar{x_0}) - f(\bar{x})$ *is small as well as the distance* $d(\bar{x}, x_0)$*.*
4. Conclusion (ii) can be interpreted as follows: The point $\bar{x}$ *is the unique global minimizer of the function* $x \to f(x) + d(x, \bar{x})$*. This observation, probably due to Clarke, is the starting point of many applications, e.g. the proof of Clarke's multiplier rule for nonsmooth optimization problems as well as the maximum principle for optimal control problems in [17], [18].*

Of course, choosing a special function f yields a special case of Corollary 21. One of them is the nice geometric result due to Penot [98] called flower petal theorem. Compare Corollary 12 for a general version. We chose $f(x) = d(x, v)$ for some $v \in X$.

Let $0 < \gamma < 1$ and $u, v \in X$ be given. The set

$$P_\gamma(u,v) := \{x \in X : \ d(x,v) + \gamma d(x,u) \leq d(u,v)\}$$

is called the **flower petal** belonging to u and v. Note that $P_\gamma(x,v)$ is a closed set.

Corollary 22 *Let (X,d) be a complete metric space, $M \subset X$, $v \in X \backslash M$ and $0 < \gamma < 1$. Then, for each $x_0 \in M$, there exists $\bar{x} \in X$ such that*

$$\bar{x} \in P_\gamma(x_0, v) \quad \text{and} \quad \{\bar{x}\} = P_\gamma(\bar{x}, v).$$

PROOF. Set $X' = M \cap P_\gamma(x_0, v)$ and replace in Corollary 21 (X,d) by $(X', \gamma d)$ as well as $f(x)$ by $d(x,v)$. ∎

Remark 22 *1. From the proofs of Corollary 21 and Corollary 22 it is clear that the relation*

$$x' \preceq x \qquad :\Longleftrightarrow \qquad x' \in P_\gamma(x,v)$$

is a partial order.
2. Remarkably, Corollary 22 is not only a special case, but also equivalent to Corollary 21. Compare [98], [38] for a proof.

The next corollary is the original version of Kirk-Caristi's fixed point theorem, see [9], [26], [25]. As in the general version (Corollary 10), we give two variants involving fixed points and stationary points of a setvalued map.

Let $T : X \to \mathcal{P}(X)$ be a set valued map. Recall that a point $\bar{x} \in X$ is a **fixed point** of T iff $\bar{x} \in T(\bar{x})$ and an **invariant point** of T iff $\{\bar{x}\} = T(\bar{x})$.

Corollary 23 *Let the following assumptions be satisfied:*
(A1) (X,d) is a complete metric space;
(A2) There exists a function $f : X \to \mathrm{I\!R} \cup \{+\infty\}$ not being identical $+\infty$ and satisfying (A2), (A3) of Corollary 21.
If $T : X \to \mathcal{P}(X)$ satisfies the weak contraction condition

$$\forall x \in X, \ \exists x' \in T(x) : \ f\left(x'\right) + d\left(x', x\right) \leq f(x), \tag{WC}$$

then it has a fixed point.
If $T : X \to \mathcal{P}(X)$ satisfies the strong contraction condition

$$\forall x \in X, \ \forall x' \in T(x) : \ f\left(x'\right) + d\left(x', x\right) \leq f(x), \tag{SC}$$

then it has an invariant point.

Proof. Define a partial order by

$$x' \preceq x \qquad :\Longleftrightarrow \qquad f\left(x'\right) + d\left(x', x\right) \leq f\left(x\right).$$

Theorem 16 (or Corollary 21) implies the existence of $\bar{x} \in X$ such that $S(\bar{x}) = \{\bar{x}\}$. Then (WC) implies $\bar{x} \in T(\bar{x})$. If (SC) is satisfied we even have $\{\bar{x}\} = T(\bar{x})$. ■

Ekeland's principle admits a reformulation as an existence principle for minimizers. This observation is due to Takahashi [113]. Compare also [96] and [48] as well as [128] for further applications of this idea.

Corollary 24 *Let the following assumptions be satisfied:*
(A1) (X, d) *is a complete metric space;*
(A2) $f: X \to \mathbb{R} \cup \{+\infty\}$ *is bounded from below;*
(A3) For each $x \in X$*, the set* $\{x' \in X : f(x') + d(x', x) \leq f(x)\}$ *is closed;*
(A4) For each $x \in X$ *with* $\inf_{u \in X} f(u) < f(x)$ *there exists* $x' \in X$ *such that*

$$x \neq x' \quad \text{and} \quad f\left(x'\right) + d\left(x', x\right) \leq f\left(x\right).$$

Then, there exists $\bar{x} \in X$ *such that* $f(\bar{x}) = \inf_{u \in X} f(u)$.

Proof. By assumption (A1), (A2), (A3) and Corollary 21, there exists $\bar{x} \in X$ such that

$$\forall x \neq \bar{x} : \ f(\bar{x}) < f(x) + d(x, \bar{x}).$$

This contradicts (A4) if $\bar{x}$ is not a minimizer of f. ■

In the spirit of Corollary 13, we state an equilibrium version of Ekeland's principle. This idea originates from [96].

Corollary 25 *Let the following assumptions be satisfied:*
(A1) (X, d) *is a complete metric space;*
(A2) $F : X \times X \to \mathbb{R} \cup \{+\infty\}$ *is a function and* $x_0 \in X$, $r \in \mathbb{R}$ *such that*
(i) $F(x_1, x_3) \leq F(x_1, x_2) + F(x_2, x_3)$ *for all* $x_1, x_2, x_3 \in X$ *and* $F(x, x) = 0$ *for all* $x \in X$*;*
(ii) $r \leq F(x_0, x)$ *for all* $x \in X$*;*
(iii) The function $x \to F(x_0, x)$ *is lower semicontinuous.*
Then, there exists $\bar{x} \in X$ *such that*

$$\begin{array}{llll} (i) & & F(x_0, \bar{x}) + d(\bar{x}, x_0) & \leq 0 \\ (ii)\ x \in X, x \neq \bar{x} & \Longrightarrow & F(\bar{x}, x) + d(x, \bar{x}) & > 0. \end{array}$$

Proof. We check the assumptions of Corollary 21 for the function $f(x) := F(x_0, x)$. (A1) and (A2) are obvious. (A3) follows from the lower semicontinuity of $F(x_0, \cdot)$. By

Corollary 21 we get some point $\bar{x} \in X$ such that $f(\bar{x}) + d(x_0, \bar{x}) = F(x_0, \bar{x}) + d(x_0, \bar{x}) \leq F(x_0, x_0) = 0$ satisfying

$$x \in X, x \neq \bar{x} \Longrightarrow f(\bar{x}) < f(x) + d(x, \bar{x}). \tag{4.14}$$

Assume that $F(\bar{x}, x) + d(x, \bar{x}) \leq 0$ for some $x \in X$, $x \neq \bar{x}$. Then we obtain from (4.14) and (A2, (i))

$$f(\bar{x}) = F(x_0, \bar{x}) < F(x_0, \bar{x}) + F(\bar{x}, x) + d(x, \bar{x}) \leq F(x_0, \bar{x}),$$

a contradiction. Hence $\bar{x}$ satisfies (ii). ■

Another proof is possible using the order relation

$$x \neq x' \quad \text{and} \quad F\left(x', x\right) + d\left(x, x'\right) \leq 0.$$

and applying Theorem 16. Also, one may specialize Corollary 13 to the case $Y = \mathbb{R} \cup \{+\infty\}$, $\Phi = d$.

Note that Corollary 25 is in fact an equivalent reformulation of Corollary 21 in the following sense. Setting $F(x_1, x_2) := f(x_2) - f(x_1)$ we obtain the assertions of Corollary 21 from those of Corollary 25.

4.6 Geometric variational principles in Banach spaces

The "grandfather" (Ekeland 1979) of all Ekeland type theorems is Lemma 1 in the paper [5] by Bishop and Phelps. Its proof already contains some essentials for the proof of Theorem 16 and its corollaries. A more general version in topological linear spaces can be found in [101]. Therefore, we formulate linear - Banach space - variants of the corollaries of the last section.

Throughout this section, $(V, \|\cdot\|)$ is a Banach space with topological dual $(V^*, \|\cdot\|_*)$. The expressions $v^*(v) = (v^*, v)$ denote the value of the linear continuous functional $v^* \in V^*$ at $v \in V$. We start with the Bishop–Phelps lemma. Actually, it is a minimal element theorem, i.e., an existence theorem for minimal points with respect to an order relation generated by a cone of $\mathcal{P}(V)$.

4.6.1 Results in Banach spaces

Corollary 26 *Let the following assumptions be satisfied:*
(A1) $(V, \|\cdot\|)$ *is a Banach space and* $M \subseteq V$ *is a nonempty and closed subset;*
(A2) $B \subseteq V$ *is nonempty, closed, bounded, convex such that* $0 \notin B$*;*
(A3) $K := \mathbb{R}_+ B = \{t \cdot b : \ t \geq 0, \ b \in B\}$*;*
(A4) $v_0 \in M$ *such that* $M \cap (\{v_0\} \oplus K)$ *is bounded.*
Then, there exists $\bar{v} \in V$ *such that*

$$\bar{v} \in M \cap (\{v_0\} \oplus K) \quad \textit{and} \quad \{\bar{v}\} = M \cap (\{\bar{v}\} \oplus K).$$

Proof. Define $V' = M \cap (\{v_0\} \oplus K)$. Then, by (A1), (A2), (A3), $(V', \|\cdot\|)$ is a complete metric space. We introduce an order relation on V' by

$$v' \preceq v \qquad :\Longleftrightarrow \qquad v' \in M \cap (\{v\} \oplus K) .$$

Since the cone K contains $0 \in V$ and is convex by construction, $\preceq$ is reflexive and transitive. The sets $S(x) = \{v' \in V' : \ v' \preceq v\} = V' \cap (\{v\} \oplus K)$ are closed, i.e., assumption (M3) of Theorem 16 is satisfied. To check (M4), take $\{v_n\}_{n \in \mathbb{N}} \subseteq V'$ such that $v_{n+1} \preceq v_n$ for all $n \in \mathbb{N}$. This especially means $v_n - v_{n+1} \in K$, hence there exist $t_n \geq 0$ and $b_n \in B$ such that $v_n - v_{n+1} = t_n b_n$.

Applying a separation theorem to B and $\{0\}$, we can find $v^* \in V^*$, $r \in \mathbb{R}$ such that

$$0 < r \leq \inf \{v^*(b) : \ b \in B\} .$$

Hence

$$r t_n \leq t_n v^*(b_n) = v^*(v_n - v_{n+1})$$

for all $n \in \mathbb{N}$. Adding up these inequalities from $n = 0$ to $n = m - 1$ we obtain

$$v^*(v_m) + r \sum_{n=0}^{m-1} t_n \leq v^*(v_0) .$$

The set of numbers $\{v^*(v_m)\}_{m \in \mathbb{N}}$ is bounded since $v_m \in V'$ and V' is a bounded subset of V. This implies $t_n \to 0$ for $n \to \infty$. Therefore,

$$\|v_n - v_{n+1}\| = \|t_n \cdot b_n\| = t_n \|b_n\|$$

tends to zero for $n \to \infty$ since B is a bounded subset of V. Hence assumption (M4) of Theorem 16 is satisfied. Applying this theorem, we arrive at the conclusions of the corollary. ■

The equivalence of Corollary 26 and Ekeland's principle (Corollary 21, V a Banach space) has been established in [38] and [3]. Especially, it is easy to prove that Corollary 26 implies a Banach space version of Corollary 21 replacing V by $V \times \mathbb{R}$ and setting $M := \mathit{epi}\, f$, $K = \{(x, r) \in V \times \mathbb{R} : \ r + \|x\| \leq 0\}$. The set B can be identified with $\{(v, r) \in V \times \mathbb{R} : \ r + \|v\| = -1\}$.

This observation gave rise to ask if this procedure can be generalized to product spaces $X \times Y$, X being a complete metric space, Y a topological linear (locally convex) space. Results in this direction have been obtained by Göpfert, Tammer and Zălinescu, compare [43], [46], [47] and also the book [44]. Even Theorem 16 implies results of this type involving the set relations $\preccurlyeq$ and $\curlyeqprec$ in $\mathcal{P}(Y)$. Compare the remarks after Corollary 17 and 18, respectively.

The so called drop theorem, established by J. Daneš in 1972 [23], is another important result being an equivalent formulation of Ekeland's principle. Moreover, the drop theorem itself is a reformulation of a renorming theorem due to Zabreiko and Krasnosel'skii from 1971, compare [131]. This observation can also be found in [23].

Let $B \subseteq V$ be closed convex set. The **drop** $D(v, B)$ generated by $v \in V \backslash B$ and B is defined to be the set

$$D(v, B) := co\{\{v\}, B\} = \{tv + (1-t)\,b : \ b \in B, t \in [0,1]\}.$$

We give a proof of the drop theorem using Theorem 16.

Corollary 27 *Let the following assumptions be satisfied:*
(A1) $(V, \|\cdot\|)$ *is a Banach space and* $M \subseteq V$ *is a nonempty and closed subset;*
(A2) $B \subseteq V$ *is a nonempty closed convex and bounded subset of* V*;*
(A3) It holds

$$0 < r := dist\,(B, M) := \inf\{\|b - x\| : \ b \in B, \ x \in M\}.$$

Then, for each $v_0 \in M$ *there exists* $\bar{v} \in V$ *such that*

$$\bar{v} \in M \cap D(v_0, B) \quad \text{and} \quad \{\bar{v}\} = M \cap D(\bar{v}, B).$$

Proof. Define $V' = M \cap D(v_0, B)$. Since B is closed, $D(v_0, B)$ is closed by construction and $(V', \|\cdot\|)$ is a complete metric space. We define a relation $\preceq$ on V' by

$$v' \preceq v \qquad :\Longleftrightarrow \qquad v' \in M \cap D(v, B).$$

Of course, $\preceq$ is reflexive. It is also transitive. To see this, take $v_3 \in D(v_2, B)$, $v_2 \in D(v_1, B)$. Then there are $t_1, t_2 \in [0,1]$ and $b_1, b_2 \in B$ such that $v_3 = t_2 v_2 + (1-t_2)\,b_2$ and $v_2 = t_1 v_1 + (1-t_1)\,b_1$. This gives

$$\begin{aligned} v_3 &= t_2\left[t_1 v_1 + (1-t_1)\,b_1\right] + (1-t_2)\,b_2 \\ &= t_1 t_2 v_1 + (1 - t_1 t_2)\left[\frac{t_2(1-t_1)}{1-t_1 t_2} b_1 + \frac{(1-t_2)}{1-t_1 t_2} b_2\right]. \end{aligned}$$

Since $\frac{t_2(1-t_1)}{1-t_1t_2} + \frac{(1-t_2)}{1-t_1t_2} = 1$, this implies that v_3 is a convex combination of v_1 and an element of b, hence $v_3 \in D(v_1, B)$ which proves the transitivity of $\preceq$.

Next, we show the regularity of $\preceq$. Let $\{v_n\}_{n\in\mathbb{N}} \subseteq V'$ be a decreasing sequence, i.e.,

$$\forall n \in \mathbb{N} : \ v_{n+1} \in M \cap D(v_n, B).$$

This means, for all $n \in \mathbb{N}$ there are $t_n \in [0,1]$, $b_n \in B$ such that

$$v_{n+1} = t_n v_n + (1-t_n)\,b_n. \tag{4.15}$$

Then, for all $b \in B$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} \|v_{n+1} - v_n\| = (1-t_n)\,\|b_n - v_n\| &\leq (1-t_n)\left(\|b - v_n\| + \|b_n - b\|\right) \\ &\leq (1-t_n)\left(\|b - v_n\| + diam\,B\right). \end{aligned}$$

Define $d_B(x) := \inf_{b\in B}\|b-v\|$. Then, d_B is convex since B is a convex set. From the latter inequality chain, we obtain

$$\|v_{n+1}-v_n\| \le (1-t_n)\left(d_B(v_n)+diam\,B\right). \tag{4.16}$$

On the other hand, the convexity of d_B implies

$$0 < d_B(v_{n+1}) = d_B(t_n v_n + (1-t_n)b_n) \le t_n d_B(v_n).$$

Therefore, $t_n \ge \frac{d_B(v_{n+1})}{d_B(v_n)}$. Invoking (4.16) we obtain

$$\begin{aligned}
\|v_{n+1}-v_n\| &\le \left(1-\frac{d_B(v_{n+1})}{d_B(v_n)}\right)\left(d_B(v_n)+diam\,B\right)\\
&= \left(d_B(v_n)-d_B(v_{n+1})\right)+\frac{d_B(v_n)-d_B(v_{n+1})}{d_B(v_n)}\,diam\,B\\
&= \left(d_B(v_n)-d_B(v_{n+1})\right)\left(1+\frac{diam\,B}{d_B(v_n)}\right).
\end{aligned}$$

Since $0<r\le d_B(v)$ for all $v\in M$ by assumption, this implies

$$\|v_{n+1}-v_n\| \le \left(d_B(v_n)-d_B(v_{n+1})\right)\left[1+\frac{diam\,B}{R}\right].$$

Setting $\alpha^{-1} := 1+\frac{diam\,B}{R}$ and adding the above inequalities from $n=0$ to $m-1$, we obtain for $m=1,2,\ldots$

$$d_B(v_m)+\alpha\sum_{n=0}^{m-1}\|v_{n+1}-v_n\| \le d_B(v_0)$$

implying $\lim_{n\to\infty}\|v_{n+1}-v_n\|=0$ which proves the regularity of $\preceq$.

