# The Geometry of Moufang sets

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# Contents

# 1 Introduction

As the title suggests, this thesis is about *Moufang sets* and some of their properties. In fact, we mainly use two structures which are very closely related: Moufang sets and *Jordan algebras*.

Moufang sets are pairs consisting of a set and some subgroups of the permutation group of the set which follow some basic operation rules: Let $M := (X, (U_x)_{x \in X})$ be a pair where $X$ is a set and $U_x$ is a subgroup of the permutation group (denoted by $\mathrm{Sym}(X)$) for all $x \in X$. We call $M$ a *Moufang set* if $U_x$ stabilizes $x$ and operates sharply transitively on $X \backslash \{x\}$ for all $x \in X$, and in addition, $U_x$ is mapped onto $U_y$ under conjugation by an element $\alpha \in U_z$ if $\alpha(x) = y$.
The groups $U_x$ are called *root groups*.

On the other hand, Jordan algebras $J$ are modules over a ring $R$ with unital element $1 \in J$ with a quadratic multiplication, denoted by $U$, which has to fulfill some equations: let $U_a b$ denote the quadratic multiplication of $a$ with $b$, then

$$U_1 = \mathrm{id}_J \quad , \quad U_x V_{y,x} = V_{x,y} U_x = U_{U_x y, x} \quad , \quad U_{U_x y} = U_x U_y U_x$$

for all $x, y \in J$ where $V_{y,x}$ denotes the linearization of $U$, $V_{x,y}(z) := \{xyz\} := U_{x,z}(y)$ (see 2.6.1 for the exact definition of the $U$-multiplication and quadratic Jordan algebras).
Well-known examples are the special Jordan algebras $A^+$ where $A$ is an associative algebra, and the $^+$ denotes the given quadratic multiplication $U_x y := xyx$, or the Jordan Clifford algebras over a given Clifford algebra. We will see that special Jordan algebras are in a one-to-one correspondence with some kinds of Moufang sets. For example, Moufang sets of the polar line can be seen as so-called *ample subspaces* of special Jordan algebras.
The main result of this thesis is the following:

**Main Result.** *Let $M$ and $M'$ be two Moufang sets of one of the following types: a Moufang set of the projective line $\mathcal{P}_1(K')$, a Moufang set of the polar line $\mathcal{PL}(K, K_0, \sigma)$, an orthogonal Moufang set $\mathcal{O}(k, L_0, q)$, or a mixed Moufang set $\mathcal{M}(k', L')$.*
*If $M \cong M'$, then all isomorphisms are known and induce – up to some exceptions – isomorphisms of the underlying skew fields.*

The motivation for this theorem was the result of my diploma thesis [8], which proved the isomorphism problem for two Moufang sets of the polar line. In my diploma thesis, I could state the result for the polar lines only when the underlying fields are neither quaternion nor biquaternion algebras. By now, we could solve these cases as well. Moreover, we have a complete solution for all Moufang sets which arise from fields and have abelian root groups.
The main result as you can see above is stated in a very general way. The isomorphisms are described more detailed in the section *Results* of the third chapter.

Besides this result we give a proof about the relation of simple Jordan algebras to Moufang sets, and we analyze the root groups of Moufang sets of the polar line so that we can prove their uniqueness.

In chapter 2, we start with the basic definitions and notations for this thesis. We explain the sesquilinear and quadratic forms as well as the polarities which we need for the main result. Furthermore, we give the definitions of some algebras like the *biquaternion algebras* and the *Clifford algebras*. We describe the *quaternion algebras* in detail and illustrate the main concepts of Moufang sets and Jordan algebras. Here, we prove for both of these basic facts which hold for arbitrary Moufang sets and Jordan algebras. We introduce the notion of a *Zelmanov polynomial* and explain at last the *special universal envelope* of a Jordan algebra.

The third chapter describes the Moufang sets which we study in the main result. We first define the underlying structures: *Skew fields* which lead to *involutory sets*; *quadratic spaces* which arise from quadratic forms, and *mixed pairs* arising from (commutative) fields. We explain what *similarity*, a weaker form of an isomorphic relation, means. With these information we create Moufang sets with abelian root groups: the Moufang sets of the *projective line* given over skew fields, Moufang sets of the *polar line* given over involutory sets, *orthogonal* Moufang sets defined over quadratic spaces and *mixed* Moufang sets given over mixed pairs. After that, we state the theorems which lead to the main result.

In chapter 4, we prove the main result step by step for the Moufang sets described above. We start with two Moufang sets of the projective line and follow the order of the theorems from the section before. The problem of Moufang sets of the projective line is already solved by Hua's well-known theorem, see [4]. We go on with the isomorphism between two mixed Moufang sets. In this case, we find a direct construction of the fields out of the data of the Moufang set. Next we investigate the isomorphism problem for Moufang sets of different types: Moufang sets of the projective line, mixed Moufang sets and orthogonal Moufang sets. These cases merely result from some lemmas. The next and more difficult problem is the isomorphism of two orthogonal Moufang sets, where the isomorphism problem can be solved in two ways; firstly via a lemma of J. Tits from [15] where we only have to show that the requirements of the lemma are indeed fulfilled, and secondly by a direct construction of the fields, similar to the one of the mixed Moufang sets.
At last we look at the Moufang sets of the polar line, being the biggest problem at all. We first investigate the cases of the isomorphism of a Moufang set of the polar line and a Moufang set of another type. Then we go on to the problem for two Moufang sets of the polar line, which – up to some exceptions – is already proved in my diploma thesis [8]. It appeared that these exceptions, when the underlying skew field is either a quaternion or a biquaternion algebra, were not easy to handle. By now, they are solved as well.

In the fifth chapter we show some further results on Moufang sets. We first prove that the special simple Jordan algebras (following the classification by K. McCrimmon and E. Zelmanov, [13]) are linked to Moufang sets, namely to all Moufang sets with abelian root groups we investigated before. This part is more a collection of facts than a concrete proof since all work has been done in the sections before. Afterwards we proof the uniqueness of the $U$-group for a Moufang set of the polar line. This proof, which is the detailed version of the proof in the paper [1], was developed during a stay in Ghent in cooperation with the other authors of the paper.

At last I want to thank all people who helped me finishing this thesis, first of all my promotor Prof. Dr. Stroth and my colleagues from Halle University, in particular Barbara Baumeister. This thesis would not have been completed like this without my several stays in Ghent, for which I want to thank Hendrik Van Maldeghem and Tom De Medts, the last one especially for his comments via email, and all the other colleagues from Ghent University who supported me during my times in Belgium.

Halle (Saale), October 6th, 2005
Rafael Knop

# 2 Basic definitions

We start with the basic definitions of structures that are important for this thesis, beginning with some basic facts about *quadratic and sesquilinear forms* and with definitions of *polarities* and *algebras*. Afterwards we look at the special structures that we need later: the definitions of a *Moufang set*, a *Jordan algebra* and the *special universal envelope* of a Jordan algebra. Before that, we start with explaining some notations and make conventions:

## 2.1 Notations and conventions

In this thesis we often look at (skew) fields, vector spaces and projective geometry. If not explained in detail, that means:

- A *skew field* or a *division ring* (usually denoted by $K$ or $F$) is a ring with unit element and inverses for all non-zero elements.
- A *field* (usually denoted by $k$ or $f$) is always commutative.
- A *vector space* over a skew field is meant as a left vector space such that the product of a matrix $M$ and a vector $v$ is written as $M \cdot v$ as usual. Scalars $\alpha$ are multiplied from the left to a vector $v$, $\alpha \cdot v$.
- In the projective geometry, classes of vectors are denoted with square brackets. $[v]$ denotes the (projective) class of vectors for which $v$ is a representative.

## 2.2 Involutions, quadratic and sesquilinear forms

**Definition 2.2.1.** Let $K$ be a skew field. An anti-automorphism $\sigma : K \to K$ is called an *involution* of $K$ if $\sigma^2 = \mathrm{id}$.

An involution $\sigma$ of $K$ is called *involution of the first kind* if $\sigma$ fixes $Z(K)$; it is called *of the second kind* otherwise.

Let $K$ be a skew field and $M_n(K)$ denote the ring of $n \times n$-matrices over $K$. Then the involutions of $M_n(K)$ are well known, see [5, p.189f]:
The transpose of matrices is an involution, usually denoted by $t$ and called the *transpose involution*. The map $L \mapsto S({}^tL)S^{-1}$ with $S = \mathrm{diag}\{Q, \dots, Q\}$ and $Q := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is an involution as well, called the *symplectic involution*.
If $A$ is a non-simple algebra, we have $A = B \oplus B^{op}$ for a simple algebra $B$ and the map $\varepsilon : (b_1, b_2) \mapsto (b_2, b_1)$ is an involution, the *exchange involution* (see [5, p.187]).

**Definition 2.2.2.** Let $k$ be a (commutative) field, $L_0$ a vector space over $k$. A *quadratic form* $q$ on $L_0$ is a function from $L_0$ to $k$ such that

1. $q(\lambda a) = \lambda^2 q(a)$ for all $\lambda \in k$ and $a \in L_0$
2. the function $f : L_0 \times L_0 \to k$ given by $f(a, b) := q(a + b) - q(a) - q(b)$ for all $a, b \in L_0$ is bilinear

The *defect* of $q$ is the set $\operatorname{def}(q) := \{v \in L_0 \mid f(v, L_0) = 0\}$.
A *quadratic space* is a triple $(k, L_0, q)$ with $k, L_0, q$ as above. The form $q$ is *anisotropic* if $q(a) = 0$ if and only if $a = 0$. $q$ is *non-defective* if $\operatorname{def}(q) = \{0\}$ and *defective* otherwise. If $\operatorname{def}(q) = L_0$, we call the form $q$ *totally defective*. Note that if the form $q$ is anisotropic and char $K \neq 2$, $q$ is always non-defective.
A quadratic space is *anisotropic* if $q$ is anisotropic, it is *non-defective* (resp. *defective* or *totally defective*) if $q$ is.
Two quadratic spaces $(k, L_0, q)$ and $(f, L_1, q^*)$ are *isomorphic* if there exists an isomorphism $\varphi : L_0 \to L_1$ of vector spaces and an isomorphism $\psi : k \to f$ of fields such that $q^* \circ \varphi = \psi \circ q$. We will mostly denote two isomorphic quadratic spaces just as $(k, q) \cong (f, q^*)$ since $L_0$ resp. $L_1$ are determined by $k$ and $q$ (resp. $f$ and $q^*$).
The *Witt index* $m$ of a quadratic form is the maximum of the dimensions of all isotropic subspaces of $L_0$:

$$m := \max\{\dim U \mid U \leq L_0 \text{ subspace, } q(U) = 0\}$$

The bilinear form $f$ is uniquely determined and called the bilinear form associated to $q$. We will sometimes denote it by $\beta q$ (especially in the cases of polarities, see 2.3).

A generalization of the bilinear forms is given by the sesquilinear forms.

**Definition 2.2.3.** Let $K$ be a skew field, $V$ be a right vector space over $K$ and $\sigma : K \to K$ an involution of $K$. A function $f : V \times V \to K$ is called a *sesquilinear form relative to* $\sigma$, if for all $u, v, x, y \in V$ and $a, b \in K$

1. $f(u + v, x + y) = f(u, x) + f(u, y) + f(v, y) + f(v, y)$
2. $f(ax, by) = a^\sigma \cdot f(x, y) \cdot b$

A sesquilinear form $f$ relative to $\sigma$ is called *reflexive* if $f(x, y) = 0 \Leftrightarrow f(y, x) = 0$. The form $f$ is called *hermitian* (resp. *skew-hermitian*), if $f(x, y)^\sigma = f(y, x)$ (resp. $f(x, y)^\sigma = -f(y, x)$).

A sesquilinear form $f$ relative to $\sigma$ is called *trace-valued* if for all $x \in V$

$$f(x, x) \in \{t + t^\sigma \mid t \in K\}$$

## 2.3 Polarities

The theory of polarities is needed to prove the main result of the isomorphism problem for orthogonal Moufang sets, see 4.6. The following notations, definitions and explanations are mostly taken from [15], chapter 8.

**Definition 2.3.1.** Let $\mathbb{P}$ be a projective space and $\pi \subset \mathbb{P} \times \mathbb{P}$ be a symmetric correspondence. Let $x \perp_\pi y$ denote $(x, y) \in \pi$. For a subset $X \subset \mathbb{P}$ we define $X^{\perp(\pi)} := \{y \in \mathbb{P} \mid x \perp_\pi y \text{ for all } x \in X\}$.
$\pi$ is a *polarity* in $\mathbb{P}$ if for all $x \in \mathbb{P}$ the set $x^{\perp(\pi)}$ is either $\mathbb{P}$ itself or a hyperplane of $\mathbb{P}$.

Let $\mathbb{P}$ be a projective space. Then the orthogonality relation with respect to a reflexive sesquilinear form $f$ in $V$ induces a polarity $\pi$ in $\mathbb{P}$. A polarity is called of *trace type* if it is represented by a trace-valued form.

Now let $\mathbb{P}$ be a projective space of a vector space $V$. A proportionality class $\kappa$ of quadratic forms in $V$ is called a *projective quadratic form* in $\mathbb{P}$. Any element $q$ of the class is said to represent $\kappa$. The Witt index and defect of $\kappa$ are those of $q$. The form $\kappa$ is *non-defective* if $q$ is. The polarity represented by $\beta q$ is called the *polarity associated to* $\kappa$.

## 2.4 Algebras

In this thesis we need some kind of algebras, namely *quaternion algebras*, *biquaternion algebras* and *Clifford algebras*. Before explaining them we need some basic algebra facts about *polynomials* and *polynomial identities*:

**Definition 2.4.1.** Let $A$ be an associative algebra. A *polynomial* $f$ in $A$ is an element of the free associative algebra $\mathcal{F}(A)$. $f$ is called a *polynomial identity* on $A$ if $f \not\equiv 0$ and $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots a_n \in A$.
$A$ is called a *PI-algebra* if it has a polynomial identity $f$ on $A$.

If $(A, \sigma)$ is an associative algebra with involution, we say that $(A, \sigma)$ satisfies a polynomial identity if we have a polynomial $f \in \mathcal{F}(A)$ such that for all $a_i \in A$ : $f(a_1, \ldots, a_n, \sigma(a_i), \ldots, \sigma(a_n)) = 0$.
The following theorems are taken from [14], p.276 and p.36:

**Theorem 2.4.2.** (Amitsur) *Let $(A, \sigma)$ be an associative algebra with involution. If $(A, \sigma)$ satisfies a polynomial identity, then $A$ satisfies a polynomial identity as well.*

**Theorem 2.4.3.** (Kaplansky) *Let $A$ be a division algebra which satisfies a polynomial identity $f$. Then $[A : Z(A)] < \infty$.*

Now, we turn to the particular algebras and start with the quaternion algebras which appear in the cases of Moufang sets of the polar line and hermitian Moufang sets. They are mostly the counterexamples or cases which we have to analyze separately:
Let $E$ be a field, $\sigma$ an automorphism on $E$ with $\sigma^2 = \mathrm{id}$ and let $K = \mathrm{Fix}_E(\sigma)$ be the fixed point set under $\sigma$. We will write $\bar{x} := x^\sigma$.
$E/K$ is a separable quadratic extension and by choosing $\beta \in K^*$, we construct a subring $Q := (E/K, \beta)$ of $M(2, E)$ consisting of the matrices

$$\begin{pmatrix} a & \beta\bar{b} \\ b & \bar{a} \end{pmatrix}$$

where $a, b \in E$. We can identify $E$ with its image in $Q$ under the map $a \mapsto \mathrm{diag}(a, \bar{a})$. Then obviously $K = Z(Q)$ and $\dim_K Q = 4$. Define

$$e_2 := \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}$$

Then every element of $Q$ can be written as $a + e_2 b$ with $a, b \in E$, and this description is unique. We define a multiplication on $Q$ by the usual rules on $E$ and via $e_2^2 := \beta$ and $ae_2 := e_2\bar{a}$.

**Definition 2.4.4.** A *quaternion algebra* is any algebra isomorphic to $Q = (E/K, \beta)$ for some separable quadratic extension $E/K$ and some $\beta \in K^*$ as defined above. A quaternion algebra has a *standard involution*, denoted by $x \mapsto \bar{x}$ and given by

$$\overline{a + e_2 b} = \bar{a} - e_2 b$$

We will now give some facts about quaternion algebras. In the following, let $Q = (E/K, \beta)$ be a quaternion algebra of characteristic 2. Let $\sigma : x \mapsto \bar{x}$ denote the standard involution. Since $\overline{a + e_2 b} = \bar{a} + e_2 b$ in $\operatorname{char} K = 2$, we have $\mathrm{Fix}_Q(\sigma) = \{d + e_2 a \mid d \in K, a \in E\}$. Obviously, all squares of fixed points $(d+e_2a)^2 = d^2 + \beta\bar{a}a$ are again fixed points, and they lie in the center $Z(Q) = K$ since $\bar{a}a \in \mathrm{Fix}_E = F = Z(Q)$.
Apparently, this only holds for the standard involution. If we have a quaternion algebra with non-standard involution $\tau$ we can always find an element fixed by $\tau$ whose square does not lie in the center. Moreover, every non-standard involution in a quaternion algebra is of the form $\tau = Int(u) \circ \sigma$, where $\sigma$ is the standard involution and $u^\sigma + u = 0$ (see [9], Proposition (2.21)). By taking a separable quadratic subfield $E/k$ which is orthogonal to $u$, the involution $\tau$ can be written as $(x + uy)^\tau = x + uy^\sigma$ where $x, y \in E$.

Next, we look at the biquaternion algebras. We just give their definition since they have only a small importance for this thesis.

**Definition 2.4.5.** A *biquaternion algebra* is an algebra which is the tensor product of two quaternion algebras.

Biquaternion algebras are the central simple algebras of degree 4 and exponent 2 or 1. They are explained in §16 of [9].

Finally, we come to the Clifford algebras. These are important examples for the Jordan algebras which we define later on. Here we distinguish between the *Clifford algebra* and the *Clifford algebra with basepoint* which is introduced by N. Jacobson and K. McCrimmon in [6].
Let $k$ be a field and $V$ a vector space over $K$. The *tensor algebra* $T(V)$ is defined as

$$T(V) := k \oplus V \oplus (V \otimes_k V) \oplus (V \otimes_k V \otimes_k V) \oplus \cdots$$

**Definition 2.4.6.**

1. Let $(k, V, q)$ be a quadratic space, $T(V)$ the tensor algebra as defined above and $I(V, q)$ the ideal of $T(V)$ given by $I(V, q) := \langle u \otimes u - q(u) \cdot 1 \mid u \in V \rangle$. Then the *Clifford algebra of* $q$ is defined as $C(V, q) := C(q) := T(V)/I(V, q)$.

2. Let $(k, V, q)$ as above, $1 \in V$ be a basepoint, that means $q(1) = 1_k$. Let $T(V)$ be again the tensor algebra and $I_B(q)$ an ideal of $T(V)$ given by $I_B(q) := \langle 1_k - 1\ ,\ x \otimes x - f(1, x)x + q(x)1 \mid x \in V \rangle$ where $f$ is the bilinear form associated to $q$. Then the *Clifford algebra with basepoint* 1 is defined as $C(q, 1) := T(V)/I_B(q)$.

## 2.5 Moufang sets

The most important structures in this thesis are the Moufang sets. They are introduced by Jacques Tits in [16]. Moufang sets are the rank-one-case of Moufang buildings. In a group theoretic sense, Moufang sets are just the *split*

*BN-pairs of rank* 1 (see for example [8] for the proof). Here, we only state the definition and some basic facts:

**Definition 2.5.1.** A *Moufang set* is a pair $(X, (U_x)_{x\in X})$, where $X$ is a set and $U_x$ is a subgroup of $\mathrm{Sym}(X)$ for all $x \in X$ such that the following holds:

(M1) $U_x$ stabilizes $x$ and operates sharply transitively on $X\backslash\{x\}$ for all $x \in X$

(M2) For all $x, y \in X$ and $\alpha \in U_y$ we have $\alpha U_x \alpha^{-1} = U_{\alpha(x)}$

The groups $U_x$ are called *root groups*. The condition (M1) is satisfied if for all fixed $x_0 \in X$ the following three identities hold:

(M1a) For all $y \in X\backslash\{x_0\} : \{u \in U_{x_0} \mid u(y) = y\} = \{\mathrm{id}_X\}$

(M1b) For all $z, y \in X\backslash\{x_0\}$ exists a $u \in U_{x_0}$ such that $u(z) = y$

(M1c) For all $u \in U_{x_0}$: $u(x_0) = x_0$

We will give examples of Moufang sets later on in section 3, where we explain several kinds of Moufang sets. More examples can be looked up in [8].
An important property of all Moufang sets is the *$\mu$-action*:
Given two elements $0, \infty \in X$, there exist unique elements $u \in U_0$ and $u', u'' \in U_\infty$ such that $u'uu''$ interchanges 0 and $\infty$ (see for example [8], (4.4)). By (M1b) there exists an element $x \in X$ such that $u(x) = \infty$. Therefore we put $\mu(x) := u'uu''$ and we call this the *simple $\mu$-action* (see also [1, p.3]). If now $x' \in X\backslash\{0, \infty\}$, the action $\mu(x, x') := \mu_{x,x'} := \mu(x)^{-1}\mu(x')$ fixes both 0 and $\infty$ and is called the *double $\mu$-action*.
In this thesis, we often say *$\mu$-multiplication* when we mean the double $\mu$-action.
In some cases we write for the $\mu$-multiplication just $\mu_x$ instead of $\mu_{1,x}$.
In [18, Chapter 33], the $\mu$-multiplication is given for all Moufang sets we are interested in. Note that [18] only deals with Moufang polygons, but since all the Moufang sets we are considering arise as residues of Moufang polygons, the formulas can be found there.

**Definition 2.5.2.** Let $(X, (U_x)_{x\in X})$ and $(Y, (U_y)_{y\in Y})$ be two Moufang sets. An *isomorphism* between $(X, (U_x)_{x\in X})$ and $(Y, (U_y)_{y\in Y})$ (or just *Moufang isomorphism* between $X$ and $Y$) is a bijection $\beta : X \to Y$ such that for all $x \in X$ the map $u_x \mapsto \beta u_x \beta^{-1}$ defines a group isomorphism from $U_x$ onto $U_{\beta(x)}$.

