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# Hedging in Incomplete Markets and Testing Compound Hypotheses via Convex Duality

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# Introduction

The motivation of this thesis was to study the problem of hedging in incomplete markets with coherent risk measures via methods of Convex Analysis. Since the method to solve this problem was under slight modifications also applicable to the problem of hedging with convex risk measures and to the closely related problem of testing compound hypotheses, the idea was born to give a theorem that unifies these different results.

### Historical Development

The problem of pricing and hedging a contingent claim with payoff $H$ is well understood in the context of arbitrage-free option pricing in complete markets (see Black and Scholes [4], Merton [30]). There, a perfect hedge is always possible, i.e., there exists a dynamic strategy such that trading in the underlying assets replicates the payoff of the contingent claim. Then, the price of the contingent claim turns out to be the expectation of $H$ with respect to the equivalent martingale measure which is unique. However, the possibility of a perfect hedge is restricted to a complete market and thus, to certain models and restrictive assumptions. In more realistic models the market will be incomplete, i.e., a perfect hedge as in the Black-Scholes-Merton model is not possible and the equivalent martingale measure is not unique any longer. Thus, a contingent claim bears an intrinsic risk that cannot be hedged away completely. Therefore, we are faced with the problem of searching strategies which reduce the risk of the resulting shortfall as much as possible.
One can still stay on the safe side using a superhedging strategy (see [13] for a survey). Then, the replicating portfolio is in any case larger than the payoff of the contingent claim. But from a practical point of view, the cost of superhedging is often too high (see for instance [21]).
For this reason, the problem of investing less capital than the superhedging price and searching strategies that minimize the risk of the shortfall is considered. An overview over the quadratic approach, where the difference between $H$ and the replicating portfolio with respect to the $L^2$-norm is minimized, can be found in [44]. This approach is symmetric since it penalizes both positive and negative differences. In this thesis, we focus on the asymmetric approach, where only the risk of the shortfall, i.e., when the replicating portfolio is less than $H$, is minimized. To do this, one

has to choose a suitable risk measure. This problem has been studied using different kinds of risk measures. Föllmer and Leukert [16] used the so called quantile hedging to determine a portfolio strategy which minimizes the probability of loss. This idea leads to partial hedges. However, in this approach, losses could be very substantial, even if they occur with a very small probability. Therefore, Föllmer and Leukert [17] proposed to use the expectation of a loss function as risk measure instead and solved the linear case in the complete market. Cvitanić [6] and Xu [48] studied the same problem in an incomplete market. Kirch [26] used a robust version of the expectation of a loss function as risk measure and Nakano [31], [32] took coherent risk measures to quantify the shortfall risk. In this thesis, we also consider the mentioned risk measures, but we use another method to solve the problem. We compare our results with the corresponding results in the literature and deduce results for further risk measures (e.g. convex risk measures).
In the above mentioned papers, nevertheless what risk measure is used, the dynamic optimization problem of finding an admissible strategy that minimizes the risk of the shortfall can be split into a static optimization problem and a representation problem. The optimal strategy consists in superhedging a modified claim $\widetilde{\varphi}H$, where $H$ is the payoff of the claim and $\widetilde{\varphi}$ is a solution of the static optimization problem, an optimal randomized test.
We prove that this decomposition of the dynamic problem is possible for any risk measure that satisfies a monotonicity property. Since for the representation problem, the results of [14] can be used (see also [15], [28]), the main topic of the above mentioned papers studying the hedging problem is how to solve the static optimization problem. This is also the central problem studied in this thesis.

Since the choice of the risk measure plays an important role in the problem of hedging in incomplete markets, we review the main recent developments in the theory of measuring risks. Risk measures should help us to rank and compare different investment possibilities or to decide if a future random monetary position is acceptable. By a monetary position we mean a payoff $Y$, modelled as a random variable on a given probability space, that will be liquidated to us at a given maturity. A traditional method to measure the risk of a position, is to calculate the variance of the payoff $\sigma^2(Y)$. This has the drawback that losses and gains are penalized in the same way. A risk measure called Value at Risk ($\mathrm{VaR}_\alpha$) at level $\alpha$ seemed to solve this problem. $\mathrm{VaR}_\alpha$ is the smallest amount of capital which, if added to a position and invested in risk-free manner, keeps the probability of a negative outcome below the level $\alpha$. Mathematically, $\mathrm{VaR}_\alpha(Y)$ is the lower $\alpha$-quantile of the distribution of $Y$ with a negative sign. This risk measure became an industry standard for risk quantification, but in the last years it has received several theoretical criticism (see for instance [1], [3]). One serious shortcoming of $\mathrm{VaR}_\alpha$ is that it takes into account only the probability of a loss and not its actual size. This leaves the position un-

protected against losses beyond the $\mathrm{VaR}_\alpha$. A further point of criticism at $\mathrm{VaR}_\alpha$ is that it may fail to measure diversification effects.
In order to develop more appropriate measures of risk, recent research has taken an axiomatic approach in which the structure of so called coherent risk measures is derived from a set of economically desirable properties, cf. Artzner et al. [3]. This set of properties consists of monotonicity, positively homogeneity, subadditivity and the translation property. In [3] representation results are deduced on a finite probability space. In [8] the theory is extended to more general spaces. In Section 1.3 we explain the concept of coherent risk measures in detail. Föllmer and Schied [18] relaxed the axioms of coherent risk measures and replaced positively homogeneity and subadditivity by the weaker condition of convexity. The corresponding risk measures are called convex risk measures (see also [19] and Section 1.2 of this thesis). Optimization problems involving extended real-valued convex risk measures on linear spaces of random variables are considered in [39]. Note that the definition of convex risk measures given in Frittelli and Gianin [20] differs from that in [18] since they do not impose the translation property. In [34] deviation measures, another generalization of coherent risk measures, are introduced. In [22] and [24] risk measures are studied from a more abstract point of view, where [22] also studies the set-valued case. In the recent years dynamic risk measures monitoring the riskiness of a final payoff not only at the beginning, but also at intermediate dates, have been introduced (for an overview over this topic see for instance [41]). For a historical overview over different risk measures see also [45] and for an introduction to the theory we refer to [9].
In this thesis we shall work mainly with coherent and convex risk measures, but also with the robust version of the expectation of a loss function, which is a risk measure that does not satisfy the translation property.

The problem of testing hypotheses is closely related to the problem of hedging in incomplete markets as we shall see in this thesis. The case of testing a compound hypothesis against a simple alternative hypothesis has been considered in a variety of papers. A good introduction to this topic can be found in Witting [47]. In Schied [42], the problem is considered in the context of risk minimization. The more general problem of testing a compound hypothesis against a compound alternative hypothesis has been studied for instance by Cvitanić and Karatzas [7], which seems to present the up to now most general result in this area. In this thesis we work as well with this general case an compare our results with [7]. In [47], the significance level $\alpha$ is generalized to be a positive, bounded and measurable function on the parameter set of the null hypothesis. In Section 3.2, we generalize this problem to the case of a compound alternative hypothesis.

### Main Results

The main contributions of this thesis are the following.

- A unified proof.
  In Chapter 2, we deduce a theorem (Theorem 2.9) that unifies the results of the closely related problem of hedging in complete and incomplete markets and the problem of testing compound hypothesis. The theory of Chapter 2 is widely applicable and yields to new results when applied to the above mentioned problems.

- A different method.
  The method used in Chapter 2 is to solve the problem by a systematic application of Convex Analysis, in particular Fenchel duality. This differs from the methods used in recent literature to solve the coherent hedging problem ([31], [32]), the problem of testing compound hypotheses ([7]) or the (robust) efficient hedging problem ([17], [26], [48]). In the mentioned papers, a dual problem had been deduced as well, but to prove existence of a dual solution, strong assumptions had to be made. The main difference to this thesis is that in our approach, the existence of a dual solution follows from the validity of strong duality. Thus, we could weaken the assumptions to get the results. Furthermore, in the case where the set $C^*$ is compact (in the hedging problem $C^*$ is the set of densities of the equivalent martingale measures which is for instance in the complete market compact and in the testing problem $C^*$ is the compound null hypothesis), it is possible to deduce with our method a result about the structure of the solution that gives more information about the solution. That is, because we work with other dual variables than in the above mentioned papers. These dual variables are finite signed measures on the set $C^*$. The structure of the solution can be deduced with respect to $C^*$ and elements from the representing set of the risk measure in the case of hedging and elements from the compound alternative hypothesis in the case of testing, whereas in the above mentioned papers, the structure of the solution is deduced with respect to elements from enlarged sets.
  In the general case we use Fenchel duality and the duality approach of [27]. A detailed discussion of the relationship between our results and the recent literature can be found in the corresponding sections in Chapter 3 and 4.

- New results in hedging problems.
  Theorem 4.1 states the decomposition of the dynamic hedging problem into a static and a representation problem for any monotone risk measure. From Theorem 2.5, the existence of a solution to the static problem follows.
  The theory of Chapter 2 gives us the possibility to solve the hedging problem for a variety of risk measures. We show which properties of a risk measure are needed to solve the problem and that important risk measures as convex and coherent risk measures and special cases of the robust version of the expectation of a loss function are included.

We consider two cases. When the set of the densities of the equivalent martingale measures is compact (this includes the case of a complete market), the problem can be solved by a systematic application of Fenchel duality. The results are new and even improve, when restricted to coherent risk measures, the results of Nakano [32].
In the general incomplete market we apply first Fenchel duality as in Chapter 2 and then, we solve the inner problem of the dual problem with a duality approach due to Kramkov and Schachermayer [27], in the version of Xu [48]. The combination of this two methods makes it possible to solve the hedging problem for a variety of risk measures which leads to new results in convex hedging (Corollary 4.14 and 4.39) and for more general risk functions (Theorem 4.9 and 4.38) and extends previous results in coherent hedging and robust efficient hedging.

- New results in testing compound hypotheses.
  We show the differences to the methods used by Cvitanić and Karatzas [7] or Witting [47] to solve the problem and show in what way our results extend previous ones.

- Risk functions on $L^p$-spaces.
  We treat in a systematic way risk functions on $L^p$-spaces. Dual representations are deduced, representations via acceptance sets are considered and the important case of $L^\infty$, endowed with the weak* topology, is studied in detail.

**Outline**

The aim of Chapter 1 is to introduce the concept of risk measures. In the literature, risk measures have been defined in different ways and on different spaces. Being aware of this, we try to study in a systematic way the basic ideas of risk measures. This is important for this thesis since we will work with risk measures on different $L^p$-spaces and with different kinds of risk measures (convex and coherent risk measures, but also with the robust version of the expectation of a loss function). To prepare a mutual basis, we consider in Section 1.1 functionals on $L^p$-spaces and define at first different properties that are important for the definition of certain classes of risk measures or for obtaining useful dual representation. Then, we study the impact of these properties to the dual representation of convex and lower semicontinuous functionals. In Section 1.1.3, we consider the special case $L^\infty$ since in this space we have to take care if a functional is lower semicontinuous with respect to the norm topology or the weak* topology (this leads to different dual representations). We provide a possibility that helps to find out if a convex functional on $L^\infty$ is weakly* lower semicontinuous. This extends Theorem 4.31 in [19]. In Section 1.1.4, we collect results about the acceptance set of a given risk measure and review how a risk measure can be defined by a given acceptance set and how these procedures are connected. In Section 1.2 we review the definition of convex risk measures and deduce their dual representation on $L^p$-spaces. Finally, in Section 1.3 we define coherent risk measures, make their relationship to convex risk measures clear and deduce their dual representation.