The conclusions of the corollary are obtained by applying Theorem 16. ■

Corollary 27 that is an equivalent formulation of the original drop theorem of [23] has been proven by J. Daneš in [24]. Besides the more or less straightforward application of Theorem 16, the proof above contains the essentials of Lemma 1 and Lemma GKZ of [24].

4.6.2 Results in locally complete locally convex spaces

Several attempts have been made to give a formulation of the drop theorem on locally convex spaces. The first one seems to be [86], in which Mizoguchi proved variational principles on complete uniform spaces among them the drop theorem in locally convex spaces. In [15] and [53] proofs for the drop theorem in sequentially complete, locally convex spaces are given which seem to be not complete. Results of Qiu [103], [104] show that it is more appropriated to assume local completeness rather than sequential completeness. Using this concept, we present the drop theorem as well as Phelps' lemma in locally complete, locally convex spaces using an idea of [53] in order to apply the Banach space versions of the corresponding theorems as the essential tool for the proof.

The following definitions as well as many results on local completeness can be found in [100]. Let $(V, +, \mathcal{T})$ be a separated, locally convex space. A bounded and absolutely convex subset $D \subseteq V$ is called a **disc**. We denote by p_D the Minkowski gauge of D, i.e.,

$$p_D(v) := \inf\{t > 0 : \; v \in tD\}, \quad v \in V.$$

The linear subspace of V spanned by D is denoted by V_D. (V_D, p_D) is a normed space. A disc $D \subseteq V$ is called a **Banach disc** if (V_D, p_D) is a Banach space. A sequence $\{v_n\}_{n \in \mathbb{N}}$ is called a **locally Cauchy** (**locally convergent**) sequence iff it is Cauchy (convergent) in (V_D, p_D) for some disc D.

The space $(V, +, \mathcal{T})$ is called **locally complete** iff every locally Cauchy sequence converges locally in V. It is well–known that every sequentially complete separated locally convex space is locally complete, but the converse is not true in general, compare [100], Corollary 5.1.8 and Example 5.1.12. The crucial result for the following proofs is the fact that a separated locally convex space is locally complete if and only if every bounded set is contained in a Banach disc, see [100], Proposition 5.1.6.

A subset $M \subseteq V$ is called **locally closed** iff for a $\{v_n\}_{n \in \mathbb{N}} \subseteq M$ converging locally to $v \in V$ we have $v \in M$.

Corollary 28 *Let the following assumptions be satisfied:*
(A1) $(V, +, \mathcal{T})$ is a locally complete, separated locally convex space and $M \subseteq V$ is a nonempty and locally closed subset;
(A2) $B \subseteq V$ is nonempty, locally closed, bounded, convex such that $0 \notin B$;
(A3) $K := \mathbb{R}_+ B = \{t \cdot b : \; t \geq 0, \; b \in B\}$;
(A4) $v_0 \in M$ such that $M \cap (\{v_0\} \oplus K)$ is bounded.
Then, there exists $\bar{v} \in V$ such that

$$\bar{v} \in M \cap (\{v_0\} \oplus K) \quad \text{and} \quad \{\bar{v}\} = M \cap (\{\bar{v}\} \oplus K).$$

Proof. Define the set

$$B_0 := B \cup \{v_0\}.$$

Since B is bounded, so is B_0. Since V is locally complete, there is a Banach disc $D \subseteq V$ such that $B_0 \subseteq D$ and hence $v_0 \in V_D$, $K \subseteq V_D$, $M \cap (\{v_0\} \oplus K) \subseteq V_D$.

In order to apply Corollary 26 in (V_D, p_D) we have to check its assumptions. First, we check (A1): The set $M \cap V_D$ is closed in (V_D, p_D). To see this, take a sequence $\{v_n\}_{n \in \mathbb{N}} \subseteq M \cap V_D$ converging to v with respect to p_D. Especially, this means $\{v_n\}$ converges locally to $v \in V$. Then $v \in V_D$ since V_D is Banach. Since M is locally closed, $v \in M$ is also true, hence $v \in M \cap V_D$.

Assumption (A2) of Corollary 26 is satisfied since obviously B as a subset of V_D has the desired properties. (A3) is clear by construction.

It remains to check (A4): Denote $M' := M \cap (\{v_0\} \oplus K) \subseteq V_D$. We shall show $M' \subseteq \{v_0\} \oplus sB$ for some fixed $s > 0$. This implies the boundedness of M' in (V_D, p_D) since $B \subseteq D$.

Assume the contrary, i.e., there are sequences $\{v_n\}_{n\in\mathbb{N}} \subseteq M'$, $\{t_n\}_{n\in\mathbb{N}} \in \mathbb{R}_+ \setminus \{0\}$, $\{b_n\}_{n\in\mathbb{N}} \subseteq B$ such that $\lim_{n\to\infty} t_n = +\infty$ and $v_n = v_0 + t_n b_n$. This implies

$$\forall n \in \mathbb{N}: \ \frac{v_n}{t_n} - \frac{v_0}{t_n} = b_n \in B.$$

Letting $n \to \infty$ we obtain $0 \in B$, a contradiction. This follows since M' is bounded in V, hence $\frac{v_n}{t_n} \to 0$ in V as well as $\frac{v_0}{t_n} \to 0$ in V if $n \to \infty$.

Applying Corollary 26 to M' in (V_D, p_D) we get a point $\bar{v} \in M \cap (\{v_0\} \oplus K)$ such that $\{\bar{v}\} = M \cap (\{\bar{v}\} \oplus K)$. Since $M \cap (\{\bar{v}\} \oplus K) \subseteq V_D$, this implies the desired result. ■

The same idea is used to establish the drop theorem in locally convex spaces. The definition of a drop as given above applies also in this case.

Corollary 29 *Let the following assumptions be satisfied:*
(A1) $(V, +, \mathcal{T})$ is a locally complete, separated locally convex space and $M \subseteq V$ is a nonempty and locally closed subset;
(A2) $B \subseteq V$ is nonempty, locally closed, bounded, convex;
(A3) If $\mathcal{N}(\theta)$ be a neighborhood base of $\theta \in V$ for $\mathcal{T}$, then there is $N \in \mathcal{N}(\theta)$ such that $M \cap (B \oplus N) = \emptyset$.
Then, there exists $\bar{v} \in V$ such that

$$\bar{v} \in M \cap D(v_0, B) \quad \textit{and} \quad \{\bar{v}\} = M \cap D(\bar{v}, B).$$

Proof. Define the set

$$B_0 := B \cup \{v_0\}.$$

Since B is bounded, so is B_0. Since V is locally complete, there is a Banach disc $D \subseteq V$ such that $B_0 \subseteq D$ and hence $v_0 \in V_D$. Define the set $M_D = M \cap V_D$ in order to apply Corollary 27 in V_D. The assumptions (A1) and (A2) of Corollary 27 can be verified with similar arguments as used in the proof of Corollary 28. It remains to check (A3) for B, M_D. We must have $p_D(b - v) \ge r > 0$ for all $b \in B$, $v \in M_D$. This is equivalent to $b - v \notin rD$ for all $b \in B$, $v \in M_D$. Assume the contrary, i.e., there are sequences $\{v_n\}_{n\in\mathbb{N}} \subseteq M_D$, $\{t_n\}_{n\in\mathbb{N}} \in \mathbb{R}_+$, $\{b_n\}_{n\in\mathbb{N}} \subseteq B$ such that $\lim_{n\to\infty} t_n = 0$ and

$$\forall n \in \mathbb{N}: \ b_n - v_n \in t_n D.$$

Take $N \in \mathcal{N}(\theta)$. Since D is bounded in V, there is $t > 0$ such that $tD \subseteq N$. Hence there is $n_N \in \mathbb{N}$ such that $b_n - v_n \in N$ for all $n \ge n_N$. Since N is arbitrary in $\mathcal{N}(\theta)$, this contradicts (A3) of the present corollary.

We may apply Corollary 27 to obtain the desired result. ■

Although the space V is not a Banach space (or a complete metric) in the last two corollaries, their proofs rely essentially on the Banach space versions presented before. Therefore, the latter results are included in this chapter.

4.7 Minimal elements on product spaces

As mentioned before, Bishop and Phelps [5], [102] as well as Ekeland [30] already observed that the so called variational principle is nothing else than a minimal element theorem for orders on $X \times \mathrm{I\!R}$ defined by

$$\left(x', r'\right) \preceq (x, r) \qquad \Longleftrightarrow \qquad r' + d\left(x', x\right) \leq r$$

for $(x', r'), (x, r) \in X \times \mathrm{I\!R}$. Applied to epigraphs of functions $f : X \to \mathrm{I\!R}$, this relation generates an order on X simply by setting

$$x' \preceq_X x \qquad \Longleftrightarrow \qquad f\left(x'\right) + d\left(x', x\right) \leq f(x).$$

The question arises if it is possible to obtain minimal element theorems on product spaces $X \times V$ where the set in question can not necessarily be interpretated as the epigraph of a function $f : X \to V$. Göpfert, Tammer and Zălinescu established results in this direction in a series of papers [43], [45], [46], [47]. Compare also the book [44], Section 3.10. A subsequent paper is [55] and more general results can be found in [50], [51].

In this section, we state a minimal element theorem on a product space $X \times V$ where X is a complete metric space and V is a Banach space. The method of proof is again an application of Theorem 16. For a more general setting, compare the following chapters.

Corollary 30 *Let the following assumptions be satisfied:*
(A1) (X, d) is a complete metric space and $(V, \|\cdot\|)$ a Banach space;
(A2) $K \subseteq V$ is a closed convex set and a cone in $\mathcal{P}(V)$ generating the quasiorder $\leq_K$ on V and $k^0 \in K\backslash -K$;
(A3) The nonempty closed set $A \subseteq X \times V$ and the topological bounded set $M \subseteq V$ are such that

$$(x, v) \in A \quad \Longrightarrow \quad v \in M \oplus K;$$

(A4) If $\{(x_n, v_n)\}_{n\in\mathrm{I\!N}} \subseteq A$ satisfies

$$\forall n \in \mathrm{I\!N}: \; v_{n+1} + k^0 d\left(x_{n+1}, x_n\right) \leq_K v_n, \tag{4.17}$$

and $\{x_n\}_{n\in\mathrm{I\!N}} \subseteq X$ is convergent, then $\{v_n\}_{n\in\mathrm{I\!N}} \subseteq V$ is asymptotic.
Then, for each $(v_0, v_0) \in A$, there exists $(\bar{x}, \bar{v}) \in A$ such that

$$\begin{array}{lll} (i) & \bar{v} + k^0 d\left(\bar{x}, v_0\right) \leq_K v_0 & \\ (ii) & (x, v) \in A, \quad v + k^0 d\left(x, \bar{x}\right) \leq_K \bar{v} & \Longrightarrow \quad (x, v) = (\bar{x}, \bar{v}). \end{array}$$

Proof. Define a relation on the complete metric space $A \subseteq X \times V$ by

$$\left(x', v'\right) \preceq (x, v) \qquad \Longleftrightarrow \qquad v' + d\left(x', x\right) k^0 \leq_K v.$$

Invoking the properties of d, we can see that $\preceq$ is reflexive and transitive. To prove regularity, take a $\preceq$–decreasing sequence $\{(x_n, v_n)\}_{n\in\mathrm{I\!N}} \subseteq A$, i.e., it satisfies (4.17). Take

$v^* \in K^+ := \{v^* \in V^* : \forall v \in K : v^*(v) \geq 0\}$ such that $v^*\left(k^0\right) = 1$. Such a v^* does exist according to classical separation arguments (see [133], Theorem 1.1.5). We obtain

$$\forall n \in \mathbb{N} : v^*(v_{n+1}) + d(x_{n+1}, x_n) \leq v^*(v_n).$$

From (A3) we obtain that v^* is bounded below on $\{v \in V : \exists x \in X : (x, v) \in M\}$. Therefore, the above relation implies that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy, hence convergent to some $\hat{x} \in X$. Assumption (A4) ensures that $\{v_n\}_{n\in\mathbb{N}}$ is asymptotic. Therefore, assumption (M3) of Theorem 16 is satisfied for $\preceq$.

To check the lower closedness, i.e., assumption (M3) of Theorem 16, take a $\preceq$-decreasing sequence $\{(x_n, v_n)\}_{n\in\mathbb{N}}$ contained in $S(v_0, v_0)$ and converging to some $(x, v) \in X \times V$. The triangle inequality for d, (4.17) and the definition of $\leq_K$ imply

$$\forall n \in \mathbb{N}, \forall m \geq n : v_n - v_m - k^0\varphi(x_m, x_n) \in K.$$

Since K is closed, we obtain via $m \to \infty$

$$\forall n \in \mathbb{N} : v_n - v - k^0 d(x, x_n) \in K.$$

The transitivity of $\preceq$ implies $(x, v) \in S(v_0, v_0)$ as desired.

We may apply Theorem 16 to obtain the result of the corollary. ■

Remark 23 *1. Corollary 30 is an example of an minimal element theorem on a product space being itself a complete metric space. More general theorems of this type, denoted as "authentic minimal point theorems", have been proven in [47] and [44] using different techniques. Compare also Section 7.1.*

2. The assumptions used in Corollary 30 are strongly related to the assumptions used in the cited references. For example, the conclusions of the corollary remain true if the set M is replaced by $\{\tilde{v}\}$, $\tilde{v} \in V$ and assumption (A4) by (SP4) of [44], p. 97, namely

(A4′) Every $\leq_K$-decreasing sequence being bounded from below is asymptotic.

3. It is possible to weaken the assumptions even in the setting of uniform spaces. This requires different methods for the proofs, namely a scalarization technique. See Section 7.1..

Using different order relations, further results can be obtained immediately. One example is the following corollary, a generalization of Theorem 8 of [61].

Corollary 31 *Let the following assumptions be satisfied:*
(A1) (X, d) is a complete metric space and $(V, \|\cdot\|)$ a Banach space;
(A2) $K \subseteq V$ is a closed convex set and a cone in $\mathcal{P}(V)$ generating the quasiorder $\leq_K$ on V and $k^0 \in K\setminus -K$;
(A3) The nonempty closed set $A \subseteq X \times V$ and the topological bounded set $M \subseteq V$ are such that

$$(x, v) \in A \quad \Longrightarrow \quad v \in M \oplus K;$$

Then, for each $(v_0, v_0) \in A$, there exists $(\bar{x}, \bar{v}) \in A$ such that

$$\begin{array}{llll} (i) & & \bar{v} + k^0 \left(d\left(\bar{x}, x_0\right) + \|\bar{v} - v_0\| \right) \leq_K v_0 & \\ (ii) & (x,v) \in M, & v + k^0 \left(d\left(x, \bar{x}\right) + \|v + \bar{v}\| \right) \leq_K \bar{v} & \Longrightarrow \quad (x,v) = (\bar{x}, \bar{v}) . \end{array}$$

PROOF. The assumptions of Theorem 16 are checked for the relation

$$\left(x', v'\right) \preceq (x, v) \qquad \Longleftrightarrow \qquad v' + k^0 \left(d\left(x', x\right) + \left\|v' - v\right\| \right) \leq_K v.$$

defined on the complete metric space $(A, d(\cdot, \cdot) + \|\cdot - \cdot\|)$. Reflexivity is obvious and transitivity can be checked straightforward.

To show regularity, take a $\preceq$–decreasing sequence $\{(x_n, v_n)\}_{n \in \mathbb{N}}$, i.e.,

$$\forall n \in \mathbb{N}: \; v_{n+1} + k^0 \left[d\left(x_{n+1}, x_n\right) + \|v_{n+1} - v_n\| \right] \leq_K v_n.$$

The transitivity of $\preceq$ yields

$$\forall m \in \mathbb{N}: \; v_0 - v_m - k^0 \sum_{n=0}^{m-1} \left(d\left(x_{n+1}, x_n\right) + \|v_{n+1} - v_n\| \right) \in K.$$

Assume that $\alpha_m := \sum_{n=0}^{m-1} \left(d\left(x_{n+1}, x_n\right) + \|v_{n+1} - v_n\| \right) \to +\infty$. Then, for sufficiently large m,

$$\frac{1}{\alpha_m} v_0 - k^0 \in \frac{1}{\alpha_m} v_m \oplus K \subseteq \frac{1}{\alpha_m} M \oplus K.$$

Letting $m \to \infty$, the contradiction $-k^0 \in K$ is obtained since M is bounded.

Finally, the lower closedness of $\preceq$ follows from the closedness of K and A.

We may apply Theorem 16 to get the desired result. ■

The preceding corollary generalizes Theorem 8 of [61] in different directions. At first, X is assumed to be a complete metric space rather than a Banach space. At second, the set M does not consist necessarily of a single element and K is not assumed to be pointed. However, one may check that, under our assumptions, if $(X, \|\cdot\|)$ is assumed to be a Banach space too, the set

$$\left\{ (x, v) \in X \times V : \; v + k^0 \left(\|x\| + \|v\| \right) \in -K \right\}$$

is a closed convex pointed cone, actually generating the relation $\preceq$ in this case.

Under the assumptions of Corollary 30, it is also possible to use order relations involving functionals $v^* \in K^+$ explicitely. Compare Theorem 3.10.7. of [44] and Theorem 4 of [47]. We establish similar results in Chapter 7 on uniform spaces.