It is well known that a Moufang isomorphism preserves the $\mu$-multiplication: $\beta(\mu_a(b)) = \mu'_{\beta(a)}(\beta(b))$ for the $\mu$-multiplications $\mu$ and $\mu'$ in the corresponding Moufang sets.
We will also need the notion of a *sub Moufang set*, in particular for the Moufang sets of the polar line. It is defined as follows:

**Definition 2.5.3.** Let $(X, (U_x)_{x\in X})$ be a Moufang set. A pair $(X', (U'_{x'})_{x'\in X'})$ is called a *sub Moufang set* if $X' \subset X$ and

$$U'_{x'} = \{u \in U_{x'} \mid \forall y \in X' : u(y) \in X'\}$$

In particular, a sub Moufang set is again a Moufang set.
Note that a Moufang set just consists of an arbitrary set $X$ which has not to be a ring or something else. As we will see in the next section, we only deal with

Moufang sets whose given set $X$ is a structure over a skew field $K$ as a vector space, fixed point set or projective line. For these Moufang sets, the following holds:

**Theorem 2.5.4.** *Let $M_1, M_2$ be two Moufang sets with elementary abelian root groups which are given over (skew) fields $K_1, K_2$. Let $T_1 := \langle \mu_{a,b} \mid a, b \in M_1 \rangle$ and $T_2 := \langle \mu_{a,b} \mid a, b \in M_2 \rangle$ denote the torus of $M_1$ resp. $M_2$, and let $M_1 \cong M_2$. Then the following holds:*

1. $\operatorname{char} K_1 = \operatorname{char} K_2$.
2. *If $\varphi : M_1 \to M_2$ is an isomorphism, then $\varphi$ maps $Z(T_1)$ onto $Z(T_2)$.*

*Proof.* 1. Let $p = \operatorname{char} K_1$ and $q = \operatorname{char} K_2$. Since the root groups are elementary abelian $p$-groups (resp. $q$-groups) and a Moufang isomorphism induces an isomorphism on the root groups, we get $p = q$.

2. Choose $a, b \in M_1$ such that $\mu_{a,b} \in Z(T_1)$. Then for all $c, d \in M_1$ we have $\varphi(\mu_{a,b}\mu_{c,d}) = \varphi(\mu_{c,d}\mu_{a,b})$ and since the $\mu$-multiplication is preserved under the Moufang isomorphism, we have $\mu_{\varphi(a),\varphi(b)}\mu_{\varphi(c),\varphi(d)} = \mu_{\varphi(c),\varphi(d)}\mu_{\varphi(a)\varphi(b)}$ and hence $\varphi(\mu_{a,b}) \in Z(T_2)$. So each element of $Z(T_1)$ is mapped onto an element of $Z(T_2)$. By looking at $\varphi^{-1}$ we get the converse as well.

□

## 2.6 Jordan algebras

In the literature you can find several definitions of Jordan algebras. The notion of a quadratic Jordan algebra was introduced by K. McCrimmon in [10] in 1966. In this thesis we follow the definition of McCrimmon and Zelmanov as in [13]. As we will see, Jordan (division) algebras and Moufang sets are closely related, see also [3] and [8]. We start with the general definition and will take a look at the Jordan algebras we are interested in afterwards.

Let $R$ be a ring, $J$ an $R$-module with an element $1 \in J^*$. Suppose that for every element $x \in J$ there exists a linear map $U_x : J \to J$ which maps an element $y \in J$ onto $U_x y$ such that $U : J \to \operatorname{End}(J), x \mapsto U_x$ is *quadratic*: For all $\lambda \in R$ and $x \in J$ we have $U_{\lambda x} = \lambda^2 U_x$, and the map

$$(x, y) \mapsto U_{x,y} := U_{x+y} - U_x - U_y$$

is bilinear for $x, y \in J$.

**Definition 2.6.1.** 1. A *unital (quadratic) Jordan algebra* $J := (J, U, 1)$ over a ring $R$ is an $R$-module $J$ with an element $1 \in J^*$ and a quadratic map $U : J \to \operatorname{End}_R(J)$ defined as above, such that in all scalar extensions $J_R$ of $J$ the following holds:

(JA1) $U_1 = \operatorname{id}_J$

(JA2) $U_x V_{y,x} = V_{x,y} U_x = U_{U(x)y,x}$

(JA3) $U_{U_y} = U_x U_y U_x$

where $V_{x,y}(z) := \{xyz\} := U_{x,z}(y)$ and $x, y, z \in J$.

2. A *(quadratic) Jordan algebra* is a subspace $J := (J, U, 2)$ of some unital Jordan algebra closed under the product $U_x y$ and the *square* $x^2 := U_x 1$.

3. A *(quadratic) Jordan division algebra* is a Jordan algebra $J$ such that all $0 \neq x \in J$ are invertible: there exists a $z \in J$ such that $U_x z = x$ (or equivalently $U_x U_z = U_z U_x = \mathrm{id}_J$).

The term of the *quadratic* Jordan algebra comes from the underlying multiplication: in former times the multiplication in a Jordan algebra was given by $a \bullet b := \frac{1}{2}(ab + ba)$, but this excluded characteristic 2. Jordan algebras with this multiplication were called *linear Jordan algebras.* Since we do not use this multiplication, we just talk about "Jordan algebras" and mean "quadratic Jordan algebras".
Note that we write $U_{U(x)y}$ instead of $U_{U_x y}$ to avoid multiple indices.

Now we can look at some interesting Jordan algebras: every associative algebra $A$ induces a Jordan algebra by

$$J := A^+ \ , \ U_x y := xyx \ , \ \{xyz\} = xyz + zyx$$

$A^+$ is unital if $A$ is. A Jordan algebra is *special* if it is isomorphic to a Jordan subalgebra for some $A^+$ and it is *i-special* if it is a homomorphic image of a special Jordan algebra.

We have two important classes of special Jordan algebras (and these two can indeed be interpreted as Moufang sets if $A$ is a division algebra):

First there is the *hermitian Jordan algebra*

$$H(A, \sigma) := \{x \in A \mid x^\sigma = x\}$$

for an associative algebra $A$ with involution (a self-inverse anti-automorphism) $\sigma$, or in general the *ample hermitian subspaces*

$$H_0 := H_0(A, \sigma) \subset H(A, \sigma)$$

where for all $a \in A$, $aH_0 a^\sigma \subset H_0$ and $a + a^\sigma, \ aa^\sigma \in H_0$.

Second there is the *Jordan Clifford algebra* which is the vector space of the Clifford algebra $C(q, 1)$ with basepoint 1 (see [6]):

$$J(q, 1) = V \subset C(q, 1), \ U_x y := f(x, \bar{y})x - q(x)\bar{y}$$

where $\bar{y} := f(y, 1)1 - y$.

We turn to some properties of the Jordan multiplication which we take from [13]. Although there exists no commutator $[x, y]$ in a Jordan algebra, we find some commutator-like elements which lie indeed in a Jordan algebra:

1. the linearized Jordan products $\{xyz\} := U_{x,z} y$ as above and $x \circ y := U_{x,y} 1$
2. the square of the commutator $[x, y]^2 = x \circ U_y x - U_x y^2 - U_y x^2$
3. the commutator $[[x, y], z] = \{xyz\} - \{yxz\}$ and its square $\{[[x, y], z]\}^2$
4. the associator $[x, y, z] = (x \circ y) \circ z - x \circ (y \circ z)$

We will see later that the Jordan Clifford algebras are related to the *orthogonal Moufang sets*, the Jordan algebra $A^+$ is related to the *Moufang sets of the projective line*, and the hermitian Jordan algebras and their ample hermitian subspaces are related to the *Moufang sets of the polar line*. Therefore, we need the notion of a *Jordan homomorphism*:

**Definition 2.6.2.** Let $J_1 := (J, U, 2)$ and $J_2 := (J', U', 2)$ be Jordan algebras. A map $\varphi : J_1 \to J_2$ is called a *Jordan homomorphism* if $\varphi$ is additive, $\varphi(U_a b) = U'_{\varphi(a)}\varphi(b)$ and $\varphi(a^2) = \varphi(a)^2$ for all $a, b \in J_1$. If $J_1, J_2$ are unital, it suffices that $\varphi(1) = 1$ instead of preserving squares.

The kernels of Jordan homomorphisms are precisely the *ideals* $I \lhd J$ which are both *inner ideals* and *outer ideals*:

$$A \lhd_i J \text{ inner if } U_A J \subset A \quad , \quad B \lhd_o J \text{ outer if } U_J B \subset B$$

A Jordan algebra is *simple* if it has no proper ideals at all. Note that a Jordan division algebra is always simple.

A *Jordan isomorphism* is a bijective Jordan homomorphism. We introduce a weaker condition with the notion of the *isotopy* of hermitian Jordan algebras (see [5, p. 246]):

Let $A$ be an associative algebra with an invertible element $c \in A$. We define a $c$-multiplication $a_c b := acb$ which has as unit $c^{-1}$ and thus defines an associative algebra $A^{(c)}$. Let $J$ be a hermitian Jordan algebra which is a subalgebra of $A^+$ such that $c, c^{-1} \in J$. Then $J$ is a subalgebra of $A^{(c)+}$: the subspace $J \subset A$ contains the element $c^{-1}$ and is closed under the Jordan multiplication $U_a^{(c)} := U_a U_c$. Hence we denote this hermitian Jordan algebra as $J^{(c)}$ and call it the *$c$-isotope* of $J$.

A Jordan isomorphism of a hermitian Jordan algebra $J$ onto the $c'$-isotope of a hermitian Jordan algebra $J'$ is called an *isotopy* of $J$ onto $J'$, and $J$ and $J'$ are called *isotopic*. In other words, a bijective additive map between two unital hermitian Jordan algebras which preserves the $U$-multiplication, but does not map the unitals onto each other, is a Jordan $c$-isotopy, where $c$ is the image of the unital.

Note that isotopy is a general algebraic concept which exists for all Jordan algebras. In this thesis we only need the isotopy for hermitian Jordan algebras and define it therefore only for them.

Another important structure in Jordan algebras are the *Jordan polynomials*, especially the *Zelmanov polynomial* $Z_{48}$ which is essential for this thesis. To construct the Jordan polynomials, we need some basic algebra knowledge:

**Remark 2.6.3.** Let $X$ be a set and $\mathcal{F}(X)$ the free associative algebra over $X$. Then there exists an involution on $\mathcal{F}(X)$, the *canonical involution* $*$, given by $(x_1 x_2 \cdots x_n)^* = x_n \cdots x_2 x_1$. Then $H(\mathcal{F}(X), *)$ is a special Jordan algebra.

The following definition is analogue to 2.4.1 for Jordan algebras:

**Definition 2.6.4.** Let $X, \mathcal{F}(X), H(\mathcal{F}(X), *)$ be as above. The subalgebra of $H(\mathcal{F}(X), *)$ generated by the $x_i \in \mathcal{F}(X)$ is a special Jordan algebra called the *free special Jordan algebra* $\mathcal{FSJ}(X)$. The elements of $\mathcal{FSJ}(X)$ are called *Jordan polynomials*.

A special Jordan algebra $J$ satisfies a *polynomial identity* if there exists a Jordan polynomial $0 \neq f \in \mathcal{FSJ}(X)$ such that $f(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in J$.

A special polynomial identity is the *standard Clifford identity*, given by

$$[[x, y]^2, z, w] = 0 \text{ if } \operatorname{char} \neq 2 \qquad \text{or} \qquad [x, y]^2 \circ z = 0 \text{ otherwise}$$

The *Zelmanov polynomial* $Z_{48}$ is a Jordan polynomial given by

$$Z_{48} = [[p_{16}(x_1, y_1, z_1, w_1), p_{16}(x_2, y_2, z_2, w_2)], p_{16}(x_3, y_3, z_3, w_3)]$$

where $p_{16}(x, y, z, w) = [[\{[[x, y], z]\}^2, [[x, y], w]], [[x, y], w]]$ (see also [13], part III). Hence by [13, (14.2) and (7.7)], if for a Jordan algebra $J$ we have $Z_{48}(J) = 0$, then $J$ satisfies the standard Clifford identity as well.

The Zelmanov polynomial is needed for the following quite essential theorem:

**Theorem 2.6.5.** *Let $K$ be a skew field, $k = Z(K)$ be its center and $\sigma$ an involution on $K$. Let $J = H_0(K, \sigma)$ be a Jordan ample subspace. If $Z_{48}(J) = 0$ then one of the following holds:*

1. $\dim_k K \leq 4$
2. *$K$ is a biquaternion division algebra and $\sigma$ is a symplectic involution.*

*Proof.* Since $Z_{48}(J) = 0$ we know by 2.4.1 that $J$ satisfies a polynomial identity $Z_{48} \equiv 0$. Thus the condition $Z_{48}(x_1 + x_1^\sigma, \ldots, x_{12} + x_{12}^\sigma) \equiv 0$ holds in $K$ since $x + x^\sigma \in J$. Now $Z_{48}(x_1 + x_1^\sigma, \ldots, x_{12} + x_{12}^\sigma)$ can be seen as a polynomial in $K[x_1, \ldots, x_{12}, x_1^\sigma, \ldots, x_{12}^\sigma]$. Thus, $(K, \sigma)$ satisfies a polynomial identity. By Amitsur's theorem 2.4.2, $K$ satisfies already a polynomial identity.
Then we can use Kaplansky's theorem 2.4.3 and we get

$$\dim[K : k] < \infty$$

Therefore, we can assume without loss of generality that $K$ is finite-dimensional. Since $Z_{48}(J) \equiv 0$, we know by the *Clifford Interconnection Theorem* (7.11) from [13], that $J$ has at most two orthogonal idempotents. By the classification of simple algebras with involution (see for example [5] or [8]), we know that only the following two cases can occur:

1. $J$ is simple algebra over $\bar{k}$, the algebraic closure of the field $k$. Then $J \cong (M_n(\bar{k}), t)$ or $J \cong (M_m(\bar{k}), t_S)$ with $m = 2r$ even. Hereby $t$ denotes the transpose involution and $t_S$ denotes the symplectic involution, see [5]. Since $J$ has at most two orthogonal idempotents, we have $n, m \leq 2$.
2. $J$ is not simple over $\bar{k}$. Then $J \cong (M_n(\bar{k}) \oplus M_n(\bar{k}), \varepsilon)$ where $\varepsilon$ is the exchange involution, and again $n \leq 2$.

Overall, there are six cases left:

1. $J \cong (M_1(\bar{k}), t)$
2. $J \cong (M_2(\bar{k}), t)$
3. $J \cong (M_2(\bar{k}), t_S)$

4. $J \cong (M_4(\bar{k}), t_S)$
5. $J \cong (M_1(\bar{k}) \oplus M_1(\bar{k}), \varepsilon)$
6. $J \cong (M_2(\bar{k}) \oplus M_2(\bar{k}), \varepsilon)$

In all cases – except case 4 – we get that $K$ is either a field or a quaternion division algebra, and all cases of involutions are possible. This means that either $\dim_k K = 1$ or $\dim_k K = 4$. In case 4 we get a biquaternion division algebra and $\sigma$ is a symplectic involution. □

## 2.7 The special universal envelope of a special Jordan algebra

The *special universal envelope* of a (special) Jordan algebra is an important concept for the isomorphism problem in the case of the Moufang set of the polar line. We start with the notion of a $\sigma$-envelope for algebras with involution:

**Definition 2.7.1.** Let $(A, \sigma)$ be an associative algebra with involution. Then $A$ is called a *$\sigma$-envelope* of the Jordan ample subspace $J \subset H(A, \sigma)$ if $J$ consists of elements fixed by $\sigma$ and $J$ generates $A$ as an associative algebra.

More important than the $\sigma$-envelopes are the special universal envelopes:

**Definition 2.7.2.** Let $J$ be a special Jordan algebra. A *special universal envelope* of $J$ is a pair $(su(J), \pi)$ where $su(J)$ is an associative algebra and $\pi$ is a Jordan homomorphism of $J$ into $su(J)^+$ such that $\pi(J)$ generates $su(J)^+$ and – if $\tau$ is any homomorphism of $J$ into $B^+$ with $B$ associative – there exists a unique homomorphism $\zeta$ of associative algebras such that

$$\begin{tikzcd} J \arrow[r, "\pi"] \arrow[d, "\tau"'] & su(J) \arrow[dl, "\zeta"] \\ B & \end{tikzcd}$$

is commutative.

**Remark 2.7.3.** Every special Jordan algebra has a special universal envelope, see [5] p.247.
If $(A, \sigma)$ is a central simple algebra with involution such that $\deg H(A, \sigma) \geq 3$, then $(A, \pi)$ with $\pi : H(A, \sigma) \to A$ the canonical injection constitute a special universal envelope for $H(A, \sigma)$. (see [5], theorem (5.11.5))
If $C(q, 1) = T(V)/I_B(q)$ is a Clifford algebra with basepoint 1 as in 2.4.6 such that $J(q, 1)$ is a Jordan Clifford algebra, then $(C(q, 1), \pi)$ with $\pi : x \mapsto x + I_B(q)$ is a special universal envelope for $J(q, 1)$. (see [6], p.10)

We just prove the universality of the special universal envelopes:

**Lemma 2.7.4.** *Let $J$ be a special Jordan algebra with special universal envelope $(su(J), \pi)$. Then $(su(J), \pi)$ is unique in the sense that, if $(su(J)', \pi')$ is another special universal envelope of $J$, there exists a unique isomorphism $\tau : su(J) \to su(J)'$ such that $\tau\pi = \pi'$.*

*Proof.* Let $(su(J), \pi)$ and $(su(J)', \pi')$ be two special universal envelopes for $J$. Since $su(J)'$ is an algebra and $\pi'$ is a Jordan homomorphism from $J$ to $su(J)'$, we can write the diagram as follows:

$$\begin{tikzcd} J \arrow[r, "\pi"] \arrow[d, "\pi'"'] & su(J) \arrow[dl, "\zeta"] \\ su(J)' & \end{tikzcd}$$

Thus $\sigma = \pi'$ and we put $\tau := \zeta$. What is left to show is that $\tau$ is indeed an isomorphism. Since we can invert the diagram above, we get with $\tau\pi = \pi'$ and $\tau'\pi' = \pi$:

$$\tau(\tau'\pi') = \pi' \Leftrightarrow (\tau\tau')\pi' = \pi' \Leftrightarrow \tau\tau' = \mathrm{id} \Leftrightarrow \tau' = \tau^{-1}$$

□

Let $Z_{48}$ be the Zelmanov polynomial as defined above. We will write $\mathcal{Z}(J)$ for the ideal generated by all values of $Z_{48}(a_1, \ldots, a_{12})$ for $a_i \in J$. We can imagine the importance of the special universal envelope when looking at the following theorem which we need for the isomorphism problems for Moufang sets of the polar line (taken from [11]):

**Theorem 2.7.5.** ($Z$-Algebra Theorem, McCrimmon, 1989) *If $J$ is a unital special Jordan algebra with $\mathcal{Z}(J) = J$, then all $\sigma$-envelopes $A$ of $J$ are isomorphic to the special universal envelope: $(A, \sigma) \cong (su(J), \pi)$. Moreover, such a $J$ is necessarily ample in $H(A, \sigma)$ : we have $J = H_0(A, \sigma)$.*

# 3 Some Moufang sets

In the beginning we give some *parameters* which we need for the construction of the Moufang sets we are interested in. These are the *Moufang sets of the projective line*, the *Moufang sets of the polar line*, the *orthogonal Moufang sets* and the *mixed Moufang sets*. For all these kinds of Moufang sets (up to some exceptions) one of the main results of this work, the isomorphism problem, can be solved as it is done in section 4. We also state the *results* here which we prove in the next section.

## 3.1 Parameters and Similarity

We start with the definitions which we need to construct Moufang sets:

**Skew Fields**

**Definition 3.1.1.** Let $K$ be a skew field and let $K^{op}$ denote its opposite. Two skew fields $K$ and $F$ are called *similar* if $K \cong F$ or $K^{op} \cong F$.

**Involutory Sets**

**Definition 3.1.2.** An *involutory set* is a triple $(K, K_0, \sigma)$, where $K$ is a field or skew field, $\sigma$ a nontrivial involution of $K$ and $K_0$ is an additive subgroup of $K$ such that

1. $K_\sigma \subset K_0 \subset \mathrm{Fix}_K(\sigma)$
2. $a^\sigma K_0 a \subset K_0$ for all $a \in K$ and
3. $1 \in K_0$

where $K_\sigma = \{a + a^\sigma \mid a \in K\}$.
Two involutory sets $(K, K_0, \sigma)$ and $(F, F_0, \tau)$ are *isomorphic* if there exists a field isomorphism $\psi : K \to F$ such that $F_0 = \psi(K_0)$ and $\psi \circ \sigma = \tau \circ \psi$.

We need some basic facts about involutory sets. The following observations are taken from [18, (11.6) and (11.8)]:

**Corollary 3.1.3.** *Let $(K, K_0, \sigma)$ be an involutory set.*

1. *$K_0^*$ is closed under inverses.*
2. *Let $t$ be a non-zero element of $K_0$, and let $\hat{K}_0 := tK_0$ and $\hat{\sigma}$ denote the anti-automorphism of $K$ given by $a^{\hat{\sigma}} = ta^\sigma t^{-1}$ for all $a \in K$. Then $\hat{K}_0 = K_0 t^{-1}$, $K_{\hat{\sigma}} = tK_\sigma$, $\mathrm{Fix}_K(\hat{\sigma}) = t\,\mathrm{Fix}_K(\sigma)$ and $(K, \hat{K}_0, \hat{\sigma})$ is an involutory set.*

**Definition 3.1.4.** The involutory set $(K, \hat{K}_0, \hat{\sigma})$ as defined above is called the *translate* of $(K, K_0, \sigma)$ with respect to $t$.
Two involutory sets are called *similar* if one is isomorphic to a translate of the other one.