In Chapter 2 an optimization problem involving randomized tests is considered. In Section 2.1, we motivate the problem and give an overview over the possible applications. In Section 2.2, we prove the existence of a primal solution and deduce in Section 2.3 the Fenchel dual problem. In Theorem 2.6 we prove that strong duality is satisfied. We show, that the problem is a saddle point problem and prove the existence of a dual solution and thus, the existence of a saddle point.
The dual problem plays an important role in solving the primal problem. In the next step as described in Section 2.4 the inner problem of the dual problem is analyzed. Again, Fenchel duality is applied. We regard the inner problem of the dual problem as a new primal problem, prove the existence of a solution, deduce the Fenchel dual problem and prove that strong duality is satisfied. The existence of a dual solution follows and it is possible to give a result about the structure of a solution with respect to the dual solution. With this result we can deduce in Section 2.5 the structure of a solution of the original problem with respect to its dual solution and obtain the main Theorem 2.9 of this thesis.

In Chapter 3 and 4 an application of the theory and the results deduced in Chapter 2 can be found. In Chapter 3, the problem of testing compound hypotheses is considered. First, we consider the classical problem of testing hypotheses, give a necessary and sufficient condition of the optimal solution and compare the obtained result with the recent literature. We show, in which cases our results extend for instance the ones of [7]. Then, we consider a more generalized test problem and solve it analogously.

In Chapter 4, we consider the problem of hedging in incomplete markets using different kinds of risk measures. In Theorem 4.1, the decomposition of the dynamic hedging problem into a static and a representation problem is proved for any risk measure that is monotone. For the representation problem there already exist results in the literature, that is why we focus in our considerations on the static optimization problem. We distinguish two cases in which different methods are used to solve the static problem. In Section 4.1, we consider the case where the set of the densities of the equivalent martingale measures is compact, which include the complete market. Then, the problem can be solves by the theory deduced in Chapter 2, i.e., by an application of Fenchel duality. In Theorem 4.9 in Section 4.1.1, a result about the structure of a solution to the static problem is proved for the most general risk function satisfying the assumptions of Chapter 2. In the following sections, we show

that this includes several well-known risk measures as convex or coherent risk measures. Section 4.1.2 summarizes the results for convex hedging. In Section 4.1.3, we outline the results for coherent hedging and compare them with results from recent literature. The problem of robust efficient hedging, i.e., the risk measure used to quantify the risk of the shortfall is a robust version of the expectation of a loss function, is discussed in Section 4.1.4. The problem can be solved for instance for Lipschitz continuous loss functions. The connection between the linear case and the problem of coherent hedging is deduced.
In Section 4.2, the problem of hedging in a general incomplete market is considered. It is solved by a combination of the results in Chapter 2 (Fenchel duality) and a duality approach due to Kramkov and Schachermayer [27], in the version of Xu [48]. This makes it possible to solve the hedging problem for a variety of risk measures. First, we deduce the result for the most general risk measure and then, in the following sections, the corresponding results for convex and coherent risk measures and for the robust version of the expectation of a loss function are deduced. A simple example is given.

In the appendix, we summarize some well-known facts, which could be useful for reading this work and prove several lemmata used in this thesis. In Section A we review some important results from Convex Analysis, among them Fenchel duality. In Section B we recall several auxiliary results from Functional Analysis and in Section C from Stochastic Finance.

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# Chapter 1

# Risk Measures

In this chapter, we shall introduce the concept of risk measures. We start in the general framework of functionals on $L^p$-spaces, discuss several important properties and their impact to the dual representation. This gives us the possibility to work not only with convex and coherent risk measures, but also with more general functions on different $L^p$-spaces that satisfy only some of the properties of e.g. a convex risk measure. This is not only helpful for the proof of Theorem 4.1 and for Section 4.1.1 and 4.1.4, but also gives a systematic insight into the relationship between duality and risk measures. We discuss the important case $L^\infty$ in Section 1.1.3 in more detail. In Section 1.2, we review the definition of convex risk measures and deduce their dual representation on $L^p$-spaces and in Section 1.3, we consider coherent risk measures.

## 1.1 Functionals on $L^p$-Spaces

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\mathcal{Y} = L^p(\Omega, \mathcal{F}, P)$ with $p \in [1, \infty]$. We write $L^p$ for $L^p(\Omega, \mathcal{F}, P)$. We want to measure the risk of a financial position with random payoff profile $Y$. In order to do this, we introduce functionals $\rho : \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$. Most of the results hold true in the general setting of an ordered separated locally convex vector space $\mathcal{Y}$, but to be more concrete and with the applications of Chapter 3 and 4 in mind, we work with $L^p$-spaces. The space $\mathcal{Y}$ is interpreted as the "habitat" of the financial positions whose riskiness have to be quantified. Let $\mathcal{Y}^*$ denote the topological dual space of $\mathcal{Y}$.
For every $p \in [1, \infty)$, $L^p$ is a Banach space whose dual can be identified, through Riesz Theorem, with $L^q$, where $\frac{1}{p} + \frac{1}{q} = 1$, $q \in (1, \infty]$. An important case in our applications in Chapter 4 will be $\mathcal{Y} = L^1$ with its dual $\mathcal{Y}^* = L^\infty$. The bilinear form between the dual spaces is $\langle Y, Y^* \rangle = E[YY^*]$ for all $Y \in L^p$ and $Y^* \in L^q$, where $E$ denotes the mathematical expectation with respect to $P$. $L^p$, $p \in [1, \infty)$, is endowed with the strong topology generated by the norm $\|Y\|_{L^p} = E[|Y|^p]^{\frac{1}{p}}$.

For $p = \infty$, we have to distinguish different cases. If we endow $L^\infty$ with the strong topology, generated by the norm $\|Y\|_{L^\infty} = \inf\{c \geq 0 : P(|Y| > c) = 0\}$, its topological dual space $\mathcal{Y}^*$ can be identified with the space $ba(\Omega, \mathcal{F}, P)$ of finitely additive set functions on $(\Omega, \mathcal{F})$ with bounded variation, absolutely continuous to $P$ (see [49], Chapter IV, 9, Example 5). The bilinear form between $L^\infty$ and $ba(\Omega, \mathcal{F}, P)$ is $\langle Y, Y^* \rangle = \int_\Omega Y dY^*$ for all $Y \in L^\infty$ and $Y^* \in ba(\Omega, \mathcal{F}, P)$.
If $L^\infty$ is endowed with the weak* topology (also called the $\sigma(L^\infty, L^1)$ topology) or with the Mackey topology (see Appendix B.1 for more explanations), this space is not a Banach space, but a separated locally convex space. Then, the topological dual space $\mathcal{Y}^*$ can be identified with $L^1$ and the bilinear form is again $\langle Y, Y^* \rangle = E[YY^*]$ for all $Y \in L^\infty$ and $Y^* \in L^1$.
In our applications in Chapter 3 and 4 we will use risk measures on $L^1$ with the norm topology, $L^\infty$ with the norm topology and $L^\infty$ with the Mackey topology.
A random variable $Y \in \mathcal{Y}$ that is $P - a.s.$ equal to a constant $c \in \mathbb{R}$, i.e., $Y(\omega) = c\ P - a.s.$, is denoted by $\mathbf{c}$. Equations and inequalities between random variables are always understood as $P - a.s.$
Let $\widehat{\mathcal{Q}}$ be the set of all probability measures on $(\Omega, \mathcal{F})$ absolutely continuous with respect to $P$. For $Q \in \widehat{\mathcal{Q}}$ we denote the expectation with respect to $Q$ by $E^Q$ and the Radon-Nikodym derivative $dQ/dP$ by $Z_Q$.

### 1.1.1 Properties and Definitions

We consider a functional $\rho : \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$. In this section, we shall introduce several important properties of $\rho$ and notations that will be used in this thesis. Some of these properties will be essential for defining certain classes of risk measures. Other properties are important to obtain useful dual representations.
We shall start with definitions of important properties for risk measures and we will give their financial motivation and interpretation (see for instance [3], [18], [19], [20]). A functional $\rho$ is said to be **monotone** iff for all $Y_1 \geq Y_2$ with $Y_1, Y_2 \in \mathcal{Y}$ we have

$$\rho(Y_1) \leq \rho(Y_2).$$

The financial interpretation of monotonicity is obvious: if the final net worth of a position $Y_2$ is $P - a.s.$ smaller than that of another position $Y_1$, which includes that possible losses are larger, then the risk of this position $Y_2$ has to be larger than the risk of $Y_1$.
A functional $\rho$ is called **convex** iff for all $\lambda \in (0, 1)$ and for all $Y_1, Y_2 \in \mathcal{Y}$ the following inequality is satisfied

$$\rho(\lambda Y_1 + (1 - \lambda)Y_2) \leq \lambda\rho(Y_1) + (1 - \lambda)\rho(Y_2).$$

If the inequality is strict for $Y_1 \neq Y_2$, then $\rho$ is called strictly convex. Convexity models diversification of risks with proportions of two positions. Since two positions

$\lambda Y_1$ and $(1-\lambda)Y_2$ can have effects on each other, the risk of this positions, when owned jointly, can only be less or equal than the weighted sum of the positions $Y_1$ and $Y_2$ taken separately. Under the assumption $\rho(\mathbf{0}) = 0$, convexity of $\rho$ implies (see [20])

$$\begin{aligned} \forall Y \in \mathcal{Y}, \lambda \in [0,1]: \quad \rho(\lambda Y) &\leq \lambda \rho(Y) \\ \forall Y \in \mathcal{Y}, \lambda \geq 1: \quad \rho(\lambda Y) &\geq \lambda \rho(Y). \end{aligned}$$

Both inequalities can be interpreted with respect to liquidity arguments. The latter is reasonable since, when $\lambda$ becomes large, the whole position $\lambda Y$ is less liquid than $\lambda$ single positions $Y$. When $\lambda$ is small, the opposite inequality must hold.
A functional $\rho$ is said to satisfy the **translation property** iff for all $c \in \mathbb{R}, Y \in \mathcal{Y}$

$$\rho(Y + c\mathbf{1}) = \rho(Y) - c.$$

The random variable $\mathbf{1}$ can be interpreted as a risk-free reference instrument. If the amount $c$ of capital is added to the position $Y$ and invested in a risk-free manner, the capital requirement is reduced by the same amount. Note that the translation property of $\rho$ implies $\rho(Y + \rho(Y)\mathbf{1}) = 0$ if $\rho(Y) < +\infty$. This means, if $\rho(Y)$ is added to the initial position $Y$, then we obtain a risk neutral position. For the financial interpretation we recall that if $\rho(Y)$ is negative, then the position $Y$ is acceptable and $\rho(Y)$ represents the maximal amount which the investor can withdraw without changing the acceptability. On the other hand, if $\rho(Y)$ is positive, then $Y$ is unacceptable and $\rho(Y)$ represents the minimal extra cash the investor has to add to the initial position $Y$ to make it acceptable. If $\rho(Y) = +\infty$, then $Y$ is not acceptable at all. We exclude the case $\rho(Y) = -\infty$, because this would mean that an arbitrary amount of capital could be withdrawn without endangering the position.
This discussion motivates us to define the **acceptance set** $\mathcal{A}_\rho$ of a risk measure $\rho$ as the set of acceptable positions

$$\mathcal{A}_\rho := \{Y \in \mathcal{Y} : \rho(Y) \leq 0\}.$$

The property $\rho(\mathbf{0}) = 0$ is called **normalization**. "Doing nothing" is not risky (but also does not "create" money in the sense of the translation property). This property is reasonable and ensures that $\rho(Y)$ can be interpreted as an risk adjusted capital requirement.
The functional $\rho$ is said to be **subadditive** iff for all $Y_1, Y_2 \in \mathcal{Y}$ it holds

$$\rho(Y_1 + Y_2) \leq \rho(Y_1) + \rho(Y_2).$$

If an investor owns two positions which jointly have a positive measure of risk, then he has to add extra cash to obtain a "neutral" position. If subadditivity did not hold, then, in order to deposit less extra cash, it would be sufficient to split the

position in two accounts.
A functional $\rho$ is called **positively homogeneous** iff for all $t > 0$ and $Y \in \mathcal{Y}$ the equality

$$\rho(tY) = t\rho(Y)$$

is satisfied. This means, the risk of a position increases in a linear way with the size of the position. If $\rho$ is positively homogeneous and $\rho(\mathbf{0}) < +\infty$, then it is normalized, i.e., $\rho(\mathbf{0}) = 0$. It holds that if $\rho$ is positively homogeneous, then $\rho$ is subadditive if and only if $\rho$ is convex. Since convexity already takes the diversification effect into account and in many situations the risk of a position might increase in a non-linear way with the size of the position, several authors (see e.g. [18], [19], [20]) propose that instead of imposing the stronger condition of $\rho$ being positively homogeneous and subadditive, it is sufficient to impose the convexity of $\rho$.