Chapter 5

Partial Minimal Element Theorems on Metric Spaces

5.1 The basic theorem on metric spaces

Let (X, d) be a complete metric space and Y a nonempty set. The goal is to extend Theorem 16 to order relations on $X \times Y$. Let us note that the assumptions to X and Y, respectively, are completely different. This is the main feature of the results of this chapter in contrast to Corollary 30 and 31. The difference to Theorem 21 and 22 is the order relation: Here we use a relation on $X \times Y$ directly, not one on $X \times \widehat{\mathcal{P}}(Y)$.

Theorem 23 *Let the following assumptions be satisfied:*
(MP1) (X, d) *is a complete metric space,* Y *a nonempty set and* $M \subseteq X \times Y$ *a nonempty set;*
(MP2) $\preceq$ *is a reflexive and transitive relation on* $X \times Y$*.*
(MP3) If $\{(x_n, y_n)\}_{n \in \mathbb{N}} \subseteq M$ *is a decreasing sequence with respect to* $\preceq$*, i.e.*

$$\forall n \in \mathbb{N}: \ (x_{n+1}, y_{n+1}) \preceq (x_n, y_n),$$

then $\lim_{n \to \infty} d(x_{n+1}, x_n) = 0$*.*
(MP4) If $\{(x_n, y_n)\}_{n \in \mathbb{N}} \subseteq M$ *is a decreasing sequence and* $\{x_n\}_{n \in \mathbb{N}}$ *converges to* $x \in X$*, then there exists* $y \in Y$ *such that* $(x, y) \in M$ *and*

$$\forall n \in \mathbb{N}: \ (x, y) \preceq (x_n, y_n).$$

Then, for each $(x_0, y_0) \in M$*, there exists* $(\bar{x}, \bar{y}) \in M$ *such that*

$$\begin{array}{llll} (i) & & (\bar{x}, \bar{y}) \preceq (x_0, y_0) & \\ (ii) & (x, y) \in M, & (x, y) \preceq (\bar{x}, \bar{y}) & \implies \quad x = \bar{x}. \end{array}$$

Proof. We may assume that d is bounded. Otherwise, it can be replaced by $d' = \frac{d}{1+d}$ observing that (MP3), (MP4) remain in force for d'. For $(x, y) \in X \times Y$, we set

$$S_X(x, y) := \left\{x' \in X : \ \exists y' \in Y : \ \left(x', y'\right) \in M, \ \left(x', y'\right) \preceq (x, y)\right\}.$$

Note that always $x \in S_X(x,y)$ whenever $(x,y) \in M$. Starting with $(x_0,y_0) \in M$, a sequence $\{(x_n,y_n)\}_{n\in\mathbb{N}} \subseteq M$ can be defined as follows: Choose $x_{n+1} \in S_X(x_n,y_n)$ such that

$$d(x_{n+1},x_n) \geq \sup_{x\in S_X(x_n,y_n)} d(x,x_n) - \frac{1}{n}$$

and $y_{n+1} \in Y$ such that $(x_{n+1},y_{n+1}) \preceq (x_n,y_n)$ as well as $(x_{n+1},y_{n+1}) \in M$.

The transitivity of $\preceq$ and the definition of $S_X(x_n,y_n)$ imply $x_m \in S_X(x_n,y_n)$ whenever $m \geq n$. Moreover,

$$d(x_m,x_n) \leq \sup_{x\in S_X(x_n,y_n)} d(x,x_n) \leq d(x_{n+1},x_n) + \frac{1}{n}.$$

Therefore, by (MP3), $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy and hence convergent to $\bar{x} \in X$. By (MP4), there is $\bar{y} \in Y$ such that $(\bar{x},\bar{y}) \in M$ and

$$\forall n \in \mathbb{N}: \ (\bar{x},\bar{y}) \preceq (x_n,y_n),$$

especially $(\bar{x},\bar{y}) \preceq (x_0,y_0)$.

Assume, there is $(x',y') \in M$ such that $(x',y') \preceq (\bar{x},\bar{y})$. Then by transitivity, $(x',y') \preceq (x_n,y_n)$ for all $n \in \mathbb{N}$ implying $x' \in S_X(x_n,y_n)$. This gives

$$d\left(x',x_n\right) \leq \sup_{x\in S_X(x_n,y_n)} d(x,x_n) \leq d(x_{n+1},x_n) + \frac{1}{n}.$$

Since the right hand side of the last expression tends to 0, we may conclude $x_n \to x'$. Since the limit in complete metric spaces is unique, we obtain $x' = \bar{x}$. This proves the theorem. ■

Theorem 23 for a separated locally convex space Y and special order relations $\preceq$ is called "non authentic minimal point theorem" in [47] and [44, Section 3.10] since conclusion (ii) of the theorem only involves the x-variable and not a true minimal element of M with respect to $\preceq$.

Theorem 16 happens to be a special case of Theorem 23. To see this, take $M = X \times \{y_s\}$ where $y_s \in Y$ is a fixed single element. In this case, the quasiorder $\preceq$ on $X \times Y$ generates a quasiorder $\preceq_X$ on X by

$$x' \preceq_X x \qquad \Longleftrightarrow \qquad \left(x',y_s\right) \preceq_X (x,y_s).$$

Therefore, (MP3) for $\preceq$ is (M3) of Theorem 16 for $\preceq_X$. The sets $S_X(x,y_s)$ coincide with the section of $\preceq_X$ at x. This means, (MP4) passes into (M4). Of course, the conclusions of Theorem 23 specialize to those of Theorem 16.

Conversely, it seems to be not possible to derive Theorem 23 directly from Theorem 16 without an additional induction argument with respect to the y -variable like in the above proof of Theorem 23.

Roughly speaking, assumption (MP4) can be understood as partial lower closedness of the sections of $\preceq$. It is a generalization of condition (H1) of [47] and [44, Section 3.10], playing in locally convex spaces Y. In [47], [44], also sufficient condition for (H1) are given.

Remark 24 *Assumption (MP3) of Theorem 23 ensures partial antisymmetry of the relation $\preceq$ in the sense of Definition 13. To see this, follow the arguments of Remark 40: If $(x, y) \preceq (x', y')$ as well as $(x', y') \preceq (x, y)$, define a sequence by*

$$\left\{(x, y), \left(x', y'\right), (x, y), \left(x', y'\right), \ldots\right\}$$

being decreasing with respect to $\preceq$. (MP3) implies $\mathrm{d}(x, x') = 0$.

In the following sections we provide Y with more algebraic and topological structure from step to step in order to obtain new as well as several known existence results for partial minimal elements of product sets.

Finally, we remark that (X, d) has not to be complete. This assumption can be slightly relaxed similar to the last chapter. However, the completeness is assumed in this chapter to make the presentation more clearly.

5.2 Results involving ordered monoids

First, we assume $(Y, \circ, \leq)$ to be a quasiordered monoid.

Corollary 32 *Let the following assumptions be satisfied:*
(A1) (X, d) is a complete metric space and $(Y, \circ, \leq)$ a quasiordered monoid;
(A2) The nonempty set $M \subseteq X \times Y$, $(x_0, y_0) \in M$ and $y \in Y$ are such that

$$(x, y) \in M \quad \Longrightarrow \quad y \leq y;$$

(A3) $\Phi : X \times X \to Y$ is a regular order premetric with respect to $y, y_0 \in Y$;
(A4) If $\{(x_n, y_n)\}_{n \in \mathbb{N}} \subseteq M$ is a sequence such that

$$\forall n \in \mathbb{N}: \ y_{n+1} \circ \Phi\left(x_{n+1}, x_n\right) \leq y_n$$

and $\{x_n\}_{n \in \mathbb{N}}$ converges to $x \in X$, then there exists $y \in Y$ such that $(x, y) \in M$ and

$$\forall n \in \mathbb{N}: \ y \circ \Phi\left(x, x_n\right) \leq y_n.$$

Then, there exists $(\bar{x}, \bar{y}) \in M$ such that

$$\begin{array}{lll} (i) & \bar{y} \circ \Phi\left(\bar{x}, x_0\right) \leq y_0 & \\ (ii) \quad (x, y) \in M, & y \circ \Phi\left(x, \bar{x}\right) \leq \bar{y} & \Longrightarrow \quad x = \bar{x}. \end{array}$$

Proof. Define a relation relation

$$\left(x', y'\right) \preceq (x, y) \qquad :\Longleftrightarrow \qquad y' \circ \Phi\left(x', x\right) \leq y$$

on $X \times Y$ and check the assumptions of Theorem 23. (MP2) is obvious. (A4) implies (MP4) directly. To check (MP3), take a decreasing sequence $\{(x_n, y_n)\}_{n \in \mathbb{N}} \subseteq M$, i.e.

$$y_{n+1} \circ \Phi\left(x_{n+1}, x_n\right) \leq y_n$$

for all $n \in \mathbb{N}$. This implies (compare the proof of Theorem 9)

$$\tilde{y} \circ \sum_{k=0}^{n} \Phi\left(x_{k+1}, x_k\right) \leq y_0.$$

The regularity of Φ ensures (MP4). The conclusions of Theorem 23 yield (i), (ii). ■

Remark 25 *Corollary 9 is a special case of Corollary 32. To see this, take*

$$M = \{(x, f(x)) \in X \times Y : \, x \in X\}.$$

One can easily check the assumptions of Corollary 32. Its conclusions passes into those of Corollary 9. Rougly speaking, this shows that a partial minimal element theorem suffices to derive a corresponding Ekeland type theorem. For special cases, this connection has been observed in [47].

5.3 Results involving power sets of ordered monoids

The set valued situation deserves special attention. Again, we consider the two order relations introduced in Section 2.2.1.

Corollary 33 *Let the following assumptions be satisfied:*
(A1) (X, d) *is a complete metric space,* $(Y, \circ, \leq)$ *an ordered monoid and* $(\mathcal{Y}, \odot, \preccurlyeq)$ *the quasiordered monoid generated by* $\mathcal{Y} := \mathcal{P}(Y)$*;*
(A2) The nonempty set $\mathcal{M} \subseteq X \times \mathcal{Y}$, $(x_0, M_0) \in \mathcal{M}$ *and* $W \in \mathcal{Y}$ *are such that*

$$(x, M) \in \mathcal{M} \quad \Longrightarrow \quad W \preccurlyeq M;$$

(A3) $\Phi : X \times X \to \mathcal{Y}$ *is a regular order premetric with respect to* $W, M_0 \in \mathcal{Y}$*;*
(A4) If $\{(x_n, M_n)\}_{n \in \mathbb{N}} \subseteq \mathcal{M}$ *is a sequence satisfying*

$$\forall n \in \mathbb{N} : \; M_{n+1} \odot \Phi\left(x_{n+1}, x_n\right) \preccurlyeq M_n$$

such that $\{x_n\}_{n \in \mathbb{N}}$ *converges to* $x \in X$*, then there exists* $M \in \mathcal{Y}$ *such that* $(x, M) \in \mathcal{M}$ *and*

$$\forall n \in \mathbb{N} : \; M \odot \Phi\left(x, x_n\right) \preccurlyeq M_n.$$

Then, there exists $\left(\bar{x}, \overline{M}\right) \in \mathcal{M}$ *such that*

$$\begin{array}{lll} (i) & \overline{M} \odot \Phi\left(\bar{x}, x_0\right) \preccurlyeq M_0 & \\ (ii) \quad (x, M) \in \mathcal{M}, & M \odot \Phi\left(x, \bar{x}\right) \preccurlyeq \overline{M} & \Longrightarrow \quad x = \bar{x}. \end{array}$$

Proof. Define a reflexive and transitive relation on $X \times \mathcal{Y}$ by

$$\left(x', M'\right) \preceq (x, M) \qquad :\Longleftrightarrow \qquad M' \odot \Phi\left(x', x\right) \preccurlyeq M.$$

(MP1) and (MP2) of Theorem 23 are obviously satisfied. (MP3) follows from (A4). The regularity of Φ and (A2) ensure (MP4). The result follows from the conclusions of Theorem 23. ■

Of course, Corollary 33 is a special case of Corollary 32. The counterpart for the relation $\preccurlyeq$ reads as follows.

Corollary 34 *Let the assumptions (A1) – (A4) of Corollary 33 be satisfied with $\preccurlyeq$ replaced by $\preccurlyeq$. Then, there exists $(\bar{x}, \bar{M}) \in \mathcal{M}$ such that*

$$\begin{array}{lll} (i) & & \overline{M} \odot \Phi(\bar{x}, x_0) \preccurlyeq M_0 \\ (ii) & (x, M) \in \mathcal{M}, & M \odot \Phi(x, \bar{x}) \preccurlyeq \overline{M} \quad \Longrightarrow \quad x = \bar{x}. \end{array}$$

Proof. Define a reflexive and transitive relation on $X \times \mathcal{Y}$ by

$$(x', M') \preceq (x, M) \qquad :\Longleftrightarrow \qquad M' \odot \Phi(x', x) \preccurlyeq M.$$

(MP1) and (MP2) of Theorem 23 are obviously satisfied. (MP4) follows from (A4). The regularity of Φ and (A2) ensure (MP3). The result follows from the conclusions of Theorem 23. ■

5.4 Results involving linear spaces

This section contains special cases of the results of the last one whereas Y is replaced by a topological linear space V and the power set of such a space, respectivly. The following two corollaries deal with $\widehat{\mathcal{P}}(V)$ involving the relations $\preccurlyeq$ and $\preccurlyeq$, respectively.

Corollary 35 *Let the following assumptions be satisfied:*
(A1) (X, d) is a complete metric space and $(V, +)$ a separated topological linear space;
(A2) $K \subseteq V$ is a convex set containing $\theta \in X$ and a cone in $\left(\mathcal{V} := \widehat{\mathcal{P}}(V), \oplus\right)$, $K^0 \subseteq K \setminus (-\operatorname{cl} K)$ is a nonempty convex and sequentially compact set;
(A3) The nonempty set $\mathcal{M} \subseteq X \times \mathcal{V}$ and the topological bounded set $W \in \mathcal{V}$ are such that

$$(x, M) \in \mathcal{M} \quad \Longrightarrow \quad W \preccurlyeq M;$$

(A4) $\varphi : X \times X \to \mathbb{R}_+$ is a regular premetric;
(A5) If $\{(x_n, M_n)\}_{n \in \mathbb{N}} \subseteq \mathcal{M}$ is a sequence satisfying

$$\forall n \in \mathbb{N}: \; M_{n+1} \oplus \varphi(x_{n+1}, x_n) K^0 \preccurlyeq M_n$$

such that $\{x_n\}_{n \in \mathbb{N}}$ converges to $x \in X$, then there exists $M \in \mathcal{V}$ such that $(x, M) \in \mathcal{M}$ and

$$\forall n \in \mathbb{N}: \; M \oplus \varphi(x, x_n) K^0 \preccurlyeq M_n.$$

Then, for each $(x_0, M_0) \in \mathcal{M}$ with $M_0 \neq \emptyset$, there exists $(\bar{x}, \overline{M}) \in \mathcal{M}$ such that

$$\begin{array}{lll} (i) & & \overline{M} \oplus \varphi(\bar{x}, x_0) K^0 \preccurlyeq M_0 \\ (ii) & (x, M) \in \mathcal{M}, & M \oplus \varphi(x, \bar{x}) K^0 \preccurlyeq \overline{M} \quad \Longrightarrow \quad x = \bar{x}. \end{array}$$

Proof. Defining the relation

$$\left(x', M'\right) \preceq (x, M) \qquad :\Longleftrightarrow \qquad M' \oplus \varphi\left(x', x\right) K^0 \preccurlyeq M,$$

on can easily see from the properties of $\preccurlyeq$ and φ that $\preceq$ is reflexive and transitive on $X \times \mathcal{V}$. Clearly, (A5) is (MP4) for this order relation. We are going to check (MP3). If for all $n \in \mathbb{N}$

$$M_{n+1} \oplus \varphi\left(x_{n+1}, x_n\right) K^0 \preccurlyeq M_n,$$

we can add $\varphi\left(x_n, x_{n-1}\right) K^0$ to both sides of this equality obtaining

$$M_{n+1} \oplus \varphi\left(x_{n+1}, x_n\right) K^0 \oplus \varphi\left(x_n, x_{n-1}\right) K^0 \preccurlyeq M_n \oplus \varphi\left(x_n, x_{n-1}\right) K^0 \preccurlyeq M_{n-1}.$$

Since K^0 is a convex subset of the linear space V, it is a convex element of $\mathcal{V}$, therefore $sK^0 \oplus tK^0 = (s+t) K^0$ whenever $s, t \geq 0$. This fact and the transitivity of $\preccurlyeq$ imply

$$M_{n+1} \oplus \left(\varphi\left(x_{n+1}, x_n\right) + \varphi\left(x_n, x_{n-1}\right)\right) K^0 \preccurlyeq M_{n-1}.$$

Continuing this process if necessary we arrive at

$$M_{n+1} \oplus \left(\sum_{k=0}^{n-1} \varphi\left(x_{k+1}, x_k\right)\right) K^0 \preccurlyeq M_0$$

for all $n \in \mathbb{N}$. Assumption (A2) implies

$$\forall n \in \mathbb{N}: \; W \oplus \left(\sum_{k=0}^{n-1} \varphi\left(x_{k+1}, x_k\right)\right) K^0 \preccurlyeq M_0.$$

Define $\alpha_n := \sum_{k=0}^{n-1} \varphi\left(x_{k+1}, x_k\right)$. Take $m_0 \in M_0$ being nonempty by assumption. The definition of $\preccurlyeq$ implies

$$\forall n \in \mathbb{N}: \; \exists w_n \in W, \; k_n^0 \in K^0: \; m_0 - \left(w_n + k_n^0 \alpha_n\right) \in K.$$

Assume $\alpha_n \to +\infty$. Then

$$\frac{1}{\alpha_n} - \frac{w_n}{\alpha_n} - k_n^0 \in K$$

for all $n \in \mathbb{N}$ sufficiently large. Since K^0 is sequentially compact, there is a subsequence of $\left\{k_n^0\right\}_{n \in \mathbb{N}}$ converging to some $k^0 \in K^0$. Since W is bounded, $\frac{1}{\alpha_n} w_n \to 0 \in V$ as well as $\frac{1}{\alpha_n} m_0 \to 0 \in V$. Hence $-k^0 \in \operatorname{cl} K$ contradicting assumption (A2). Therefore, the α_n's remain bounded and $\varphi\left(x_{n+1}, x_n\right) \to 0$. Hence $d\left(x_{n+1}, x_n\right) \to 0$ by regularity of φ. This shows that (MP3) is satisfied. The conclusions of Theorem 23 yield the conclusions the present theorem. ■

As before (compare Corollary 17 and 18, respectively), the assumptions involving the set K^0, M_0 and W are different if $\preccurlyeq$ is replaced by $\curlyeqprec$.