Involutory sets often arise as quaternion algebras with involution. The following lemmas about involutory sets from quaternion algebras are important for the isomorphism problem:

**Lemma 3.1.5.** *Let $K$ be a quaternion division algebra over its center $k$, and let $\sigma$ be its standard involution. Suppose* $\operatorname{char} k = 2$. *Let $(K, K_0, \sigma)$ be an involutory set with $K_0 \neq k$. Then there exist $x, y \in K_0$ with $xy \neq yx$. In particular, $E = k + kxy$ is a separable quadratic extension and $K \cong (E, x^2)$.*

*Proof.* Let $x \in K_0 \backslash k$, and $C_K(x) = \{t \in K \mid tx = xt\}$ be the centralizer of $x$ in $K$. Obviously $C_K(x) = k[x]$, hence $\dim_k C_K(x) = 2$.
Since $\dim_k \operatorname{Fix}_K(\sigma) = 3$ there exists a $z \in \operatorname{Fix}_K(\sigma)$ such that $xz \neq zx$. If $z \notin K_0$, then $zxz \in K_0$ by definition of $K_0$ and $xzxz \neq zxzx$: Assume $xzxz = zxzx$, then $xzxzx^{-1}z^{-1}x^{-1}z^{-1} = 1$. Multiplying from the left side with $x^4z^4$ yields $(xz)^4 = x^4z^4$ since the squares $x^2, z^2$ of fixed points are central. This means that $xz$ is a zero of the polynomial $f(t) = t^4 + x^4z^4$ in $k[t]$. On the other hand, $xz$ is zero of the polynomial $g(t) = t^2 + (xz + zx)t + x^2z^2$, which is its minimal polynomial. But obviously $\gcd(f, g) = 1$, a contradiction since both polynomials have a common zero. Thus, $xzxz \neq zxzx$.
So we can find a $y \in K_0$ with $xy \neq yx$ by putting $y := z$ if $z \in K_0$ and $y := zxz$ otherwise. As we have seen above, the minimal polynomial of $xy$ over $k$ is of degree 2. Hence $E = k + kxy$ is a separable quadratic extension, and since all squares of fixed points lie in the center of the quaternion algebra, $x^2 \in k$ and by Definition 2.4.4, $K \cong (E, x^2)$. □

**Lemma 3.1.6.** *Let $(K, K_0, \tau)$ be an involutory set of a quaternion algebra $K = E/F$ where $\tau$ is a non-standard involution. Then there exists an involutory set $(K, \widetilde{K_0}, \sigma)$, where $\sigma$ is the standard involution, which is a translate of $(K, K_0, \tau)$.*

*Proof.* Remember that every non-standard involution $\tau$ is of the form $(x + uy)^\tau = x + uy^\sigma$ with $u^\sigma + u = 0$ and $x, y \in E$. Hence for every element $z \in K$ we have $z^\sigma = uz^\tau u^{-1}$ since we can write $z = x + uy$ and then
$uz^\tau u^{-1} = u(x + uy)^\tau u^{-1} = uxu{-}1 + u^2y^\sigma u^{-1} = x^\sigma + uy = x^\sigma + y^\sigma u^\sigma = (x + uy)^\sigma = z^\sigma$ since $au = ua^\sigma$ for all $a \in E$.
Hence $(K, \widetilde{K_0}, \sigma)$ with $\widetilde{K_0} = u^{-1}K_0$ is a pre-involutory set and a translate of $(K, K_0, \tau)$ if $u^{-1} \in \operatorname{Fix}_K(\tau)$ by the remark in section 3.1 (which is proved as (11.8) in [18]). Since $u^\tau = (0 + u1)^\tau = u1^\sigma = u$ and fixed points are closed under inverses, this holds indeed. $(K, \widetilde{K_0}, \sigma)$ is an involutory set if and only if $u^{-1} \in K_0$ (see [18]). By $K_\tau = \{u(y + y^\sigma) \mid y \in E\} = uK_\sigma$ we have $K_\sigma = u^{-1}K_\tau$ and since $1 \in K_\sigma, u^{-1} \in K_\tau \subset K_0$ as well. □

**Quadratic Spaces**

Remember the notion of a quadratic space as given in section 2.2. We define as above a similarity relation on the set of quadratic spaces:

**Definition 3.1.7.** Let $(k, L_0, q)$ be a quadratic space over the field $k$ and let $\gamma \in k^*$. Then $\gamma q$ denotes the quadratic form on $L_0$ given by $(\gamma q)(u) = \gamma \cdot q(u)$ for all $u \in L_0$. The quadratic space $(k, L_0, \gamma q)$ is called a *translate* of $(k, L_0, q)$. Two quadratic spaces $(k, L_0, q)$ and $(f, L_1, q^*)$ are *similar* if one is isomorphic to a translate of the other.

### Mixed Pairs

**Definition 3.1.8.** A pair $(k, L)$ consisting of a field $k$ of characteristic 2 and a subset $L \subset k$ is called a *mixed pair* if $k^2 \subset L$, $L$ is a vector space over $k^2$ containing $1_k$ and $\langle L \rangle = k$.
Two mixed pairs $(k, L)$ and $(f, L^*)$ are called *isomorphic* if there exists a field isomorphism $\alpha : k \to f$ with $\alpha(L) = L^*$.

## 3.2 An inventory of Moufang sets

We define the four kinds of Moufang sets we are interested in:

### Projective Lines

*Moufang sets of the projective line* are the standard examples of Moufang sets since they are easy to construct. The following results are all due to chapter 4 in [8], where we also find the proofs of the corollaries.

**Definition 3.2.1.** Let $K$ be a skew field. The *projective line* $PG_1(K)$ is the pair $(K, K^2)$ such that we identify every element $x \in K$ with

$$x := \begin{bmatrix} x \\ 1 \end{bmatrix} := \left\{ \alpha \begin{pmatrix} x \\ 1 \end{pmatrix} \mid \alpha \in K \right\} \subset K^2$$

Apparently, we can obtain every element of $K^2$ by this identification except $\begin{bmatrix} 1 \\ 0 \end{bmatrix} := \left\{ \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mid \alpha \in K \right\}$. Thus we denote $\begin{bmatrix} 1 \\ 0 \end{bmatrix} =: \infty$.

**Corollary 3.2.2.** *Let $PG_1(K)$ be the projective line over a skew field $K$ as defined above. Then $PG_1(K)$ induces a Moufang set $(X, (U_x)_{x \in X})$ given by*

- $X := PG_1(K)$ *with* $x = \begin{bmatrix} x \\ 1 \end{bmatrix}$ *and* $\infty = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- $U_\infty := \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \mid a \in K \right\}, U_0 := \left\{ \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \mid b \in K \right\}$
- *for* $x \neq 0 : U_x := \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} u_0 \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} \mid u_0 \in U_0 \right\}$

**Definition 3.2.3.** The Moufang set defined in the corollary above is called the *Moufang set of the projective line* $\mathcal{P}_1(K)$.

The following observation about the $\mu$-multiplication is taken from [8] and can also be found in [18]:

**Corollary 3.2.4.** *Let $K$ be a skew field. Then $\mathcal{P}_1(K)$ is a Moufang set with $\mu$-multiplication $\mu_{b,a}$ given by*

$$\mu_{\bar{b},\bar{a}}(\bar{x}) := \begin{bmatrix} 1 & ab^{-1}xb^{-1}a \\ 0 & 1 \end{bmatrix}$$

*where* $\bar{b} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}, \bar{a} := \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \in U_0, \bar{x} := \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \in U_\infty.$

To simplify the terms, the $\mu$-multiplication is sometimes written as $\mu_{b,a}x = ab^{-1}xb^{-1}a$. If we put $b = 1$ we just write $\mu_a x = axa$.

**Polar Lines**

With the knowledge of the Moufang sets of the projective line, we can now define the *Moufang sets of the polar line*. They are given by a projective line structure with an involution on the skew field $K$. Again, the following results are due to chapter 4 in [8]:

**Lemma 3.2.5.** *Let $K$ be a skew field such that $\mathcal{P}_1(K) = (K, (U_x)_{x\in K})$ is a Moufang set of the projective line. Then $(K_0, (U'_x)_{x\in K_0})$ is a sub Moufang set of $\mathcal{P}_1(K)$ if and only if $K_0$ is closed under addition and inverses, and for all $x, y \in K_0 : yxy \in K_0$.*

*Proof.* Remember that by definition $(K_0, (U'_x)_{x\in K_0})$ is a sub Moufang set if $U'_{K_0} = \{u \in U_{K_0} \mid \forall y \in K_0 : u(y) \in K_0\}$, where $U$ are the root groups of $\mathcal{P}_1(K)$ and $U'$ are the new root groups.
"$\Rightarrow$:" Let $K_0$ be a sub Moufang set. Then for all $b, x \in K_0$ we have

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} x+b \\ 1 \end{bmatrix}$$

such that $x+b \in K_0$, thus $K_0$ is closed under addition. Since $\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ b \end{bmatrix} = \begin{bmatrix} b^{-1} \\ 1 \end{bmatrix}$, $K_0$ must be closed under inverses, too. At last, the $\mu$-multiplication in $K_0$ is given by $\mu_y x = yxy \in K_0$.
"$\Leftarrow$:" Assume that $K_0$ is closed under addition and inverses, and $yxy \in K_0$ for $x, y \in K_0$. Then the $\mu$-multiplication lies in $K_0$ and we get the root groups $U'_0$ and $U'_\infty$ as expected if $\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$ is again in $K_0$ (the other cases are proved above). But $\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ bx+1 \end{bmatrix} = \begin{bmatrix} x(bx+1)^{-1} \\ 1 \end{bmatrix}$, and $x(bx+1)^{-1} = \frac{x}{bx+1} = \frac{1}{b+x^{-1}} = (b + x^{-1}) \in K_0$ since $b, x \in K_0$. Now the other root groups are defined by conjugation, so $U'_z = \{u \in U_z \mid \forall y \in K_0 : u(y) \in K_0\}$ holds as well. $\square$

**Lemma 3.2.6.** *Let $(K, K_0, \sigma)$ be an involutory set. Then $K_0 \cup \{\infty\}$ is a sub Moufang set of $\mathcal{P}_1(K)$.*

*Proof.* Since $K_0$ is an additive subgroup, closed under inverses and $yxy \in K_0$ for $x, y \in K_0$, the result follows from the lemma above. $\square$

**Definition 3.2.7.** The Moufang set defined in the lemma above is called the *Moufang set of the polar line* $\mathcal{PL}(K, K_0, \sigma)$.

Obviously, the $\mu$-multiplication is the same as in the projective line case (up to the restriction).

**Orthogonal Sets**

We turn to the *orthogonal Moufang set* $\mathcal{O}(k, L_0, q)$:

**Lemma 3.2.8.** *Let $(k, L_0, q)$ be an anisotropic quadratic space and define a new vector space $V := L_0 \oplus k^2$ with a new quadratic form $q_v(x, \alpha, \beta) := q(x) + \alpha\beta$, with $x \in L_0, \alpha, \beta \in k$. Then the following holds:*

1. *$q_v : V \to k$ is a quadratic form of Witt index 1.*
2. *Let $X := \{W \subset V \mid W$ subspace, $\dim W = 1, q_v(W) = 0\}$. All elements of $X$ are represented by an element of the form $\begin{bmatrix} a \\ 1 \\ -q(a) \end{bmatrix}$ with $a \in L_0$. There is also an element $\infty := \begin{bmatrix} 0_{L_0} \\ 0 \\ 1 \end{bmatrix}$.*
   *The subgroups $(U_W)_{W \in X}$ are defined as follows:*
   - *For $W = \langle \begin{pmatrix} 0_{L_0} \\ 0 \\ 1 \end{pmatrix} \rangle$, $U_W =: U_\infty$ consists of the maps $u_v, v \in L_0$ fixing $\infty$ and mapping $\begin{bmatrix} a \\ 1 \\ -q(a) \end{bmatrix}$ onto $\begin{bmatrix} a+v \\ 1 \\ -q(a+v) \end{bmatrix}$.*
   - *For $W = \langle \begin{pmatrix} 0_{L_0} \\ 1 \\ 0 \end{pmatrix} \rangle$, $U_W =: U_0$ consists of the maps $u_v, v \in L_0$ fixing $\begin{bmatrix} 0_{L_0} \\ 1 \\ 0 \end{bmatrix}$ and mapping $\begin{bmatrix} a \\ -q(a) \\ 1 \end{bmatrix}$ onto $\begin{bmatrix} a+v \\ -q(a+v) \\ 1 \end{bmatrix}$ (obviously, all elements except 0 can be written in such a way).*
   - *For $W = \langle \begin{pmatrix} w \\ 1 \\ -q(w) \end{pmatrix} \rangle$ with $w \neq 0_{L_0}$, $U_W$ is given by conjugation: $U_W = u_{\overline{w}} \circ U_0 \circ u_{-\overline{w}}$, with $u_{\overline{w}}, u_{-\overline{w}} \in U_\infty$ mapping 0 on $\overline{w} = \begin{bmatrix} w \\ 1 \\ -q(w) \end{bmatrix}$.*

   *Then $(X, (U_W)_{W \in X})$ is a Moufang set.*

*Proof.* All elements of $X$ are isotropic, and it is obvious that there cannot exist more isotropic elements. Hence $q_v$ is of Witt Index 1 and it is quadratic, too. For every isotropic subspace $W$, $U_W$ stabilizes $W$ and operates sharply transitive on $X \backslash W$. Thus, $(X, (U_W)_{W \in X})$ is a Moufang set if for $W, S \in X$ and $\alpha \in U_S : \alpha U_W \alpha^{-1} = U_{\alpha(W)}$. This holds since $\alpha(W) \xmapsto{\alpha^{-1}} W \xmapsto{U_W} W \xmapsto{\alpha} \alpha(W)$, and by this $(X, (U_W)_{W \in X})$ is indeed a Moufang set. $\square$

**Definition 3.2.9.** The Moufang set defined in the lemma above is called the *orthogonal Moufang set* $\mathcal{O}(k, L_0, q)$.

The following observation about the $\mu$-multiplication is again taken from [18, (33.11)]:

**Corollary 3.2.10.** *Let $(k, L_0, q)$ be an anisotropic quadratic space such that $\mathcal{O}(k, L_0, q)$ is an orthogonal Moufang set. Then the $\mu$-multiplication is given by*

$$\mu_{a,b}(v) = \frac{q(a)}{q(b)}\pi_a\pi_b(v)$$

*for given $a, b \in L_0$ and all $v \in L_0$, where $\pi_a(v) := v - \frac{f(a,v)}{q(a)}a$.*

With the knowledge of the $\mu$-multiplication in an orthogonal Moufang set, we can prove the next theorem:

**Theorem 3.2.11.** *Let $(k, L_0, q)$ be an anisotropic quadratic space such that $\mathcal{O}(k, L_0, q)$ is an orthogonal Moufang set with the $\mu$-multiplication defined as above. If $\dim L_0 \geq 3$ and $T = \langle \mu_{a,b} \mid a, b \in L_0^* \rangle$, we have*

$$\mu_{a,b} \in Z(T) \Longleftrightarrow b \in \langle a \rangle \text{ or } a, b \in \operatorname{def}(q)$$

For the $\mu$-multiplication, we have

$$\begin{aligned} \mu_{a,b}(v) &= \tfrac{q(a)}{q(b)}\pi_a\pi_b(v) = \tfrac{q(a)}{q(b)} \ \pi_a\left(v - \tfrac{f(b,v)}{q(b)}b\right) \\ &= \tfrac{q(a)}{q(b)}\left[\ v - \tfrac{f(b,v)}{q(b)}b - \tfrac{f(a,v)}{q(a)}a + \tfrac{f(b,v)f(a,b)}{q(a)q(b)}a \ \right] \end{aligned}$$

To prove this theorem, we need the following lemma:

**Lemma 3.2.12.** *Let $a, b, c \in L_0^*$. Then $\pi_a\pi_b\pi_c = \pi_{\pi_a\pi_b(c)}\pi_a\pi_b$.*

*Proof.* Since $\pi$ is a reflection, it is well-known that we get for arbitrary $\pi_a$ by conjugation with $\pi_b$

$$\pi_b\pi_a\pi_b = \pi_{\pi_b(a)}$$

Hence we have $\pi_a\pi_b\pi_c\pi_b\pi_a = \pi_{\pi_a\pi_b(c)} \Leftrightarrow \pi_a\pi_b\pi_c = \pi_{\pi_a\pi_b(c)}\pi_a\pi_b$. $\square$

*Proof.* (Theorem 3.2.11) Let $a, b \in L_0^*$.

- If $b \in \langle a \rangle$, then $\mu_{a,b}(v) = \frac{1}{\lambda^2}v$ if $b = \lambda a$.
- Let $a, b \in \operatorname{def}(q) \Rightarrow \mu_{a,b}(v) = \frac{q(a)}{q(b)}v$ and hence $\mu_{a,b} \in Z(T)$.
- Let $b \notin \langle a \rangle$ and not both $a, b \in \operatorname{def}(q)$. Assume $\mu_{a,b} \in Z(T)$.
  Then for all $c, d \in L_0$ :

  $$\mu_{a,b}\mu_{c,d} = \mu_{c,d}\mu_{a,b} \Leftrightarrow \pi_a\pi_b\pi_c\pi_d = \pi_c\pi_d\pi_a\pi_b$$

  Now put $d = b$, then we have

  $$\pi_a\pi_b\pi_c\pi_b = \pi_c\pi_b\pi_a\pi_b \Leftrightarrow (\pi_a\pi_b)\pi_c(\pi_a\pi_b)\pi_c = 1$$

  because $\pi_v$ is inverse to itself.
  Now put $\varphi = \pi_a\pi_b$. Then $\varphi\pi_c\varphi\pi_c = 1$ and by the lemma above $\varphi\pi_v = \pi_{\varphi(v)}\varphi$ for all $v \in V$. Thus:

  $$\varphi\pi_c\varphi\pi_c = 1 \Leftrightarrow \pi_{\varphi(c)}\varphi\varphi\pi_c = 1 \Leftrightarrow \varphi^2 = \pi_{\varphi(c)}\pi_c \qquad \forall\ c \in L_0$$

  We distinguish two cases:

1. $q$ is defective (or even totally defective). Then there is a $w \in \operatorname{def}(q)^* \Rightarrow \varphi(w) \in \operatorname{def}(q)^* \Rightarrow \pi_{\varphi(w)}\pi_w = \mathrm{id}$.
2. $q$ is non-defective. Because of $\dim L_0 \geq 3$ there exists a $w \in L_0$ which is orthogonal to $a$ and $b$, hence $\varphi(w) = w \Rightarrow \pi_{\varphi(w)}\pi_w = \pi_w\pi_w = \mathrm{id}$.

In both cases we get $\varphi^2 = \mathrm{id}$. So $\pi_{\varphi(v)} = \pi_v$ for all $v \in L_0$, hence

$$\forall\ w \in L_0 : w + \frac{f(\varphi(v), w)}{q(\varphi(v))}\varphi(v) = w + \frac{f(v,w)}{q(v)}v \Rightarrow \varphi(v) = \lambda v$$

for some $\lambda \in K$ and all $v \in L_0$. But then clearly $b \in \langle a \rangle$ or $a, b \in \operatorname{def}(q)$, what means a contradiction.

□

**Mixed Sets**

Let $(k, L)$ be a mixed pair. The next type of Moufang sets are the *mixed Moufang sets* $\mathcal{M}(k, L)$. They are as usual the rank-one-case of the mixed buildings defined in [18]. A mixed Moufang set is a subset of the projective line in characteristic 2. It is defined as follows:

**Definition 3.2.13.** Let $k$ be a field of characteristic 2. If $M := L \cup \{\infty\}$ is a Moufang set such that $k^2 \subset L \subset k$ and $\langle L \rangle = k$, then $M$ is called a *mixed Moufang set* $\mathcal{M}(k, L)$.

The existence of such Moufang sets is proved by the following lemma:

**Lemma 3.2.14.** *Let $(k, L_0, q)$ be an anisotropic quadratic space with a totally defective form $q$. Then the Moufang set $\mathcal{O}(k, L_0, q)$ is isomorphic to the mixed Moufang set $\mathcal{M}(\langle q(L_0)\rangle, q(L_0))$.*

*Proof.* Let $(k, L_0, q)$ be an anisotropic quadratic space with totally defective form $q$ (so $\operatorname{char}(k) = 2$). Put $L := q(L_0)$. We get $k^2 \subset L$ since $q(\lambda v) = \lambda^2 v \Rightarrow k^2 \cdot L \subset L$. Since $q$ is totally defective, it is even additive, so $L$ is an additive subgroup of $k$. The map

$$\varphi : L_0 \to L, v \mapsto q(v)$$

is an isomorphism of vector spaces, since it is obviously surjective and injective by

$$q(v) = q(w) \Rightarrow q(v + w) = q(v) + q(w) = 2q(v) = 0 \Rightarrow v + w = 0$$

since $q$ is anisotropic and linear. But then $v = w$ in $\operatorname{char}(k) = 2$, so $\varphi$ is injective, too.
$\varphi$ is an isomorphism because of the linearity of $q$.
If we now put $\varphi(\infty) := \infty$, we have $\mathcal{O}(k, L_0, q) \cong \mathcal{M}(\langle q(L_0)\rangle, L)$ for totally defective $q$, as desired. □

## 3.3 Results

The following results are all proved in chapter 4. Theorem 3.3.1 is essentially due to L.K. Hua, see [4].

**Theorem 3.3.1.** *Let $K$ and $F$ be two skew fields and $\alpha : PG_1(K) \to PG_1(F)$ an isomorphism from $\mathcal{P}_1(K)$ onto $\mathcal{P}_1(F)$ such that $\alpha(0_K) = 0_F$ and $\alpha(\infty) = \infty$. Then $\alpha|_K$ is an isomorphism or anti-isomorphism from $K$ onto $F$.*

**Corollary 3.3.2.** *Let $K$ and $F$ be two skew fields. If $\mathcal{P}_1(K) \cong \mathcal{P}_1(F)$, then $K$ and $F$ are similar.*

**Theorem 3.3.3.** *Let $(k, L)$ and $(f, L')$ be two mixed pairs. Then $\mathcal{M}(k, L) \cong \mathcal{M}(f, L')$ if and only if $(k, L) \cong (f, L')$.*

**Theorem 3.3.4.** *Let $F$ be a skew field and $(k, L)$ a mixed pair. Then $\mathcal{P}_1(F) \cong \mathcal{M}(k, L)$ if and only if $L = k$ or $L = k^2$ and $F \cong k$.*

**Theorem 3.3.5.** *Let $K$ be a skew field and $(k, L_0, q)$ an anisotropic quadratic space. Then $\mathcal{P}_1(K) \cong \mathcal{O}(k, L_0, q)$ if and only if one of the following holds:*

1. $\dim_k L_0 = 1$ *and* $k \cong K$
2. $\dim_k L_0 = 2$ *and* $K \cong E$ *where* $E$ *is a suitable quadratic extension of* $k$.
3. $\dim_k L_0 = 4$, $\operatorname{def}(q) = 0$ *and* $K$ *is a quaternion division algebra.*
4. $\operatorname{def}(q) = L_0$, $\langle q(L_0) \rangle$ *is a subfield of* $k$ *and* $K \cong \langle q(L_0) \rangle$.