**Remark 1.1.** A "good" risk measure should satisfy certain reasonable properties. In Section 1.2 and 1.3 we shall introduce the concept of convex and coherent risk measures, that are defined in that way.
By contrast, several well-known risk measures fail to satisfy important properties. For instance, the widely used risk measure $\mathrm{VaR}_\alpha$ is positively homogeneous, but not subadditive and thus, not convex. As a consequence, $\mathrm{VaR}_\alpha$ may fail to measure diversification effects. Examples that demonstrate this effect are given in [3], [19].
The risk measure variance $\sigma^2(Y)$ as well as risk measures defined by $\rho(Y) = -E[Y] + \alpha\sigma(Y)$ for $\alpha > 0$ are not monotone. Semi-variance type risk measures defined by $\rho(Y) = -E[Y] + \sigma((Y - E[Y])^-)$ are not subadditive (see [3]).

Now, we will give several definitions and introduce properties that are important for the deduction of dual representations.
A functional $\rho$ is said to be **proper** if $\operatorname{dom}\rho \neq \emptyset$, where $\operatorname{dom}\rho := \{Y \in \mathcal{Y} : \rho(Y) < +\infty\}$ denotes the **effective domain** of $\rho$.

**Definition 1.2.** *$\rho$ is called* **lower semicontinuous** *if and only if one of the following equivalent conditions is satisfied (see [11], Section I.2.2).*

*(i) The* **epigraph** $\operatorname{epi}\rho := \{(Y, r) \in \mathcal{Y} \times \mathbb{R} : \rho(Y) \leq r\}$ *is closed with respect to the product topology on* $\mathcal{Y} \times \mathbb{R}$.

*(ii) The* **sublevel set** $N_a := \{Y \in \mathcal{Y} : \rho(Y) \leq a\}$ *is closed for every* $a \in \mathbb{R}$.

*(iii) For every net* $\{Y_\alpha\}_{\alpha \in D} \subseteq \mathcal{Y}$ *(see Appendix B.3) converging to* $Y$ *we have*

$$\rho(Y) \leq \liminf_{\alpha\to\infty} \rho(Y_\alpha).$$

If $\mathcal{Y}$ is a Banach space, nets can be replaced by sequences $\{Y_n\}_{n\in\mathbb{N}}$. By lower semicontinuity we will always understand lower semicontinuity with respect to the

topology on $\mathcal{Y}$. If we work with lower semicontinuity with respect to another topology, we will explicitly say so.
We will also use a property of $\rho$ which we call lower semicontinuity with respect to $P-a.s.$ convergent sequences since it is defined similarly: A functional is said to be **lower semicontinuous with respect to P-a.s. convergent sequences** iff for all sequences $\{Y_n\}_{n\in\mathbb{N}} \subset \mathcal{Y}$ with $Y_n \to Y \ P-a.s.$, we have

$$\rho(Y) \leq \liminf_{n\to\infty} \rho(Y_n).$$

In analogy to lower semicontinuity with respect to a topology on $\mathcal{Y}$, we give an equivalent characterization for lower semicontinuity with respect to $P-a.s.$ convergent sequences in terms of the closedness of $\operatorname{epi}\rho$ with respect to $P-a.s.$ convergent sequences that is sometimes easier to verify.

**Lemma 1.3.** *Let $\rho : \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$. The following properties are equivalent:*

*(i) $\rho$ is lower semicontinuous with respect to $P-a.s.$ convergent sequences.*

*(ii) $\operatorname{epi}\rho$ is closed with respect to $P-a.s.$ convergent sequences, i.e., for all sequences $\{(Y_n, r_n)\}_{n\in\mathbb{N}} \subset \operatorname{epi}\rho$ with $Y_n \to Y \ P-a.s.$ and $r_n \to r$, it holds that $(Y, r) \in \operatorname{epi}\rho$.*

*Proof.* (i)⇒(ii): Let (i) be satisfied. Consider a sequence $\{(Y_n, r_n)\}_{n\in\mathbb{N}} \subset \operatorname{epi}\rho$ with $Y_n \to Y \ P-a.s.$ and $r_n \to r$. Then, for all $n \in \mathbb{N}$ it holds $\rho(Y_n) \leq r_n$. Since $\rho$ is assumed to satisfy (i), we obtain

$$\rho(Y) \leq \liminf_{n\to\infty} \rho(Y_n) \leq \lim_{n\to\infty} r_n = r.$$

Hence, $\operatorname{epi}\rho$ is closed with respect to $P-a.s.$ convergent sequences.
(ii)⇒(i): Let (ii) be satisfied. Suppose, $\rho$ is not lower semicontinuous with respect to $P-a.s.$ convergent sequences. This means, there exists a sequence $\{Y_n\}_{n\in\mathbb{N}} \subset \mathcal{Y}$ with $Y_n \to Y \ P-a.s.$ satisfying $\rho(Y) > \liminf_{n\to\infty} \rho(Y_n)$. Hence, there exists a subsequence $\{Y_{n_k}\}_{k\in\mathbb{N}}$ with $Y_{n_k} \to Y \ P-a.s.$ satisfying $Y_{n_k} \in \operatorname{dom}\rho$ for all $k \in \mathbb{N}$ and $\rho(Y_{n_k}) \to \alpha < \rho(Y)$.
Take $r \in \mathbb{R}$ with $\alpha < r < \rho(Y)$. Since $\rho(Y_{n_k}) \to \alpha$, there exists $k_0 \in \mathbb{N}$, such that

$$\forall k > k_0 : \quad \rho(Y_{n_k}) \leq r < \rho(Y).$$

It follows, $(Y_{n_k}, r) \in \operatorname{epi}\rho$ for all $k > k_0$. Condition (ii) implies $(Y, r) \in \operatorname{epi}\rho$. Thus, $\rho(Y) \leq r$, a contradiction to $r < \rho(Y)$. Thus, (i) follows. □

A functional $\rho$ is said to satisfy the **Fatou property** iff for any bounded sequence $\{Y_n\}_{n\in\mathbb{N}} \subset \mathcal{Y}$ with $Y_n \to Y \ P-a.s.$,

$$\rho(Y) \leq \liminf_{n\to\infty} \rho(Y_n).$$

It is obvious, that lower semicontinuity with respect to $P-a.s.$ convergent sequences of a functional $\rho$ implies the Fatou property of $\rho$.
The function $\rho$ is called **continuous from above** iff for all $\{Y_n\}_{n\in\mathbb{N}} \subseteq \mathcal{Y}$ with

$$Y_n \searrow Y \Rightarrow \rho(Y_n) \nearrow \rho(Y),$$

where $Y_n \searrow Y$ denotes a nonincreasing sequence $\{Y_n\}_{n\in\mathbb{N}} \subseteq \mathcal{Y}$ converging to $Y$ $P-a.s.$ and analogously $\rho(Y_n) \nearrow \rho(Y)$ denotes a nondecreasing sequence converging to $\rho(Y)$.
Next, we shall introduce several properties of sets and further notations.
Let us denote the indicator function of a set $\Omega_1 \subseteq \Omega$ by $1_{\Omega_1}(\omega)$. It is defined as

$$1_{\Omega_1}(\omega) := \begin{cases} 1 & : \quad \omega \in \Omega_1 \\ 0 & : \quad \omega \notin \Omega_1. \end{cases}$$

In contrast to this we define the indicator function of a set $A \subseteq \mathcal{Y}$ by

$$\mathcal{I}_A(Y) := \begin{cases} 0 & : \quad Y \in A \\ +\infty & : \quad Y \notin A. \end{cases}$$

A set $K \subseteq \mathcal{Y}$ is called a **cone** if $Y \in K$ implies $tY \in K$ for all $t > 0$.
Let $\emptyset \neq K \subseteq \mathcal{Y}$ be a cone. The set $K^*$ defined by $K^* := \{Y^* \in \mathcal{Y}^* : \forall Y \in K : \langle Y, Y^* \rangle \leq 0\}$ is a convex, weakly* closed cone with $\mathbf{0} \in K^*$ and is called the **negative dual cone** of $K$.
Let us define the cone $\mathcal{Y}_+ := \{Y \in \mathcal{Y} : Y \geq \mathbf{0}\ P - a.s.\}$ and its negative dual cone $(\mathcal{Y}_+)^* := \{Y^* \in \mathcal{Y}^* : \forall Y \in \mathcal{Y}_+ : \langle Y, Y^* \rangle \leq 0\}$.

**Lemma 1.4.** *It holds* $(\mathcal{Y}_+)^* = \mathcal{Y}^*_-$, *where* $\mathcal{Y}^*_- = \{Y^* \in \mathcal{Y}^* : Y^* \leq 0\ P - a.s.\}$ *for* $\mathcal{Y}^* = L^q, q \in [1,\infty]$ *and* $\mathcal{Y}^*_- = \{Y^* \in \mathcal{Y}^* : \forall A \in \mathcal{F} : Y^*(A) \leq 0\}$ *for* $\mathcal{Y}^* = ba(\Omega, \mathcal{F}, P)$.

*Proof.* Let us first consider the case $\mathcal{Y}^* = L^q, q \in [1,\infty]$. Take $Y^* \in (\mathcal{Y}_+)^*$ and suppose that $Y^* \notin \mathcal{Y}^*_-$. Then, there exists a set $\Omega_1 \in \mathcal{F}$ with $P(\Omega_1) > 0$ and $Y^*(\omega) > 0$ for $\omega \in \Omega_1$. Consider $\overline{Y} \in \mathcal{Y}_+$ defined by $\overline{Y}(\omega) := 1_{\Omega_1}(\omega)$. Then, $\langle \overline{Y}, Y^* \rangle > 0$, a contradiction to $Y^* \in (\mathcal{Y}_+)^*$. Thus, $(\mathcal{Y}_+)^* \subseteq \mathcal{Y}^*_-$. Vice versa, take $Y^* \in \mathcal{Y}^*_-$. Then for all $Y \in \mathcal{Y}_+$ it holds $\langle Y, Y^* \rangle \leq 0$. Thus, $Y^* \in (\mathcal{Y}_+)^*$.
The proof is analogous for the case $\mathcal{Y}^* = ba(\Omega, \mathcal{F}, P)$. □

From Lemma 1.4, it follows that $-(\mathcal{Y}_+)^* = \mathcal{Y}^*_+$, which is used in the next section.