Corollary 36 *Let the following assumptions be satisfied:*
(A1) (X, d) is a complete metric space and $(V, +)$ a separated topological linear space;
(A2) $K \subseteq V$ is a convex set containing $\theta \in V$ and a cone in $\left(\mathcal{V} := \widehat{\mathcal{P}}(V), \oplus\right)$, $K^0 \subseteq K \setminus (-\operatorname{cl} K)$ is a nonempty convex set;
(A3) The nonempty set $\mathcal{M} \subseteq X \times \mathcal{V}$ and the nonempty set $W \in \mathcal{V}$ are such that

$$(x, M) \in \mathcal{M} \quad \Longrightarrow \quad W \preccurlyeq M;$$

(A4) $\varphi : X \times X \to \mathbb{R}_+$ is a regular premetric;
(A5) If $\{(x_n, W_n)\}_{n \in \mathbb{N}} \subseteq \mathcal{M}$ is a sequence satisfying

$$\forall n \in \mathbb{N} : \ M_{n+1} \oplus \varphi(x_{n+1}, x_n) K^0 \preccurlyeq M_n$$

such that $\{x_n\}_{n \in \mathbb{N}}$ converges to $x \in X$, then there exists $M \in \mathcal{V}$ such that $(x, M) \in \mathcal{M}$ and

$$\forall n \in \mathbb{N} : \ M \oplus \varphi(x, x_n) \preccurlyeq M_n.$$

Then, for each $(x_0, M_0) \in \mathcal{M}$ such that $M_0 \subseteq V$ is nonempty and topological bounded, there exists $\left(\bar{x}, \overline{M}\right) \in \mathcal{M}$ such that

$$\begin{array}{lll} (i) & \overline{M} \oplus \varphi(\bar{x}, x_0) K^0 \preccurlyeq M_0 & \\ (ii) \quad (x, M) \in \mathcal{M}, & M \oplus \varphi(x, \bar{x}) K^0 \preccurlyeq \overline{M} & \Longrightarrow \quad x = \bar{x}. \end{array}$$

Proof. Theorem 23 should be applied to the order relation

$$\left(x', M'\right) \preceq (x, M) \qquad :\Longleftrightarrow \qquad M' \oplus \varphi\left(x', x\right) K^0 \preccurlyeq M.$$

Again, only (MP3) calls for a proof. By similar considerations as in the proof of Corollary 35, we arrive at

$$W \oplus \left(\sum_{k=0}^{n-1} \varphi(x_{k+1}, x_k)\right) K^0 \preccurlyeq M_0$$

for all $n \in \mathbb{N}$. Define $\alpha_n := \sum_{k=0}^{n-1} \varphi(x_{k+1}, x_k)$. Choose $w \in W$ and $k^0 \in K^0$ which is possible by (A2), (A3). According to the definition of $\preccurlyeq$, for each $n \in \mathbb{N}$, there is $(m_0)_n \in M_0$ such that

$$(m_0)_n - w - \alpha_n k^0 \in K.$$

Assume that $\alpha_n \to +\infty$. Then we may conclude $k^0 \in \operatorname{cl} K$, a contradiction. Hence $\lim_{n \to +\infty} \varphi(x_{n+1}, x_n) = 0$. The regularity of φ ensures that (MP3) is satisfied. Applying Theorem 23 gives the desired result. ∎

The next corollary involves a subset $M \subseteq X \times V$ rather than $\mathcal{M} \subseteq X \times \mathcal{P}(V)$. However, it is still more general than Theorem 1 of [47]. This is due to the sets W, K^0 not necessarily containing just a single point.

Corollary 37 *Let the following assumptions be satisfied:*
(A1) (X,d) *is a complete metric space and* $(V,+)$ *a topological linear space;*
(A2) $K \subseteq V$ *is a convex set containing* $\theta \in V$ *and a cone in* $\left(\mathcal{V} := \widehat{\mathcal{P}}(V), \oplus\right)$ *generating the quasiorder* $\leq_K$ *on* V *and* $k^0 \in K \setminus (-\operatorname{cl} K)$*;*
(A3) The nonempty set $M \subseteq X \times V$ *and the topological bounded set* $W \subseteq V$ *are such that*

$$(x,v) \in M \quad \Longrightarrow \quad v \in W \oplus K;$$

(A4) $\varphi : X \times X \to \mathrm{I\!R}_+$ *is a regular premetric;*
(A5) If $\{(x_n, v_n)\}_{n\in\mathrm{I\!N}} \subseteq M$ *is a sequence satisfying*

$$\forall n \in \mathrm{I\!N}: \ v_{n+1} + k^0\varphi(x_{n+1}, x_n) \leq_K v_n$$

such that $\{x_n\}_{n\in\mathrm{I\!N}}$ *converges to* $x \in X$*, then there exists* $v \in V$ *such that* $(x,v) \in M$ *and*

$$\forall n \in \mathrm{I\!N}: \ v + k^0\varphi(x, x_n) \leq_K v_n.$$

Then, for each $(x_0, v_0) \in M$*, there exists* $(\bar{x}, \bar{v}) \in M$ *such that*

$$\begin{array}{lll} (i) & \bar{v} + k^0\varphi(\bar{x}, x_0) \leq_K v_0 & \\ (ii) \quad (x,v) \in M, & v + k^0\varphi(x, \bar{x}) \leq_K \bar{v} & \Longrightarrow \quad x = \bar{x}. \end{array}$$

Proof. Setting $\mathcal{M} := \{(x, \{v\}) : \ (x,v) \in M\}$, $K^0 = \left\{k^0\right\}$ and observing that $\{v'\} \preccurlyeq \{v\}$ if and only if $v' \leq_K v$ for $v \in V$ one can see that Corollary 37 is a special case of Corollary 35 (as well as of Corollary 36). ■

Of course, using the relation

$$\left(x', v'\right) \preceq (x,v) \qquad :\Longleftrightarrow \qquad v' + k^0\varphi\left(x', x\right) \leq_K v$$

one can also prove Corollary 37 by an application of Theorem 23.

The results of this subsection are essentially due to Hamel and Löhne [50], [52]. Let us note that Corollary 32 can be applied especially to conlinear subspaces of $\widehat{\mathcal{P}}(V)$, where $(V, +, \mathcal{T})$ is a topological linear space. Possible candidates are for example the set of all closed sets or the set of all closed convex sets with suitable addition. Compare Example 12.

Chapter 6

Variational Principles on Complete Uniform Spaces

This chapter is devoted to variational principles on complete uniform spaces. The main difference to the metric case is the appearance of a transfinite induction argument such as Zorn's lemma. Without additional assumptions, i.e., simply transforming Theorem 16 into the context of uniform spaces we are not able to avoid such an argument. The situation completely changes if a scalarization functional is present or can be constructed. This is the theme of the next chapter.

Minimal element theorem on uniform spaces are a common generalization of Phelps' lemma (Lemma 1 in [101] from 1963) on the one hand and Ekeland's variational principle from 1972 ([28], [30]) on the other hand. The former is in toplogical linear spaces, the latter in metric spaces, both classes of spaces belong to the class of uniform spaces. The first result in this direction is Theorem 1 in [8] due to Brønstedt. A very general result has been given by Vályi in the 1985 paper [124]. Besides, he proved also the first so called vector valued version of Ekeland's principle on uniform spaces (Theorem 5 of [124]).

6.1 The basic theorem on complete uniform spaces

6.1.1 Preliminaries

Let $(X,\mathcal{U})$ be a uniform space with uniformity $\mathcal{U} \subseteq \mathcal{P}(X \times X)$.

Let $\preceq$ be a quasiorder on X, i.e., a reflexive and transitive relation. As before, we denote the lower sections $S_l(x) = \{x' \in X : x' \preceq x\}$ by $S(x)$ for $x \in X$, compare Definiton 12.

Let $(A,\succ)$ be a directed set (compare [72], p. 65). A net $\{x_\alpha\}_{\alpha\in A} \subseteq X$ is said to be **decreasing** with respect to $\preceq$ iff

$$\forall \alpha, \beta \in A,\ \alpha \succ \beta :\ x_\alpha \preceq x_\beta.$$

In this chapter, $(X,\mathcal{U})$ is assumed to be complete. We note that the results can be modified in order to replace the completeness by $\preceq$–completeness as in Chapter 4.

A quasiorder $\preceq$ is called **regular** iff every decreasing sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is asymptotic, i.e.,

$$\forall E \in \mathcal{U},\ \exists n_E \in \mathbb{N},\ \forall n \geq n_E :\ (x_{n+1}, x_n) \in E.$$

As in the case of a metric space, regularity forces antisymmetry.

Proposition 42 *A regular quasiorder $\preceq$ on a separated uniform space X is antisymmetric.*

PROOF. Take $x, x' \in X$ such that $x \preceq x' \preceq x$. Then, the sequence $\{x, x', x, x', \ldots\}$ is decreasing. Regularity implies

$$\forall D \in \mathcal{U} :\ \left(x, x'\right), \left(x', x\right) \in E.$$

Hence $x = x'$ since X is separated. ■

A quasiorder $\preceq$ is called **lower closed** iff for any decreasing net $\{x_\alpha\}_{\alpha\in A} \subseteq X$ converging to some $x \in X$

$$\forall \alpha \in A :\ x \preceq x_\alpha$$

holds true. A quasiorder is lower closed if and only if the sections $S(x)$ are closed with respect to decreasing nets, i.e. if $\{x_\alpha\}_{\alpha\in A} \subset S(x)$ and $\lim_\alpha x_\alpha = x$, then $x \in S(x)$.

6.1.2 The basic theorem

The following theorem is parallel to Theorem 16. A theorem of this type has been established by Vályi in [124].

Theorem 24 *Let the following assumptions be satisfied:*
(M1) $(X,\mathcal{U})$ is a complete uniform space;
(M2) $\preceq$ is a reflexive and transitive relation on X;
(M3) $\preceq$ is regular;
(M4) $\preceq$ is lower closed.
Then, for each $x_0 \in X$ there exists $\bar{x} \in X$ such that

$$\bar{x} \in S(x_0) \qquad \textit{and} \qquad \{\bar{x}\} = S(\bar{x}).$$

PROOF. Consider the set $S(x_0) := \{x \in X :\ x \preceq x_0\}$. Let $S_0 \subseteq S(x_0)$ be a totally ordered subset of $S(x_0)$. Consider S_0 to be a decreasing net,

$$S_0 = \{x_\alpha\}_{\alpha\in A}, \quad x_\alpha \preceq x_{\alpha'} \quad \text{for} \quad \alpha \succ \alpha'$$

for some index set A, directed by $\succ$. We claim that $\{x_\alpha\}_{\alpha\in A}$ is Cauchy. Assume the contrary. Then there exist $E \in \mathcal{U}$ and $\{x_n\}_{n\in\mathbb{N}} \subseteq \{x_\alpha\}_{\alpha\in A} \subseteq S_0$ such that

$$(x_{n+1}, x_n) \notin E \quad \text{for} \quad n \in \mathbb{N}.$$

Indeed, if $\{x_\alpha\}_{\alpha\in A}$ is not Cauchy, there is $E\in\mathcal{U}$ such that

$$\forall\alpha\in A\ \exists\alpha_2\succ\alpha_1\succ\alpha:\ (x_{\alpha_1},x_{\alpha_2})\notin E.$$

Hence we can find $\alpha_1,\alpha_2\in A$ such that $\alpha_2\succ\alpha_1$ and $(x_{\alpha_1},x_{\alpha_2})\notin E$. Set $x_1:=x_{\alpha_1}$, $x_2:=x_{\alpha_2}$. Similiarly, $\alpha_3,\alpha_4\in A$ can be found such that $\alpha_4\succ\alpha_3\succ\alpha_2$ and $(x_{\alpha_3},x_{\alpha_4})\notin E$. Set $x_3:=x_{\alpha_3}$, $x_4:=x_{\alpha_4}$ and continue this procedure. A decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ is obtained being not asymptotic. This contradicts (M3).

Since X is complete, $\{x_\alpha\}_{\alpha\in A}$ converges to some $\bar{x}_0\in X$. From (M3) we obtain that $\bar{x}_0\in S(x_\alpha)$ for each $\alpha\in A$, especially $\bar{x}_0\in S(x_0)$. Hence $\bar{x}_0\preceq x_\alpha$ for each $\alpha\in A$, i.e., $\bar{x}_0$ is a lower bound of S_0.

By Zorn's lemma, there exists a minimal element $\bar{x}$ in $S(x_0)$. Moreover, $\{\bar{x}\}=S(\bar{x})$ because if $x\neq\bar{x}$, $x\preceq\bar{x}$ we obtain by transitivity $x\preceq\bar{x}\preceq x_0$ contradicting the minimality of $\bar{x}$ in $S(x_0)$. ■

Remark 26 *The uniform structure $\mathcal{U}$ on X can be equivalently generated by a family of pseudometrics $\{p_\lambda\}_{\lambda\in\Lambda}$ according to Definition 23. This means, each $E\in\mathcal{U}$ contains a set of the form*

$$E_{\lambda,r}:=\left\{(x,x'):\ d_\lambda(x,x')<r\right\},\ \lambda\in\Lambda,\ r>0.$$

The sets $E_{\lambda,r}$, $\lambda\in\Lambda$, $r>0$ form a base of the uniform structure $\mathcal{U}$ on X. Hence, a sequence $\{x_n\}_{n\in\mathbb{N}}\subset X$ is asymptotic if and only if

$$\forall r>0,\ \forall\lambda\in\Lambda,\ \exists n_{r,\lambda}\in\mathbb{N},\ \forall n\geq n_{r,\lambda}:\ p_\lambda(x_{n+1},x_n)<r.$$

Similarly, the property of being asymptotic can be described by quasimetrics (see Definition 23) or an order metric (see Definition 25).

6.1.3 Equivalent formulations of the basic theorem

Without serious difficulties it is possible to transform the equivalent formulations of the basic minimal element theorem for metric spaces to the case of uniform spaces. We shall give the statements and refer for the details of the proofs to Section 4.1.3.

Theorem 25 *Let (M1) through (M4) of Theorem 24 be in force and, additionally, $T: X\to\mathcal{P}(X)$ be a set-valued mapping. If T satisfies*

$$\forall x\in X,\ \exists x'\in T(x):\ x'\preceq x, \tag{WC}$$

then there is $\bar{x}\in X$ such that $\bar{x}\in T(\bar{x})$, i.e. $\bar{x}$ is a fixed point of T. If T satisfies

$$\forall x\in X,\ \forall x'\in T(x):\ x'\preceq x, \tag{SC}$$

then there is $\bar{x}\in X$ such that $\{\bar{x}\}=T(\bar{x})$, i.e. $\bar{x}$ is an invariant point of T.

Proof. Each point $\bar{x}$ satisfying the conclusions of Theorem 24 does the job. ■

Theorem 26 *Let (M1) through (M4) of Theorem 24 be in force and, additionally:*
(M5) The set $M \subseteq X$ *satisfies*

$$\forall x \in S(x_0) \setminus M \quad \exists x' \in S(x) \setminus \{x\}.$$

Then, there exists $\bar{x} \in S(x_0) \cap M$.

PROOF. By Theorem 24, there exists $\bar{x} \in S(x_0)$ such that $\{\bar{x}\} = S(\bar{x})$. By assumption (M5), $\bar{x} \in M$, hence $\bar{x} \in M \cap S(x_0)$. ■

Theorem 24 can be derived from Theorem 25 and Theorem 26 in the same way as Theorem 16 from Theorem 17 and Theorem 18, respectively.

6.1.4 Set relation formulation

In this section, the analogues to the Theorems 21 and 22 shall be established.