**Theorem 3.3.6.** *Let $(k, L_0, q)$ be an anisotropic quadratic space and $(f, L)$ a mixed pair. Then $\mathcal{O}(k, L_0, q) \cong \mathcal{M}(f, L)$ if and only if one of the following holds:*

1. $def(q) = L_0$, $\langle q(L_0) \rangle \cong f$ *and* $(\langle q(L_0) \rangle, L_0, q)$ *is similar to the anisotropic quadratic space* $(f^2, L, x \mapsto x^2)$
2. $\dim_k L_0 = 2$, $E$ *is a suitable quadratic extension of* $k$ *as in 3.3.5 and* $E \cong f$.

**Theorem 3.3.7.** *Let $(k, L_0, q)$ and $(f, L_1, q^*)$ be two anisotropic quadratic spaces. Then $\mathcal{O}(k, L_0, q) \cong \mathcal{O}(f, L_1, q^*)$ if and only if one of the following holds:*

1. $(k, L_0, q)$ *and* $(f, L_1, q^*)$ *are similar.*
2. *There exists a field* $k'$ *such that* $\mathcal{O}(k, L_0, q) \cong \mathcal{P}_1(k') \cong \mathcal{O}(f, L_1, q^*)$.
3. *There exists a mixed pair* $(k', L')$ *such that* $\mathcal{O}(k, L_0, q) \cong \mathcal{M}(k', L') \cong \mathcal{O}(f, L_1, q^*)$.

**Theorem 3.3.8.** *Let $F$ be a skew field and $(K, K_0, \sigma)$ an involutory set. Then $\mathcal{P}_1(F) \cong \mathcal{PL}(K, K_0, \sigma)$ if and only if $F$ is already a field, $K_0$ is a subfield of $Z(K)$ (where $Z(K) = K_0$ is possible), and $F \cong K_0$.*

**Theorem 3.3.9.** *Let $(K, K_0, \sigma)$ be an involutory set and $(f, L)$ a mixed pair. Then $\mathcal{PL}(K, K_0, \sigma) \cong \mathcal{M}(f, L)$ if and only if there exists a field $k$ such that $\mathcal{PL}(K, K_0, \sigma) \cong \mathcal{P}_1(k) \cong \mathcal{M}(f, L)$.*

**Theorem 3.3.10.** *Let $(K, K_0, \sigma)$ be an involutory set and $(f, L_0, q)$ an anisotropic quadratic space. Then $\mathcal{PL}(K, K_0, \sigma) \cong \mathcal{O}(f, L_0, q)$ if and only if one of the following holds:*

1. *There is a field* $k$ *such that* $\mathcal{O}(f, L_0, q) \cong \mathcal{P}_1(k) \cong \mathcal{PL}(K, K_0, \sigma)$.

2. *$K$ is a quaternion division algebra with norm form $N$.*
   *If* $\mathrm{char}\, K \neq 2$ *then* $\mathcal{PL}(K, K_0, \sigma) \cong \mathcal{O}(Z(K), K_0, N|_{K_0})$.
   *If* $\mathrm{char}\, K = 2$ *then there exists a subfield $E$ of $Z(K)$ with $Z(K)^2 \subset E \subset Z(K)$ and $\mathcal{PL}(K, K_0, \sigma) \cong \mathcal{O}(E, K_0, N|_{K_0})$.*

3. $\dim_f L_0 = 6$*, $K$ is a biquaternion division algebra, $K_0 = K_\sigma$ and $\sigma$ is a symplectic involution.*

**Theorem 3.3.11.** *Let $(K, K_0, \sigma)$ and $(F, F_0, \tau)$ be two involutory sets. Then $\mathcal{PL}(K, K_0, \sigma) \cong \mathcal{PL}(F, F_0, \tau)$ if and only if one of the following holds:*

1. *$(K, K_0, \sigma)$ and $(F, F_0, \tau)$ are similar.*

2. *$K$ and $F$ are fields such that $K/K_0$ and $F/F_0$ are separable quadratic extensions with $K_0 \cong F_0$.*

3. *There exists a field $k$ such that $\mathcal{PL}(K, K_0, \sigma) \cong \mathcal{P}_1(k) \cong \mathcal{PL}(F, F_0, \tau)$.*

4. *There exists a quadratic space $(k, L_0, q)$ such that*
   $\mathcal{PL}(K, K_0, \sigma) \cong \mathcal{O}(k, L_0, q) \cong \mathcal{PL}(F, F_0, \tau)$

# 4 The isomorphism problem for Moufang sets

In this section we give the proof for the theorems stated in the section before. We can summarize them in the following main result:

**Main Result.** *Let $M$ and $M'$ be two Moufang sets of one of the following types: a Moufang set of the projective line $\mathcal{P}_1(K')$, a Moufang set of the polar line $\mathcal{PL}(K, K_0, \sigma)$, an orthogonal Moufang set $\mathcal{O}(k, L_0, q)$, or a mixed Moufang set $\mathcal{M}(k', L')$.*
*If $M \cong M'$, then all isomorphisms are known and induce – up to some exceptions – isomorphisms of the underlying skew fields.*

## 4.1 $M = \mathcal{P}_1(K)$ and $M' = \mathcal{P}_1(K')$

Let $K$ and $K'$ be skew field such that the Moufang sets of the projective line $\mathcal{P}_1(K)$ and $\mathcal{P}_1(K')$ are isomorphic. The isomorphism problem for Moufang sets of the projective line is solved since 1949 when L.K. Hua presented his well-known theorem in [4] – even without knowing the terminology of a Moufang set. This version of Hua's theorem is taken from [15] and can also be found in [8]:

**Theorem 4.1.1.** (Hua, 1949) *Let $L$ and $L'$ be two projective lines over skew fields $K$ resp. $K'$ which have the same translations. Then there exists a bijection $\alpha : K \to K'$ with $\alpha(a + b) = \alpha(a) + \alpha(b)$ and $\alpha(aba) = \alpha(a)\alpha(b)\alpha(a)$ for all $a, b \in K$.*
*The bijection $\alpha$ is an isomorphism or anti-isomorphism of $K$ onto $K'$.*

Since the "translations" of the projective line are nothing else than the group $U_\infty$ of the Moufang set of the projective line, two isomorphic Moufang sets of the projective line indeed have "the same translations". So there exists an isomorphism or anti-isomorphism between $K$ and $K'$, and $\mathcal{P}_1(K) \cong \mathcal{P}_1(K') \Leftrightarrow$ $K$ and $K'$ are similar.

## 4.2 $M = \mathcal{M}(k, L)$ and $M' = \mathcal{M}(f, L')$

Next we look at the mixed Moufang sets. Remember that they are defined as $\mathcal{M}(k, L) := L \cup \{\infty\}$, where $k^2 \subset L \subset k$, $\langle L \rangle = k$ and char $k = 2$. We will show, that we can already reconstruct the field $k$ only by knowledge of the Moufang set $\mathcal{M}(k, L)$:
Let $(k, L)$ be a mixed pair such that $\mathcal{M}(k, L)$ is a mixed Moufang set. Since $\langle L \rangle = k$ it is obvious that also $\langle L^2 \rangle = k^2$. In $\mathcal{M}(k, L)$ we have two opposite elements 0 and $\infty$. Since $U_\infty$ operates sharply transitively on $L$, we can define the addition "+" for $a, b \in L$ as follows:
$a + b := u_a \circ u_b(0)$, where $u_i \in U_\infty, u_i(0) = i$
We also have an element $1 \in \mathcal{M}(k, L)$, since $\mathcal{M}(k, L)$ is a subset of the projective line and we can without loss of generality assume the existence of the 1 (otherwise just shift $\mathcal{M}(k, L)$ to an isomorphic Moufang set containing the 1).
Then $L^2 = \{\mu_a 1 \mid a \in L\}$. The addition + is in $L^2$ the same as in $L$.
Define the multiplication "$\cdot$" for $a^2, b^2 \in L^2$ as follows:
$a^2 \cdot b^2 := \mu_a(\mu_b 1)$, where $a^2 := \mu_a 1, b^2 := \mu_b 1$.

Now $\langle L^2 \rangle = k^2$, so arbitrary elements of $k^2$ are of the form

$$s^2 = \mu_{a_1}1 + \cdots + \mu_{a_n}1, \;\; t^2 = \mu_{b_1}1 + \cdots + \mu_{b_m}1$$

with $a_i, b_j \in L$ for $1 \le i \le n, 1 \le j \le m$. We can define addition and multiplication as expected by

$$s^2 + t^2 := \mu_{a_1}1 + \cdots + \mu_{a_n}1 + \mu_{b_1}1 + \cdots + \mu_{b_m}1$$

and

$$s^2 \cdot t^2 := \mu_{(\mu(a_1)1 + \cdots + \mu(a_n)1)}(\mu_{(\mu(b_1)1 + \cdots + \mu(b_m)1)}1)$$

where we write $\mu_{\mu(a_i)1}$ instead of $\mu_{\mu_{a_i}1}$ to avoid multiple indices. By this, we know the elements of $k^2$, and addition and multiplication is given as well. But in char $k = 2$ we obviously have $k^2 \cong k$, so we can reconstruct $k$ from $L$.

With the observations above we have:

**Lemma 4.2.1.** *Let $(k, L)$ and $(f, L')$ be two mixed pairs with corresponding mixed Moufang sets $\mathcal{M}(k, L)$ and $\mathcal{M}(f, L')$ such that $k^2 \subset L \subset k$ and $f^2 \subset L' \subset f$. If $\mathcal{M}(k, L) \cong \mathcal{M}(f, L')$, then we also have $(k, L) \cong (f, L')$.*

*Proof.* Let $\varphi : \mathcal{M}(k, L) \to \mathcal{M}(f, L')$ be a Moufang isomorphism. Because of $\varphi(\infty) = \infty$, the restriction $\varphi|_L : L \to L'$ must be bijective, too. In addition, $\varphi(\mu_a b) = \mu'_{\varphi(a)}\varphi(b)$, so in particular, elements of $L^2$ are mapped on elements of $L'^2$. We can extend $\varphi|_L$ to a mapping $\psi$ from $k^2$ to $f^2$ by setting:

$$\psi(a^2) := \begin{cases} \varphi(\mu_a 1) & \text{if } a^2 = \mu_a 1 \in L^2 \\ \varphi(\mu_{a_1}1) + \cdots + \varphi(\mu_{a_n}1) & \text{if } a^2 = \mu_{a_1}1 + \cdots + \mu_{a_n}1 \in k^2 \backslash L^2 \end{cases}$$

We now have to show that $\psi$ is indeed an isomorphism of fields.
Therefore, let $a^2, b^2 \in k^2$ be defined as above.

- $\psi$ is additive:
$$\begin{aligned} \psi(a^2 + b^2) &= \psi\left((a+b)^2\right) \\ &= \varphi(\mu_{a_1}1) + \cdots + \varphi(\mu_{a_n}1) + \varphi(\mu_{b_1}1) + \cdots + \varphi(\mu_{b_m}1) \\ &= \psi(a^2) + \psi(b^2) \end{aligned}$$

- $\psi$ is multiplicative:
$$\begin{aligned} \psi(a^2 \cdot b^2) &= \psi(\mu_a(\mu_b 1)) \\ &= \varphi\left(\mu_{(\mu(a_1)1 + \cdots + \mu(a_n)1)}(\mu_{(\mu(b_1)1 + \cdots + \mu(b_m)1)}1)\right) \\ &= \mu'_{\varphi(\mu(a_1)1 + \cdots + \mu(a_n)1)}\mu'_{\varphi(\mu(b_1)1 + \cdots + \mu(b_m)1)}1' \\ &= \mu'_{(\varphi(\mu(a_1)1) + \cdots + \varphi(\mu(a_n)1))}\mu'_{(\varphi(\mu(b_1)1) + \cdots + \varphi(\mu(b_m)1))}1' \\ &= \mu'_{\psi(a^2)}\mu'_{\psi(b^2)}1' \\ &= \psi(a^2) \cdot \psi(b^2) \end{aligned}$$

So $k \cong k^2 \cong f^2 \cong f$, as expected. Since $\psi|_L = L'$, we get $(k, L) \cong (f, L')$. □

## 4.3 $M = \mathcal{P}_1(K)$ and $M' = \mathcal{M}(k, L)$

Let $K$ be a skew field such that $\mathcal{P}_1(K)$ is a Moufang set of the projective line, and $(k, L)$ be a mixed pair such that $\mathcal{M}(k, L)$ is a mixed Moufang set. Since all mixed Moufang sets are given in characteristic 2, the Moufang set of the

projective line must be defined over a skew field of characteristic 2 as well by Theorem 2.5.4.
The $\mu$-multiplication of the mixed Moufang set is given by $\mu_{a,1}b = q(a)\pi_a(b) = q(a)b$. The torus $T'$ of the mixed Moufang set is hence obviously commutative. Since the Moufang isomorphism maps the torus $T$ of $\mathcal{P}_1(K)$ onto the torus $T'$ of $\mathcal{M}(k, L)$, we know that $T$ must be commutative, too. We start with the following lemma:

**Lemma 4.3.1.** *A projective Moufang set $\mathcal{P}_1(K)$ over a skew field $K$ has an abelian torus if and only if $K$ is a commutative field.*

*Proof.* The direction "$\Leftarrow$" is clear.
"$\Rightarrow$": Let $T$ be abelian. Remember that the $\mu$-multiplication of the projective line is given by $\mu_a b = aba$. We define $\rho_a(x) := ax + xa$ and observe that for all $a, x \in K$ we can describe $\rho_a(x)$ as a sum of the given $\mu_a b$ since $\rho_a(x) = \mu_{a+x}1 - \mu_a 1 - \mu_x 1$. By the commutativity of $T$ we have $\rho_a\rho_b(x) = \rho_b\rho_a(x)$ for all $a, b \in K$, hence

$$\begin{aligned} & a(bx + xb) + (bx + xb)a = b(ax + xa) + (ax + xa)b \\ \Leftrightarrow\quad & abx - bax = xab - xba \\ \Leftrightarrow\quad & (ab - ba)x = x(ab - ba) \end{aligned}$$

thus $ab - ba \in Z(K)$ for all $a, b \in K$.
Now assume there exist some $x, y \in K$ such that $x, y \notin Z(K)$. Then $(xy-yx)x = xyx - yx^2 = x(yx) - (yx)x \in Z(K)$ since $ab - ba \in Z(K)$ for all $a, b \in K$. Hence $(xy - yx)x \in Z(K)$ and since $(xy - yx) \in Z(K)$ it follows that also $x \in Z(K)$, a contradiction.
Thus all elements $a$ are in the center of $K \Rightarrow K$ is commutative. □

By the lemma above we know that $\mathcal{P}_1(K)$ can only be isomorphic to $\mathcal{M}(k, L)$ if $K$ is already commutative. We go ahead with the next lemma:

**Lemma 4.3.2.** *Let $K$ be a commutative field such that $\mathcal{P}_1(K) \cong \mathcal{M}(k, L)$. Then $K \cong k$ and either $L = k$ or $L = k^2$.*

*Proof.* Clearly, $K^2 \subset K \subset K$ and hence $\mathcal{P}_1(K) = \mathcal{M}(K, K)$ and so $\mathcal{M}(k, L) \cong \mathcal{M}(K, K)$. Then by Lemma 4.2.1, we get $k \cong K$. Since $k^2 \cong K \cong k$ and we have a Moufang isomorphism mapping $L$ onto $k$, $L$ must be a field as well. So $k^2 \subset L \subset k$ leads to either $k = L$ or $k^2 = L$. □

## 4.4 $M = \mathcal{P}_1(K)$ and $M' = \mathcal{O}(k, L_0, q)$

Let $K$ be a skew field such that $\mathcal{P}_1(K)$ is a Moufang set of the projective line, and let $(k, L_0, q)$ be an anisotropic quadratic space such that $\mathcal{O}(k, L_0, q)$ is an orthogonal Moufang set. Assume that $\mathcal{P}_1(K) \cong \mathcal{O}(k, L_0, q)$.
We start with some observations for a low-dimensional vector space $L_0$ over $k$:

If $\dim_k L_0 = 1$ then clearly $\mathcal{O}(k, L_0, q) \cong \mathcal{P}_1(k)$.

If $\dim_k L_0 = 2$ then the form $q$ defines a field extension $E$ of $k$: Since $q$ is anisotropic, we can look at it like on a quadratic polynomial in two variables with only zero $(0, 0)$. Then there exists an element $t \in \overline{k}\backslash k$, $\overline{k}$ the algebraic closure of $k$, such that $q(1,t) = 0$. Hence $q$ defines a field $E := k \cup t \cdot k$ and $E$ consists of all "imaginary" zeros of $q$:

$$q(a,b) = 0 \Rightarrow a^2 q(1, \underbrace{a^{-1}b}_{=t}) = 0 \quad \Longrightarrow \quad q(a,b) = 0 \Leftrightarrow b = at$$

where we can assume $a \neq 0$, since otherwise also $b = 0$ (and the condition $b = at$ holds as well). Obviously, $[E : k] = 2$ and the orthogonal Moufang set is isomorphic to the Moufang set $\mathcal{P}_1(E)$: for $\mathcal{O}(k, L_0, q)$ we can consider the set $X_1$ as $L_0 \cup \{\infty\}$ and for $\mathcal{P}_1(E)$ we have $X_2 := F \cup \{\infty\}$. So a bijection $\varphi : X_1 \to X_2$ is given by

$$\varphi(\infty) = \infty, \qquad \varphi((a,b)) = a + tb$$

$\varphi$ is a Moufang isomorphism if for all $x \in X_1$ a group isomorphism is given by $\varphi u_x \varphi^{-1}$. This holds since the $U$-groups of the Moufang sets of the projective line and the orthogonal Moufang sets are defined in the same way.

We may now concentrate on the case $\dim_k L_0 \geq 3$.

**Lemma 4.4.1.** *Let $\mathcal{O}(k, L_0, q)$ be an orthogonal Moufang set with* $\dim_k L_0 \geq 3$. *Then $\mathcal{P}_1(k) \subset \mathcal{O}(k, L_0, q)$ is a sub Moufang set of $\mathcal{O}(k, L_0, q)$.*

*Proof.* Remember that we may identify the elements of $\mathcal{O}(k, L_0, q)$ with $L_0 \cup \{\infty\}$. Take a subspace $V$ of $L_0$ given by

$$V := \{x \in L_0 \mid x = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}, a \in k\}$$

and let $V^* := V \cup \{\infty\}$. Then, $V^*$ is a sub Moufang set of $\mathcal{O}(k, L_0, q)$: obviously, $V^* \subset L_0 \cup \{\infty\}$, and the only thing left to show is that the new $U$-groups map every element of $V^*$ onto another element of $V^*$. But this holds as well since the $U$-groups act by addition of elements (see 3.2.8), and $V$ is closed under addition.

Moreover, we have $V^* = \mathcal{P}_1(k)$: the elements of $V^*$ and $\mathcal{P}_1(k)$ are the same (since we can identify an element of $V$ just with $a \in k$. We have to show that the $\mu$-multiplication is the same as well: in $\mathcal{P}_1(k)$, we have $\mu_{a,1} b = \mu_a b = a^2 b$. In $V^*$, we have $\mu_{a,1} b = \frac{q(a)}{q(1)} \pi_a \pi_1(b)$. Since $a \in \langle 1 \rangle$ and $\dim_k L_0 \geq 3$ by assumption, we know by theorem 3.2.11 that $\mu_{a,1} b$ is in the center of the torus of $\mathcal{O}(k, L_0, q)$ and $\mu_{a,1} b = a^2 b$. □

We know that $\mathcal{O}(k, L_0, q)$ corresponds to a Jordan Clifford algebra $J(L_0, q)$. By [6, Theorem 1], every element of $J(L_0, q)$ satisfies a quadratic equation over $k$. Since $\mathcal{P}_1(K) \cong \mathcal{O}(k, L_0, q)$, all elements of $K$ must satisfy a quadratic equation over $Z(K)$ as well. But then it is well known that $K$ must be either a field or a quaternion algebra with standard involution.

If $K$ is a quaternion algebra, it is a 4-dimensional vector space over its center. Hence $\dim_k L_0 = 4$ and $q$ is the norm form of the quaternion algebra $L_0/k$ which is non-defective.

If $K$ is a field then the torus of $\mathcal{P}_1(K)$ is commutative. Thus the torus of $\mathcal{O}(k, L_0, q)$ must be commutative as well. By Theorem 3.2.11 this can only hold if one of the following cases occurs:

- $\dim_k L_0 = 1$. In this case we know that $\mathcal{O}(k, L_0, q) \cong \mathcal{P}_1(k)$, hence $\mathcal{P}_1(k) \cong \mathcal{P}_1(K)$ and $k \cong K$.
- $\dim_k L_0 = 2$. Again we know that $\mathcal{O}(k, L_0, q) \cong \mathcal{P}_1(E)$ for a suitable quadratic extension $E$ of $k$, and $E \cong K$.
- $\dim_k L_0 \geq 3$ and $\mathrm{def}(q) = L_0$. Then $\mathcal{O}(k, L_0, q)$ is isomorphic to $\mathcal{M}(\langle q(L_0)\rangle, q(L_0))$. Hence $K$ is commutative and isomorphic to the subfield $\langle q(L_0)\rangle$ of $k$, as proved above in 4.3.