### 1.1.2 Dual Representation

In this section, we shall deduce representations of functionals $\rho : \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$ by means of elements of $\mathcal{Y}^*$. It is well known that every convex and lower semicontinuous functional $\rho$ admits a dual representation via the biconjugation theorem

(Theorem A.5) of Convex Analysis. We discuss the impact of additional properties of $\rho$ to this representation. Dual representations are discussed in [3], [8] for coherent risk measures and in [18], [19] for convex risk measure. Since we shall work in this thesis also with risk measures on different $L^p$-spaces that satisfy only some of the properties of e.g. a convex risk measure, we describe in the following theorem the impact of several properties of $\rho$ to this representation separately. Note that in the last item (e) of the following theorem we exclude the case $\mathcal{Y} = L^\infty$ endowed with the norm topology, since this allows us to work with probability measures as dual elements. We denote by $\rho^*$ the conjugate function of $\rho$ (see Definition A.3).

**Theorem 1.5.** *Let $\mathcal{Y} = L^p, p \in [1, +\infty]$ and let $\rho : \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$ be a convex and lower semicontinuous function with $\rho(\mathbf{0}) < +\infty$. Then, $\rho$ can be represented in the following way*

$$\rho(Y) = \sup_{Y^* \in \mathcal{Y}^*} \{\langle Y, Y^* \rangle - \rho^*(Y^*)\}. \tag{1.1}$$

*The impact of additional properties of $\rho$ to the dual representation (1.1) is as follows:*

*(a) The following conditions are equivalent:*
*(i) $\rho(\mathbf{0}) = 0$.*
*(ii) $\inf_{Y^* \in \mathcal{Y}^*} \rho^*(Y^*) = 0$.*
*In this case, it holds $\rho^*(Y^*) \geq 0$ for all $Y^* \in \mathcal{Y}^*$.*

*(b) The following conditions are equivalent:*
*(i) The functional $\rho$ is* **monotone**.
*(ii)* $\operatorname{dom} \rho^* \subseteq \mathcal{Y}^*_-$.
*(iii) It holds*

$$\rho(Y) = \sup_{Y^* \in \mathcal{Y}^*_+} \{\langle Y, -Y^* \rangle - \rho^*(-Y^*)\}.$$

*(c) The following conditions are equivalent:*
*(i) $\rho$ satisfies the* **translation property**.
*(ii)* $\operatorname{dom} \rho^* = \{Y^* \in \mathcal{Y}^* : \langle Y^*, \mathbf{1} \rangle = -1$ *and* $\sup_{Y \in \mathcal{A}_\rho} \langle Y, Y^* \rangle < +\infty\}$
*(iii) It holds*

$$\rho(Y) = \sup_{\{Y^* \in \mathcal{Y}^* : \langle Y^*, \mathbf{1} \rangle = -1\}} \{\langle Y, Y^* \rangle - \sup_{\widetilde{Y} \in \mathcal{A}_\rho} \langle \widetilde{Y}, Y^* \rangle\}.$$

*In this case, it holds for all $Y^*$ with $\langle Y^*, \mathbf{1} \rangle = -1$*

$$\rho^*(Y^*) = \sup_{Y \in \mathcal{A}_\rho} \langle Y, Y^* \rangle. \tag{1.2}$$

*(d) The following conditions are equivalent:*
*(i) $\rho$ is* **positively homogeneous**.
*(ii) $\rho$ is* **subadditive** *and $\rho(\mathbf{0}) = 0$.*

*(iii)* $\rho^*(Y^*) = \mathcal{I}_{\operatorname{dom}\rho^*}(Y^*)$.
*(iv) It holds*

$$\rho(Y) = \sup_{Y^* \in \operatorname{dom}\rho^*} \langle Y, Y^* \rangle .$$

(e) *If* $\rho$ *is* **monotone** *and satisfies the* **translation property** *and* $\mathcal{Y}^* = L^q$, $q \in [1, \infty]$, *then* $\operatorname{dom}\rho^* \subseteq \{-Z_Q \in L^q : Q \in \widehat{\mathcal{Q}}\}$. *The dual representation reduces to*

$$\rho(Y) = \sup_{Q \in \mathcal{Q}} \{E^Q[-Y] - \sup_{\widetilde{Y} \in \mathcal{A}_\rho} E^Q[-\widetilde{Y}]\},$$

*where* $\mathcal{Q} := \{Q \in \widehat{\mathcal{Q}} : Z_Q \in L^q\}$.

*Proof.* Let $\rho$ be convex and lower semicontinuous with $\rho(\mathbf{0}) < +\infty$. Since $\rho$ is proper, the biconjugation theorem (Theorem A.5) yields (1.1).

(a) With (1.1) we obtain $\rho(\mathbf{0}) = \sup_{Y^* \in \mathcal{Y}^*}\{\langle \mathbf{0}, Y^* \rangle - \rho^*(Y^*)\} = -\inf_{Y^* \in \mathcal{Y}^*} \rho^*(Y^*)$ and the equivalence of (i) and (ii) follows. The last assertion is obvious.

(b) Let $\rho$ be monotone. Take $Y \in \mathcal{Y}_+ \setminus \{\mathbf{0}\}$, i.e., $Y \geq \mathbf{0}$. This implies $\rho(Y) \leq \rho(\mathbf{0})$. Because of (1.1), we have

$$\forall Y^* \in \mathcal{Y}^* : \quad \langle Y, Y^* \rangle - \rho^*(Y^*) \leq \rho(Y) \leq \rho(\mathbf{0}).$$

Hence, $\langle Y, Y^* \rangle \leq \rho^*(Y^*) + \rho(\mathbf{0})$ for all $Y^* \in \mathcal{Y}^*$. Since $\mathcal{Y}_+$ is a cone, the last inequality is also satisfied for $tY$ for all $t > 0$. Thus, for all $Y^* \in \mathcal{Y}^*$ we have

$$\forall t > 0 : \quad t \langle Y, Y^* \rangle \leq \rho^*(Y^*) + \rho(\mathbf{0}).$$

For $Y^*$ in $\operatorname{dom}\rho^*$ this is only possible if $Y^* \in (\mathcal{Y}_+)^* = \mathcal{Y}^*_-$ (Lemma 1.4) since $\rho(\mathbf{0}) < +\infty$. Hence, $\operatorname{dom}\rho^* \subseteq \mathcal{Y}^*_-$, i.e., (ii) is satisfied, which implies the dual representation in (iii). Vice versa, let (iii) be satisfied. Consider $Y_1 \geq Y_2$. Then $Y_1 - Y_2 \in \mathcal{Y}_+$. By definition of $\mathcal{Y}^*_- = (\mathcal{Y}_+)^*$ we obtain $\langle Y_1, Y^* \rangle \leq \langle Y_2, Y^* \rangle$ for all $Y^* \in \mathcal{Y}^*_-$ and therefore

$$\forall Y^* \in \mathcal{Y}^*_+ : \quad \langle Y_1, -Y^* \rangle \leq \langle Y_2, -Y^* \rangle .$$

Thus, $\rho(Y_1) \leq \rho(Y_2)$, i.e., (i) is satisfied.

(c) Let (i) be satisfied. Denote $M^* := \{Y^* \in \mathcal{Y}^* : \langle Y^*, \mathbf{1} \rangle = -1, \sup_{Y \in \mathcal{A}_\rho} \langle Y, Y^* \rangle < +\infty\}$. Take $Y^* \in \operatorname{dom}\rho^*$. Because of (1.1), we have for all $Y \in \mathcal{Y}$

$$\forall Y^* \in \operatorname{dom}\rho^* : \quad \langle Y, Y^* \rangle - \rho^*(Y^*) \leq \rho(Y).$$

Consider this inequality with $Y = \mathbf{0} + c\mathbf{1}$ for an arbitrary $c \in \mathbb{R}$. Then, because of the translation property of $\rho$, we obtain

$$\forall c \in \mathbb{R}, \forall Y^* \in \operatorname{dom}\rho^* : \quad c(1 + \langle \mathbf{1}, Y^* \rangle) \leq \rho^*(Y^*) + \rho(\mathbf{0}).$$

This is only possible if $\langle \mathbf{1}, Y^* \rangle = -1$ for all $Y^* \in \operatorname{dom} \rho^*$ since $\rho(\mathbf{0}) < +\infty$ and $\rho^*(Y^*) < +\infty$ for $Y^* \in \operatorname{dom} \rho^*$. For $Y^* \in \operatorname{dom} \rho^*$ we can choose $a \in \mathbb{R}$ with $a \geq \rho^*(Y^*)$ and $Y \in \mathcal{A}_\rho$. Then we obtain from (1.1)

$$\forall Y \in \mathcal{A}_\rho : \quad \langle Y, Y^* \rangle \leq a,$$

hence $\operatorname{dom} \rho^* \subseteq M^*$. To prove the reverse take $Y^* \in M^*$ and $Y \in \operatorname{dom} \rho$. Then $Y + \rho(Y)\mathbf{1} \in \mathcal{A}_\rho$ and by definition of $M^*$,

$$\exists a \in \mathbb{R} : \quad \langle Y + \rho(Y)\mathbf{1}, Y^* \rangle = \langle Y, Y^* \rangle - \rho(Y) \leq a.$$

This inequality is trivially satisfied for $Y \notin \operatorname{dom} \rho$. Hence for all $Y \in \mathcal{Y}$ it holds $\langle Y, Y^* \rangle - \rho(Y) \leq a$. Taking the supremum over all $Y \in \mathcal{Y}$ yields $\rho^*(Y^*) \leq a$, hence $Y^* \in \operatorname{dom} \rho^*$ and $\operatorname{dom} \rho^* = M^*$. Thus, (ii) is satisfied.
To prove (1.2) we observe that for $Y \in \mathcal{A}_\rho$ we have $\rho(Y) \leq 0$ and it follows that

$$\forall Y^* \in \mathcal{Y}^* : \quad \rho^*(Y^*) \geq \sup_{Y \in \mathcal{A}_\rho} \{\langle Y, Y^* \rangle - \rho(Y)\} \geq \sup_{Y \in \mathcal{A}_\rho} \langle Y, Y^* \rangle . \tag{1.3}$$

To show the reverse, take $Y^*$ with $\langle Y^*, \mathbf{1} \rangle = -1$. Consider $Y \in \operatorname{dom} \rho$, then $Y + \rho(Y)\mathbf{1} \in \mathcal{A}_\rho$, hence

$$\sup_{\widetilde{Y} \in \mathcal{A}_\rho} \langle \widetilde{Y}, Y^* \rangle \geq \langle (Y + \rho(Y)\mathbf{1}), Y^* \rangle = \langle Y, Y^* \rangle - \rho(Y). \tag{1.4}$$

This inequality is trivially satisfied for $Y \notin \operatorname{dom} \rho$. Thus, we can take the supremum over all $Y \in \mathcal{Y}$ in (1.4), which gives $\rho^*$ on the right hand side and leads together with (1.3) to equation (1.2).
The equation $\operatorname{dom} \rho^* = M^*$ together with (1.2) leads to the dual representation of $\rho$ in (iii). Vice versa, if $\rho$ admits a dual representation as in (iii), then for all $Y \in \mathcal{Y}$ and $c \in \mathbb{R}$

$$\rho(Y + c\mathbf{1}) = \sup_{\{Y^* \in \mathcal{Y}^* : \langle Y^*, \mathbf{1} \rangle = -1\}} \{\langle Y, Y^* \rangle + c \langle 1, Y^* \rangle - \sup_{\widetilde{Y} \in \mathcal{A}_\rho} \langle \widetilde{Y}, Y^* \rangle\} = \rho(Y) - c.$$

Hence, $\rho$ satisfies the translation property, i.e., (i) is satisfied.