Let $(X,\mathcal{U})$ be a uniform space and Y as well as $M \subseteq X \times Y$ be nonempty sets. For $x \in X$, let us define $M(x) := \{(x', y) \in X \times Y : x' = x, (x', y) \in M\} \in \widehat{\mathcal{P}}(X \times Y)$ and $M_Y(x) := \{y \in Y : (x, y) \in M\} \in \widehat{\mathcal{P}}(Y)$. Let $\preceq$ be a quasiorder on M. Then, $(\{M(x) : x \in X\}, \preccurlyeq)$ as well as $(\{M(x) : x \in X\}, \curlyeqprec)$ is quasiordered. As in Section 4.1.6 we have $M(x') \preccurlyeq M(x)$ if and only if

$$\forall y \in M_Y(x), \exists y' \in M_Y(x') : (x', y') \preceq (x, y) \tag{6.1}$$

and $M(x') \curlyeqprec M(x)$ if and only if

$$\forall y' \in M_Y(x'), \exists y \in M_Y(x) : (x', y') \preceq (x, y). \tag{6.2}$$

Theorem 27 *Let the following assumptions be satisfied:*
(M1') $(X,\mathcal{U})$ is a uniform space and X, Y as well as $M \subseteq X \times Y$ are nonempty sets;
(M2') $\preceq$ is a quasiorder, i.e. a reflexive and transitive relation on $X \times Y$;
(M3') If $\{M(x_\alpha)\}_{\alpha \in A}$ is a decreasing net with respect to $\preccurlyeq$, i.e.

$$\alpha, \beta \in A,\ \alpha \succ \beta \quad \Longrightarrow \quad \forall y_\beta \in M_Y(x_\beta), \exists y_\alpha \in M_Y(x_\alpha) : (x_\alpha, y_\alpha) \preceq (x_\beta, y_\beta)$$

and the net $\{x_\alpha\}_{\alpha \in A}$ converges to $x \in X$, then

$$\forall \alpha \in A : M(x) \preccurlyeq M(x_\alpha);$$

(M4') If $\{(x_n, y_n)\}_{n \in \mathbb{N}} \subseteq M$ is a decreasing sequence with respect to $\preceq$, then $\{x_n\}_{n \in \mathbb{N}}$ is asymptotic.
Then, for each $x_0 \in X$ with $M_Y(x_0) \neq \emptyset$, there exists $\bar{x} \in X$ such that $M_Y(\bar{x}) \neq \emptyset$ and

$$\begin{array}{ll} (i) & M(\bar{x}) \preccurlyeq M(x_0) \\ (ii) & M(x) \preccurlyeq M(\bar{x}) \quad \Longrightarrow \quad x = \bar{x}. \end{array}$$

PROOF. We define a binary relation on X be setting

$$x' \preceq_X x \quad \Longleftrightarrow \quad M\left(x'\right) \preccurlyeq M\left(x\right).$$

in order to apply Theorem 24. With the help of (6.1), one can see that $\preceq_X$ is reflexive and transitive. (MP3) gives the lower closedness of $\preceq_X$. It remains to show the regularity. This can be done in the same way as the regularity for $\preccurlyeq$ in the proof of Theorem 21. Finally, a straightforward application of Theorem 24 yields (i) and (ii). ■

Note that the closedness assumption (M3') can not be formulated merely in terms of the order relation on $X \times Y$ whereas the regularity assumption (M4') can. This is due to the fact that closedness in uniform spaces requires nets whereas regularity involves only sequences. Compare the proof of Theorem 21.

The corresponding result for $\preccurlyeq$ reads as follows.

Theorem 28 *Let the following assumptions be satisfied:*
(M1') $(X,\mathcal{U})$ is a uniform space and X, Y as well as $M \subseteq X \times Y$ are nonempty sets;
(M2') $\preceq$ is a quasiorder, i.e., a reflexive and transitive relation on $X \times Y$;
(M3') If $\{M\left(x_\alpha\right)\}_{\alpha\in A} \subseteq M$ is an increasing net with respect to $\preccurlyeq$, i.e.

$$\alpha, \beta \in A,\ \alpha \succ \beta \quad \Longrightarrow \quad \forall y_\beta \in M\left(x_\beta\right)\ \exists y_\alpha \in M\left(x_\alpha\right):\ \left(x_\beta, y_\beta\right) \preceq \left(x_\alpha, y_\alpha\right)$$

and the net $\{x_\alpha\}_{\alpha\in A}$ converges to $x \in X$, then

$$\forall \alpha \in A:\ M\left(x_\alpha\right) \preccurlyeq M\left(x\right);$$

(M4') If $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq M$ is a increasing sequence with respect to $\preceq$, then $\{x_n\}_{n\in\mathbb{N}}$ is asymptotic.
Then, for each $x_0 \in X$ with $M_Y\left(x_0\right) \neq \emptyset$, there exists $\bar{x} \in X$ such that $M_Y\left(\bar{x}\right) \neq \emptyset$ and

$$\begin{array}{ll} (i) & M\left(x_0\right) \preccurlyeq M\left(\bar{x}\right) \\ (ii) & M\left(\bar{x}\right) \preccurlyeq M\left(x\right) \quad \Longrightarrow \quad x = \bar{x}. \end{array}$$

PROOF. The proof is an application of Theorem 27 using the same arguments as in the proof of Theorem 22 applying Theorem 21. ■

Remark 27 *As in the metric case, one can consider the special $Y = \{y_S\}$, a singleton. In this case, Theorem 27 reduces to Theorem 24 (as well as Theorem 28 to a maximal element reformulation of Theorem 24). On the other hand, Theorem 27 (as well as Theorem 28) are proven using Theorem 24 without any reference to the constructions in the proof of Theorem 24, especially not to Zorn's lemma. In this sense, the theorems are equivalent.*

6.1.5 Special cases of Theorem 24

In this section, we shall show that the fundamental lemma of Phelps (Lemma 1 of [101]) as well as its generalizations of Brønsted (Theorem 1 of [8]) and Mizoguchi (the lemma in [86]) are special cases of Theorem 24.

To begin with, we reformulate Brønsted's theorem.

Corollary 38 *Let the following assumptions be satisfied:*
(A1) $(X,\mathcal{U})$ *is a complete uniform space;*
(A2) $\preceq$ *is a quasiorder on* X *with lower closed section* $S(x) = \{x' \in X : \ x' \preceq x\}$*;*
(A3) The function $f : X \to \mathrm{I\!R} \cup \{+\infty\}$ *is bounded below and monotone with respect to* $\preceq$*, i.e.,*

$$x_1 \preceq x_2 \quad \Longrightarrow \quad f(x_1) \leq f(x_2);$$

(A4) For each $E \in \mathcal{U}$*, there is* $\delta > 0$ *such that* $x_1 \preceq x_2$ *and* $f(x_2) - f(x_1) < \delta$ *implies* $(x_1, x_2) \in E$*.*
Then, for each $x_0 \in X$ *with* $f(x_0) \in \mathrm{I\!R}$*, there is* $\bar{x} \in X$ *such that*

$$\bar{x} \in S(x_0) \quad \text{and} \quad \{\bar{x}\} = S(\bar{x}).$$

Proof. It suffices to verify the regularity of $\preceq$ in order to apply Theorem 24. Take a decreasing sequence $\{x_n\}_{n\in\mathrm{I\!N}}$, i.e.

$$\forall n \in \mathrm{I\!N} : \ x_{n+1} \preceq x_n.$$

Fix $E \in \mathcal{U}$ and take $\delta > 0$ from assumption (A4). Since f is monotone and bounded below, the sequence $\{f(x_n)\}_{n\in\mathrm{I\!N}}$ is convergent. Hence, there is $n_\delta \in \mathrm{I\!N}$ such that

$$\forall n \geq n_\delta : \ f(x_n) - f(x_{n+1}) < \delta.$$

Assumption (A4) implies $(x_n, x_{n+1}) \in E$ for all $n \geq n_\delta$, hence $\{x_n\}_{n\in\mathrm{I\!N}}$ is asymptotic. This proves the regularity of $\preceq$. The assertions of the theorem follow from those of Theorem 24. ■

The original lemma of Phelps (Lemma 1 in [101]) is a consequence of Corollary 38. The details are not repeated here and can be found in [8].

Corollary 39 *Let the following assumptions be satisfied:*
(A1) $(X,\mathcal{U})$ *is a uniform space and* $\{p_\lambda\}_{\lambda\in\Lambda}$ *a family of pseudometrics generating the uniformity;*
(A2) $\preceq$ *is a quasiorder on* X *with lower closed section* $S(x) = \{x' \in X : \ x' \preceq x\}$*;*
(A3) $\{f_\lambda\}_{\lambda\in\Lambda}$ *is a family of functions* $f_\lambda : X \to \mathrm{I\!R}$ *such that each* f_λ *is bounded below on* X *and monotone with respect to* $\preceq$*, i.e.*

$$\forall \lambda \in \Lambda : \ (x_1 \preceq x_2 \implies f_\lambda(x_1) \leq f_\lambda(x_2));$$

(A4) For each $\lambda \in \Lambda$ and each $\varepsilon > 0$, there is $\delta_\lambda > 0$ such that $x_1 \preceq x_2$ and $f_\lambda(x_2) - f_\lambda(x_1) < \delta_\lambda$ implies $p_\lambda(x_1, x_2) < \varepsilon$.
Then, for each $x_0 \in X$, there is $\bar{x} \in X$ such that

$$\bar{x} \in S(x_0) \quad \textit{and} \quad \{\bar{x}\} = S(\bar{x}).$$

Proof. Again, it suffices to verify the regularity of $\preceq$ in order to apply Theorem 24. Let $\{x_n\}_{n\in\mathbb{N}}$ be a decreasing sequence with respect to $\preceq$. Repeating the arguments from the proof of Corollary 38 with f replaced by f_λ, we obtain

$$\forall \lambda \in \Lambda : \exists n_\lambda : \forall n \geq n_\lambda : p_\lambda(x_{n+1}, x_n) < \varepsilon.$$

Since the sets

$$E_{\lambda,\varepsilon} = \left\{ (x, x') \in X \times X : p_\lambda(x, x') < \varepsilon \right\}, \ \lambda \in \Lambda, \varepsilon > 0$$

form a base of the uniformity, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is asymptotic. Hence Theorem 24 can be applied to finish the proof. ∎

In view of Proposition 33, the above corollary can be formulated replacing the family of pseudometrics by a family of quasimetrics. Finally, we give a formulation with a function $f : X \to Y$, $(Y, \leq, \circ)$ a normally ordered, topological Abelian group. If X is separated uniform, such a group exists and additionally an order metric $D : X \times X \to Y$ generating the uniform structure and the topology on X, cf. Section 2.2.3.

To formulate the result, an additional condition is needed. Let X, Y as above such that the following condition is satisfied:

(R) Every sequence $\{y_n\}_{n\in\mathbb{N}}$, that is decreasing with respect to $\leq$ and bounded from below, is asymptotic, i.e.,

$$\forall B \in \mathcal{B}(\theta), \exists n_B \in \mathbb{N}, \forall n \geq n_B : y_n \circ (y_{n+1})^{-1} \in B,$$

where $\mathcal{B}(\theta)$ is a neighborhood base of $\theta \in Y$ consisting of full sets.

Corollary 40 *Let the following assumptions be satisfied:*
(A1) $(X, \mathcal{U})$ is a complete uniform space, $(Y, \circ, \leq)$ is a normally ordered, topological Abelian group satisfying condition (R) above;
(A2) $\preceq$ is a quasiorder on X with lower closed sections $S(x) = \{x' \in X : x' \preceq x\}$;
(A3) The function $f : X \to Y$ is bounded below and monotone with respect to $\preceq$, i.e.,

$$x_1 \preceq x_2 \quad \Longrightarrow \quad f(x_1) \leq f(x_2);$$

(A4) For all $E \in \mathcal{U}$ there is $B \in \mathcal{B}(\theta)$ such that

$$x_1 \preceq x_2, \ f(x_2) \circ (f(x_1))^{-1} \in B \quad \Longrightarrow \quad (x_1, x_2) \in E.$$

Then, for each $x_0 \in X$, there is $\bar{x} \in X$ such that

$$\bar{x} \in S(x_0) \quad \textit{and} \quad \{\bar{x}\} = S(\bar{x}).$$

Proof. Again, the only thing te check is the regularity of $\preceq$. Take a decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$. Then, the sequence $\{f(x_n)\}_{n\in\mathbb{N}} \subseteq Y$ is decreasing with respect to $\preccurlyeq$. Take $E \in \mathcal{U}$ and consider $B \in \mathcal{B}(\theta)$ from (A4). There is $n_B \in \mathbb{N}$ such that

$$\forall n \geq n_B: \; x_{n+1} \preceq x_n, \; f(x_n) \circ (f(x_{n+1}))^{-1} \in B,$$

since $\{f(x_n)\}_{n\in\mathbb{N}}$ is asymptotic according to (R). (A4) implies $(x_n, x_{n+1}) \in E$ for all $n \geq n_B$. Hence $\{x_n\}_{n\in\mathbb{N}}$ is asymptotic as desired and we may apply Theorem 24 to obtain the assertions of the corollary. ■

Of course, Corollary 40 is a generalization of Corollary 38. A related result is Theorem 5 of [124].

6.2 Results with functions into ordered monoids

6.2.1 Ekeland's principle over quasiordered monoids

It is possible to give a uniform space formulation of all results of Section4.2. We pick out three main theorems to show the principal procedure at work, namely Ekeland's principle and its equilibrium version as well as Caristi's fixed point theorem.

We start with Ekeland's principle for functions mapping a uniform space into a quasiordered monoid. The first result of this type for extended real valued function on uniform spaces can be found in [8]. Therein, Brønsted proved a common generalization of Ekeland's theorem [30], Theorem 1.1, and Lemma 1 of [101] due to Phelps playing in linear topological spaces. Mizoguchi [86] gave a slight generalization of Brønsted's results as well as a fixed point theorem of Kirk–Caristi type for uniform spaces and a drop theorem in locally convex spaces. Moreover, she established the equivalence of these results. In several papers [89], [91], [90], [92], [93], Nemeth generalized Ekeland's principle to functions mapping a uniform space into an ordered topological Abelian group. Also, Khanh [73] dealt with functions mapping so called L-spaces into ordered linear spaces. Finally, in [51] set valued variants of Ekeland's principle and fixed point theorems for uniform spaces have been proven.

The following results involve order premetrics on uniform spaces in the sense of Definition 30.

Corollary 41 *Let the following assumptions be satisfied:*
(A1) $(X, \mathcal{U})$ *is a complete uniform space and* $(Y, \circ, \leq)$ *a quasiordered monoid;*
(A2) $\Phi : X \times X \to Y$ *is an order premetric;*
(A3) The function $f : X \to Y$ *and* $\tilde{y} \in Y$, $x_0 \in X$ *are such that*
(i) $\tilde{y} \leq f(x)$ *for all* $x \in X$*;*
(ii) Φ *is regular with respect to* $\tilde{y}, f(x_0) \in Y$*;*
(A4) If the net $\{x_\alpha\}_{\alpha\in A} \subseteq X$ *converges to* $x \in X$ *and*

$$\forall \alpha, \beta \in A, \; \alpha \succ \beta: \; f(x_\alpha) \circ \Phi(x_\alpha, x_\beta) \leq f(x_\beta),$$

then $f(x) \circ \Phi(x_\alpha, x) \le f(x_\alpha)$ *for all* $\alpha \in A$.
Then, there is $\bar{x} \in X$ *such that*

$$\begin{array}{llcll} (i) & f(\bar{x}) \circ \Phi(\bar{x}, x_0) & \le & f(x_0) & \\ (ii) & f(x) \circ \Phi(x, \bar{x}) & \le & f(\bar{x}) & \Longrightarrow \quad x = \bar{x}. \end{array}$$

Proof. The proof is by checking the assumptions of Theorem 24 for the relation

$$x' \preceq x \qquad :\Longleftrightarrow \qquad f\left(x'\right) \circ \Phi\left(x', x\right) \le f(x).$$

The relation $\preceq$ is reflexive since $\le$ is reflexive and (P1) of Definition 30 holds. It is transitive by (P2) and the transitivity of $\le$. (M4) follows directly from assumption (A4). It remains to check the regularity of $\preceq$. Let $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ be such that $x_{n+1} \preceq x_n$ for all $n \in \mathbb{N}$, i.e.,

$$f(x_{n+1}) \circ \Phi(x_{n+1}, x_n) \le f(x_n).$$

The transitivity of $\le$ implies

$$f(x_{n+1}) \circ \Phi(x_{n+1}, x_n) \circ \Phi(x_n, x_{n-1}) \le f(x_n) \circ \Phi(x_n, x_{n-1}) \le f(x_{n-1}).$$

Continuing this process, we obtain for each $n \in \mathbb{N}$

$$f(x_{n+1}) \circ \sum_{k=0}^{n} \Phi(x_{k+1}, x_k) \le f(x_0).$$

Since $\tilde{y} \le f(x_m)$ for each $m \in \mathbb{N}$ by (A2), it follows

$$\tilde{y} \circ \sum_{k=0}^{n} \Phi(x_{k+1}, x_k) \le f(x_0).$$

Since by (A3) Φ is regular with respect to $\tilde{y}, f(x_0)$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is asymptotic. Applying Theorem 24 yields the desired result. ■

We consider a set valued mapping $T : X \to \widehat{\mathcal{P}}(X)$ in order to prove a fixed point theorem of Kirk–Caristi type.

Corollary 42 *Let the following assumptions be satisfied:*
Let (A1) to (A4) of Corollary 41 be in force. If the mapping $T : X \to \widehat{\mathcal{P}}(X)$ *satisfies the weak contraction condition*

$$\forall x \in X,\ \exists x' \in T(x) :\ f\left(x'\right) \circ \Phi\left(x', x\right) \le f(x), \tag{WC}$$

then T *has a fixed point, i.e., there is* $\bar{x} \in X$ *such that* $\bar{x} \in T(\bar{x})$.
If the mapping $T : X \to \mathcal{P}(X)$ *satisfies the strong contraction condition*

$$\forall x \in X,\ \forall x' \in T(x) :\ f\left(x'\right) \circ \Phi\left(x', x\right) \le f(x), \tag{SC}$$

then T *has an invariant point, i.e., there is* $\bar{x} \in X$ *such that* $\{\bar{x}\} = T(\bar{x})$.