## 4.5 $M = \mathcal{O}(k, L_0, q)$ and $M' = \mathcal{M}(f, L)$

Let $(k, L_0, q)$ be an anisotropic quadratic space such that $\mathcal{O}(k, L_0, q)$ is an orthogonal Moufang set, and let $(f, L)$ be a mixed pair such that $\mathcal{M}(f, L)$ is a mixed Moufang set. Assume $\mathcal{O}(k, L_0, q) \cong \mathcal{M}(f, L)$. By Theorem 2.5.4 we have $\mathrm{char}\, k = \mathrm{char}\, f = 2$. We start with the following observation.

**Lemma 4.5.1.** *Let $\mathcal{O}(k, L_0, q) \cong \mathcal{M}(f, L)$ as above. Then either $\dim_k L_0 = 2$ or $\mathrm{def}(q) = L_0$.*

*Proof.* If $\dim_k L_0 = 1$, the form $q$ is given by $q(a) = a^2 q(1)$, which is obviously totally defective.
Now assume $\dim_k L_0 \geq 3$. Since the torus of a mixed Moufang set is commutative, so must be the torus of $\mathcal{O}(k, L_0, q)$. By Theorem 3.2.11, this can only hold if $q$ is totally defective. □

Firstly assume that $\dim_k L_0 = 2$. By the result of section 4.4 we know that there exists a suitable quadratic extension $E$ of $k$ such that $\mathcal{P}_1(E) \cong \mathcal{O}(k, L_0, q)$. Thus we also have $\mathcal{P}_1(E) \cong \mathcal{M}(f, L)$ and by the result of 4.3 we get $E \cong f$.

Secondly assume that $q$ is totally defective. Then we know that $\mathcal{O}(k, L_0, q)$ is isomorphic to the Moufang set $\mathcal{M}(\langle q(L_0)\rangle, q(L_0))$ and hence by Lemma 4.2.1 $\langle q(L_0)\rangle \cong f$. Moreover, we can identify $\mathcal{M}(f, L)$ with the orthogonal Moufang set $\mathcal{O}(f^2, L, x \mapsto x^2)$.

## 4.6 $M = \mathcal{O}(k, L_0, q)$ and $M' = \mathcal{O}(F, L_1, q^*)$

Let $(k, L_0, q)$ and $(F, L_1, q^*)$ be two anisotropic quadratic spaces such that $\mathcal{O}(k, L_0, q)$ and $\mathcal{O}(F, L_1, q^*)$ are the corresponding orthogonal Moufang sets. We denote the second field with $F$ instead of $f$ to avoid confusion with the bilinear form $f$ of $q$. Assume that $\mathcal{O}(k, L_0, q) \cong \mathcal{O}(F, L_1, q^*)$.
In this section we may assume that both quadratic forms $q$ and $q^*$ are not totally defective, since otherwise the orthogonal Moufang sets corresponds to a mixed Moufang set as proved above. We may also assume that $\dim_k L_0 \geq 3$ (and for the same reason, $\dim_F L_1 \geq 3$), since otherwise the orthogonal Moufang set is isomorphic to a Moufang set of the projective line as proved in section 4.4.

We start with the following lemma about the defect:

**Lemma 4.6.1.** *Let $(k, L_0, q)$ be an anisotropic quadratic space such that $\mathcal{O}(k, L_0, q)$ is an orthogonal Moufang set, and $\dim_k L_0 \geq 3$.*
*Then* $\mathrm{def}(q) = \{b \in L_0 \mid (\mu_{a,b})^2 = \lambda\mathrm{id}$ with $\lambda \in k\ \forall a \in L_0\}$.
*Moreover, if $(f, L_1, q^*)$ is another anisotropic quadratic space such that $\mathcal{O}(f, L_1, q^*)$ is an orthogonal Moufang set with $\dim_f L_1 \geq 3$ and $\varphi : \mathcal{O}(k, L_0, q) \to \mathcal{O}(f, L_1, q^*)$ is a Moufang isomorphism, then $\varphi(\mathrm{def}(q)) = \mathrm{def}(q^*)$.*

*Proof.* We may assume that char $k = 2$ since otherwise the defect contains only the zero.
"$\subset$:" $b \in \mathrm{def}(q) \Rightarrow \mu_{a,b}\mu_{a,b} = \mu_{a,a} = \lambda\mathrm{id}$ for a $\lambda \in k$.
"$\supset$:" Let $b \in L_0$ fixed and $\mu_{a,b}\mu_{a,b} = \lambda\mathrm{id}$ for all $a \in L_0$ and a $\lambda \in k$. This means that $\frac{q(a)^2}{q(b)^2}\pi_a\pi_b\pi_a\pi_b = \lambda\mathrm{id}$, hence by Lemma 3.2.12, $\frac{q(a)^2}{q(b)^2}\pi_{\pi(a)b}\pi_b = \lambda\mathrm{id}$. Now, $\pi_a(b) = b - \frac{f(a,b)}{q(a)}a \in L_0$ and $\pi_a(b) \notin \langle b\rangle$ since the dimension of $L_0$ is at least 3. Hence by Theorem 3.2.11, $\mu_{\pi_a b,b} \in Z(T)$ and thus $b \in \mathrm{def}(q)$.
Now assume $\varphi : (k, L_0, q) \to (f, L_1, q^*)$ is an isomorphism. Since $\varphi$ preserves the $\mu$-multiplication and maps $Z(T_k)$ onto $Z(T_f)$ by Theorem 2.5.4, we have $(\mu_{\varphi(a),\varphi(b)})^2 \in Z(T_f)$. Thus $\varphi(b) \in \mathrm{def}(q^*)$ as proved above. $\square$

We will now give two different solutions for the isomorphism problem: the first one is based on a lemma by Jacques Tits, the second one uses the theorem above to reconstruct the field out of the data of the Moufang set. The advantage of the first solution is the fact that we get in this case the similarity of $(k, L_0, q)$ and $(F, L_1, q^*)$ for the anisotropic, not totally defective quadratic spaces, while the second solution only leads to an isomorphism between the underlying fields. We use the notations and definitions of section 2.3:

For the first solution we need the following lemma by Tits from 1974 (see [15], Lemma 8.18):

**Lemma 4.6.2.** (J.Tits, 1974) *For $i = 1, 2$, let $P_i$ be a projective space of dimension at least 3 over a division ring $k_i$ and $\pi_i$ a polarity of trace type in $P_i$. Let $\xi_i$ denote either $\pi_i$ or a projective quadratic form such that $\beta\xi_i = \pi_i$. Set*

$$S_i = S_{\xi_i} = \{x \in P_i \mid x \perp_{\xi_i} x\}$$

*Assume that the codimension of $P_1^{\perp(\pi_1)} \cap S_1$, where*

$$P_1^{\perp(\pi_1)} := \{y \in P_1 \mid x \perp_{\pi_1} y \text{ for all } x \in P_1\}$$

*in the maximal subspaces of $S_1$ (Witt index of $\xi_1$ if $\xi_1$ is not degenerate) is $\geq 1$, and that it is $\geq 2$ if $k_1 \cong \mathbb{F}_2$. Let $Q$ be a subspace of $P_1^{\perp(\pi_1)}$ whose codimension in $P_1$ is $\geq 3$. Finally, let $\varphi : S_1 \to S_2$ be a bijection mapping the subspaces of $S_1$ onto the subspaces of $S_2$. Suppose that for any plane $R$ of $P_1$ spanned by its intersection with $S_1$ and disjoint from $Q$, the set $\varphi(S_1 \cap R)$ is the intersection of $S_2$ and a plane of $P_2$.*
*Then, $\varphi$ extends to an isomorphism of the projective spaces $\psi : P_1 \to P_2$ and this extension is unique.*
*Suppose further that the codimension of $P_1^{\perp(\pi_1)}$ in $P_1$ is at least three. Then, one has $\psi \circ \pi_1 = \pi_2$ and if $\xi_1$ and $\xi_2$ are both projective quadratic forms, $\psi \circ \xi_1 = \xi_2$.*

We cannot solve the case $k = \mathbb{F}_2$ with this lemma, but all other cases are fine. Hence we have to show that we fulfill the requirements of the lemma if

$k \neq \mathbb{F}_2$:
Let $(k, L_0, q)$ and $(F, L_1, q^*)$ be two anisotropic quadratic spaces with dimension $L_0, L_1 \geq 3$. Let $V := L_0 \oplus k^2$ and $W := L_1 \oplus F^2$ be the vector spaces with quadratic forms $q_v(x, \alpha, \beta) := q(x) - \alpha\beta$ and $q_w(y, \gamma, \delta) := q^*(y) - \gamma\delta$ as defined in 3.2.8.
Then obviously the projective spaces $P(V)$ and $P(W)$ have dimension at least 3 over $k$ resp. $F$ and we have polarities of trace type given by $\pi_1 := f_v$ and $\pi_2 := f_w$ where the $f_i$ are the bilinear forms associated to $q_v$ resp. $q_w$. Let $\xi_i$ denote the forms $q_v$ resp. $q_w$.
We also have $S_1 = \{x \in P_1 \mid q_v(x) = 0\}$ and $S_2 = \{x \in P_2 \mid q_w(x) = 0\}$. These are just the orthogonal Moufang sets, see 3.2.8.
Since $P_1^{\perp(\pi_1)} = \{y \in P_1 \mid f_v(x, y) = 0 \text{ for all } x \in P_1\} = \operatorname{def}(q_v)$ we know that $P_1^{\perp(\pi_1)} \cap S_1 = \{0\}$: if $a, b \in S_1$ it follows that $q_v(a+b) \neq 0$ since otherwise there would be a 2-dimensional subspace in $S_1$ generated by $a, b$. But this would be a contradiction to Witt Index $q_v = 1$. Hence $f_v(a, b) = q_v(a+b) - q_v(a) - q_v(b) = q_v(a+b) \neq 0$ and an element of $S_1$ cannot be an element of the defect as well. The maximal subspaces of $S_1$ are of dimension 1 (by the Witt index). Thus the codimension of $P_1^{\perp(\pi_1)} \cap S_1$ in the maximal subspaces is equal to 1 (and this is why we have to look at the case $k = \mathbb{F}_2$ separately).

As a subspace $Q$ of $\operatorname{def}(q_v)$ whose codimension in $P_1$ is $\geq 3$ we choose $Q = \{0\}$. As the bijection between $S_1$ and $S_2$ we take the Moufang isomorphism $\varphi$.

What is left to show is that any plane $R$ of $P_1$ spanned by its intersection with $S_1$ is mapped on the intersection of $S_2$ and a plane of $P_2$ and that the bijection $\varphi$ maps subspaces of $S_1$ onto subspaces of $S_2$:

Let $R = \langle a, b, c\rangle$ be a plane with $a, b, c \in S_1$. Since Moufang sets are doubly transitive, we can put without loss of generality $a = 0$ and $b = \infty$. Hence all elements in $R \cap S_1$ are of the form $\begin{bmatrix} tc \\ 1 \\ -t^2 q(c) \end{bmatrix}$ with $c \in L_0$ as chosen, $t \in k$ arbitrary. In addition there exists an element $\begin{bmatrix} 0_{L_0} \\ 0 \\ 1 \end{bmatrix}$.
The $L_0$-component of the vectors is generated only by $c$. Therefore, we are finished if we can show that the bijection $\varphi$ maps one-dimensional subspaces of $L_0$ onto one-dimensional subspaces of $L_1$. But in this case we have also proved that $\varphi$ maps subspaces of $S_1$ onto subspaces of $S_2$: since the Witt index of the quadratic form $q$ is just 1, all subspaces of $S_1$ are one-dimensional. Assume that $\varphi$ maps one-dimensional subspaces of $L_0$ onto one-dimensional subspaces of $L_1$. But the subspaces of $S_1$ are uniquely determined by the underlying subspaces of $L_0$, hence the subspaces of $S_1$ are mapped onto subspaces of $S_2$.

To prove that $\varphi$ maps one-dimensional subspaces onto one-dimensional subspaces we have to distinguish between two cases: is the one-dimensional subspace generated by an element of $\operatorname{def}(q)$ or not?

- Let $a, b \in L_0^* \setminus \operatorname{def}(q)$ such that $a \in \langle b\rangle$. Then by Theorem 3.2.11, $\mu_{a,b}$ is trivial. Remember that a Moufang isomorphism preserves the $\mu$-multiplication:
$$\varphi(\mu_{a,b}(v)) = \mu'_{\varphi(a),\varphi(b)}(\varphi(v))$$

and since $\mu_{a,b}$ is trivial so must be $\mu'_{\varphi(a),\varphi(b)}$. Hence by Theorem 3.2.11 either $\varphi(a) \in \langle\varphi(b)\rangle$ or $\varphi(a), \varphi(b) \in \operatorname{def}(q_w)$. But the second case is a contradiction to Lemma 4.6.1.

- On the other hand we might have $a \in L_0^*$ with $a \in \operatorname{def}(q)$. Let $x \in L_0^*$ be an arbitrary element not in the defect (and such an element exists since $q$ shall not be totally defective). We know that $\mu_{a,x}(a)$ is mapped onto $\mu'_{\varphi(a),\varphi(x)}(\varphi(a))$ under the Moufang isomorphism $\varphi$. Now

$$\mu_{a,x}(a) = \frac{q(a)}{q(x)}\left(\pi_a\left(a + \frac{f(a,x)}{q(x)}a\right)\right) = \frac{q(a)}{q(x)}\left(a + \frac{f(a,a)}{q(a)}a\right) = \frac{q(a)}{q(x)}a$$

and on the other side

$$\mu'_{\varphi(a),\varphi(x)}(\varphi(a)) = \frac{q^*(\varphi(a))}{q^*(\varphi(x))}\varphi(a)$$

since by Lemma 4.6.1 elements of the defect are mapped on elements of the defect. But by this, $\lambda a \mapsto \widetilde{\lambda}\varphi(a)$ and hence the subspace generated by $a$ is mapped on the subspace generated by $\varphi(a)$.

Overall, $\varphi$ maps subspaces of $S_1$ onto subspaces of $S_2$ and planes of $P_1$ spanned by their intersection with $S_1$ are mapped onto the intersection of $S_2$ and planes of $P_2$.

Thus the requirements of Tits' Lemma are fulfilled and we get an isomorphism between the projective spaces $P(V)$ and $P(W)$. Now we use the main theorem of the projective geometry:

**Theorem 4.6.3.** (Main Theorem of the projective geometry) *Let $K, F$ be skew fields, $V$ resp. $W$ vector spaces over $K$ resp. $F$ with dimension at least 3. Let $\alpha : P(V) \to P(W)$ be an isomorphism of the projective spaces of $V$ and $W$. Then there exists an isomorphism $\beta : K \to F$.*

Hence the isomorphism problem for orthogonal Moufang sets is solved if the quadratic forms are not totally defective, $k \neq \mathbb{F}_2$ and the dimension of the vector spaces is at least 3.
In these cases we even get by Tits' Lemma 4.6.2 that the quadratic forms are mapped onto each other; therefore we need $\operatorname{codim}_{P_1} \operatorname{def}(q_v) \geq 3$. As shown above, $\operatorname{def}(q_v) \cap S_1 = \{0\}$, hence we just need three linearly independent elements of $S_1$ which will generate a 3-dimensional subspace of $P_1$ disjoint with $\operatorname{def}(q_v)$. But such elements exist since $\dim L_0 \geq 3$ and every element of $S_1 \backslash \{0, \infty\}$ is uniquely determined by an element of $L_0$.
Note that the notation $\psi \circ \xi_1 = \xi_2$ in Tits' lemma means $\psi \circ \xi_1 = \xi_2 \circ \kappa$ for a given field isomorphism $\kappa$ which we get by the main theorem of the projective geometry.

The case $k = \mathbb{F}_2$ can be solved easily: since the quadratic form $q$ on $L_0$ is anisotropic, it is uniquely determined by $L_0$: we have

$$q(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases}$$

The only case where we cannot fulfill the requirements of Lemma 4.6.2 is when $k = F = \mathbb{F}_2$: if $k = \mathbb{F}_2$ and $F$ is not, we can just exchange the roles of $k$ and $F$ and look at the Moufang isomorphism $\varphi^{-1}$. But when $k = F = \mathbb{F}_2$ and there is a bijection mapping $L_0 \cup \{\infty\}$ onto $L_1 \cup \{\infty\}$ with $\varphi(\infty) = \infty$ we must have $\dim L_0 = \dim L_1$. Since there exists only one possible quadratic form on $L_0$ resp. $L_1$ as seen above, we have $(k, L_0, q) \cong (F, L_1, q^*)$.

Now we look at the second possible solution. With Theorem 3.2.11 we can reconstruct the field $k$ only by the knowledge of the Moufang set and its $\mu$-multiplication. Remember that $T = \langle \mu_{a,b} \mid a, b \in L_0^* \rangle$:

Assume first that $\operatorname{char} k \neq 2$. Fix $e \in L_0^*$ and define

$$Y := \{a \in L_0 \mid \mu_{a,e} \in Z(T)\}$$

Put $M_a := \mu_{a,e}$ (so $M_e = \mathrm{id}$) and $M_{a,b} := M_{a+b} - M_a - M_b$
Define a multiplication for $a, b \in Y$ by

$$a \cdot b := \frac{1}{2} M_{a,e} b$$

By Theorem 3.2.11, we have $a = se$ and $b = te$. Then we have for arbitrary $x \in Y$

$$\begin{aligned} M_a x &= \mu_{a,e}(x) = \mu_{se,e}(x) = \tfrac{q(se)}{q(e)} \pi_{se} \pi_e(x) = s^2 \pi_{se} \left(x - \tfrac{f(e,x)}{q(e)} e\right) \\ &= s^2 \left(x - \tfrac{f(e,x)}{q(e)} e - \tfrac{f(se,x)}{q(se)} se + \underbrace{\tfrac{f(e,e) f(e,x)}{q(se) q(e)} se}_{\frac{2s^2 q(e) f(e,x)}{s^2 q(e)^2} e}\right) = s^2 x \end{aligned}$$

Hence $a \cdot b = \frac{1}{2} M_{a,e} b = \frac{1}{2}\left(M_{a+e}(b) - M_a b - M_e b\right) = \frac{1}{2}\big((s+1)^2 b - s^2 b - b\big) = \frac{1}{2}(2sb) = sb = ste$, and we found the multiplication on $k$.
As usual, we get the addition by $a + b := u_a u_b(0)$ where $u_i \in U_\infty$ with $u_i(0) = i$.

Now assume $\operatorname{char} k = 2$ and $q$ not totally defective.
Fix $e \in L_0^* \setminus \operatorname{def}(q)$ and define $Y := \{a \in L_0^* \mid \mu_{a,e} \in Z(T)\}$ as above. Put again $M_a := \mu_{a,e}$. We now want to define the multiplication for $a, b \in L_0$ by $a \cdot b := M_a M_b(e)$. Therefore we have to show that there exists exactly one $c \in Y$ such that $M_a M_b = M_c$ for all $a, b \in Y$.
Since $M_a M_b(e) = \mu_{a,e} \mu_{b,e}(e) = s^2 t^2 e = (st)^2 e$ by Theorem 3.2.11, such a $c$ exists. This $c$ is indeed unique:
Assume that there is a $d \neq c$ such that $M_d = M_c$. Then $d = re$ for some $r \in K$. We know that $M_c(e) = (st)^2 e$, so $(st)^2 e = \mu_{re,e}(e) = r^2 e \Rightarrow r^2 = (st)^2 \Leftrightarrow r = st$ since $\operatorname{char} K = 2$. Again it follows that $d = c$.
By this we found again the multiplication on $k$: since we have

$$a \cdot b = M_a M_b(e) = s^2 t^2 e$$

we reconstructed the multiplication on $k^2$. But in $\operatorname{char} k = 2$, $k^2 \cong k$, hence we found the multiplication on $k$ as well.
The addition we get as above by $a + b := u_a u_b(0)$.

By this we have reconstructed the field $k$ only by the data of the Moufang set. Now we can show the following:

**Lemma 4.6.4.** *Let $\mathcal{O}(k, L_0, q)$ and $\mathcal{O}(F, L_1, q^*)$ be two isomorphic orthogonal Moufang sets over anisotropic quadratic spaces. If both $\dim L_0$ and $\dim L_1 \geq 3$ and the forms $q, q^*$ are not totally defective, then the fields $k$ and $F$ are already isomorphic.*

*Proof.* As shown above, we have $k = (Y, +, \cdot)$ and $F = (Y', +, \cdot)$. Let $\varphi : \mathcal{O}(k, L_0, q) \to \mathcal{O}(F, L_1, q^*)$ be a Moufang isomorphism. Take $e \in Y$ and choose $e' := \varphi(e)$. By Lemma 4.6.1 we know that $\varphi(e) \notin \operatorname{def}(q^*)$ if $e \notin \operatorname{def}(q^*)$. Define the map

$$\psi : Y \to Y' \text{ by } \psi(a) := \varphi(\mu_{a,e} e)$$

We have to show the following:

- $\psi\mid_Y = Y'$
  Let $a \in Y$ arbitrary. Then:
  $\psi(a) = \varphi(\mu_{a,e}(e)) = \mu'_{\varphi(a),\varphi(e)}\varphi(e) = \mu'_{\varphi(a),e'}e' \in Y'$.
- $\psi(a + b) = \psi(a) + \psi(b)$
  Let $a, b \in Y$ arbitrary. Then:
  $\psi(a+b) = \varphi(u_{\mu_{a,e}e}u_{\mu_{b,e}e}(0)) = u'_{\varphi(\mu_{a,e}e)}u'_{\varphi(\mu_{b,e}e)}(\varphi(0)) = u'_{\psi(a)}u'_{\psi(b)}(0') = \psi(a) + \psi(b)$ since $\varphi(0) = 0'$.
- $\psi(a \cdot b) = \psi(a) \cdot \psi(b)$
  We distinguish the cases whether char $K = 2$ or not:
  Let char $K = 2$.
  $\psi(ab) = \varphi(\mu_{a,e}\mu_{b,e}e) = \mu'_{\varphi(a),\varphi(e)}\mu'_{\varphi(b),\varphi(e)}(\varphi(e)) = \mu'_{\varphi(a),e'}\mu'_{\varphi(b),e'}(e') = \psi(a)\psi(b)$ since $e' = \varphi(e)$ as chosen above.
  Now let char $K \neq 2$.
  $\psi(ab) = \varphi\left(\frac{1}{2}(\mu_{a+e,e}(b) - \mu_{a,e}(b) - \mu_{e,e}(b))\right) = \frac{1}{2}\varphi(\mu_{a+e,e}(b) - \mu_{a,e}(b) - \mu_{e,e}(b)) = \frac{1}{2}\left(\mu'_{\varphi(a+e),\varphi(e)}\varphi(b) - \mu'_{\varphi(a),\varphi(e)}(\varphi(b)) - \mu'_{\varphi(e),\varphi(e)}(\varphi(b))\right) = \frac{1}{2}\left(\mu'_{\varphi(a)+\varphi(e),e'}(\varphi(b)) - \mu'_{\varphi(a),e'}(\varphi(b)) - \mu'_{e',e'}(\varphi(b))\right) = \psi(a)\psi(b)$.