(d) Convexity and positive homogeneity of $\rho$ imply subadditivity: $\frac{1}{2}\rho(Y_1 + Y_2) = \rho(\frac{1}{2}Y_1 + \frac{1}{2}Y_2) \leq \frac{1}{2}\rho(Y_1) + \frac{1}{2}\rho(Y_2)$ for all $Y_1, Y_2 \in \mathcal{Y}$. From positive homogeneity of $\rho$ we obtain $\rho(\mathbf{0}) = \rho(t\mathbf{0}) = t\rho(\mathbf{0})$ for all $t > 0$. Since $\rho(\mathbf{0})$ is finite by assumption, we have $\rho(\mathbf{0}) = 0$. Vice versa, convexity, subadditivity of $\rho$ and $\rho(\mathbf{0}) = 0$ imply the positive homogeneity of $\rho$. To show this, we first prove that $\rho(tY) \leq t\rho(Y)$ for all $t > 0$. Let $t \in [0, 1]$. Because of the convexity of $\rho$ and $\rho(\mathbf{0}) = 0$ we obtain

$$\forall t \in [0, 1] : \quad \rho(tY) = \rho(tY + (1 - t)\mathbf{0}) \leq t\rho(Y). \tag{1.5}$$

For $t > 1$, we write $t = n + s$ with $n \in \mathbb{N}$ and $s \in [0,1)$. Then, because of the subadditivity and (1.5), it follows

$$\rho(tY) = \rho((n+s)Y) = \rho(nY+sY) \leq \rho(nY)+\rho(sY) \leq n\rho(Y)+s\rho(Y) = t\rho(Y).$$

Thus,

$$\forall t > 0 : \quad \rho(tY) \leq t\rho(Y). \tag{1.6}$$

To show the reverse, take $t > 0$. Then, we apply (1.6) for $\frac{1}{t} > 0$ and obtain

$$\forall t > 0 : \quad \rho(Y) = \rho(\frac{1}{t}tY) \leq \frac{1}{t}\rho(tY).$$

Thus, $\rho$ is positively homogeneous.
We now verify the equivalence between positive homogeneity and the dual representation of $\rho$ in (iv). Let $\rho$ be positively homogeneous. Then, $\rho(\mathbf{0}) = 0$ and $\rho^*(Y^*) \geq 0$ for all $Y^* \in \mathcal{Y}^*$ by (a). On the other hand, if $Y^* \in \mathcal{Y}^*$ and $\widehat{Y} \in \mathcal{Y}$ satisfy $\langle \widehat{Y}, Y^* \rangle - \rho(\widehat{Y}) > 0$, then

$$\begin{aligned} \rho^*(Y^*) &= \sup_{Y \in \mathcal{Y}} \{\langle Y, Y^* \rangle - \rho(Y)\} \geq \sup_{\lambda > 0} \{\langle \lambda\widehat{Y}, Y^* \rangle - \rho(\lambda\widehat{Y})\} \\ &= \sup_{\lambda > 0} \{\lambda[\langle \widehat{Y}, Y^* \rangle - \rho(\widehat{Y})]\} = +\infty. \end{aligned}$$

Thus, $Y^* \in \operatorname{dom}\rho^*$ implies $\langle Y, Y^* \rangle - \rho(Y) \leq 0$ for all $Y \in \mathcal{Y}$. We obtain $\rho^*(Y^*) \leq 0$ for all $Y^* \in \operatorname{dom}\rho^*$. Hence, $\rho^*(Y^*) = 0$ for all $Y^* \in \operatorname{dom}\rho^*$. This is $\rho^*(Y^*) = \mathcal{I}_{\operatorname{dom}\rho^*}(Y^*)$, which is equivalent to the dual representation of $\rho$ in (iv). Vice versa, if $\rho$ admits a dual representation as in (iv), then $\rho(\lambda Y) = \sup_{Y^* \in \operatorname{dom}\rho^*} \langle \lambda Y, Y^* \rangle = \lambda \sup_{Y^* \in \operatorname{dom}\rho^*} \langle Y, Y^* \rangle$. Thus, $\rho$ is positively homogeneous.

(e) Let $\mathcal{Y}^* = L^q, q \in [1, \infty]$ and let $\rho$ be monotone and satisfying the translation property. We can define for every $Y^* \in \operatorname{dom}\rho^*$ a measure $Q$, absolutely continuous to $P$ by $\frac{dQ}{dP} = -Y^*$. We can show that $Q$ is a probability measure: for all $A \subseteq \Omega$ it holds $Q(A) \geq 0$ since by the monotonicity of $\rho$ we obtain with (b) $-Y^* \in \mathcal{Y}^*_+$. Furthermore, $Q(\Omega) = 1$, since $E[-Y^*] = 1$ by the translation property of $\rho$ (see (c)). Hence, $Q$ is a probability measure and we obtain the dual representation in (e) for $\rho$.

$\square$

In this thesis, Theorem 1.5 is applied in Section 1.2 and 1.3, where we work with convex and coherent risk measures, but also in Section 4.1.1 and 4.1.4 where more general risk functions are used.

### 1.1.3 The Case $\mathcal{Y} = L^\infty$

In the case of $\mathcal{Y} = L^\infty$, the supremum in the dual representation (1.1) of a lower semicontinuous, convex and proper functional $\rho : L^\infty \to \mathbb{R} \cup \{+\infty\}$ is taken over the dual space $\mathcal{Y}^* = ba(\Omega, \mathcal{F}, P)$

$$\rho(Y) = \sup_{Y^* \in ba(\Omega,\mathcal{F},P)} \{\langle Y, Y^* \rangle - \rho^*(Y^*)\},$$

where $\langle Y, Y^* \rangle = \int_\Omega Y dY^*$.
If $\rho$ is additionally lower semicontinuous with respect to the weak* topology, the supremum in the dual representation can be taken over the smaller, more convenient space $L^1 \subset ba(\Omega, \mathcal{F}, P)$ and $\langle Y, Y^* \rangle$ reduces to $E[YY^*]$ for $Y^* \in L^1$. In this case, the dual representation of $\rho$ is

$$\rho(Y) = \sup_{Y^* \in L^1} \{E[YY^*] - \rho^*(Y^*)\}.$$

This follows immediately from the biconjugation theorem (Theorem A.5) for the dual pair $(L^\infty, L^1)$, i.e., $L^\infty$ is endowed with the weak* topology. To verify the additional property, that $\rho$ is weakly* lower semicontinuous, we have to work with nets, since $L^\infty$, endowed with the weak* topology, is not metrizable (see Appendix, Section B.3). It turns out that for a convex function $\rho$, weak* lower semicontinuity is equivalent to the Fatou property. The Fatou property can be verified by using $P - a.s.$ convergent sequences instead of nets. Furthermore, the closedness of epi $\rho$ with respect to $P - a.s.$ convergent sequences implies the weak* lower semicontinuity of $\rho$.
The following theorem generalizes Theorem 4.31 in [19] that treats convex risk measures $\rho$ and Theorem 3.2 in [8] that deals with coherent risk measures $\rho$. The following theorem allows us to find properties for a convex functional on $L^\infty$ that ensure $\rho$ to admit a dual representation with elements of the space $L^1$.

**Theorem 1.6.** *Let $\rho : L^\infty \to \mathbb{R} \cup \{+\infty\}$ be a convex function. Then, the following properties are equivalent:*

*(i) $\rho$ admits a dual representation*

$$\rho(Y) = \sup_{Y^* \in L^1} \{E[YY^*] - \rho^*(Y^*)\}.$$

*(ii) $\rho$ is lower semicontinuous with respect to the weak* topology.*

*(iii) $\rho$ satisfies the Fatou property.*

*Proof.* Suppose $\rho$ is not proper. Since $\rho$ maps into $\mathbb{R} \cup \{+\infty\}$, $\rho \equiv +\infty$. Thus, $\rho^* \equiv -\infty$ and $\rho^{**} \equiv +\infty$. Then, (i), (ii) and (iii) are trivially satisfied. In the

following we suppose that $\rho$ is proper.
(ii)$\Rightarrow$(i): Let (ii) be satisfied. $\rho$ is convex, weakly* lower semicontinuous and proper. Thus, we can apply the biconjugation theorem (Theorem A.5) for the dual pair $(L^\infty, L^1)$, i.e., $L^\infty$ is endowed with the weak* topology, and obtain (i).
(i)$\Rightarrow$(iii): Let (i) be satisfied. If $\{Y_n\}_{n\in\mathbb{N}} \subset L^\infty$ is a bounded sequence converging to $Y$ $P-a.s.$, then by Corollary B.20, $E[Y_nY^*] \to E[YY^*]$ for all $Y^* \in L^1$. Since for all $n \in \mathbb{N}, Y^* \in L^1$

$$E[Y_nY^*] - \rho^*(Y^*) \leq \sup_{Z\in L^1} \{E[ZY_n] - \rho^*(Z)\} = \rho(Y_n),$$

we obtain for all $Y^* \in L^1$

$$E[YY^*] - \rho^*(Y^*) = \lim_{n\to\infty} \{E[Y_nY^*] - \rho^*(Y^*)\} \leq \liminf_{n\to\infty} \rho(Y_n).$$

Hence,

$$\rho(Y) = \sup_{Y^*\in L^1} \{E[YY^*] - \rho^*(Y^*)\} \leq \liminf_{n\to\infty} \rho(Y_n).$$

(iii)$\Rightarrow$(ii): Let (iii) be satisfied. $\rho$ is lower semicontinuous with respect to the weak* topology if and only if for each $a \in \mathbb{R}$ the sublevel set $N_a := \{Y \in L^\infty : \rho(Y) \leq a\}$ is weakly* closed (see Definition 1.2). For $r > 0$ let

$$B_r := \{Y \in L^\infty : \|Y\|_{L^\infty} \leq r\}.$$

Since $\rho$ is convex, $N_a$ is convex for each $a \in \mathbb{R}$. Fix $a \in \mathbb{R}$. By Theorem A.64, [19], $N_a$ is weakly* closed if $C_r := N_a \cap B_r$ is closed in $L^1$ for each $r > 0$.
Take an arbitrary $r > 0$. Consider a sequence $\{Y_n\}_{n\in\mathbb{N}} \subset C_r$ converging in $L^1$ to some $Y \in L^1$. Then, there is a subsequence $\{Y_{n_k}\}_{k\in\mathbb{N}}$ converging to $Y$ $P-a.s.$ (see Theorem 12.38 and 12.39 in [2]). We have $Y \in B_r$ since $Y_{n_k} \in B_r$ for each $k \in \mathbb{N}$ and the set

$$A := \{\omega \in \Omega : |Y(\omega)| > r\} \subseteq \bigcup_{k\in\mathbb{N}} \{\omega \in \Omega : |Y_{n_k}(\omega)| > r\} \cup \{\omega \in \Omega : Y_{n_k}(\omega) \nrightarrow Y(\omega)\}$$

satisfies $P(A) = 0$. Moreover $Y \in N_a$ since $\rho(Y) \leq \liminf_{k\to\infty} \rho(Y_{n_k}) \leq a$ by (ii). Thus, $Y \in C_r$, i.e., $C_r$ is closed in $L^1$ for all $r > 0$.
Thus, $N_a$ is weakly* closed for all $a \in \mathbb{R}$ and (ii) is satisfied. $\square$

Lower semicontinuity with respect to $P-a.s.$ convergent sequences of $\rho$ implies by definition the Fatou property of $\rho$. In Lemma 1.3 we proved the equivalence between lower semicontinuity with respect to $P-a.s.$ convergent sequences and the closedness of $\operatorname{epi}\rho$ with respect to $P-a.s.$ convergent sequences. The following theorem results.