Proof. Use Corollary 41 instead of Corollary 9 in the proof of Corollary 10. ■

Conversely, Corollary 41 can be proven using the fixed point result above. To see this, one has to proceed along the same lines as in the metric case. Compare the remarks after Corollary 10.

The Theorems 6 and 8 of [124] are also fixed point theorems of Kirk–Caristi type on uniform spaces. They involve real valued functions and a family of pseudometrics generating the uniformity, respectively.

The next result deals with a function $F : X \times X \to Y$ instead of $f : X \to Y$.

Corollary 43 *Let the following assumptions be satisfied:*
(A1) $(X, \mathcal{U})$ *is a complete uniform space and* $(Y, \circ, \leq)$ *a quasiordered monoid;*
(A2) The function $F : X \times X \to Y$, $\tilde{y} \in Y$ *and* $x_0 \in X$ *are such that*
(i) $F(x_1, x_3) \leq F(x_1, x_2) \circ F(x_2, x_3)$ *for all* $x_1, x_2, x_3 \in X$*;*
(ii) $\tilde{y} \leq F(x_0, x)$ *for all* $x \in X$*;*
(A2) $\Phi : X \times X \to Y$ *is a regular order premetric with respect to* $\tilde{y}, \theta \in Y$*;*
(A4) If the net $\{x_\alpha\}_{\alpha \in A} \subseteq X$ *converges to* $x \in X$ *and*

$$\forall \alpha, \beta \in A, \ \alpha \succ \beta : \ F(x_\alpha, x_\beta) \circ \Phi(x_\beta, x_\alpha) \leq \theta,$$

then $F(x_\alpha, x) \circ \Phi(x, x_\alpha) \leq \theta$ *for all* $\alpha \in A$*.*
Then, there exists $\bar{x} \in X$ *such that*

$$\begin{array}{lllll} (i) & F(x_0, \bar{x}) \circ \Phi(\bar{x}, x_0) & \leq & \theta & \\ (ii) & F(\bar{x}, x) \circ \Phi(x, \bar{x}) & \leq & \theta & \Longrightarrow \quad x = \bar{x}. \end{array}$$

First Proof. We check the assumptions of Theorem 16 for the relation

$$x' \preceq x \qquad :\Longleftrightarrow \qquad x' = x \quad \text{or} \quad F\left(x, x'\right) \circ \Phi\left(x', x\right) \leq \theta.$$

being reflexive and transitive by the properties of Φ, F and $\leq$. (M4) follows directly from assumption (A4). The regularity of $\preceq$ can be seen in the same way as in the proof of Corollary 13. Therefore, we may apply Theorem 24 obtaining the desired result. ■

We shall indicate a sufficient condition for (A4) of Corollary 41. A function $f : X \to Y$ is called **lower monotone** iff for each net $\{x_\alpha\}_{\alpha \in A} \subseteq X$ converging to some $x \in X$ and satisfying $x_\alpha \preceq x_\beta$ for $\alpha \succ \beta$ the inequality $f(x) \leq f(x_\alpha)$ holds true for all $\alpha \in A$. Compare [93] for this kind of condition. It can be interpreted as a generalization of lower semicontinuity.

Moreover, an order premetric $\Phi : X \times X \to Y$ is called **lower monotone** with respect to the first variable iff for each net $\{x_\alpha\}_{\alpha \in A} \subseteq X$ converging to $x \in X$ and satisfying $x_\alpha \preceq x_\beta$ for $\alpha \succ \beta$ we have for all $x' \in X$

$$y_1, y_2 \in Y, \quad \forall \alpha \in A : \ y_1 \circ \Phi\left(x_\alpha, x'\right) \leq y_2 \quad \Longrightarrow \quad y_1 \circ \Phi\left(x, x'\right) \leq y_2.$$

Lemma 8 *Let X, Y be as in Corollary 41, the function $f : X \to Y$ be lower monotone and the order premetric Φ be lower monotone with respect to the first variable. Then (A4) of Corollary 41 is satisfied.*

Proof. Take a net $\{x_\alpha\}_{\alpha \in A} \subseteq X$ converging to $x \in X$ such that

$$\forall \alpha, \beta \in A,\ \alpha \succ \beta :\ f(x_\alpha) \circ \Phi(x_\alpha, x_\beta) \leq f(x_\beta).$$

Then, since $\theta \leq \Phi(x_\alpha, x_\beta)$,

$$f(x_\alpha) \leq f(x_\alpha) \circ \Phi(x_\alpha, x_\beta) \leq f(x_\beta)$$

and therefore $f(x_\alpha) \leq f(x_\beta)$ for all $\alpha \succ \beta$ since $\leq$ is transitive. The lower monotonicity of f implies $f(x) \leq f(x_\alpha)$ for all $\alpha \in A$. For $\alpha, \beta \in A$, $\alpha \succ \beta$ we obtain

$$f(x) \circ \Phi(x_\alpha, x_\beta) \leq f(x_\alpha) \circ \Phi(x_\alpha, x_\beta) \leq f(x_\beta).$$

Since Φ is lower monotone with respect to the first variable, this implies

$$f(x) \circ \Phi(x, x_\beta) \leq f(x_\beta)$$

as desired. ■

6.2.2 Power sets of quasiordered monoids

This subsection contains results parallel to those of Section 1.3.1.

Corollary 44 *Let the following assumptions be satisfied:*
(A1) $(X, \mathcal{U})$ is a complete uniform space, $(Y, \circ, \leq)$ an ordered monoid and $(\mathcal{Y}, \odot, \preccurlyeq)$ the ordered monoid generated by $\mathcal{Y} := \widehat{\mathcal{P}}(Y)$;
(A2) The function $f : X \to \mathcal{Y}$ and $W \in \mathcal{Y}$ are such that

$$\forall x \in X :\ W \preccurlyeq f(x);$$

(A3) $\Phi : X \times X \to \mathcal{Y}$ is a regular order premetric with respect to $W, f(x_0) \in \mathcal{Y}$;
(A4) If the net $\{x_\alpha\}_{\alpha \in A} \subseteq X$ converges to $x \in X$ and

$$\forall \alpha, \beta \in A,\ \alpha \succ \beta :\ f(x_\alpha) \circ \Phi(x_\alpha, x_\beta) \preccurlyeq f(x_\beta),$$

then $f(x) \circ \Phi(x_\alpha, x) \preccurlyeq f(x_\alpha)$ for all $\alpha \in A$.
Then, there exists $\bar{x} \in X$ such that

$$\begin{array}{lll} (i) & & f(\bar{x}) \odot \Phi(\bar{x}, x_0) \preccurlyeq f(x_0) \\ (ii) & x \in X, & f(x) \odot \Phi(x, \bar{x}) \preccurlyeq f(\bar{x}) \quad \Longrightarrow \quad x = \bar{x}. \end{array}$$

Proof. By Theorem 11, $\left(\widehat{\mathcal{P}}(Y), \odot, \preccurlyeq\right)$ is a quasiordered monoid. Defining the relation

$$x' \preceq x \qquad :\Longleftrightarrow \qquad f\left(x'\right) \odot \Phi\left(x', x\right) \preccurlyeq f(x)$$

on X, the assumptions of Corollary 41 are easy to check. Its conclusions yield the desired result. ■

Corollary 45 *Let the assumptions of Corollary 44 be satisfied with $\preccurlyeq$ replaced by $\prec$. Then, there exists $\bar{x} \in X$ such that*

$$\begin{array}{lll} (i) & & f(\bar{x}) \odot \Phi(\bar{x}, x_0) \prec f(x_0) \\ (ii) & x \in X, & f(x) \odot \Phi(x, \bar{x}) \prec f(\bar{x}) \quad \Longrightarrow \quad x = \bar{x}. \end{array}$$

Proof. Replace $\preccurlyeq$ by $\prec$ in the proof of Corollary 44. ■

6.2.3 Single valued functions

We show that Nemeth's results in [93] follow from Theorem 24. They involve a function f mapping a uniform space into a toplogical Abelian group. Compare Corollary 46.

In the following corollary, Y is an ordered group not order complete in general. As usual, we can adjoin a largest element y_l obtaining an ordered monoid.

Corollary 46 *Let the following assumptions be satisfied: (A1) $(X, \mathcal{U})$ is a complete uniform space and $(Y, \circ, \leq)$ an ordered topological Abelian group;*
(A2) The function $f : X \to Y \cup \{y_l\}$ and $\tilde{y} \in Y$ are such that $\tilde{y} \leq f(x)$ for all $x \in X$;
(A3) $\Phi : X \times X \to Y$ is a regular order premetric with respect to $\tilde{y}$, $f(x_0)$ for $x_0 \in X$;
(A4) If the net $\{x_\alpha\}_{\alpha \in A} \subseteq X$ converges to $x \in X$ and

$$\forall \alpha, \beta \in A,\ \alpha \succ \beta :\ f(x_\alpha) \circ \Phi(x_\alpha, x_\beta) \leq f(x_\beta),$$

then $f(x) \circ \Phi(x, x_\alpha) \leq f(x_\alpha)$ for all $\alpha \in A$.
Then, there is $\bar{x} \in X$ such that

$$\begin{array}{llll} (i) & f(\bar{x}) \circ \Phi(\bar{x}, x_0) & \leq & f(x_0) \\ (ii) & f(x) \circ \Phi(x, \bar{x}) & \leq & f(\bar{x}) \quad \Longrightarrow \quad x = \bar{x}. \end{array}$$

Proof. Apply Corollary 41 to $(Y \cup \{y_l\}, \circ, \leq)$. ■

Taking $Y = V$ a topological linear space and corresponding order premetrics we are led to further results parallel to those of Section 4.3, 4.4 and 4.5. The results of Section 4.7 also have counterparts for a uniform space X. Let us mention that Theorem 1 of Brønsted in [8] is also a special case of Corollary 46.

Similar results with a family of quasimetrics instead of an order premetric can be found in [49].

6.3 A partial minimal element theorem on complete uniform spaces

Let $(X,\mathcal{U})$ be a complete uniform space and Y a nonempty set. The goal is to extend Theorem 24 to order relations on $X \times Y$. The result is parallel to Theorem 23. Let $\preceq$ be a quasiordering on $X \times Y$. A net $\{(x_\alpha, y_\alpha)\}_{\alpha\in A} \subseteq X \times Y$ is said to be **decreasing** iff

$$\forall \alpha, \beta \in A,\ \alpha \succ \beta :\ (x_\alpha, y_\alpha) \preceq (x_\beta, y_\beta)\,.$$

A quasiorder on $X \times Y$ is called **regular on** $M \subseteq X \times Y$ iff for each decreasing sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq M$ the sequence $\{x_n\}_{n\in\mathbb{N}}$ is asymptotic, i.e.,

$$\forall E \in \mathcal{U},\ \exists n_E \in \mathbb{N},\ \forall n \geq n_E :\ (x_{n+1}, x_n) \in E.$$

Remark 28 *Let $(X,\mathcal{U})$ be a uniform space and Y a nonempty set. A regular quasiordering $\preceq$ on $X \times Y$ is partially antisymmetric. To see this, proceed as in Remark 24.*

We state a result for complete uniform spaces parallel to Theorem 23.

Theorem 29 *Let the following assumptions be satisfied:*
(MP1) $(X,\mathcal{U})$ is a complete uniform space, Y and $M \subseteq X \times Y$ are nonempty sets;
(MP2) $\preceq$ is a quasiorder on $X \times Y$;
(MP3) The quasiorder $\preceq$ is regular on M;
(MP4)If $\{(x_\alpha, y_\alpha)\}_{\alpha\in A} \subseteq M$ is a decreasing net such that $\{x_\alpha\}_{\alpha\in A}$ converges to $x \in X$, then there exists $y \in Y$ such that $(x, y) \in M$ and

$$\forall \alpha \in A :\ (x, y) \preceq (x_\alpha, y_\alpha)\,.$$

Then, for each $(x_0, y_0) \in M$, there exists $(\bar{x}, \bar{y}) \in M$ such that

$$\begin{array}{lll} (i) & & (\bar{x}, \bar{y}) \preceq (x_0, y_0) \\ (ii) & (x, y) \in M, & (x, y) \preceq (\bar{x}, \bar{y}) \quad \Longrightarrow \quad x = \bar{x}. \end{array}$$

Proof. Consider the section $S(x_0, y_0) := \{(x, y) \in M :\ (x, y) \preceq (x_0, y_0)\}$. Let $S_0 \subseteq S(x_0, y_0)$ be a totally ordered subset of $S(x_0, y_0)$, namely a decreasing net $\{(x_\alpha, y_\alpha)\}_{\alpha\in A}$ with some directed index set A. Then $\{x_\alpha\}_{\alpha\in A}$ is a Cauchy net by (MP3). To see this, one can argue in the same way as in the proof of Theorem 23. By completeness, $\{x_\alpha\}_{\alpha\in A}$ is convergent to some $\hat{x} \in X$, hence there is $\hat{y} \in Y$ such that

$$\forall \alpha \in A :\ (\hat{x}, \hat{y}) \preceq (x_\alpha, y_\alpha)\,.$$

This shows that S_0 is bounded below in $S(x_0, y_0)$. Zorn's lemma, applied to the partially antisymmetric quasiorder $\preceq$, ensures the existence of a partial minimal point $(\bar{x}, \bar{y}) \in S(x_0, y_0)$. This completes the proof. ■

Using Theorem 29 we may obtain results which are the analogues to those of Chapter 5 in complete uniform space. We do not go into the details, but switch to the case of sequentially complete uniform spaces.

Chapter 7

Variational Principles on Sequentially Complete Uniform Spaces

Two observations gave rise to the developments of this chapter. First, there are two proofs for the central result in Brønsted's paper [8], the first one involves Zorn's lemma, the second one does not, but only a countable induction argument. The question arises, in which cases the countable induction argument is sufficient. Secondly, minimal element theorems on sequentially complete uniform spaces have been established by A. Löehne and the author in [51] using a scalarization technique and the Brézis–Browder theorem [6]. The proof of the latter involves a countable induction argument only. Again, the question is under which assumptions this is sufficient. The results of this chapter show that in presence of a monotone real valued function with suitable porperties linking the order and the uniform structure it is not necessary to use full versions of Zorn's lemma. Since only sequences are involved, the completeness assumption of the last chapter can be weakend to sequential completeness.

7.1 The basic theorem with sequential completeness

7.1.1 Preliminaries

Let $(X,\mathcal{U})$ be uniform space. The quasiorder $\preceq$ on X is called **sequentially lower closed** iff the section $S(x) = \{x' \in X : x' \preceq x\}$ is sequentially lower closed, i.e., if $\{x_n\}_{n\in\mathbb{N}} \subseteq S(x)$ is decreasing with respect to $\preceq$ and convergent to $\bar{x} \in X$, then $\bar{x} \in S(x)$.

7.1.2 The basic theorem

Theorem 30 *Let the following assumptions be satisfied:*
(M1) $(X,\mathcal{U})$ *is a separated, sequentially complete uniform space;*
(M2) $\preceq$ *is a quasiorder on* X*;*

(M3) The function $f : X \to \mathrm{I\!R} \cup \{+\infty\}$ is proper, bounded from below on X and monotone with respect to $\preceq$, i.e.

$$x_1 \preceq x_2 \quad \Longrightarrow \quad f(x_1) \leq f(x_2),$$

moreover, for each $E \in \mathcal{U}$, there is $\delta > 0$ such that

$$x_1 \preceq x_2,\ f(x_2) - f(x_1) < \delta \quad \Longrightarrow \quad (x_1, x_2) \in E;$$

(M4) The quasiorder $\preceq$ is sequentially lower closed.
Then, for each $x_0 \in \operatorname{dom} f$ there exists $\bar{x} \in \operatorname{dom} f$ such that

$$\bar{x} \in S(x_0) \qquad \text{and} \qquad \{\bar{x}\} = S(\bar{x}).$$

Proof. Starting with x_0 we choose a sequence according to

$$x_{n+1} \in S(x_n), \qquad f(x_{n+1}) \leq \inf_{x \in S(x_n)} f(x) + \frac{1}{n}.$$

The monotonicity of f implies $f(x_{n+1}) \leq f(x_n)$ for all $n \in \mathrm{I\!N}$. Since f is bounded from below, the sequence $\{f(x_n)\}_{n \in \mathrm{I\!N}}$ converges to some $r \in \mathrm{I\!R}$.

Take $E \in \mathcal{U}$ and choose $n_E \in \mathrm{I\!N}$ such that $f(x_{n_E}) < r + \delta$ with δ from (M3). This implies for $n \geq n_E$

$$f(x_{n_E}) - f(x_n) < r + \delta - r = \delta.$$

Hence, for all $m \geq n \geq n_E$

$$f(x_m) - f(x_n) < \delta$$

holds true implying $(x_n, x_m) \in E$. Therefore, $\{x_n\}_{n \in \mathrm{I\!N}}$ is a Cauchy sequence. Since X is sequentially complete, it converges to some $\bar{x} \in X$ and by (M4), $\bar{x} \in S(x_n)$ for each $n \in \mathrm{I\!N}$. Especially, $\bar{x} \in S(x_0)$ holds true.