The only thing left to check is that $\operatorname{char} k = 2$ and $\operatorname{char} F \neq 2$ cannot occur when $\mathcal{O}(k, L_0, q) \cong \mathcal{O}(F, L_1, q^*)$. But this is the conclusion of Theorem 2.5.4. The proof is complete. □

## 4.7 $M = \mathcal{P}_1(F)$ and $M' = \mathcal{PL}(K, K_0, \sigma)$

Let $F$ be a skew field and $(K, K_0, \sigma)$ be an involutory set such that $\mathcal{P}_1(F)$ is a Moufang set of the projective line and $\mathcal{PL}(K, K_0, \sigma)$ is a Moufang set of the polar line. Assume $\mathcal{P}_1(F) \cong \mathcal{PL}(K, K_0, \sigma)$. The following theorem is a result from [18, (21.14)] and proved in [18, (23.23)]:

**Theorem 4.7.1.** *Let $(K, K_0, \sigma)$ be an involutory set, $k = Z(K)$ denote the center of $K$ and suppose that $K \neq \langle K_0 \rangle$. Then $K_0 = k$ or $K_0$ is a subfield of $k$.*

We distinguish two cases: either $K_0$ is a commutative field or $\langle K_0 \rangle = K$.

If $K_0$ is already a field, the Moufang set $\mathcal{PL}(K, K_0, \sigma)$ is isomorphic to $\mathcal{P}_1(K_0)$, and by the result of section 4.1, we get a field isomorphism between $K_0$ and $F$. In particular, $F$ must already be commutative.

Next assume that $\langle K_0 \rangle = K$. Let $\varphi : \mathcal{P}_1(F) \to \mathcal{PL}(K, K_0, \sigma)$ denote a Moufang isomorphism. Obviously, it preserves the addition and the *aba*-multiplication. Thus $\varphi$ is an isomorphism in respect to the addition between $F$ and $K_0$. Then $\varphi$ induces a monomorphism $\psi : F \to K$ (in respect to the addition) which preserves the *aba*-multiplication as well. By a result of Jacobson and Rickart in [7], $\psi$ is thus either a skew field monomorphism or a skew field anti-monomorphism and $F$ is thereby mapped onto a subfield of $K$. Now, $\langle K_0 \rangle = K$ and hence the subfield of $K$ must already be the whole skew field. But this means that $K_0 = K$ and $\sigma = \mathrm{id}$, a contradiction to our definition of an involutory set.

## 4.8 $M = \mathcal{PL}(K, K_0, \sigma)$ and $M' = \mathcal{M}(f, L)$

Let $(K, K_0, \sigma)$ be an involutory set and $(f, L)$ be a mixed pair such that $\mathcal{PL}(K, K_0, \sigma)$ and $\mathcal{M}(f, L)$ are the corresponding Moufang sets.
Assume $\mathcal{PL}(K, K_0, \sigma) \cong \mathcal{M}(f, L)$.
We start with the same observations as in section 4.3: the Moufang set of the polar line must be defined over a skew field of characteristic 2. Again we know that the torus of $\mathcal{PL}(K, K_0, \sigma)$ must be commutative. That does not mean that already the whole field is commutative. But we can restrict the possibilities for $K$ by the following lemma:

**Lemma 4.8.1.** *Let $M$ be a Moufang set which induces a special Jordan algebra $J$ such that $J$ satisfies the standard Clifford identity (see section 2.6.1), hence in particular, $M = \mathcal{O}(f, L_0, q)$ orthogonal or $M = \mathcal{M}(f, L)$ mixed. Assume that $\bar{M} = \mathcal{PL}(K, K_0, \sigma)$ is a Moufang set of the polar line such that $M \cong \bar{M}$.
Then $Z_{48}(\bar{J}) \equiv 0$ where $\bar{J}$ is the Jordan ample subspace induced by $\bar{M}$.*

*Proof.* The Moufang isomorphism induces a Jordan isomorphism between $J$ and $\bar{J}$. Assume $Z_{48}(\bar{J}) \not\equiv 0$. Thus by the *Clifford Interconnection Theorem* (7.11) in [13], $\bar{J}$ must have at least three orthogonal idempotents. These are preserved by a Jordan isomorphism, and hence $J$ must have at least three orthogonal idempotents as well. But this means that $J$ cannot satisfy any Clifford identity (again by [13, (7.11)]), a contradiction. □

Thus by the lemma above, the Zelmanov polynomial $Z_{48}$ must vanish on $\mathcal{M}(f, L)$. Hence by Theorem 2.6.5 the underlying skew field $K$ of $\mathcal{PL}(K, K_0, \sigma)$ is either commutative, a quaternion division algebra or a biquaternion division algebra. We look at the cases one after another:

If $K$ is commutative we know that $K_\sigma = K_0 = \mathrm{Fix}_K(\sigma)$ is a subfield of $K'$ (see [18, (11.3)]) and hence $\mathcal{PL}(K, K_0, \sigma) \cong \mathcal{P}_1(K_0) \cong \mathcal{M}(f, L)$.

If $K$ is a quaternion division algebra, we know by [18, (11.4)], that $K_\sigma = Z(K)$.
If $K_\sigma = K_0$, then $\mathcal{PL}(K, K_0, \sigma) \cong \mathcal{P}_1(K_0) \cong \mathcal{M}(f, L)$.
If $Z(K) \subset K_0$ there exists elements $x, y \in K_0 \backslash Z(K)$ such that $xy \neq yx$ and the torus is not commutative. Hence this case cannot occur.

If $K'$ is a biquaternion division algebra, then $K_\sigma = K_0$ and hence by Theorem 4.7.1 $K_0$ generates $K$. Thus $K_0$ cannot be central, and the torus of $M$ is not commutative. This case cannot occur.

## 4.9 $M = \mathcal{PL}(K, K_0, \sigma)$ and $M' = \mathcal{O}(f, L_0, q)$

Let $(K, K_0, \sigma)$ be an involutory set such that $M := \mathcal{PL}(K, K_0, \sigma)$ is a Moufang set of the polar line, and $(f, L_0, q)$ be an anisotropic quadratic space such that $\mathcal{O}(f, L_0, q)$ is an orthogonal Moufang set. Assume $M \cong M'$.
We know by Lemma 4.8.1 that the Zelmanov polynomial $Z_{48}$ vanishes on $\mathcal{PL}(K, K_0, \sigma)$. Hence by Theorem 2.6.5 $K$ must either be a field, a quaternion division algebra or a biquaternion division algebra.
We start with the case that $(K, \sigma)$ is a quaternion algebra with standard involution and $K_0 \neq K_\sigma$ (hence in particular, $\operatorname{char} K = 2$). We show that a Moufang isomorphism between $M$ and $M'$ maps the center $k$ of $K$ onto the defect of $q$. Therefore we need to localize the center of the set $K_0$:

**Definition 4.9.1.** Let $\mathcal{PL}(K, K_0, \sigma)$ be a Moufang set of the polar line with $(K, \sigma)$ a quaternion algebra with standard involution as above, i.e. $\operatorname{char} K = 2$ and $K_0 \neq K_\sigma$. Then the *center* of $K_0$, denoted by $Z(K_0)$, is given by

$$Z(K_0) := \{a \in K_0 \mid \forall b \in K_0 : (\mu_{b,a})^2 \in Z(T)\}$$

where $T = \langle \mu_{b,a} \mid a, b \in K_0 \rangle$ is the torus of the Moufang set.

**Lemma 4.9.2.** *Let $\mathcal{PL}(K, K_0, \sigma)$ be as in the definition above. Then $Z(K) = Z(K_0)$.*

*Proof.* Note that by the definition of $\mu_{b,a}$ we have

$$(\mu_{b,a})^2 \mu_{c,d}(x) = ab^{-1}ab^{-1}cd^{-1}xd^{-1}cb^{-1}ab^{-1}a$$

and

$$\mu_{d,c}(\mu_{a,b})^2(x) = cd^{-1}ab^{-1}ab^{-1}xb^{-1}ab^{-1}ad^{-1}c$$

We will show the identity $a \in Z(K) \Leftrightarrow a \in Z(K_0)$:
"$\Rightarrow$": Let $a \in Z(K)$, then $(\mu_{b,a})^2\mu_{d,c}(x) = a^2(b^{-1})^2cd^{-1}xd^{-1}c(b^{-1})^2a^2 = cd^{-1}a^2(b^{-1})^2x(b^{-1})^2a^2 = \mu_{d,c}(\mu_{b,a})^2(x)$ since we know that squares of fixed points lie in the center, i.e. $(b^{-1})^2 \in Z(K)$. Thus, $(\mu_{b,a})^2 \in Z(T)$ for all $b \in K_0$ and thereby $a \in Z(K_0)$.
"$\Leftarrow$": Let $a \in Z(K_0)$. Put $d = 1$. Then

$$\begin{aligned} ab^{-1}ab^{-1}cxcb^{-1}ab^{-1}a = cab^{-1}ab^{-1}xb^{-1}ab^{-1}ac \quad &\Leftrightarrow \\ xcb^{-1}ab^{-1}ac^{-1}a^{-1}ba^{-1}b = c^{-1}ba^{-1}ba^{-1}cab^{-1}ab^{-1}x & \end{aligned}$$

Put $b := c$, then $xac^{-1}ac^{-1}a^{-1}ca^{-1}c = a^{-1}ca^{-1}cac^{-1}ac^{-1}x$, and multiplication from the left with $a^2c^2a^2c^2$, which are all central since they are squares of fixed points, leads to

$$x(ac)^4 = (ac)^4x$$

Since $x \in K_0$ is arbitrary and $K_0$ generates $K$, $(ac)^4$ must be central. Now assume $ac \neq ca$. Then $m(t) = t^2 + (ac + ca)t + a^2c^2$ is the minimal polynomial of $ac$ over $Z(K)$, and $g(t) = t^4 + (ac)^4$ is another polynomial with zero $ac$ over $Z(K)$. But obviously, $\gcd(m, g) = 1$, a contradiction.
Hence $ac = ca$ for all $c \in K_0$, and since $K_0$ generates $K$, we must have $a \in Z(K)$. $\square$

With the observation above, we can solve the problem for the case that $(K, \sigma)$ is a quaternion division algebra with standard involution. Remember that we may assume that $\sigma$ is standard, since otherwise we just take a translate of the involutory set which has the standard involution by Lemma 3.1.6.

**Lemma 4.9.3.** *Let $(K, \sigma)$ be a quaternion algebra with standard involution $\sigma$ and $N$ its norm form, given by $N(x) = xx^\sigma$. Let $k$ denote the center of $K$.*
*If $\operatorname{char} K \neq 2$ and $(K, K_0, \sigma)$ is an involutory set, then $K_0 = \operatorname{Fix}_K(\sigma)$ and $\mathcal{PL}(K, K_0, \sigma) \cong \mathcal{O}(k, K_0, N|_{K_0})$.*
*If $\operatorname{char} K = 2$ and $\sigma$ the standard involution (which we may assume), then there exists a field $E$ with $k^2 \subset E \subset k$ and $x, y \in \operatorname{Fix}_K(\sigma)$ such that $xy + yx \in E$ and $K_0 = k + Ex + Ey$. In this case we have $\mathcal{PL}(K, K_0, \sigma) \cong \mathcal{O}(E, K_0, N|_{K_0})$.*

*Proof.* We start with the case $\operatorname{char} K \neq 2$. We know that $\operatorname{Fix}_K(\sigma)$ is a vector space over $k$ and $K_0 = \operatorname{Fix}_K(\sigma)$ by [18, (11.2)]. We show that the $\mu$-multiplication $\mu_{a,1}(x) = axa$ of the Moufang set $\mathcal{PL}(K, K_0, \sigma)$ can be described as $\frac{N(a)}{N(1)}\pi_a\pi_1(x)$ which is the $\mu$-multiplication of the Moufang set $\mathcal{O}(k, K_0, N|_{K_0})$:

$$\begin{aligned} \tfrac{N(a)}{N(1)}\pi_a\pi_1(x) &= N(a)\pi_a\left(x - \tfrac{f(1,x)}{N(1)}1\right) = N(a)\pi_a(-x) = N(a)\left(-x + \tfrac{f(a,x)}{N(a)}a\right) \\ &= -N(a)x + (ax + xa)a = -N(a)x + axa + xN(a) = axa \end{aligned}$$

since $N(a)$ is central. Hence the $\mu$-multiplications coincide and the sets are the same. We have $\mathcal{PL}(K, K_0, \sigma) \cong \mathcal{O}(k, K_0, N|_{K_0})$.

Now let $\operatorname{char} K = 2$. We assume that we have a given involutory set $(K, K_0, \sigma)$ and an anisotropic quadratic space $(f, L_0, q)$ such that $\mathcal{PL}(K, K_0, \sigma) \cong \mathcal{O}(f, L_0, q)$. We show that $K_0$ can be explained as $k + Ex + Ey$ for a subfield $E$ of $k$. By Lemma 4.9.2 we know that $k = \{a \in K_0 \mid \forall b \in K_0 : (\mu_{a,b})^2 \in Z(T)\}$. On the other hand, by Lemma 4.6.1 we have $\operatorname{def}(q) = \{b \in L_0 \mid \forall a \in L_0 : (\mu_{a,b})^2 = \lambda \mathrm{id}\}$ and since a Moufang isomorphism preserves the $\mu$-multiplication and its squares, we can identify $\operatorname{def}(q)$ with $k$. Moreover, $\operatorname{def}(q) \supset f$ and hence we can identify the sought subfield $E$ with $f$:

- Let $z = \lambda y \in K_0$ for a $\lambda \in k$. Then $\lambda \in E$: we use the $\mu$-multiplication in $\mathcal{O}(f, L_0, q)$. We know by Theorem 3.2.11 that $\mu_{\lambda y, y} \in Z(T)$ and therefore $\lambda y \in \langle y \rangle$. Then $\lambda \in f$, thus $\lambda \in E$. In the same way we get $\lambda x \in K_0 \Rightarrow \lambda \in E$.

- $xy + yx \in E$. We have $xy + yx = U_{x,y}1$ in the corresponding Jordan ample subspace $K_0 = H_0(K, \sigma)$. The orthogonal Moufang set $\mathcal{O}(f, L_0, q)$ corresponds to a Jordan Clifford algebra $J(L_0, q)$, and the Moufang isomorphism satisfies the requirements of a Jordan isomorphism.
  Let $\psi : H_0(K, \sigma) \to J(L_0, q)$ be a Jordan isomorphism, then $\psi(U_{x,y}1) = U'_{\psi(x),\psi(y)}\psi(1)$. In $J(L_0, q)$, the $U$-multiplication is given by

  $$U_a b = f(a, \bar{b}), a + q(a)\bar{b} \text{ with } \bar{b} = f(b, 1)1 - b$$

  see section 2.6. We may assume that $\varphi(1)$ is the $1'$ in $J(L_0, q)$, thus

  $$\begin{aligned} U'_{\psi(x),\psi(y)}1' &= f(\psi(x) + \psi(y), 1')(\psi(x) + \psi(y)) + q(\psi(x) + \psi(y))1' + \\ &\quad f(\psi(x), 1')\psi(x) + q(\psi(x))1' + f(\psi(y), 1')\psi(y) + q(\psi(y))1' \\ &= f(\psi(x), 1')\psi(y) + f(\psi(y), 1')\psi(x) + f(\psi(x), \psi(y))1' \end{aligned}$$

Remember that $1 \in k$ and elements of $k$ are mapped on elements of $\operatorname{def}(q)$ as noted above. Hence $1' \in \operatorname{def}(q)$, and $U'_{\psi(x),\psi(y)}1' = f(\psi(x),\psi(y))1'$. In particular, we showed that $(xy+yx)1$ is mapped onto $f(\psi(x),\psi(y))1'$, and since $f(\psi(x),\psi(y))$ is an element of the field $f$, $xy+yx \in E$.

- $z = \alpha x + \beta y \in K_0$ for $\alpha, \beta \in k$. Then already $\alpha, \beta \in E$. We have $\alpha x + \beta y = \alpha((xy+yx)y + y^2x) + \beta y^2 y = \alpha y^2 x + (\alpha(xy+yx) + \beta y^2)y = \alpha x + \left(\frac{\alpha(xy+yx)}{y^2} + \beta\right) y$ which is apparently in $K_0$. If we now subtract the first and the last term, we get $\left(\frac{\alpha(xy+yx)}{y^2}\right) y \in K_0$, hence $\alpha \in K_0$. In the same way we get $\beta \in K_0$.
- $yxy \in K_0$: We have $yxy = yxyxx^{-2}x = ((xy+yx)yx + x^2y^2)x^{-2}x = (xy+yx)y + y^2x$ and both $(xy+yx) \in E$ and $y^2 \in E$.

Now $N(a) = aa^\sigma = a^2$ for all $a \in K_0$ is an anisotropic quadratic form, and thus $\mathcal{PL}(K, K_0, \sigma) \cong \mathcal{O}(E, K_0, N|_{K_0})$. □

We can state the result of this section:

**Theorem 4.9.4.** *Let $(K, K_0, \sigma)$ be an involutory set and $(f, L_0, q)$ be an anisotropic quadratic space. Let $\mathcal{PL}(K, K_0, \sigma)$ and $\mathcal{O}(f, L_0, q)$ be the corresponding Moufang sets and assume $\mathcal{PL}(K, K_0, \sigma) \cong \mathcal{O}(f, L_0, q)$. Then one of the following holds:*

1. *There exists a field $E$ such that $\mathcal{PL}(K, K_0, \sigma) \cong \mathcal{P}_1(E) \cong \mathcal{O}(f, L_0, q)$*
2. *$K$ is a quaternion division algebra over its center $k = Z(K)$.*
   *If $\operatorname{char} K = 2$ and $\sigma$ is standard (otherwise take a translate of $(K, K_0, \sigma)$), then there exists a subfield $E$ of $k$ with $k^2 \subset E \subset k$ and $x, y \in \operatorname{Fix}_K(\sigma)$ such that $xy + yx \in E$ and $K_0 = k + Ex + Ey$. Moreover, $\mathcal{O}(f, L_0, q) \cong \mathcal{O}(E, K_0, N|_{K_0})$.*
   *If $\operatorname{char} K \neq 2$ and $\sigma$ is standard (otherwise take a translate of $(K, K_0, \sigma)$), then $K_0 = \operatorname{Fix}_K(\sigma)$ and $\mathcal{O}(f, L_0, q) \cong \mathcal{O}(k, K_0, N|_{K_0})$.*
3. *$K$ is a biquaternion division algebra, $\sigma$ is symplectic and $K_0 = K_\sigma$.*

*Proof.* We know by Lemma 4.8.1 that $K$ can only be a field, a quaternion division algebra or a biquaternion division algebra. If $K$ is a field, then $K_0$ is a subfield and we get (1). If $K$ is a quaternion division algebra, then (2) follows from Lemma 4.9.3.
(3) is a result of Theorem 2.6.5. What is left to prove is that $K_0 = K_\sigma$: If $\operatorname{char} K \neq 2$ or $\sigma$ is an involution of second kind, then $K_\sigma = K_0$ by [18, (11.2) and (11.5)]. Thus we may concentrate on the case $\operatorname{char} K = 2$ and $\sigma$ an involution of first kind.
Assume $K_\sigma \subset K_0$. Then by [9, (2.7)(1)] there exists an involution $\tau = Int(u) \circ \sigma$ of the first kind which is not symplectic by [9, (2.7)(3)]. Furthermore, $(K, uK_0, \tau)$ is a translate of $(K, K_0, \sigma)$, and the Moufang set $\mathcal{PL}(K, uK_0, \tau)$ has a corresponding Jordan ample subspace $J'$. Since $Z_{48}(J) = 0$ by assumption and the underlying involutory sets are isotopic, we also have $Z_{48}(J') = 0$. But by Theorem 2.6.5 this can only occur if $\tau$ is symplectic, a contradiction. □

## 4.10 $M = \mathcal{PL}(K, K_0, \sigma)$ and $M' = \mathcal{PL}(F, F_0, \tau)$

As mentioned above, the solution of the isomorphism problem for Moufang sets of the polar line was the aim of my diploma thesis. We were only able to prove this case for Moufang sets of the polar line which are neither fields nor quaternion or biquaternion algebras, as in 4.10.2. But in the meantime we could extend the proof that it holds for all Moufang sets of the polar line. We start with the following observation:

**Lemma 4.10.1.** *Let $(K, K_0, \sigma)$ and $(F, F_0, \tau)$ be two involutory sets with corresponding Moufang sets of the polar line $\mathcal{PL}(K, K_0, \sigma)$ and $\mathcal{PL}(F, F_0, \tau)$. Then every Moufang isomorphism $\varphi : \mathcal{PL}(K, K_0, \sigma) \to \mathcal{PL}(F, F_0, \tau)$ induces a Jordan isotopy $\psi : J \to J'$ on the corresponding Jordan ample subspaces $J$ and $J'$.*
*Let $c' \in \mathcal{PL}(F, F_0, \tau)$ such that $\varphi(1) = c'^{-1}$. Then $\varphi$ induces a Jordan isomorphism $\psi_{c'} : J \to J^{(c')'}$ where $J^{(c')'}$ is a $c'$-isotope of $J'$.*