**Theorem 1.7.** *Let $\rho : L^\infty \to \mathbb{R} \cup \{+\infty\}$ be convex and* $\operatorname{epi}\rho$ *closed with respect to $P - a.s.$ convergent sequences. Then $\rho$ satisfies the Fatou property, is weakly* lower semicontinuous and admits a dual representation with elements from $L^1$*

$$\rho(Y) = \sup_{Y^* \in L^1} \{E[YY^*] - \rho^*(Y^*)\}.$$

*Proof.* The closedness of $\operatorname{epi}\rho$ with respect to $P - a.s.$ convergent sequences is by Lemma 1.3 equivalent to the lower semicontinuity of $\rho$ with respect to $P - a.s.$ convergent sequences, which imply the weaker condition that $\rho$ satisfies the Fatou property. Theorem 1.6 shows that this property is equivalent to the weakly* lower semicontinuity of $\rho$ and with the biconjugation theorem (Theorem A.5) follows the dual representation of $\rho$ with elements from $L^1$. □

### 1.1.4 Acceptance Sets

In this section, we review the relationships between a function $\rho : \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$ and its acceptance set $\mathcal{A}_\rho$. Furthermore, we review how a function $\rho_\mathcal{A}$ can be defined by a given set $\mathcal{A} \subseteq \mathcal{Y}$ and which properties of $\mathcal{A}$ lead to which properties of $\rho_\mathcal{A}$. Finally, we state the relationship between the function $\rho$ and the function $\rho_{\mathcal{A}_\rho}$ defined by $\mathcal{A}_\rho$. Analogously the relationship between a set $\mathcal{A}$ and the acceptance set $\mathcal{A}_{\rho_\mathcal{A}}$ of the function $\rho_\mathcal{A}$ can be studied.
A discussion of these questions for different kinds of risk measures and on different spaces can be found in several papers. For instance, [3] works with coherent risk measures on a finite probability space $\Omega$ and [19] considers in Section 4.1 this topic for finite valued risk measures that act on the space of bounded functionals on $\Omega$.
To my knowledge, the most general results regarding these questions can be found in [22] where extended real-valued translative functions on linear spaces and translative sets are studied from a much more general point of view than in this thesis. In this section, we summarize the results that are of interest for our case.
Therefore, we give the following definitions. For $A, B \subseteq \mathcal{Y}$ we understand $A + B$ to be the Minkowski sum of two subsets defined by $A + B = \{a + b : a \in A, b \in B\}$. A set $A \subseteq \mathcal{Y}$ is **closed under addition** iff $A + A \subseteq A$ and **convex** iff $t \in (0, 1), Y_1, Y_2 \in A$ imply $tY_1 + (1 - t)Y_2 \in A$.
Let $C \subseteq \mathcal{Y}$ be a nonempty set. A set $A \subseteq \mathcal{Y}$ is called $C-$**upward** iff $A + C \subseteq A$ ([22], Definition 7). A set $A \subseteq \mathcal{Y}$ is called **translative** with respect to $\mathbf{1}$ and $\mathbb{R}_+$ iff for all $Y \in A$ and $s \geq 0$ it holds $Y + s\mathbf{1} \in A$ or, formulated in another way, iff $A + \mathbb{R}_+\mathbf{1} \subseteq A$ ([22], Definition 1). Note that if $A$ is $\mathcal{Y}_+-$upward, then $A$ is automatically translative with respect to $\mathbf{1}$ and $\mathbb{R}_+$ since $\mathbb{R}_+\mathbf{1} \subseteq \mathcal{Y}_+$.
A set $A \subseteq \mathcal{Y}$ is said to be **radially closed** with respect to $\mathbf{1}$ iff

$$Y \in A, \{s_n\}_{n \in \mathbb{N}} \subset \mathbb{R}, \lim_{n \to \infty} s_n = s \in \mathbb{R}, Y + s_n\mathbf{1} \in A \Rightarrow Y + s\mathbf{1} \in A,$$

see Definition 3, [22]. Note that if $A \subseteq \mathcal{Y}$ is closed, then it is radially closed with respect to any $\mathbf{k} \in \mathcal{Y}\backslash\{\mathbf{0}\}$ ([22], Section 3.2). In this thesis, we call a set $A \subseteq \mathcal{Y}$ that is translative with respect to $\mathbf{1}$ and $\mathbb{R}_+$ just translative and a set that is radially closed with respect to $\mathbf{1}$ just radially closed.
For a set $A \subseteq \mathcal{Y}$, the intersection of all translative sets containing $A$ is called the **translative hull** of $A$ and is denoted by $\operatorname{tr} A$([22], Definition 2).
For a set $A \subseteq \mathcal{Y}$, the intersection of all subsets of $\mathcal{Y}$ which contain $A$ and are radially closed and translative is called the **radially closed, translative hull** of $A$. It is denoted by $\operatorname{rt} A$([22], Definition 5).
We recall the definition of the acceptance set of $\rho$

$$\mathcal{A}_\rho = \{Y \in \mathcal{Y} : \rho(Y) \leq 0\}.$$

First, we want to study the relationship between $\rho$ and its acceptance set $\mathcal{A}_\rho$. The following proposition is due to [22], Proposition 3, 5-8, Corollary 6 and Theorem 1.

**Proposition 1.8.** *Consider a function $\rho : \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$.*

*(i) $\rho$ is convex. $\Rightarrow$ $\mathcal{A}_\rho$ is convex.*

*(ii) $\rho$ is positively homogeneous. $\Rightarrow$ $\mathcal{A}_\rho$ is a cone.*

*(iii) $\rho$ is subadditive. $\Rightarrow$ $\mathcal{A}_\rho$ is closed under addition.*

*(iv) $\rho$ is monotone. $\Rightarrow$ $\mathcal{A}_\rho$ is $\mathcal{Y}_+-$upward.*

*(v) $\rho$ satisfies the translation property. $\Rightarrow$ $\mathcal{A}_\rho$ is translative and radially closed and the sublevel sets of $\rho$ satisfy $N_a = \mathcal{A}_\rho + \{-a\mathbf{1}\}$ for all $a \in \mathbb{R}$.*

*(vi) Let $\rho$ satisfy the translation property. Then, the function $\rho$ is lower semicontinuous if and only if $\mathcal{A}_\rho$ is closed.*

Conversely, one can take a given set $\mathcal{A} \subseteq \mathcal{Y}$ of acceptable positions as the primary object. For a position $Y \in \mathcal{Y}$, we can define the capital requirement as the minimal amount $t$ for which $Y + t\mathbf{1}$ becomes acceptable. This means, we can define the function $\rho_\mathcal{A} : \mathcal{Y} \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\rho_\mathcal{A}(Y) := \inf\{t \in \mathbb{R} : Y + t\mathbf{1} \in \mathcal{A}\}$$

agreeing on $\inf \emptyset = +\infty$ and $\inf \mathbb{R} = -\infty$. The relationship between $\mathcal{A}$ and $\rho_\mathcal{A}$ is as follows, due to [22], Proposition 3, 5-9 and Corollary 6.

**Proposition 1.9.** *Consider a set $\mathcal{A} \subseteq \mathcal{Y}$.*

*(i) $\mathcal{A}$ is convex. $\Rightarrow$ $\rho_\mathcal{A}$ is convex.*

*(ii) $\mathcal{A}$ is a cone. $\Rightarrow$ $\rho_\mathcal{A}$ is positively homogeneous.*

*(iii)* *$\mathcal{A}$ is closed under addition. $\Rightarrow$ $\rho_{\mathcal{A}}$ is subadditive.*

*(iv)* *$\mathcal{A}$ is $\mathcal{Y}_+$−upward. $\Rightarrow$ $\rho_{\mathcal{A}}$ is monotone.*

*(v)* *$\rho_{\mathcal{A}}$ satisfies the translation property and it holds $\mathcal{A} \subseteq \mathcal{A}_{\rho_{\mathcal{A}}}$.*

*(vi)* *Let $\mathcal{A}$ be translative and radially closed. Then, $\mathcal{A}$ is closed if and only if $\rho_{\mathcal{A}}$ is lower semicontinuous.*

*(vii)* $\inf\{t \in \mathrm{I\!R} : t\mathbf{1} \in \mathcal{A}\} = 0$. $\Rightarrow \rho_{\mathcal{A}}(0) = 0$.

*(viii)* *If $\mathcal{A} \neq \emptyset$ and for all $Y \in \mathcal{Y}$, there exists $t \in \mathrm{I\!R}$ such that $Y + t\mathbf{1} \notin \operatorname{tr} \mathcal{A}$, then $\rho_{\mathcal{A}}$ is proper.*

*(ix)* *If for all $Y \in \mathcal{Y}$, there exists an $t \in \mathrm{I\!R}$ such that $Y + t\mathbf{1} \notin \operatorname{tr} \mathcal{A}$ and $\mathcal{Y} = \mathcal{A} + \mathrm{I\!R}\{\mathbf{1}\}$, then $\rho_{\mathcal{A}}$ is finite valued.*

We now discuss the relationship between $\rho$ and $\rho_{\mathcal{A}_\rho}$ and the relationship between $\mathcal{A}$ and $\mathcal{A}_{\rho_{\mathcal{A}}}$.

**Corollary 1.10** ([22], Proposition 3)**.**
*(i) Let $\rho$ satisfies the translation property. Then $\mathcal{A}_\rho$ is translative and radially closed and $\rho = \rho_{\mathcal{A}_\rho}$.*
*(ii) Let $\mathcal{A}$ be translative and radially closed. Then $\rho_{\mathcal{A}}$ satisfies the translation property and $\mathcal{A} = \mathcal{A}_{\rho_{\mathcal{A}}}$.*

**Remark 1.11.** In [19], the properties of $\mathcal{A}$ that lead to (ii) in Corollary 1.10 are formulated in a different way. There, $\mathcal{A}$ is assumed to satisfy

$$Y_1 \in \mathcal{A}, Y_2 \in \mathcal{Y}, Y_2 \geq Y_1 \Rightarrow Y_2 \in \mathcal{A} \tag{1.7}$$

and the following closure property: For $Y_1 \in \mathcal{Y}$ and $Y_2 \in \mathcal{Y}$,

$$\{\lambda \in [0,1] : \lambda Y_1 + (1-\lambda)Y_2 \in \mathcal{A}\}$$

is closed in $[0,1]$. Then, $\mathcal{A} = \mathcal{A}_{\rho_{\mathcal{A}}}$ holds true.
Condition (1.7) is equivalent to $\mathcal{A} + \mathcal{Y}_+ \subseteq \mathcal{A}$, which means $\mathcal{A}$ is $\mathcal{Y}_+$−upward. This implies that $\mathcal{A}$ is translative. Thus, the assumption in Corollary 1.10 (ii) concerning the translation property of $\mathcal{A}$ are weaker than in [19].

Proposition 1.9 (v) states that $\rho_{\mathcal{A}}$ satisfies the translation property whether or not $\mathcal{A}$ is translative and radially closed. Also, $\mathcal{A} \subseteq \mathcal{A}_{\rho_{\mathcal{A}}}$ is always true. This gives rise to ask for the relationship between a set $\mathcal{A}$ and $\mathcal{A}_{\rho_{\mathcal{A}}}$ and between $\rho_{\mathcal{A}}$ and $\rho_{\mathcal{A}_{\rho_{\mathcal{A}}}}$ if we do not impose any conditions on $\mathcal{A}$.

**Proposition 1.12** ([22], Proposition 4)**.** *Let $A \subseteq \mathcal{Y}$ be a nonempty set. Then, $\mathcal{A}_{\rho_{\mathcal{A}}} = \operatorname{rt} \mathcal{A}$ and $\rho_{\mathcal{A}} = \rho_{\operatorname{rt} \mathcal{A}}$.*

Conditions (vii) of Proposition 1.9 can be equivalently formulated in terms of $\operatorname{tr} \mathcal{A}$.