Let $x \preceq \bar{x}$, $x \neq \bar{x}$. Since f is monotone, the yields $f(x) \leq f(\bar{x})$. On the other hand, the transitivity of $\preceq$ implies $x \preceq \bar{x} \preceq x_n$ for each $n \in \mathrm{I\!N}$. The rules for the choice of x_{n+1} yield

$$f(\bar{x}) \leq f(x_{n+1}) \leq f(x) + \frac{1}{n}.$$

This yields $f(\bar{x}) \leq r \leq f(x)$, hence $f(\bar{x}) = r = f(x)$.

Again, take an arbitrary $E \in \mathcal{U}$. Since $\{f(x_n)\}_{n \in \mathrm{I\!N}}$ converges to $f(x)$, there is $n_E \in \mathrm{I\!N}$ such that

$$\forall n \geq n_E : \ f(x) - f(x_n) < \delta$$

This implies $(x_n, x) \in E$. Since $(X, \mathcal{U})$ is separated, we can conclude $x = \bar{x}$. ■

7.1.3 Equivalent formulations of the basic theorem

Without serious difficulties it is possible to transform the equivalent formulations of the basic minimal element theorem for metric spaces to the case of sequentially complete uniform spaces. We shall give the statements and refer for the details of the proofs to Section 4.1.3 and 7.1.3.

Theorem 31 *Let (M1) through (M4) of Theorem 30 be in force and, additionally,* $T : X \to \mathcal{P}(X)$ *be a set-valued mapping. If* T *satisfies*

$$\forall x \in X,\ \exists x' \in T(x) :\ x' \preceq x, \tag{WC}$$

then there is $\bar{x} \in X$ *such that* $\bar{x} \in T(\bar{x})$*, i.e.,* $\bar{x}$ *is a fixed point of* T*. If* T *satisfies*

$$\forall x \in X,\ \forall x' \in T(x) :\ x' \preceq x, \tag{SC}$$

then there is $\bar{x} \in X$ *such that* $\{\bar{x}\} = T(\bar{x})$*, i.e.,* $\bar{x}$ *is an invariant point of* T*.*

Proof. Each point $\bar{x}$ satisfying the conclusions of Theorem 30 does the job. ■

Theorem 32 *Let (M1) through (M4) of Theorem 30 be in force and, additionally:*
(M5) The set $M \subseteq X$ *satisfies*

$$\forall x \in S(x_0) \setminus M \quad \exists x' \in S(x) \setminus \{x\}.$$

Then, there exists $\bar{x} \in S(x_0) \cap M$*.*

Proof. By Theorem 30, there exists $\bar{x} \in S(x_0)$ such that $\{\bar{x}\} = S(\bar{x})$. By assumption (M5), $\bar{x} \in M$, hence $\bar{x} \in M \cap S(x_0)$. ■

Again, Theorem 30 can be derived from Theorem 31 and Theorem 32 in the same way as Theorem 16 from Theorem 17 and Theorem 18, respectively.

7.1.4 Set relation ordering principle

In this section, the analogues to Theorems 21 and 22 shall be established.

Let X, Y as well as $M \subseteq X \times Y$ be nonempty sets. Again, we define as in Section 4.1.6, $M(x) := \{(x', y) \in X \times Y :\ x' = x,\ (x', y) \in M\} \in \widehat{\mathcal{P}}(X \times Y)$ and $M_Y(x) := \{y \in Y :\ (x, y) \in M\} \in \widehat{\mathcal{P}}(Y)$ for $x \in X$.

Let $\preceq$ be a quasiorder on M. Then, $(\{M(x) :\ x \in X\}, \preccurlyeq)$ as well as $(\{M(x) :\ x \in X\}, \curlyeqprec)$ is quasiordered, compare Section 4.1.6.

Theorem 33 *Let the following assumptions be satisfied:*
(M1') $(X, \mathcal{U})$ *is a separated, sequentially complete uniform space and* Y *as well as* $M \subseteq X \times Y$ *are nonempty sets;*
(M2') $\preceq$ *is a quasiorder on* $X \times Y$*;*

(M3') The function $g : Y \to \mathbb{R} \cup \{+\infty\}$ *is proper, bounded from below on* Y *and satisfies the monotonicity condition*

$$(x_1, y_1) \preceq (x_2, y_2) \quad \Longrightarrow \quad g(y_1) \leq g(y_2),$$

moreover, for each $E \in \mathcal{U}$*, there is* $\delta > 0$ *such that*

$$(x_1, y_1) \preceq (x_2, y_2), \; g(y_2) - g(y_1) < \delta \quad \Longrightarrow \quad (x_1, x_2) \in E;$$

(M4') If $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq M$ *is a decreasing sequence with respect to* $\preceq$ *such that* $\{x_n\}_{n\in\mathbb{N}}$ *is converges to* $x \in X$*, then there is* $y \in Y$ *such that* $(x, y) \in M$ *and*

$$\forall n \in \mathbb{N}: \; (x, y) \preceq (x_n, y_n).$$

Then, for each $x_0 \in X$ *with* $M_Y(x_0) \cap dom\, g \neq \emptyset$*, there exists* $\bar{x} \in X$ *such that* $M_Y(\bar{x}) \neq \emptyset$ *and*

$$\begin{array}{ll} (i) & M(\bar{x}) \preccurlyeq M(x_0) \\ (ii) & M(x) \preccurlyeq M(\bar{x}) \quad \Longrightarrow \quad x = \bar{x}. \end{array}$$

Proof. Define a binary relation on X be setting

$$x' \preceq_X x \quad \Longleftrightarrow \quad M(x') \preccurlyeq M(x).$$

in order to apply Theorem 30. From the definition of $\preccurlyeq$, one can see that $\preceq_X$ is reflexive and transitive.

To check (M3) of Theorem 30, define a function $f : X \to \mathbb{R} \cup \{+\infty\}$ by

$$f(x) := \inf_{y \in M_Y(x)} g(y).$$

Then f is proper and bounded below on X since g is proper and bounded below on Y. Moreover, $(x, y) \in M$, $y \in dom\, g$ implies $x \in dom\, f$. Assume that $x_1 \preceq_X x_2$, i.e., $M(x_1) \preccurlyeq M(x_2)$. The definition of $\preccurlyeq$ and the monotonicity property of g yield

$$\forall y_2 \in M_Y(x_2) \; \exists y_1 \in M_Y(x_1): \; g(y_1) \leq g(y_2)$$

implying $f(x_1) \leq f(x_2)$, i.e., f is monotone with respect to $\preceq_X$. Fix $E \in \mathcal{U}$ and take $\delta > 0$ from (M3'). Then there is $y_2 \in M_Y(x_2)$ such that $g(y_2) \leq f(x_2) + \frac{\delta}{2}$ according to the definition of f. Since for each $y_1 \in M_Y(x_1)$ we have $f(x_1) \leq g(y_1)$, this implies

$$g(y_2) - g(y_1) \leq f(x_2) - f(x_1) + \frac{\delta}{2}.$$

Hence from $x_1 \preceq_X x_2$, $f(x_2) - f(x_1) < \frac{\delta}{2}$ we may conclude $(x_1, x_2) \in E$.

It remains to show the sequential lower closedness of $\preceq_X$. This is straightforward by taking a $\preceq_X$-decreasing and converging sequence and construct a $\preceq$-decreasing sequence in the same way as in the proof of Theorem 21. Then (M4') is sufficient for (M4) of Theorem 30.

Finally, an application of the latter theorem yields the desired results. ■

Again, the corresponding result for $\preccurlyeq$ has to be formulated as a maximal element result for the same reasons as Theorem 22.

Theorem 34 *Let the following assumptions be satisfied:*
(M1') $(X,\mathcal{U})$ is a separated, sequentially complete uniform space and Y as well as $M \subseteq X \times Y$ are nonempty sets;
(M2') $\preceq$ is a quasiorder on $X \times Y$;
(M3') The function $g : Y \to \mathrm{I\!R} \cup \{+\infty\}$ is proper, bounded from below on Y and satisfies the monotonicity condition

$$(x_1, y_1) \preceq (x_2, y_2) \quad \Longrightarrow \quad g(y_1) \geq g(y_2),$$

moreover, for each $E \in \mathcal{U}$, there is $\delta > 0$ such that

$$(x_1, y_1) \preceq (x_2, y_2),\ g(y_1) - g(y_2) < \delta \quad \Longrightarrow \quad (x_1, x_2) \in E;$$

(M4') If $\{(x_n, y_n)\}_{n\in\mathrm{I\!N}} \subseteq M$ is a increasing sequence with respect to $\preceq$ such that $\{x_n\}_{n\in\mathrm{I\!N}}$ is converges to $x \in X$, then there is $y \in Y$ such that $(x, y) \in M$ and

$$\forall n \in \mathrm{I\!N} :\ (x_n, y_n) \preceq (x, y).$$

Then, for each $x_0 \in X$ with $M_Y(x_0) \cap \mathrm{dom}\, g \neq \emptyset$, there exists $\bar{x} \in X$ such that $M_Y(\bar{x}) \neq \emptyset$ and

$$\begin{array}{ll} (i) & M(x_0) \preccurlyeq M(\bar{x}) \\ (ii) & M(\bar{x}) \preccurlyeq M(x) \quad \Longrightarrow \quad x = \bar{x}. \end{array}$$

Proof. The proof is an application of Theorem 33 using the same arguments as in the proof of Theorem 22 applying Theorem 21. Thus, the relation $\preceq'$ and $\preccurlyeq'$ are defined by

$$(x_1, y_1) \preceq' (x_2, y_2) \quad \Longleftrightarrow \quad (x_2, y_2) \preceq (x_1, y_1)$$

and

$$M(x_1) \preccurlyeq' M(x_2) \quad \Longleftrightarrow \quad M(x_2) \preccurlyeq M(x_1),$$

respectively. Assumption (M3) of Theorem 33 is satisfied for g and $\preceq'$. For more details compare the proof of Theorem 22. ■

Of course, assumption (M3') of Theorem 34 can be formulated using a function $g : Y \to \mathrm{I\!R} \cup \{-\infty\}$ being bounded from above and satisfying

$$(x_1, y_1) \preceq (x_2, y_2) \quad \Longrightarrow \quad g(y_1) \leq g(y_2).$$

Then, one has to define $g' := -g$ using $(-1)(-\infty) = +\infty$ in order to apply Theorem 33.

7.2 The basic theorem on a product space

As in Chapter 5 for the case of a metric space X we establish a partial minimal element theorem for subsets $M \subseteq X \times Y$.

Theorem 35 *Let the following assumptions be satisfied:*
(A1) $(X, \mathcal{U})$ *is a sequentially complete, separated uniform space;* Y *is a nonempty set and* $M \subseteq X \times Y$ *is also nonempty;*
(A2) $\preceq$ *is a quasiorder on* $X \times Y$*;*
(A3) The function $f : Y \to \mathrm{I\!R} \cup \{+\infty\}$ *is bounded from below on*

$$Y_M := \{y \in Y : \exists x \in X : (x, y) \in M\}$$

and monotone with respect to $\preceq$*, i.e.,*

$$(x_1, y_1) \preceq (x_2, y_2) \quad \Longrightarrow \quad f(y_1) \leq f(y_2),$$

moreover, for each $E \in \mathcal{U}$*, there is* $\delta > 0$ *such that*

$$(x_1, y_1) \preceq (x_2, y_2), \; f(y_2) - f(y_1) < \delta \quad \Longrightarrow \quad (x_1, x_2) \in E;$$

(A4) If the sequence $\{(x_n, y_n)\}_{n \in \mathrm{I\!N}} \subseteq M$ *is decreasing with respect to* $\preceq$ *and* $\{x_n\}_{n \in \mathrm{I\!N}}$ *converges to some* $x \in X$*, then there is* $y \in Y$ *such that* $(x, y) \in M$ *and* $(x, y) \preceq (x_n, y_n)$ *for each* $n \in \mathrm{I\!N}$*.*
Then, for each $(x_0, y_0) \in M$ *with* $f(y_0) \in \mathrm{I\!R}$*, there exists* $(\bar{x}, \bar{y}) \in M$ *such that*

$$\begin{array}{lll} (i) & (\bar{x}, \bar{y}) \preceq (x_0, y_0) & \\ (ii) \quad (x, y) \in M, & (x, y) \preceq (\bar{x}, \bar{y}) & \Longrightarrow \quad x = \bar{x}, \; f(y) = f(\bar{y}). \end{array}$$

If, additionally, the assumption
(A5) $(x_1, y_1) \preceq (x_2, y_2)$*,* $y_1 \neq y_2$ *implies* $f(y_1) < f(y_2)$*;*
holds true, then $(\bar{x}, \bar{y})$ *can be chosen to be a minimal point of* M *with respect to* $\preceq$*, i.e.,*

$$\{(\bar{x}, \bar{y})\} = S(\bar{x}, \bar{y}) \cap M.$$

Proof. Starting with (x_0, y_0) we choose a sequence according to

$$(x_{n+1}, y_{n+1}) \in S(x_n, y_n) \cap M, \qquad f(y_{n+1}) \leq \inf_{(x,y) \in S(x_n, y_n) \cap M} f(y) + \frac{1}{n}.$$

The monotonicity of f implies $f(y_{n+1}) \leq f(y_n)$ for all $n \in \mathrm{I\!N}$. Since f is bounded from below on Y_M, the sequence $\{f(y_n)\}_{n \in \mathrm{I\!N}}$ converges to some $r \in \mathrm{I\!R}$.

Take $E \in \mathcal{U}$ and choose $n_E \in \mathrm{I\!N}$ such that $f(y_{n_E}) < r + \delta$ with δ from (A3). This implies for $n \geq n_E$

$$f(y_{n_E}) - f(y_n) < r + \delta - r = \delta.$$

Hence, for all $m \geq n \geq n_E$

$$f(y_m) - f(y_n) < \delta$$

holds true implying $(x_n, x_m) \in E$. Therefore, $\{x_n\}_{n \in \mathrm{I\!N}}$ is a Cauchy sequence converging to some $\bar{x} \in X$ by completeness. By (A4), there is $\bar{y} \in Y$ such that $(\bar{x}, \bar{y}) \in M$ and

$$\forall n \in \mathrm{I\!N} : \; (\bar{x}, \bar{y}) \preceq (x_n, y_n).$$

Let $(x, y) \preceq (\bar{x}, \bar{y})$. The monotonicity property of f implies $f(y) \leq f(\bar{y})$. On the other hand, the transitivity of $\preceq$ implies $(x, y) \preceq (\bar{x}, \bar{y}) \preceq (x_n, y_n)$ for each $n \in \mathbb{N}$. The rules for the choice of y_{n+1} gives

$$f(\bar{y}) \leq f(y_{n+1}) \leq f(y) + \frac{1}{n}.$$

This yields $f(\bar{y}) \leq r \leq f(y)$, hence $f(\bar{y}) = r = f(y)$.

Again, take an arbitrary $E \in \mathcal{U}$. Since $\{f(y_n)\}_{n \in \mathbb{N}}$ converges to $f(y)$, there is $n_E \in \mathbb{N}$ such that

$$\forall n \geq n_E: \; f(y) - f(y_n) < \delta$$

implying $(x_n, x) \in E$. Since $(X,\mathcal{U})$ is separated, we may conclude $x = \bar{x}$. Since $f(\bar{y}) = f(y)$, $(\bar{x}, \bar{y}) \in M$ is minimal with repect to $\preceq$ if (A5) is satisfied. This completes the proof of the theorem. ■

Before investigating some special cases of Theorem 35, we mention a version where the order relation is defined in terms involving the function f. Consider X, Y, M, $\preceq$, f as in Theorem 35 and define an order relation by

$$(x_1, y_1) \preceq_f (x_2, y_2) \quad :\Longleftrightarrow \quad \begin{cases} (x_1, y_1) = (x_2, y_2) \text{ or} \\ (x_1, y_1) \preceq (x_2, y_2) \text{ and } f(y_1) < f(y_2). \end{cases}$$

Obviously, $(x_1, y_1) \preceq_f (x_2, y_2)$ implies $(x_1, y_1) \preceq (x_2, y_2)$. Therefore, it is easily seen that the assumptions (A1) to (A5) of Theorem 35 are satisfied for $\preceq_f$ if (A1) to (A4) are satisfied for $\preceq$. This is the idea of the proof of the following theorem.

Theorem 36 *Let the assumptions (A1) through (A4) of Theorem 35 be in force. Then, for each $(x_0, y_0) \in M$ with $y_0 \in \operatorname{dom} f$ there exists $(\bar{x}, \bar{y}) \in M$ such that*

$$\begin{array}{lll} (i) & (\bar{x}, \bar{y}) \preceq_f (x_0, y_0) & \\ (ii) \quad (x, y) \in M, & (x, y) \preceq_f (\bar{x}, \bar{y}) & \Longrightarrow \quad (x, y) = (\bar{x}, \bar{y}), \end{array}$$

i.e., $(\bar{x}, \bar{y})$ is a minimal point of M with respect to $\preceq_f$.

Proof. According to the remarks above, an obvious application of Theorem 35. ■

Next, we produce a series of corollaries from Theorem 35 by special choices of Y and the order relation $\preceq$. Thereby, many recent results can be proven, e.g. minimal point theorems from [47] and [44] as well as results from [51] and [50].

The first case involves a locally convex space Y and f is replaced by a continuous linear functional.