*Proof.* We can identify the elements of $\mathcal{PL}(K, K_0, \sigma)$ (resp. $\mathcal{PL}(F, F_0, \tau)$) with elements of $J$ (resp. $J'$) by omitting $\infty$ (resp. $\infty'$). Hence we have a bijective map $\psi : J \to J'$ with $\psi(a) := \varphi(a)$ for the corresponding element $a \in \mathcal{PL}(K, K_0, \sigma)$.
Since $\varphi$ preserves the $\mu$-multiplication, we have $\psi(aba) = \psi(a)\psi(b)\psi(a)$ as well. For an element $u_a \in U_\infty$ we have $u_a(b) = a + b$, hence $\varphi(a + b) = \varphi(u_a(b)) = u'_{\varphi(a)}(\varphi(b)) = \varphi(a) + \varphi(b)$. So $\psi : J \to J'$ is a Jordan isotopy.
Since $c'^{-1}$ is the unit element of $J^{(c')'}$, $\psi$ fulfills the requirements of a Jordan isomorphism between $J$ and $J^{(c')'}$. □

Let $\mathcal{PL}(K, K_0, \sigma)$ and $\mathcal{PL}(F, F_0, \tau)$ be two Moufang sets of the polar line. Thus if $\mathcal{PL}(K, K_0, \sigma) \cong \mathcal{PL}(F, F_0, \tau)$, we get a Jordan isotopy $\psi : J \to J'$. Remember that $Z_{48}$ denotes the Zelmanov polynomial as described in section 2.6. Then we can use the following theorem about isotopic Jordan ample subspaces over skew field with involutions (see also [8, (7.2.1)]):

**Theorem 4.10.2.** *Let $(K_1, \sigma_1)$ and $(K_2, \sigma_2)$ be two skew fields with involutions with special Jordan algebras $J_1 := H(K_1, \sigma_1)$ and $J_2 := H(K_2, \sigma_2)$. Moreover, let $J_{01} \subset J_1, J_{02} \subset J_2$ be two Jordan ample subspaces such that there exists an isotopy $\varphi : J_{01} \to J_{02}$.*
*Suppose that either $Z_{48}(J_{01}) \neq 0$ or $Z_{48}(J_{02}) \neq 0$.*
*Then the corresponding involutory sets $(K_1, K_{01}, \sigma_1)$ and $(K_2, K_{02}, \sigma_2)$ are similar, and if $\varphi(1) = 1$ (hence the Jordan ample subspaces are even isomorphic) we even have $(K_1, K_{01}, \sigma_1) \cong (K_2, K_{02}, \sigma_2)$. In both cases, $K_1 \cong K_2$.*

*Proof.* Obviously, $Z_{48}(J_{01}) \neq 0$ implies that $Z_{48}(J_{02}) \neq 0$ and vice versa since we have a Jordan isotopy between $J_{01}$ and $J_{02}$. We distinguish the cases whether the Jordan ample subspaces are isomorphic or not:

1. $J_{01}$ and $J_{02}$ are isomorphic, i.e. $\varphi(1) = 1$. Then it follows by $Z_{48}(J_{0i}) \neq 0$ that $\mathcal{Z}(J_{0i}) = J_{0i}$ since Jordan division algebras are always simple. Thus we may apply McCrimmon's $\mathcal{Z}$-algebra Theorem 2.7.5 and we get $(su(J_{01}), \sigma_1) \cong (K_1, \sigma_1)$ and $(su(J_{02}), \sigma_2) \cong (K_2, \sigma_2)$.
   Now we have $J_{01} \cong J_{02}$ and by this, $(su(J_{01}), \sigma_1)$ is a special universal envelope of $J_{02}$ as well. Similarly, $(su(J_{02}), \sigma_2)$ is a special universal envelope of $J_{01}$. Then we get by Theorem 2.7.5 and Lemma 2.7.4:
$$(K_1, \sigma_1) \cong (su(J_{01}), \sigma_1) \cong (su(J_{02}), \sigma_2) \cong (K_2, \sigma_2)$$

2. $J_{01}$ and $J_{02}$ are isotopic, but not isomorphic, i.e. $\varphi(1) = c$ for a $c \in J_{02}$. We look at the translate $(K_2, \hat{K_{02}}, \hat{\sigma_2})$ of the involutory set $(K_2, K_{02}, \sigma_2)$ with $\hat{K_{02}} = cK_{02}$ as defined in 3.1.4. This involutory set induces a Jordan isotope $\hat{J_{02}}$ with unital $c$.
Thus, $\varphi$ maps 1 onto the unital of $\hat{J_{02}}$ and it is hence a Jordan isomorphism. So we can go back to case 1, and the involutory sets $(K_1, K_{01}, \sigma_1)$ and $(K_2, K_{02}, \sigma_2)$ are thus similar. Obviously, $K_1 \cong K_2$ since the similarity of involutory sets does not alter the base fields.

□

Hence the isomorphism problem for Moufang sets of the polar line is proved when we investigate the cases of Jordan ample subspaces in which the Zelmanov polynomial vanishes. By Theorem 2.6.5 this happens if $K_i$ is either a field, a quaternion division algebra or a biquaternion division algebra.
We start with investigating the cases that both $K_i$ are of the same type. Assume first that both $K_i$ are fields:

By [18, (11.3)], we know that if $\sigma$ is non-trivial and $K$ is commutative, $K_\sigma = K_0 = \mathrm{Fix}_K(\sigma)$ is a subfield of $K$ such that $K/K_0$ is a separable quadratic extension. Thus since both $\sigma_1$ and $\sigma_2$ are non-trivial, we get a Moufang isomorphism between two projective lines over subfields $K_{01}$ and $K_{02}$. Thus by 4.1 we get an isomorphism between $K_{01}$ and $K_{02}$.

Let us next look at the case of both $K_i$ being quaternion division algebras. The solution of the isomorphism problem in the case $\mathrm{char}\, K_i \neq 2$ follows by a result of N. Jacobson from [5, p.257]:

**Theorem 4.10.3.** (N. Jacobson, 1996) *Assume* $\mathrm{char}\, K \neq 2$ *and let* $(A_i, J_i)$ *be central simple algebras with involutions* $J_i$ *such that* $A_i$ *is simple and exclude* $(A_i, J_i)$ *with* $J_i$ *of symplectic type if* $\deg H(A_i, J_i) = 2$.
*Then* $H(A_1, J_1) \cong H(A_2, J_2) \Leftrightarrow (A_1, J_1) \cong (A_2, J_2)$.

The degree $\deg H(A_i, J_i)$ is meant as the number of orthogonal idempotents of the Jordan algebra.
In addition to the restriction of the characteristic, there seem to occur two cases where the theorem fails: first the case $(A_i, J_i)$ symplectic in degree 2 and second the case where we have proper Jordan subspaces of the special Jordan algebra. In fact both cases cannot occur: note that the case $(A_i, J_i)$ symplectic and $\deg H(A_i, J_i) = 2$ is just the case of the biquaternion algebras, and that in $\mathrm{char}\, K \neq 2$ the Jordan ample subspace is always the whole special Jordan algebra, since $H_\sigma \subset H_0 \subset H$ and every element $t \in H$ can be written as $\frac{1}{2}(t + t^\sigma)$, so $H_\sigma = H_0 = H$.

Now assume $(K, K_0, \sigma)$ and $(F, F_0, \tau)$ are involutory sets over quaternion algebras such that the Moufang sets $\mathcal{PL}(K, K_0, \sigma)$ and $\mathcal{PL}(F, F_0, \tau)$ are isomorphic, and $\mathrm{char}(K) \neq 2$ (hence $\mathrm{char}(F) \neq 2$ as well).
By the theorem above we get that $(K, \sigma) \cong (F, \tau)$, and the involutory sets are even similar, since both $K_0$ and $F_0$ must be the whole fixed point set.

We concentrate on the case of $\operatorname{char} K = 2$:
We have to distinguish the cases of $\sigma$ being the standard involution of the quaternion algebra or not. Remember that by Lemma 3.1.6 every involutory set $(K, K_0, \tau)$, with $\tau$ not the standard involution, is a translate of $(K, \widetilde{K_0}, \sigma)$ where $\sigma$ is the standard involution. We handle the case of a non-standard involution with the following lemma:

**Lemma 4.10.4.** *Let $(K, K_0, \tau)$ and $(K', K_0', \tau')$ be involutory sets over quaternion algebras with $\tau, \tau'$ non-standard involutions. Let $(K, \widetilde{K_0}, \sigma)$ and $(K', \widetilde{K_0'}, \sigma')$ be their translates such that $\sigma, \sigma'$ are the standard involutions. Assume that $\mathcal{PL}(K, K_0, \tau) \cong \mathcal{PL}(K', K_0', \tau')$.*
*Then $\mathcal{PL}(K, \widetilde{K_0}, \sigma) \cong \mathcal{PL}(K', \widetilde{K_0'}, \sigma')$. In particular, if $(K, \widetilde{K_0}, \sigma)$ and $(K', \widetilde{K_0'}, \sigma')$ are similar, so are $(K, K_0, \tau)$ and $(K', K_0', \tau')$.*

*Proof.* By Lemma 3.1.6 we know that $\tau$ is of the form $(x+uy)^\tau = x+uy^\sigma$, with $u^{-1} \in K_0$. In the same way, $\tau'$ is of the form $(x'+u'y')^{\tau'} = x'+u'y'^{\tau'}$ with $u'^{-1} \in K_0'$. Let $\varphi : \mathcal{PL}(K, K_0, \tau) \to \mathcal{PL}(K', K_0', \tau')$ be a Moufang isomorphism. Let $\alpha : u^{-1}K_0 \to K_0$ be the isomorphism defined by $\alpha : u^{-1}x \mapsto x$, and $\beta : K_0' \to u'^{-1}K_0'$ be the isomorphism defined by $\beta : x \mapsto u'^{-1}x$. Then, $\psi := \beta \circ \varphi \circ \alpha$ is a Moufang isomorphism between $\mathcal{PL}(K, \widetilde{K_0}, \sigma)$ and $\mathcal{PL}(K', \widetilde{K_0'}, \sigma')$.
The second statement follows since similarity of involutory sets is an equivalence relation. $\square$

We may go on like this:

Let $(K, K_0, \sigma)$ and $(K', K_0', \sigma')$ be involutory sets of quaternion algebras $K, K'$ with standard involutions $\sigma, \sigma'$ such that $\mathcal{PL}(K, K_0, \sigma) \cong \mathcal{PL}(K', K_0', \sigma')$. Then $K = L/k$ and $K' = L'/k'$ for separable quadratic extensions and $K_\sigma = Z(K) = k$, $K'_{\sigma'} = Z(K') = k'$, see [18, (11.4)].
Assume first that $Z(K) \neq K_0$. By Lemma 3.1.5 we know that there exists elements $x, y \in K_0$ with $xy \neq yx$. Since $xy \notin \operatorname{Fix}_K(\sigma)$ and $K$ is a 4-dimensional vector space over $k$, $\{1, x, y, xy\}$ is a basis of $K$ over $k$.

**Lemma 4.10.5.** *Let $\mathcal{PL}(K, K_0, \sigma)$, $\mathcal{PL}(K', K_0', \sigma')$ be as above and let $\varphi : \mathcal{PL}(K, K_0, \sigma) \to \mathcal{PL}(K', K_0', \sigma')$ be a Moufang isomorphism. Then $\varphi|_k = k'$.*

*Proof.* We can in fact localize the center $k$ from the data of the Moufang set, see Lemma 4.9.2. Thus let $a \in k$ be fixed, $b \in K_0$ be arbitrary. Then $(a+b)^2 = a^2+ab+ba+b^2$ is mapped under $\varphi$ onto $(a^\varphi+b^\varphi)^2 = (a^\varphi)^2+b^\varphi a^\varphi+b^\varphi a^\varphi+(b^\varphi)^2$ and since $ab = ba$ we must have $a^\varphi b^\varphi + b^\varphi a^\varphi = 0$ as well. Assume $a^\varphi \notin k'$. For all $b' \in K_0'$, $a^\varphi b' = b' a^\varphi$. Thus there must exist an element $t$ in $K' \backslash K_0'$ such that $a^\varphi t \neq t a^\varphi$. Without loss of generality we may assume $t = a^\varphi z'$ for suitable $z' \in K_0'$ (since $\langle 1', a^\varphi, z', a^\varphi z' \rangle = K'$). But then $a^\varphi(a^\varphi z') = a^\varphi(z' a^\varphi) = a^\varphi z' a^\varphi = (a^\varphi z') a^\varphi$, a contradiction. Thus $a^\varphi \in k$, and $\varphi(k) = k'$. $\square$

Put $1' := \varphi(1)$, $x' := \varphi(x)$ and $y' := \varphi(y)$. Since $xy + yx \in K_0$, we have $(xy+yx)^\varphi = x^\varphi y^\varphi + y^\varphi x^\varphi$ and it follows that $x^\varphi y^\varphi + y^\varphi x^\varphi \neq 0$ since $\varphi(0) = 0$. Put $x'y' := x^\varphi y^\varphi$, then $\{1', x', y', x'y'\}$ is a basis of $K'$ over $k'$.
We have to find a way to extend $\varphi$ such that it maps $xy$ onto $x'y'$ and gets the properties of a skew field isomorphism. Therefore we look at the minimal polynomial of $xy$ in $k$:

Obviously, $xy$ is a zero of the polynomial $t^2 + (xy + yx)t + yxxy$ and it cannot be a zero of a polynomial of degree 1. We have to show that $xy + yx$ and $xyyx$ are indeed elements of $k$: as noted in 2.4, squares of fixed points are in the center of a quaternion algebra with standard involution. Thus with $x, y \in K_0$ we have $xyyx = xy^2x = x^2y^2$ which is in the center as well. On the other hand, $(xy)^\sigma = yx$ and hence $xy + yx \in K_\sigma = k$ as noted above.
Moreover, the minimal polynomial of $xy$ is mapped onto the minimal polynomial of $x'y'$ by $\varphi$. Now let $E := k(xy)$ and $E' := k'(x'y')$ be two field extensions. Since we have an isomorphism $\varphi\mid_k$ between $k$ and $k'$ which maps the minimal polynomial of $xy$ onto $x'y'$, we can extend it to an isomorphism $\psi : E \to E'$.
Since we know that $K = \langle 1, x, y, xy\rangle$, every element of $K$ can be written as $\alpha 1 + \beta x + \gamma y + \delta xy$ with $\alpha, \beta, \gamma, \delta \in k$. We can prove the following:

**Lemma 4.10.6.** *Let $K = \langle 1, x, y, xy\rangle$ and $K' = \langle 1', x', y', x'y'\rangle$ be two quaternion division algebras and $\varphi, \psi$ as above. Then we get a skew field isomorphism $\Psi : K \to K'$ by $\Psi(\alpha 1 + \beta x + \gamma y + \delta xy) = \varphi(\alpha)1' + \varphi(\beta)x' + \varphi(\gamma)y' + \varphi(\delta)x'y'$ which extends $\varphi$ and $\psi$.*
*Especially, $\Psi(1) = 1$, $\Psi(x) = x'$, $\Psi(y) = y'$ and $\Psi(xy) = x'y'$.*

*Proof.* $\Psi$ is indeed an isomorphism: obviously, it is additive by definition and surjective since $\{1', x', y', x'y'\}$ is a basis of $K'$. $\Psi$ is injective since we just extend two given injections. The only thing left to prove is that $\Psi$ is multiplicative:

$$
\begin{aligned}
&\Psi([\alpha_1 1 + \beta_1 x + \gamma_1 y + \delta_1 xy] \cdot [\alpha_2 1 + \beta_2 x + \gamma_2 y + \delta_2 xy])\\
&= \Psi(\alpha_1\alpha_2 1 + \alpha_1\beta_2 x + \alpha_1\gamma_2 y + \alpha_1\delta_2 xy +\\
&\quad \beta_1\alpha_2 x + \beta_1\beta_2 x^2 + \beta_1\gamma_2 xy + \beta_1\gamma_2 x^2 y +\\
&\quad \gamma_1\alpha_2 y + \gamma_1\beta_2 yx + \gamma_1\gamma_2 y^2 + \gamma_1\delta_2 yxy +\\
&\quad \delta_1\alpha_2 xy + \delta_1\beta_2 xyx + \delta_1\gamma_2 xy^2 + \delta_1\delta_2 (xy)^2)\\
&= \varphi(\alpha_1\alpha_2)1' + \varphi(\alpha_1\beta_2)x' + \varphi(\alpha_1\gamma_2)y' + \varphi(\alpha_1\delta_2)x'y' +\\
&\quad \varphi(\beta_1\alpha_2)x' + \Psi(\beta_1\beta_2 x^2) + \varphi(\beta_1\gamma_2)x'y' + \Psi(\beta_1\delta_2 x^2 y) +\\
&\quad \varphi(\gamma_1\alpha_1)y' + \Psi(\gamma_1\beta_2 yx) + \Psi(\gamma_1\gamma_2 y^2) + \Psi(\gamma_1\delta_2 yxy) +\\
&\quad \varphi(\delta_1\alpha_2)x'y' + \Psi(\delta_1\beta_2 xyx) + \Psi(\delta_1\gamma_2 xy^2) + \Psi(\delta_1\delta_2 (xy)^2)
\end{aligned}
$$

Thus $\Psi$ is multiplicative, if we can show the following:
$\Psi(x^2) = (x')^2$, $\Psi(x^2y) = (x')^2y'$, $\Psi(yx) = y'x'$, $\Psi(y^2) = (y')^2$, $\Psi(yxy) = y'x'y'$, $\Psi(xyx) = x'y'x'$, $\Psi(xy^2) = x'(y')^2$ and $\Psi((xy)^2) = (x'y')^2$.
Since $\Psi$ is an extension of $\varphi$ and $\varphi(xyx) = x'y'x'$, $\varphi(yxy) = y'x'y', \varphi(x^2) = \varphi(x1x) = x'1'x' = (x')^2$ and in the same way $\varphi(y^2) = (y')^2$, these equations hold for $\Psi$ as well. Moreover, by $\psi(xy) = x'y'$ and the fact that $\psi$ is a field isomorphism, $\psi((xy)^2) = (x'y')^2$ and hence $\Psi((xy)^2) = (x'y')^2$.
As noted above, squares of fixed elements lie in the center of $K$. Hence $x^2, y^2 \in k$ and thereby $\Psi(x^2y) = \varphi(x^2)y' = (x')^2y'$ and $\Psi(xy^2) = \Psi(y^2x) = \varphi(y^2)x' = (y')^2x' = x'(y')^2$ as well. Finally, since $\varphi(xy + yx) = x'y' + y'x'$ we have $\Psi(xy + yx) = x'y' + y'x'$ and by the additivity of $\Psi$ and $\Psi(xy) = x'y'$ we get $\Psi(yx) = y'x'$.
Thus $\Psi(a \cdot b) = \Psi(a) \cdot \Psi(b)$ for all elements $a, b \in K$ and $\Psi$ is a skew field isomorphism between $K$ and $K'$. $\square$

Since the skew field isomorphism $\Psi$ is an extension of the Moufang isomorphism $\varphi$, the underlying involutory sets are similar.
Now the only case left of the quaternion algebras is $Z(K) = K_0$. This is solved by the following lemma:

**Lemma 4.10.7.** *Let $\mathcal{PL}(K, K_0, \sigma)$ be a Moufang set of the polar line where $K$ is a quaternion division algebra and $\sigma$ its standard involution. If $Z(K) = K_0$ then $\mathcal{PL}(K, K_0, \sigma) = \mathcal{M}(K, K_0)$ is already a mixed Moufang set.*
*Moreover, $\mathcal{PL}(K, K_0, \sigma) = \mathcal{P}_1(K_0)$ is a projective Moufang set as well.*

*Proof.* In this case the $\mu$-multiplication is given by $\mu_a b = aba = a^2 b$. Since the norm form of the quaternion algebra is given by $N(u) = u^2$ for $u \in Z(K)$, we can write $\mu_a b = N(a)b$. Thus the $\mu$-multiplication depends on an anisotropic quadratic form $N$. By $N(a, b) = N(a + b) - N(a) - N(b) = (a + b)^2 - a^2 - b^2 = 2ab = 0$, the form $N$ is totally defective. Hence this Moufang set of the polar line is a mixed Moufang set.
Since $Z(K) = K_0$, $\mathcal{PL}(K, K_0, \sigma)$ can be seen as a projective Moufang set over the field $K_0$. □

We have proved the following:

**Corollary 4.10.8.** *Let $\mathcal{PL}(K, K_0, \sigma)$ and $\mathcal{PL}(F, F_0, \tau)$ be two Moufang sets of the polar line over quaternion division algebras $K$ and $F$ with* $\operatorname{char} K = \operatorname{char} F = 2$. *Let $\mathcal{PL}(K, K_0, \sigma) \cong \mathcal{PL}(F, F_0, \tau)$.*
*If $Z(K) \neq K_0$, then $Z(F) \neq F_0$ as well, and $(K, K_0, \sigma)$ and $(F, F_0, \tau)$ are similar.*
*If $Z(K) = K_0$, then there exists a field $k$ such that $\mathcal{PL}(K, K_0, \sigma) \cong \mathcal{P}_1(k) \cong \mathcal{PL}(F, F_0, \tau)$.*

At last we look at the case of both $K_i$ being biquaternion division algebras.

Let $K$ be a biquaternion algebra with center $Z(K) =: k$. Assume that $(K, K_0, \sigma)$ is an involutory set and $\mathcal{PL}(K, K_0, \sigma)$ is a Moufang set of the polar line such that the Zelmanov polynomial $Z_{48}$ vanishes on the corresponding Jordan algebra. Then we know that $K_0 = K_\sigma = \{t + t^\sigma \mid t \in K\}$ and $\sigma$ is a symplectic involution, see 4.9.4.
By [9, (16.3)], the Moufang set $\mathcal{PL}(K, K_\sigma, \sigma)$ with $\sigma$ a symplectic involution of $K$ is isomorphic to the Moufang set $\mathcal{O}(k, K_\sigma, q)$ where $q$ is the Albert quadratic form of the biquaternion algebra $K$.

We have proved the following:

**Corollary 4.10.9.** *Let $\mathcal{PL}(K, K_0, \sigma)$ and $\mathcal{PL}(F, F_0, \tau)$ be two Moufang sets of the polar line over biquaternion algebras $K$ and $F$ with centers $k$ and $f$. Let $J_{K_0}$ and $J_{F_0}$ denote the corresponding Jordan ample subspaces and assume $Z_{48}(J_{K_0}) = Z_{48}(J_{F_0}) = 0$.*
*If $\mathcal{PL}(K, K_0, \sigma) \cong \mathcal{PL}(F, F_0, \tau)$, then $\sigma$ and $\tau$ are both symplectic involutions, $K_0 = K_\sigma$ and $F_0 = F_\tau$, and $\mathcal{PL}(K, K_0, \sigma) \cong \mathcal{O}(k, K_\sigma, q_K) \cong \mathcal{O}(f, F_\tau, q_F) \cong \mathcal{PL}(F, F_0, \tau)$.*

What is left to show is the case when $K_1$ is not of the same type as $K_2$.