**Proposition 1.13** ([22], Corollary 4)**.** *Let $\mathcal{A} \subseteq \mathcal{Y}$. The condition $\mathbb{R}_+ \boldsymbol{1} \cap (-\operatorname{tr} \mathcal{A}) = \{0\}$ implies $\rho_{\mathcal{A}}(0) = 0$.*

Last but not least, we give a useful property of functions satisfying the translation property.

**Proposition 1.14** ([22], Proposition 1)**.** *If $\rho : \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$ satisfies the translation property and $\rho(0) = 0$, then $\rho$ is linear on the one dimensional subspace $L(\boldsymbol{1})$ spanned by $\boldsymbol{1}$.*

## 1.2 Convex Risk Measures

In this section, we shall introduce the concept of convex risk measure.

**Definition 1.15.** *A functional $\rho : \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$ is called a* **convex risk measure** *if it is convex, monotone, satisfies the translation property and $\rho(\boldsymbol{0}) = 0$.*

This kind of risk measure was introduced by Föllmer and Schied in 2002 in [18] (see also [19]) and in a slightly different way (without imposing the translation property) at the same time by Frittelli and Gianin in [20]. We follow the definition of [18], but in contrast to [18], we allow $\rho$ also to attain the value $+\infty$ and work on $L^p$-spaces. Lower semicontinuous convex risk measures have a dual representation. First, we consider the case $\mathcal{Y}^* = L^q, q \in [1, \infty]$. This means, $\mathcal{Y} = L^p, p \in [1, \infty)$, endowed with the norm topology or $\mathcal{Y} = L^\infty$, endowed with the weak* topology or the Mackey topology.

**Theorem 1.16.** *Let $\mathcal{Y}^* = L^q, q \in [1, \infty]$. A function $\rho : \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous, convex risk measure if and only if there exists a representation of the form*

$$\rho(Y) = \sup_{Q \in \mathcal{Q}} \{E^Q[-Y] - \sup_{\widetilde{Y} \in \mathcal{A}_\rho} E^Q[-\widetilde{Y}]\}, \tag{1.8}$$

*where $\mathcal{Q} := \{Q \in \widehat{\mathcal{Q}} : Z_Q \in \mathcal{Y}^*\}$ and $\inf_{Q \in \mathcal{Q}} \sup_{Y \in \mathcal{A}_\rho} E^Q[-Y] = 0$. The conjugate function $\rho^*$ of $\rho$ is nonnegative, convex, proper, weakly* lower semicontinuous, satisfies for all $Y^*$ with $E[Y^*] = -1$*

$$\rho^*(Y^*) = \sup_{Y \in \mathcal{A}_\rho} E[YY^*]$$

*and it holds*

$$\operatorname{dom} \rho^* \subseteq \{-Z_Q : Q \in \mathcal{Q}\}.$$

*Proof.* The "if"-part follows from Theorem 1.5 (e) and (a).
The inverse implication is straightforward: The function

$$Y \mapsto \sup_{Q \in \mathcal{Q}} \{E^Q[-Y] - \sup_{\widetilde{Y} \in \mathcal{A}_\rho} E^Q[-\widetilde{Y}]\}$$

is convex, monotone, lower semicontinuous (Lemma 2.38, [2]), satisfies the translation property and $\rho(\mathbf{0}) = 0$.
If $\rho$ is a lower semicontinuous, convex risk measure, then $\rho^*$ is convex, weak* lower semicontinuous and proper (see Theorem A.4). From Theorem 1.5 (a) we conclude, that $\rho^*$ is nonnegative. The remaining results follow from Theorem 1.5 (e). □

**Remark 1.17.** Föllmer and Schied [19] called $\rho^*(-Z_Q) =: \alpha_{min}(Q)$ for $Q \in \mathcal{Q}$ the "minimal penalty function". In this notation (1.8) becomes

$$\rho(Y) = \sup_{Q \in \mathcal{Q}} \{E^Q[-Y] - \alpha_{min}(Q)\}.$$

**Remark 1.18.** Let $\alpha(Q) : \mathcal{Q} \to \mathrm{I\!R} \cup \{+\infty\}$ be a functional with $\inf_{Q \in \mathcal{Q}} \alpha(Q) = 0$. Then,

$$\rho(Y) := \sup_{Q \in \mathcal{Q}} \{E^Q[-Y] - \alpha(Q)\} \tag{1.9}$$

is a convex risk measure. The functional $\alpha$ is called a penalty function and $\alpha_{min}$ (Remark 1.17) is the minimal penalty function on $\mathcal{Q}$ that represents $\rho$ (see [19]).
The penalty function $\alpha$ can describe how seriously the probabilistic model $Q \in \mathcal{Q}$ is taken. The value of the convex risk measure $\rho(Y)$ is the worst case of the expected loss $E^Q[-Y]$, taken over all models $Q \in \mathcal{Q}$, but reduced by $\alpha(Q)$ ([19], Section 3.4).

The case $\mathcal{Y}^* = ba(\Omega, \mathcal{F}, P)$, hence the case $\mathcal{Y} = L^\infty$, endowed with the norm topology, was excluded in Theorem 1.16. In this case, we have to work with finitely additive measures and obtain an analogous theorem.

**Theorem 1.19.** *Let $\mathcal{Y} = L^\infty$, endowed with the norm topology. A function $\rho : L^\infty \to \mathrm{I\!R} \cup \{+\infty\}$ is a lower semicontinuous, convex risk measure if and only if there exists a representation of the form*

$$\rho(Y) = \sup_{Y^* \in \mathcal{M}} \{\langle -Y, Y^* \rangle - \sup_{\widetilde{Y} \in \mathcal{A}_\rho} \langle -\widetilde{Y}, Y^* \rangle\}, \tag{1.10}$$

*where $\mathcal{M} := \{Y^* \in ba(\Omega, \mathcal{F}, P)_+ : Y^*(\Omega) = 1\}$. The conjugate function $\rho^*$ of $\rho$ is nonnegative, convex, proper, weakly* lower semicontinuous, satisfies for all $Y^*$ with $Y^*(\Omega) = -1$*

$$\rho^*(Y^*) = \sup_{Y \in \mathcal{A}_\rho} \langle Y, Y^* \rangle$$

*and we have*

$$\operatorname{dom} \rho^* \subseteq \{Y^* \in ba(\Omega, \mathcal{F}, P)_- : Y^*(\Omega) = -1\}.$$

*Proof.* We obtain the results analogously to the proof of Theorem 1.16. □

If $\rho$ is additionally lower semicontinuous with respect to the weak* topology, the supremum in the dual representation (1.10) can be taken over the smaller, more convenient space $L^1 \subset ba(\Omega, \mathcal{F}, P)$ and $\langle Y, Y^* \rangle$ reduces to $E[YY^*]$ for $Y^* \in L^1$. We give some equivalent conditions that ensure the weak* lower semicontinuity of $\rho$, and thus a dual representation with elements of $L^1$. The first part of the theorem corresponds to Theorem 4.31 in [19].

**Theorem 1.20.** *Let $\rho : L^\infty \to \mathbb{R} \cup \{+\infty\}$ be a convex risk measure. Then the following conditions are equivalent.*

*(i) $\rho$ can be represented by elements of $L^1$*

$$\rho(Y) = \sup_{Q \in \widehat{\mathcal{Q}}} \{E^Q[-Y] - \sup_{\widetilde{Y} \in \mathcal{A}_\rho} E^Q[-\widetilde{Y}]\}.$$

*(ii) $\rho$ is weakly* lower semicontinuous.*

*(iii) $\rho$ satisfies the Fatou property.*

*(iv) The acceptance set $\mathcal{A}_\rho$ of $\rho$ is weakly* closed.*

*(v) $\rho$ is continuous from above.*

*Furthermore, if* epi $\rho$ *is closed with respect to $P - a.s.$ convergent sequences, then $\rho$ is weakly* lower semicontinuous and the above properties are satisfied.*

*Proof.* (i)⇔(ii) follows from Theorem 1.6 and 1.5 (e).
(ii)⇔(iii) because of Theorem 1.6.
(ii)⇔(iv) because of Definition 1.2 (ii) and because the translation property of $\rho$ implies the weak* closedness of all sublevel sets if the sublevel set to level zero $\mathcal{A}_\rho = N_0$ is weakly* closed (see Proposition 1.8 (v)).
(iii)⇒(v): Take $Y_n \searrow Y$. $Y_n$ is a bounded sequence converging to $Y$ $P-a.s.$ Because of the monotonicity of $\rho$ we have for all $n \in \mathbb{N}$ that $\rho(Y_n) \leq \rho(Y_{n+1})$ and together with (iii) we obtain that $\rho(Y_n) \leq \rho(Y) \leq \liminf_{n\to\infty} \rho(Y_n)$ and thus, $\rho(Y_n) \nearrow \rho(Y)$.
(v)⇒(iii): Let $\{Y_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{Y}$ converging $P-a.s.$ to $Y$. Define $Z_m := \sup_{n\geq m} Y_n \in \mathcal{Y}$. Then $Z_m$ decreases $P - a.s.$ to $Y$. Since $Z_m \geq Y_m$ for all $m \in \mathbb{N}$, we obtain by the monotonicity of $\rho$ that $\rho(Y_m) \geq \rho(Z_m)$ and thus, with (v)

$$\liminf_{m\to\infty} \rho(Y_m) \geq \lim_{m\to\infty} \rho(Z_m) = \rho(Y).$$

The last part follows from Theorem 1.7. □

As shown in Section 1.1.4, a risk measure can be constructed via a given set $\mathcal{A} \subseteq \mathcal{Y}$ of acceptable positions by defining

$$\rho_{\mathcal{A}}(Y) := \inf\{t \in \mathbb{R} : Y + t\mathbf{1} \in \mathcal{A}\}. \tag{1.11}$$

If the set $\mathcal{A}$ is convex and $\mathcal{Y}_+$-upward, then $\rho_{\mathcal{A}}$ is convex, monotone and satisfies the translation property (Proposition 1.9). To obtain a convex risk measure, $\mathcal{A}$ has additionally to satisfy $\inf\{t \in \mathbb{R} : t\mathbf{1} \in \mathcal{A}\} = 0$ or the equivalent condition in Proposition 1.13. Then, $\rho_{\mathcal{A}}(0) = 0$. Alternatively, the normalization $\widetilde{\rho}(Y) := \rho_{\mathcal{A}}(Y) - \rho_{\mathcal{A}}(0)$ yields a convex risk measure if $\inf\{t \in \mathbb{R} : t\mathbf{1} \in \mathcal{A}\} \in \mathbb{R}$.
Another possibility is to start with a given penalty function $\alpha : \mathcal{Q} \to \mathbb{R} \cup \{+\infty\}$ on $\mathcal{Q} := \{Q \in \widehat{\mathcal{Q}} : Z_Q \in \mathcal{Y}^*\}$ with $\inf_{Q\in\mathcal{Q}} \alpha(Q) \in \mathbb{R}$ (cf. Remark 1.18) and to define

$$\rho\,(Y) := \sup_{Q\in\mathcal{Q}}\{E^Q[-Y] - \alpha(Q)\}. \tag{1.12}$$

If $\inf_{Q\in\mathcal{Q}} \alpha(Q) = 0$ holds, then $\rho$ is a convex risk measure. Otherwise, the normalized function $\widetilde{\rho}(Y) := \rho(Y) - \rho(0)$ is a convex risk measure on $\mathcal{Y}$. We give now some examples.