Corollary 47 *Let the following assumptions be satisfied:*
(A1) $(X,\mathcal{U})$ is a sequentially complete, separated uniform space and $\{p_\lambda\}_{\lambda \in \Lambda}$ a family of pseudometrics generating the uniformity; Y is a locally convex space and $M \subseteq X \times Y$ is a nonempty set;

(A2) $\leq_K$ *is a quasiorder on* Y *with* $K \subseteq Y$ *being a convex set containing* $\theta \in Y$ *and a cone in* $\mathcal{P}(Y)$*, further, let* $k \in K \setminus -\operatorname{cl} K$*; a relation* $\preceq$ *on* $X \times Y$ *is defined via*

$$(x_1, y_1) \preceq (x_2, y_2) \quad :\Longleftrightarrow \quad \forall \lambda \in \Lambda : \ y_1 + p_\lambda(x_1, x_2)\, k \leq_K y_2;$$

(A3) There is a bounded set $W \subseteq Y$ *such that*

$$Y_M := \{y \in Y : \ \exists x \in X : \ (x, y) \in M\} \subseteq W \oplus K;$$

(A4) If the sequence $\{(x_n, y_n)\}_{n \in \mathbb{N}} \subseteq M$ *is decreasing with respect to* $\preceq$ *and* $\{x_n\}_{n \in \mathbb{N}}$ *converges to some* $x \in X$*, then there is* $y \in Y_M$ *such that* $(x, y) \preceq (x_n, y_n)$ *for each* $n \in \mathbb{N}$*.*
Then, for each $(x_0, y_0) \in M$ *there exists* $(\bar{x}, \bar{y}) \in M$ *such that*

$$\begin{array}{lll} (i) & (\bar{x}, \bar{y}) \preceq (x_0, y_0) & \\ (ii) \quad (x, y) \in M, & (x, y) \preceq (\bar{x}, \bar{y}) & \Longrightarrow \quad x = \bar{x}. \end{array}$$

If, additionally,
(A5) $K^+ := \{y^* \in Y^* : \ \forall y \in K \setminus \{0\} : \ y^*(y) > 0\} \neq \emptyset$*;*
is satisfied, then $(\bar{x}, \bar{y})$ *can be chosen to be a minimal point of* M *with respect to* $\preceq$*, i.e.*

$$\{(\bar{x}, \bar{y})\} = S(\bar{x}, \bar{y}) \cap M.$$

Proof. Of course, $\preceq$ is a quasiorder. A standard separation argument, applied to $\{-k\}$ and K yields a continuous linear functional $y^* \in Y^*$ such that $y^*(k) = 1$ and

$$\forall y \in K : \ y^*(y) \geq 0.$$

We set $f(y) := y^*(y)$ and check assumption (M3) of Theorem 35. Since W is bounded and f nonnegative on K, f is bounded below on $W \oplus K$ and all the more on Y_M. Take $E \in \mathcal{U}$. Then there are $r > 0$, $\lambda \in \Lambda$ such that

$$E_{r,\lambda} := \{(x_1, x_2) \in X \times X : \ p_\lambda(x_1, x_2) < r\} \subseteq E$$

since the sets $E_{r,\lambda}$ form a base of the uniformity $\mathcal{U}$. If $(x_1, y_1) \preceq (x_2, y_2)$ and $f(y_2) - f(y_1) < r$, then

$$y^*(y_1) + p_\lambda(x_1, x_2) = y^*(y_1 + k p_\lambda(x_1, x_2)) \leq y^*(y_2)$$

since f is linear and nonnegative on K. Therefore

$$p_\lambda(x_1, x_2) \leq y^*(y_2) - y^*(y_1) \leq f(y_2) - f(y_1) < r,$$

hence $(x_1, x_2) \in E_{r,\lambda} \subseteq E$ as desired.

Since assumption (A4) coincides with assumption (M4) of Theorem 35 we can apply the latter to get (i) and (ii). If, additionally, $K^+ \neq \emptyset$, then simply take $y^* \in K^+$ to ensure (M5). This completes the proof. ■

Corollary 47 is a generalization of Theorem 1 in [47] and Theorem 3.10.4 of [44]. Replacing the relation $\preceq$ by $\preceq_{y^*}$ in the sense of Theorem 36, we obtain generalizations of Theorem 4 in [47] and Theorem 3.10.7 of [44].

The generalizations mainly concern the space X and the boundedness assumption: We deal with sequentially complete uniform spaces instead of complete metric spaces. Moreover, a single element $\widehat{y}$ is replaced by a bounded set W in assumption (A2).

Note that Corollary 47 has a counterpart using quasimetrics instead of pseudometrics. Also, a formulation with an order metric is possible.

The reach of Corollary 47 is limited by the appearence of a nontrivial continuous linear functional on Y, i.e., as a rule, Y has to be a locally convex space. Moreover, it has been observed in [47], [44] as well as in [51] and [50] that the boundedness assumption can be relaxed by a weaker one. In the following corollary, a sublinear functional on a linear space being linear only on a one dimensional subspace is used as a substitute for the continuous linear functional in Corollary 47. This allows to deal with merely linear spaces and to replace the boundedness assumption by a weaker one. Functionals of this type has been introduced and investigated by C. Tammer and P. Weidner in [39], [40], [41] and extensively in [125]. Compare also Theorem 2.3.1. in [44].

Corollary 48 *Let the following assumptions be satisfied:*
(A1) $(X,\mathcal{U})$ is a sequentially complete, separated uniform space and $\{p_\lambda\}_{\lambda\in\Lambda}$ a family of pseudometrics generating the uniformity; Y is a linear space and $M \subseteq X \times Y$ is a nonempty set;
(A2) $\leq_K$ is a quasiorder on Y with $K \subseteq Y$ being a convex set containing $\theta \in Y$ and a cone in $\mathcal{P}(Y)$, further, let $k \in K\backslash -K$; a relation $\preceq$ on $X \times Y$ is defined via

$$(x_1, y_1) \preceq (x_2, y_2) \quad :\Longleftrightarrow \quad \forall \lambda \in \Lambda : \; y_1 + p_\lambda(x_1, x_2)\, k \leq_K y_2;$$

(A3) There exist $\widehat{y} \in Y$ and $\widehat{t} \in \mathbb{R}$ such that

$$\{y \in Y : \; \exists x \in X : \; (x, y) \in M\} \cap \left(\{\widehat{y} - \widehat{t}k\} \oplus (-K)\right) = \emptyset;$$

(A4) If the sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ is decreasing with respect to $\preceq$ and $\{x_n\}_{n\in\mathbb{N}}$ converges to some $x \in X$, then there is $y \in Y$ such that $(x, y) \in M$ and $(x, y) \preceq (x_n, y_n)$ for each $n \in \mathbb{N}$.
Then, for each $(x_0, y_0) \in M$ with $y_0 \in \mathbb{R}\{k\} \oplus (-K)$ there exists $(\bar{x}, \bar{y}) \in M$ such that

$$\begin{array}{lll} (i) & & (\bar{x}, \bar{y}) \preceq (x_0, y_0) \\ (ii) & (x, y) \in M, & (x, y) \preceq (\bar{x}, \bar{y}) \quad \Longrightarrow \quad x = \bar{x}. \end{array}$$

Proof. We are going to apply Theorem 35. Of course, (M1) and (M2) of this theorem are satisfied. We check (M3) using the function

$$f(y) := \inf\{t \in \mathbb{R} : \; y - \widehat{y} \in \{tk\} \oplus (-K)\}.$$

Since the function $\varphi(y) := \inf\{t \in \mathbb{R} : y \in \{tk\} \oplus (-K)\}$ is monotone with respect to $\leq_K$ and subadditive, f is monotone as well. Note that φ satisfies

$$\forall s \in \mathbb{R},\ y \in Y :\ \varphi(y + sk) = \varphi(y) + s.$$

This property is called translation property, see [44], Section 2.3. Further, f is bounded below on $Y_M := \{y \in Y : \exists x \in X : (x, y) \in M\}$. To see this, assume the contrary, i.e., there is a $\tilde{y} \in Y_M$ such that $f(\tilde{y}) < -\hat{t}$ and hence there is $\tilde{t} \in \mathbb{R}$, $\tilde{t} < -\hat{t}$ such that $\tilde{y} - \hat{y} \in \tilde{t}\{k\} \oplus (-K)$. This implies

$$\tilde{y} \in \left\{\hat{y} + \tilde{t}k\right\} \oplus (-K) = \left\{\hat{y} - \hat{t}k\right\} \oplus \left\{\left(\tilde{t} + \hat{t}\right)k\right\} \oplus (-K) \subseteq \left\{\hat{y} - \hat{t}k\right\} \oplus (-K)$$

contradicting (A3). Hence f is bounded below on Y_M.

Take $E \in \mathcal{U}$ and $r > 0$, $\lambda \in \Lambda$ such that $E_{r,\lambda} \subseteq E$. Assuming $(x_1, y_1) \preceq (x_2, y_2)$ and $f(y_2) - f(y_1) < r$ we obtain by monotonicity and the translation property of φ

$$p_\lambda(x_1, x_2) \leq \varphi(y_2 - \hat{y}) - \varphi(y_2 - \hat{y}) = f(y_2) - f(y_1) < r,$$

hence $(x_1, x_2) \in E_{r,\lambda} \subseteq E$. Therefore, the assumptions of Theorem 35 are satisfied. Note that $y_0 \in \mathbb{R}\{k\} \oplus (-K)$ implies $f(y_0) \in \mathbb{R}$.

The conclusions (i) and (ii) of Theorem 35 yield (i) and (ii) above. ■

Corollary 48 produces a generalization of Corollary 47: Y can be replaced by a linear space and the boundedness assumption can be weakened.

Note that within the setting of Corollary 48 it is difficult to give a sufficient condition for (A5) of Theorem 35, i.e., for the existence of a minimal point with respect to $\preceq$. Usually, topological properties are used as in part (g) of Theorem 2.3.1. in [44]. Note also that $f(y) = f(\bar{y})$ can be added in (ii) of Corollary 48.

The set Y in Theorem 35 is arbitrary, hence the possibility of choosing $Y \subseteq \mathcal{P}(V)$, V being a quasiordered linear space, is not excluded. We turn to this case in order to derive results similar to those of [50]. Since sets are compared, the order relations $\preccurlyeq$ and $\curlyeqprec$ appear.

Corollary 49 *Let the following assumptions be satisfied:*
(A1) $(X, \mathcal{U})$ is a sequentially complete, separated uniform space and $\{p_\lambda\}_{\lambda\in\Lambda}$ a family of pseudometrics generating the uniformity; V is a linear space and $\mathcal{M} \subseteq X \times \widehat{\mathcal{P}}(V)$ is a nonempty set;
(A2) $\leq$ is a quasiorder on V with $K \subseteq V$ being a convex set containing $\theta \in V$ and a cone in $\mathcal{P}(V)$ and $k \in K \backslash -K$; a relation $\preceq$ is defined via

$$(x_1, W_1) \preceq (x_2, W_2) \quad :\Longleftrightarrow \quad \forall \lambda \in \Lambda :\ W_1 \oplus \{p_\lambda(x_1, x_2)\, k\} \preccurlyeq W_2;$$

(A3) There exist $\hat{v} \in V$ and $\hat{t} \in \mathbb{R}$ such that

$$\left(\bigcup_{(x,W)\in\mathcal{M}} W\right) \bigcap \left(\left\{\hat{v} - \hat{t}k\right\} \oplus (-K)\right) = \emptyset;$$

(A4) If the sequence $\{(x_n, W_n)\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$ *is decreasing with respect to* $\preceq$ *and* $\{x_n\}_{n\in\mathbb{N}}$ *converges to some* $x \in X$*, then there is* $W \in \widehat{\mathcal{P}}(V)$ *such that* $(x, W) \in \mathcal{M}$ *and* $(x, W) \preceq (x_n, W_n)$ *for each* $n \in \mathbb{N}$*.*
Then, for each $(x_0, W_0) \in \mathcal{M}$ *with* $\mathbb{R}\{k\} \oplus \{\widehat{v}\} \cap (W_0 \oplus K) \neq \emptyset$ *there exists* $(\bar{x}, \bar{W}) \in \mathcal{M}$ *such that*

$$\begin{array}{lll} (i) & & (\bar{x}, \bar{W}) \preceq (x_0, W_0) \\ (ii) & (x, W) \in \mathcal{M}, & (x, W) \preceq (\bar{x}, \bar{W}) \quad \Longrightarrow \quad x = \bar{x}. \end{array}$$

Proof. Again, we wish to apply Theorem 35. It is not hard to verify that (M1) and (M2) of this theorem are matched as well as (M4). To verify (M3), we define a function $f : \widehat{\mathcal{P}}(V) \to \mathbb{R} \cup \{\pm\infty\}$ by

$$f(W) := \inf\{t \in \mathbb{R} : \ tk + \widehat{v} \in W \oplus K\}.$$

Using this definition, the monotonicity property of f and the translation property

$$\forall W \in \widehat{\mathcal{P}}(V), \forall s \in \mathbb{R} : \ f(W \oplus \{sk\}) = f(W) + s$$

can be proven straighforward. Let us show that f is bounded below on

$$\left\{W \in \widehat{\mathcal{P}}(V) : \ \exists x \in X : \ (x, W) \in \mathcal{M}\right\}.$$

Assume the contrary. Then there is $(x, W) \in \mathcal{M}$ such that $f(W) < -\widehat{t}$. Hence there is $s \in \mathbb{R}$, $s < \widehat{t}$ such that

$$sk + \widehat{v} \in W \oplus K.$$

Especially, $W \neq \emptyset$. Take $w \in W$. Then

$$w \in \widehat{v} + sk \oplus (-K) = \widehat{v} + \left(s + \widehat{t}\right) k - \widehat{t}k \oplus (-K) \subseteq \widehat{v} - \widehat{t}k \oplus (-K)$$

contradicting (A3). The last part of assumption (M3) can be proven as in the proof of Corollary 48. We may apply Theorem 35 to obtain (i) and (ii) above from its conclusions. ■

Corollary 50 *Let (A1) and (A2) of Corollar49 be satisfied with* $\preccurlyeq$ *replaced by* $\preccurlyeq$*. Moreover, assume:*
(A3) There exist $\widehat{v} \in V$ *and* $\widehat{t} \in \mathbb{R}$ *such that*

$$(x, W) \in \mathcal{M} \quad \Longrightarrow \quad W \not\subseteq \left\{\widehat{v} - \widehat{t}k\right\} \oplus (-K)$$

(A4) If the sequence $\{(x_n, W_n)\}_{n\in\mathbb{N}} \subset \mathcal{M}$ *is decreasing with respect to* $\preceq$ *and* $\{x_n\}_{n\in\mathbb{N}}$ *converges to some* $x \in X$*, then there is* $W \in \widehat{\mathcal{P}}(V)$ *such that* $(x, W) \in \mathcal{M}$ *and* $(x, W) \preceq (x_n, W_n)$ *for each* $n \in \mathbb{N}$*.*
Then, for each $(x_0, W_0) \in \mathcal{M}$ *with* $W_0 \subseteq \{t_0 k + \widehat{v}\} \oplus (-K)$ *for some* $t_0 \in \mathbb{R}$*, there exists* $(\bar{x}, \bar{W}) \in \mathcal{M}$ *such that*

$$\begin{array}{lll} (i) & & (\bar{x}, \bar{W}) \preceq (x_0, W_0) \\ (ii) & (x, W) \in \mathcal{M}, & (x, W) \preceq (\bar{x}, \bar{W}) \quad \Longrightarrow \quad x = \bar{x}. \end{array}$$

PROOF. As in the proof of Corollary 50, the only problem is to verify (M3) of Theorem 35 involving the function $f : \widehat{\mathcal{P}}(V) \to \mathrm{I\!R} \cup \{\pm\infty\}$ defined by

$$f(W) := \inf\{t \in \mathrm{I\!R} : \ W \subseteq \{tk + \widehat{v}\} \oplus (-K)\}.$$

Using this definition, the monotonicity property of f and the translation property

$$\forall W \in \widehat{\mathcal{P}}(V), \forall s \in \mathrm{I\!R} : \ f(W \oplus \{sk\}) = f(W) + s$$

can be proven straighforward. Let us show that f is bounded below on

$$\left\{W \in \widehat{\mathcal{P}}(V) : \ \exists x \in X : \ (x, W) \in \mathcal{M}\right\}.$$

Assume the contrary. Then there is $(x, W) \in \mathcal{M}$ such that $f(W) < -\widehat{t}$. Hence there is $s \in \mathrm{I\!R}$, $s < \widehat{t}$ such that

$$W \subseteq \{sk + \widehat{v}\} \oplus (-K) = \left\{\left(s + \widehat{t}\right) k - \widehat{t}k + \widehat{v}\right\} \oplus (-K) \subseteq \left\{-\widehat{t}k + \widehat{v}\right\} \oplus (-K)$$

contradicting (A3). The last part of assumption (M3) of Theorem 35 can be proven as in the proof of Corollary 48. We may apply Theorem 35 to obtain (i) and (ii) above from its conclusions. ■

Both of Corollary 49 and Corollary 50 imply Corollary 48 by setting

$$\mathcal{M} = \{(x, \{v\}) : \ (x, v) \in M\}.$$

Note the complete symmetry of the constructions in Corollary 49 and 50, respectively. Again, a sufficient condition for (A5) is a difficult task and requires additional topological assumptions such as compactness. We refer to [52].

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Selbständigkeitserklärung

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Variational Principles on Metric and Uniform Spaces

selbständig und ohne fremde Hilfe verfaßt zu haben. Andere als die angegebenen Quellen und Hilfsmittel habe ich nicht benutzt und die den benutzten Werken wörtlich oder inhaltlich entnommenen Stellen habe ich als solche kenntlich gemacht.

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