If $K_1$ is a field, the torus $T_1$ of the Moufang set $\mathcal{PL}(K_1, K_{01}, \sigma_1)$ must be commutative. By Theorem 2.5.4, the torus $T_2$ of $\mathcal{PL}(K_2, K_{02}, \sigma_2)$ must then be commutative as well, so $K_2$ cannot be a biquaternion algebra or a quaternion algebra with $K_{02} \neq Z(K_2)$ as explained in section 4.8. Assume that $K_2$ is a quaternion algebra with $Z(K) = K_0$. Then by Lemma 4.10.7 $\mathcal{PL}(K_1, K_{01}, \sigma_1) \cong \mathcal{M}(K_1, K_{01}) \cong \mathcal{PL}(K_2, K_{02}, \sigma_2)$.

If $K_1$ is a quaternion algebra and $K_2$ is a biquaternion algebra, then $\mathcal{PL}(K_2, K_{02}, \sigma_2) \cong \mathcal{O}(Z(K_2), K_{\sigma_2 2}, q_{K_2})$ as explained above, so $\mathcal{PL}(K_1, K_{01}, \sigma_1)$ is isomorphic to an orthogonal Moufang set. But by the result of section 4.9, this case is not possible.

The isomorphism problem is thereby solved.

# 5 Further results on Moufang sets

In the last section we give some more results on Moufang sets. We look at the results of a joint work with Tom De Medts, Fabienne Haot and Hendrik Van Maldeghem (all from Ghent University), see [1]. We were able to prove the uniqueness of the normal nilpotent transitive subgroups of the group $G_x = \{g \in \mathrm{Sym}(X) \mid g(x) = x\}$ of a Moufang set $(X, (U_x)_{x \in X})$ for several kinds of Moufang sets.

But first we turn to the relation of Moufang sets and Jordan algebras. By the first studies of the isomorphism problem for my diploma thesis, the interest was focused on the correspondence between Jordan ample subspaces and Moufang sets of the polar line. In fact we found that special Jordan algebras always induce Moufang sets:

## 5.1 Simple Jordan algebras and Moufang sets

For this section we introduce the notion of a *classical abelian Moufang set*: any Moufang set that is either orthogonal, a projective line, or a polar line will be called a classical abelian Moufang set.
As noted earlier, there exists a correspondence between some Moufang sets and some Jordan algebras. In this section we will state this result more precisely and give the proof for it. Before starting with this, we give a short version of McCrimmon's and Zelmanov's classification theorem for simple Jordan algebras (see [13], p.202):

**Theorem 5.1.1.** (Simple Structure Theorem, McCrimmon/Zelmanov 1988) *A simple Jordan algebra $J$ is isomorphic to one of the following types:*

1. *an ample subspace $H_0(A, \sigma)$ in a hermitian algebra $H(A, \sigma)$ for a $\sigma$-simple associative algebra $A$, which has one of the two forms*
   (a) *$A^+$ for simple $A$*
   (b) *$H_0(A, \sigma)$ for simple $A$ with involution $\sigma$*
2. *an ample outer ideal in a Clifford algebra $J(q_0, 1) \lhd_o J \lhd_o J(q_1, 1)$ for non-degenerate quadratic forms $q_i$ with* $\dim J = 3$
3. *a Clifford algebra $J(q, 1)$ of a non-degenerate quadratic form $q$ with* $\dim J \geq 4$
4. *an Albert algebra $J(N, 1)$.*

Since we will not need the explicit description of Albert algebras, we refer for an explanation of them to [13]. The only thing we need to know is the following which is taken from [20]:

**Theorem 5.1.2.** (Prime Dichotomy Theorem, 1979) *A prime and non-degenerate Jordan algebra is either i-special or an Albert form.*

Since *prime* means that there exist no orthogonal ideals (i.e. ideals $I, K$ such that $U_I K = 0$) and *non-degenerate* means that there exist no trivial elements $z \neq 0$ with $U_z = 0$, all simple Jordan algebras are prime and non-degenerate.

Hence Albert algebras are not special, but hermitian and Clifford algebras are, as stated in the definition of Jordan algebras. We start with concentrating on the first case:

Let $M := \mathcal{PL}(K, K_0, \sigma)$ be a Moufang set of the polar line. We allow $\sigma$ to be the identity and hence $M$ to be a Moufang set of the projective line. The $\mu$-multiplication for this Moufang set is given by $\mu_a b = aba$, see section 3.2.
Assume first that $\sigma = \mathrm{id}$. Then the Moufang set is given by $K \cup \{\infty\}$ and if we omit $\infty$, we can identify the rest with the Jordan algebra $K^+$ which is defined in section 2.6: the Jordan multiplication $U_a b = aba$ is given by the $\mu$-multiplication for all elements except 0. Thus we define $\mu_0 a := \mu_a 0 := 0$ and we get the Jordan multiplication.
Assume next that $\sigma \neq \mathrm{id}$. Then the Moufang set is given by $K_0 \cup \{\infty\}$ and again we can omit $\infty$ to identify the rest with the Jordan ample subspace $H_0(K, \sigma)$. The Jordan multiplication can be obtained in the same way as above.

Now we look at the Jordan Clifford algebras. We will see that they are related to the orthogonal Moufang sets:
A Jordan Clifford algebra is given as the vector space $V$ embedded in the Clifford algebra with basepoint $C(V, q, 1)$ with Jordan multiplication $U_a b = q(a, \bar{b})a - q(a)\bar{b}$, where $\bar{b} = q(b, 1)1 - b$. On the other hand, orthogonal Moufang sets are given as vector spaces $V$ with an element $\infty$ and the $\mu$-multiplication $\mu_{a,b} c = \frac{q(a)}{q(b)} \pi_a \pi_b(c)$. As above we can identify the two vector spaces by omitting $\infty$. What is left to show that we have a correspondence is the agreement of the multiplication. Let $\mathcal{O}(k, V, q)$ be an orthogonal Moufang set. Since by definition the quadratic form $q$ has to be anisotropic, we have an element $\epsilon \in V$ such that $q(\epsilon) \neq 0$. Look at the quadratic form $q^*$ defined as $q/q(\epsilon)$: for this form, $\epsilon$ is a basepoint. We can define a new $\mu^*$-multiplication by $\mu^*_{a,b}(c) = \frac{q^*(a)}{q^*(b)} \pi^*_a \pi^*_b(c)$ where $\pi^*$ also uses the $q^*$ instead of $q$. This multiplication obviously maintains the properties of the $\mu$-multiplication. Then we have $U_a b = \mu^*_{a,\epsilon} b$:

$$\mu^*_{a,\epsilon} b = \frac{q^*(a)}{q^*(\epsilon)} \pi_a \pi_\epsilon b = q^*(a)\pi_a(b - q^*(b, \epsilon)\epsilon) = q^*(a, \bar{b})a - q^*(a)\bar{b}$$

with $\bar{b} = q^*(b, \epsilon)\epsilon - b$ as above. Hence we obtain the Jordan multiplication out of the data of the $\mu$-multiplication.
As we see in the Simple Structure Theorem, Jordan Clifford algebras occur only in dimension at least 3. The reason of this is already proved in section 4.6: anisotropic quadratic spaces – and hence the corresponding Jordan Clifford algebras – of dimension 1 or 2 are cases of the Moufang sets of the projective line. So a Jordan Clifford algebra of dimension 1 or 2 can be seen as a hermitian Jordan algebra $A^+$ for a suitable algebra $A$. Again, these Jordan Clifford algebras are induced by classical abelian Moufang sets.

We have proved the following corollary:

**Corollary 5.1.3.** *Every classical abelian Moufang set induces a special simple Jordan algebra.*

Since all the Moufang sets are defined over (skew) fields, the induced special simple Jordan algebras must already be division algebras. Hence the converse of the corollary does not hold: a hermitian Jordan algebra $A^+$ which is given as an algebra and is not a skew field cannot be induced by a classical abelian Moufang set.

## 5.2 The uniqueness of $U$-groups in Moufang sets of the polar line

As noted above, the $U$-groups of some Moufang sets are unique (see [1]). Here, we will give a detailed proof for the case of the Moufang set of the polar line. In this section we will sometimes denote the action of a permutation $g$ on a point $a$ as $a^g$. Note that we will also use the usual notation for the conjugation of a permutation $g$ with another permutation $h$ as $h^g$.

**Theorem 5.2.1.** *Let $(X, (U_x)_{x\in X})$ be a Moufang set of the polar line over an involutory set $(K, K_0, \sigma)$ as defined in 3.2. Then for each $x \in X$ the root group $U_x$ is unique as a transitive normal nilpotent subgroup of $G_x$.*

Let $M = (X, (U_x)_{x\in X})$ be a Moufang set of the polar line and $(K, K_0, \sigma)$ the corresponding involutory set. We will prove the theorem for the root group $U_\infty =: U_+$. Then by the Moufang condition (M2) the result follows for all other root groups as well. Let $B = G_\infty$ denote the stabilizer of $\infty$, hence $U_+$ is normal and nilpotent in $B$.

We can identify $U_+$ with the additive group $K_0$ and an element $u_a \in U_+$ operates as $x \mapsto x + a$ with $x, a \in K_0$; $u_a$ fixes $\infty$. Now let $U$ be another transitive normal nilpotent subgroup of $G_x$. Then $U$ and $U_+$ normalize each other and we can replace $U$ by $UU_+$. In particular we can assume $U_+ \leq U$ without loss of generality.
We have $[U_+, U] \leq U_+$ hence there exists a non-trivial element in $U_+$ which is in the center of $U$ (since $U_+ \leq U$ normal and $U$ is nilpotent, it is a well known group theoretic result that $Z(U) \cap U_+ \neq \{1\}$). Let this element be $u_1$ (we can define this element to be the 1 in our skew field by arranging the coordinates). We now assume $U_+ \neq U$ and try to obtain a contradiction.

Since $U_+ \leq U$, $U_+$ sharply transitive, there exists an element $\varphi \in U$ (which is not in $U_+$) such that $\varphi$ fixes an element of $K_0$ which is not $\infty$. We may assume that $\varphi$ fixes 0: take an arbitrary $\psi \in U\backslash U_+$ which maps 0 onto $\psi(0)$ and then compose $\psi$ with the map in $U_+$ mapping $\psi(0)$ back to 0.
We will finish the proof by showing that $\varphi$ cannot be non-trivial, thus $U = U_+$: Let $Z$ denote the center of $U$. Since for all $z \in Z$

$$\varphi(z.0) = z.(\varphi(0)) = z.0$$

$\varphi$ fixes the orbit $0^Z = \{z.0 \mid z \in Z\}$ pointwise. The subgroup $U$ is normal in $B$, hence so is $Z$. Look at the map

$$\mu_a : x \mapsto axa$$

for $a, x \in K_0$ which fixes 0 and $\infty$ (the $\mu$-multiplication of the Moufang set). Obviously, the $\mu$-maps lie in $B$ and hence they normalize $U$ (and by this, they also normalize $Z$). It is well-known (see [18], p.27) that for all $g \in B$

$$\mu_x(1^g) = (\mu_x(1))^g$$

Since $u_1 \in Z$ and $\mu_a(1) = a^2$ we hence have $u_{a^2} \in Z$ for all $a \in K_0$. Then $a^2 = u_{a^2}(0) \in 0^Z$ and we know that $\varphi$ fixes $0^Z$ pointwise. Hence $\varphi$ fixes all squares.

Assume char $K \neq 2$. We can write an arbitrary element $a \in K_0$ as $a = \frac{1}{4}((a+1)^2 - (a-1)^2)$. Since $a+1, a-1 \in K_0$ ($K_0$ is an additive subgroup of $K$ containing 1), $u_{(a+1)^2-(a-1)^2} \in Z$. By this,

$$\mu_{\frac{1}{2}}((a+1)^2 - (a-1)^2) = \frac{1}{2}((a+1)^2 - (a-1)^2)\frac{1}{2} = a$$

and hence $u_a \in 0^Z$ for all $a \in K_0$. This means that $\varphi$ fixes $K_0$ pointwise and must be the identity. We found a contradiction to $\varphi$ non-trivial.

Now assume char $K = 2$. We will use the following properties of $\varphi$:

- $\varphi$ is additive: we have $(u_a u_b)^\varphi = u_a^\varphi u_b^\varphi$ and $u_a^\varphi = u_{\varphi(a)}$ since $u_a^\varphi(0) = \varphi u_a \varphi^{-1}(0) = \varphi(0+a) = \varphi(a) = a^\varphi + 0 = u_{a^\varphi}(0)$ and $\varphi$ normalizes $U_+$. Thus we have $(u_{a+b})^\varphi = (u_a u_b)^\varphi = u_a^\varphi u_b^\varphi = u_{\varphi(a)} u_{\varphi(b)}$ and plugging in 0 we get $\varphi(a+b) = \varphi(u_{a+b})\varphi^{-1}(0) = (u_{a+b})^\varphi(0)$ and $u_{a^\varphi} u_{b^\varphi}(0) = \varphi(a) + \varphi(b)$. Thus $\varphi(a+b) = \varphi(a) + \varphi(b)$.
- $\varphi(1) = 1$ since $\varphi$ fixes all squares as proved above.
- $(aba)^\varphi = a^\varphi b^\varphi a^\varphi$. By definition of the $\mu$-multiplication (see section 2.5), $\mu_a : x \mapsto axa$ is equal to the product $u_1 u_1' u_1 u_a u_a' u_a$ where $u_x'$ denotes the mapping $a \mapsto \infty$ in the group $U_0$. As above we have $u_a^\varphi = u_{a^\varphi}$ and similarly $u_a'^\varphi = u_{a^\varphi}'$. Then:

$$\begin{aligned}(aba)^\varphi &= b^{\mu_a \varphi} = b^{u_1' u_1 u_1' u_a' u_a u_a' \varphi} = b^{\varphi\varphi^{-1} u_1' u_1 u_1' u_a' u_a u_a' \varphi} \\ &= (b^\varphi)^{(u_1' u_1 u_1' u_a' u_a u_a')^\varphi} = (b^\varphi)^{u_1' u_1 u_1' u_{a^\varphi}' u_{a^\varphi} u_{a^\varphi}'} \\ &= (b^\varphi)^{\mu_{a^\varphi}} = a^\varphi b^\varphi a^\varphi\end{aligned}$$

Define $U^{[i]} := [U, U^{[i-1]}]$ with $U^{[1]} := [U, U]$ and let $j$ be such that $U^{[j]} \not\leq U_+$ but $U^{[j+1]} \leq U_+$. Such a $j$ must exist since $U$ is nilpotent. Choose $\varphi \in U^{[j]} \backslash U_+$. We can still clearly assume that $\varphi$ fixes 0.
For all $b \in K_0$ we have $[\varphi, u_b] \in U_+$ by choice of $\varphi$ and

$$\begin{aligned}[\varphi, u_b](x) &= \varphi^{-1} u_b^{-1} \varphi u_b(x) = \varphi^{-1} u_b^{-1} \varphi(b+x) = \varphi^{-1}(b^{-1} + \varphi(b) + \varphi(x)) \\ &= \varphi^{-1}(b + \varphi(b) + \varphi(x)) = b^{\varphi^{-1}} + b + x = u_{b+b^{\varphi^{-1}}}(x)\end{aligned}$$

There exists a $b \in K_0$ such that $b \neq b^{\varphi^{-1}}$ (and so $b \neq b^\varphi$), and $[\varphi, u_{b+b^{\varphi^{-1}}}] = 1$: as shown above, $[\varphi, u_{b+b^{\varphi^{-1}}}] = [\varphi, [\varphi, u_b]]$. Assume $[\varphi, u_{b+b^{\varphi^{-1}}}] \neq 1$. Put $z := b + b^{\varphi^{-1}}$, hence $z^{\varphi^{-1}} = b^{\varphi^{-1}} + b^{\varphi^{-2}}$. Then $[\varphi, u_{z+z^{\varphi^{-1}}}] = [\varphi, [\varphi, [\varphi, u_b]]]$. If this is 1, we have found $z \in K_0$ as desired. If not, we can go on like this. Since $U$ is nilpotent, this chain will come to an end.

With the computation above we get

$$[\varphi, u_{b+b^{\varphi^{-1}}}](a) = (b + b^{\varphi^{-1}})^{\varphi^{-1}} + (b + b^{\varphi^{-1}}) + a = b^{\varphi^{-2}} + b + a$$

By this, we have $[\varphi, u_{b+b^{\varphi^{-1}}}] = u_{b^{\varphi^{-2}}+b} = 1$ and thus

$$u_{b+b^{\varphi^{-2}}} = 1 \Longleftrightarrow b + b^{\varphi^{-2}} = 0 \Longleftrightarrow b^{\varphi^{-2}} = b \Longleftrightarrow b = b^{\varphi^2}$$

by applying $\varphi$ twice.
Now let $a \in K_0$ be arbitrary and remember the $\mu$-multiplication $x \mapsto axa$. By the choice of $\varphi$ we know that $[\varphi, U^{[j]}] \subset U_+$ and since $\varphi \in U^{[j]}$ and $U$ is normal, we have $[\varphi, \mu_a^{-1}\varphi\mu_a] \in U_+$. It is

$$\begin{aligned}
[\varphi, \mu_a^{-1}\varphi\mu_a](t) &= \varphi^{-1}\mu_a^{-1}\varphi^{-1}\mu_a\varphi\mu_a^{-1}\varphi\mu_a(t) = \varphi^{-1}\mu_a^{-1}\varphi^{-1}\mu_a\varphi\mu_a^{-1}\varphi(ata) \\
&= \varphi^{-1}\mu_a^{-1}\varphi^{-1}\mu_a\varphi(a^{-1}a^{\varphi}t^{\varphi}a^{\varphi}a^{-1}) \\
&= \varphi^{-1}\mu_a^{-1}\varphi^{-1}\left(a(a^{-1})^{\varphi}a^{\varphi^2}t^{\varphi^2}a^{\varphi^2}(a^{-1})^{\varphi}a\right) \\
&= \varphi^{-1}\left(a^{-1}a^{\varphi^{-1}}a^{-1}a^{\varphi}t^{\varphi}a^{\varphi}a^{-1}a^{\varphi^{-1}}a^{-1}\right) \\
&= (a^{-1})^{\varphi^{-1}}a^{\varphi^{-2}}(a^{-1})^{\varphi^{-1}}ata(a^{-1})^{\varphi^{-1}}a^{\varphi^{-2}}(a^{-1})^{\varphi^{-1}} \qquad (*)
\end{aligned}$$

Remember that $\varphi$ fixes all squares. Plugging in $t = 1$ we get

$$\begin{aligned}
[\varphi, \mu_a^{-1}\varphi\mu_a](1) &= \varphi^{-1}\mu_a^{-1}\varphi^{-1}\mu_a\varphi\mu_a^{-1}\varphi(a^2) = \varphi^{-1}\mu_a^{-1}\varphi^{-1}\mu_a\varphi\mu_a^{-1}(a^2) \\
&= \varphi^{-1}\mu_a^{-1}\varphi^{-1}\mu_a\varphi(1) = \varphi^{-1}\mu_a^{-1}\varphi^{-1}\mu_a(1) = \varphi^{-1}\mu_a^{-1}\varphi^{-1}(a^2) \\
&= \varphi^{-1}\mu_a^{-1}(a^2) = \varphi^{-1}(1) = 1
\end{aligned}$$

Since $[\varphi, \mu_a^{-1}\varphi\mu_a] \subset U_+$ and elements of $U_+$ are of the type $x \mapsto x + z$, $[\varphi, \mu_a^{-1}\varphi\mu_a]$ must already be the identity: it fixes 1 as shown above.
Now look at $(*)$ and put $c := (a^{-1})^{\varphi^{-1}}a^{\varphi^{-2}}(a^{-1})^{\varphi^{-1}}$. Then $[\varphi, \mu_a^{-1}\varphi\mu_a](t) = catac$ and by the arguments above follows $catac = t$. Put $t := a^{-1}$. Then

$$cac = a^{-1} \Rightarrow (ca)^2 = 1 \Rightarrow ca = 1$$

So $c = a^{-1}$ and thus $(a^{-1})^{\varphi^{-1}}a^{\varphi^{-2}}(a^{-1})^{\varphi^{-1}} = a^{-1}$. We can apply $\varphi^2$ to both sides and get $(a^{-1})^{\varphi}a(a^{-1})^{\varphi} = (a^{-1})^{\varphi^2}$, and this equation leads to

$$a^{-\varphi}a = a^{-\varphi^2}a^{\varphi}$$

All computations above hold for arbitrary $a \in K_0$. We turn back to our $b \in K_0$ with $b \neq b^{\varphi}$ and $b = b^{\varphi^2}$.

$$\begin{aligned}
b^{-\varphi}b &= b^{-\varphi^2}b^{\varphi} & \Leftrightarrow \\
b^{\varphi}b^{-\varphi}bb^{\varphi} &= b^{\varphi}b^{-\varphi^2}b^{\varphi}b^{\varphi} & \Leftrightarrow \\
bb^{\varphi} &= b^{\varphi}b^{-1}b^2 & \Leftrightarrow \\
bb^{\varphi} &= b^{\varphi}b &
\end{aligned}$$

On the other hand we have $(b + b^{\varphi})^2 = b^2 + bb^{\varphi} + b^{\varphi}b + (b^{\varphi})^2$. We know that $b^2 = (b^2)^{\varphi} = (b^{\varphi})^2$ since $\varphi$ fixes all squares and $b^{\varphi}1^{\varphi}b^{\varphi} = (b1b)^{\varphi}$, so with $bb^{\varphi} = b^{\varphi}b$ as shown above, $(b + b^{\varphi})^2$ must be 0, thus $b + b^{\varphi}$ must be 0.
But this is a contradiction to $b \neq b^{\varphi}$. Hence $b = b^{\varphi}$ for all $b \in K_0$, and $\varphi$ must be the identity. $\square$

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