**Example 1.21.** [[19], Example 4.9] Let $\mathcal{Q} \subseteq \widehat{\mathcal{Q}}$. Consider a map $\gamma : \mathcal{Q} \to \mathbb{R}$ satisfying $\sup_{Q\in\mathcal{Q}} \gamma(Q) < \infty$, which specifies for each $Q \in \mathcal{Q}$ some threshold $\gamma(Q)$. Suppose that a position $Y$ is acceptable if

$$\forall Q \in \mathcal{Q} : \quad E^Q[Y] \geq \gamma(Q).$$

The set $\mathcal{A}$ of all acceptable positions is convex, satisfies $\inf\{t \in \mathbb{R} : t\mathbf{1} \in \mathcal{A}\} \in \mathbb{R}$ and is $L^\infty_+$-upward since it satisfies (1.7). Thus, $\rho = \rho_{\mathcal{A}}$ defined by (1.11) is convex, monotone and translation invariant and the normalization of $\rho$ yields a convex risk measure on $L^\infty$ that is weakly* lower semicontinuous.
Alternatively, $\rho$ can be written as

$$\rho(Y) = \sup_{Q\in\widehat{\mathcal{Q}}}\{E^Q[-Y] - \alpha(Q)\},$$

where the penalty function $\alpha : \widehat{\mathcal{Q}} \to (-\infty, \infty]$ is defined by $\alpha(Q) = -\gamma(Q)$ for $Q \in \mathcal{Q}$ and $\alpha(Q) = +\infty$ otherwise.

**Example 1.22.** [[19], Example 4.33] Consider the penalty function $\alpha : \widehat{\mathcal{Q}} \to [0, \infty]$ defined by

$$\alpha(Q) := \frac{1}{\beta} H(Q|P),$$

where $\beta > 0$ is a given constant and $H(Q|P) = E^Q[\log \frac{dQ}{dP}]$ is the relative entropy of $Q$ with respect to $P$. The corresponding entropic risk measure $\rho$ on $L^\infty$ is given by

$$\rho(Y) = \sup_{Q\in\widehat{\mathcal{Q}}}\{E^Q[-Y] - \frac{1}{\beta} H(Q|P)\}.$$

The functional $\alpha$ is the minimal penalty function representing $\rho$. The function $\alpha(Q)$ penalizes the model $Q \in \mathcal{Q}$ proportional to the deviation of $Q$ from $P$, measured by the relative entropy. Thus, the given model $P$ is the one taken most seriously. The entropic risk measure can be written as

$$\rho(Y) = \frac{1}{\beta} \log E[e^{-\beta Y}]$$

and is a weakly* lower semicontinuous convex risk measure on $L^\infty$.

## 1.3 Coherent Risk Measures

In this section, we shall introduce the concept of coherent risk measure. Coherent risk measures are a subclass of convex risk measures and have been introduced in the seminal paper of Artzner et al. [3] in 1999 on finite probability spaces. Delbaen [8] extended the results to more general spaces.

**Definition 1.23.** *A functional $\rho : \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$ is called a* **coherent risk measure** *if it is subadditive, positively homogeneous, monotone, satisfies the translation property and $\rho(\mathbf{0}) = 0$.*

The relationships between convex and coherent risk measures are explained by the following theorem.

**Theorem 1.24.** *Let $\rho : \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex risk measure. The following conditions are equivalent.*

*(i) $\rho$ is a coherent risk measure*

*(ii) The acceptance set $\mathcal{A}_\rho$ is a cone.*

*(iii) $\rho$ is subadditive.*

*(iv) $\rho$ is positively homogeneous.*

*(v) $\rho^*(Y^*) = \mathcal{I}_{M^*}(Y^*)$, where $M^* \subseteq \mathcal{Y}^*$.*

*Proof.* Let $\rho$ be a lower semicontinuous convex risk measure.
(i) $\Rightarrow$ (ii): Since $\rho$ is positively homogeneous, we have for all $t > 0$ and all $Y \in \mathcal{A}_\rho$ that $\rho(tY) = t\rho(Y) \leq 0$. Thus, $tY \in \mathcal{A}_\rho$. This means, $\mathcal{A}_\rho$ is a cone.
(ii) $\Rightarrow$ (i): From Theorem 1.5 (c) we obtain that $\rho^*(Y^*) = \sup_{Y \in \mathcal{A}_\rho} \langle Y, Y^* \rangle$ for all $Y^* \in \mathcal{Y}^*$ with $\langle Y^*, \mathbf{1} \rangle = -1$ and $+\infty$ else. Since $\rho(\mathbf{0}) = 0$, $\mathcal{A}_\rho$ is a cone with $\mathbf{0} \in \mathcal{A}_\rho$ and we obtain with Example A.11

$$\rho^*(Y^*) = \mathcal{I}_{\mathcal{A}_\rho^* \cap \{Y^* \in \mathcal{Y}^* : \langle Y^*, \mathbf{1} \rangle = -1\}}(Y^*).$$

Thus, $\rho(Y) = \sup_{\mathcal{A}_\rho^* \cap \{Y^* \in \mathcal{Y}^* : E[Y^*] = -1\}} \langle Y, Y^* \rangle$ and is as a support function positively homogeneous and subadditive (Example A.6), thus a coherent risk measure.
(iii) $\Leftrightarrow$ (iv): This was shown in Theorem 1.5 (d).
(i) $\Leftrightarrow$ (iii): Obvious since (iii) is equivalent to (iv).
(i) $\Leftrightarrow$ (v): Follows from Theorem 1.5 (d). □

Next, we shall show that a lower semicontinuous coherent risk measure admits a dual representation as a support function. We start with the case $\mathcal{Y}^* = L^q, q \in [1, \infty]$, that means, $\mathcal{Y} = L^p, p \in [1, \infty)$, endowed with the norm topology or $\mathcal{Y} = L^\infty$, endowed with the weak* topology or with the Mackey topology. In this cases, the bilinear form between the dual spaces is the expectation $\langle Y, Y^* \rangle = E[YY^*]$ for all $Y \in \mathcal{Y}$ and $Y^* \in \mathcal{Y}^*$. In the following theorem we shall show that the dual elements in the representation of $\rho$ are probability measures.

**Theorem 1.25.** *Let $\mathcal{Y}^* = L^q, q \in [1, \infty]$. A function $\rho : \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous coherent risk measure if and only if there exists a subset $\mathcal{Q}$ of $\widehat{\mathcal{Q}}$ such that $\{Z_Q : Q \in \mathcal{Q}\} \subseteq \mathcal{Y}^*$ and*

$$\rho(Y) = \sup_{Q \in \mathcal{Q}} E^Q[-Y]. \tag{1.13}$$

*The maximal representing set is $\mathcal{Q}_{\max} = \overline{\text{co}}^* \mathcal{Q}$, i.e., it holds*

$$\rho(Y) = \sup_{Q \in \overline{\text{co}}^* \mathcal{Q}} E^Q[-Y],$$

*where $\overline{\text{co}}^* \mathcal{Q}$ is the weak* closure of the convex hull of of the densities $Z_Q$ of $\mathcal{Q}$. It holds $-\overline{\text{co}}^* \mathcal{Q} = \mathcal{A}_\rho^* \cap \{Y^* \in \mathcal{Y}^* : E[Y^*] = -1\}$ and*

$$\rho^*(Y^*) = \mathcal{I}_{-\overline{\text{co}}^* \mathcal{Q}}(Y^*).$$

**Remark 1.26.** Obviously, a coherent risk measure $\rho$ with dual representation (1.13) can be represented with every set $\mathcal{Q}'$ satisfying $\overline{\text{co}}^* \mathcal{Q}' = \overline{\text{co}}^* \mathcal{Q}$ and $\mathcal{Q}' = \overline{\text{co}}^* \mathcal{Q}$ is the maximal one.

**Remark 1.27.** Except in the case $\mathcal{Y} = L^1$, we can replace the weak* closure of $\text{co}\, \mathcal{Q}$ in Theorem 1.25 with the closure of $\text{co}\, \mathcal{Q}$ with respect to the norm topology. Let us explain this in more detail.
First, we consider the case $\mathcal{Y} = L^p, p \in (1, \infty)$, endowed with the norm topology or $\mathcal{Y} = L^\infty$, endowed with the weak* topology or with the Mackey topology. This means, we exclude for a moment the case $\mathcal{Y} = L^1$. We have $\{Z_Q : Q \in \mathcal{Q}\} \subseteq L^q, q \in [1, \infty)$. Then, the weak* closure of the convex hull of the densities of $\mathcal{Q}$ coincides with the closure of the convex hull of the densities of $\mathcal{Q}$ with respect to the norm topology. This follows since a convex subset of $L^q$ is weakly closed if and only if it is closed with respect to the norm topology ([19], Theorem A.59) and since in

$L^q, q \in [1, \infty)$ the weak and the weak* topology coincide.
Note that in the case of $\mathcal{Y} = L^1$, i.e., $\{Z_Q : Q \in \mathcal{Q}\} \subseteq L^\infty$, this is not true. In this case, the norm closure of $\operatorname{co} \mathcal{Q}$ coincides with the weak closure of $\operatorname{co} \mathcal{Q}$, but this is in general only a subset of the weak* closure of $\operatorname{co} \mathcal{Q}$.

*Proof of Theorem 1.25.* The "if" part follows from Theorem 1.5. The inverse implication follows since the function $\rho : \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$ defined by $\rho(Y) := \sup_{Q \in \mathcal{Q}} E^Q[-Y]$ satisfies the translation property, is lower semicontinuous, monotone, positively homogeneous and convex with $\rho(\mathbf{0}) = 0$ and thus, subadditive.
Let $\rho$ be a lower semicontinuous coherent risk measure with dual representation (1.13). Consider a space $\mathcal{Z}$ with the topology $\tau$ such that its topological dual space $\mathcal{Z}^*$ satisfies $\mathcal{Z}^* = \mathcal{Y}$. For $\mathcal{Y} = L^p, p \in [1, \infty]$ we have $\mathcal{Z} = L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$, endowed with the weak* topology (cf. Remark 1.27). Since in our cases $\mathcal{Z} = \mathcal{Y}^*$, we can regard $\mathcal{Q}$ as a subset of $\mathcal{Z}$. Since $\rho$ is a support function of $-\mathcal{Q}$, it follows from Example A.7 that $\rho$ can also be represented in terms of $\mathcal{Q}_{\max} = \overline{\operatorname{co}}^* \mathcal{Q}$

$$\rho(Y) = \sup_{Q \in \mathcal{Q}} E^Q[-Y] = \sup_{Q \in \overline{\operatorname{co}}^* \mathcal{Q}} E^Q[-Y],$$

where the closure is taken with respect to the topology $\tau$ on $\mathcal{Z}$ and thus, with respect to the weak* topology on $\mathcal{Y}^*$. With Example A.10 we obtain

$$\rho^*(Y^*) = \mathcal{I}_{-\overline{\operatorname{co}}^* \mathcal{Q}}(Y^*).$$

From Theorem 1.5 (c) we obtain that $\rho^*(Y^*) = \sup_{Y \in \mathcal{A}_\rho} E[YY^*]$ for all $Y^*$ with $E[Y^*] = -1$ and $+\infty$ else. From Theorem 1.24 (i) and (ii) and $\rho(\mathbf{0}) = 0$, it follows that $\mathcal{A}_\rho$ is a cone containing $\mathbf{0} \in \mathcal{Y}$. Together with Example A.11 we obtain

$$\rho^*(Y^*) = \mathcal{I}_{\mathcal{A}_\rho^* \cap \{Y^* \in \mathcal{Y}^* : E[Y^*] = -1\}}(Y^*).$$

Thus, $-\overline{\operatorname{co}}^* \mathcal{Q} = \mathcal{A}_\rho^* \cap \{Y^* \in \mathcal{Y}^* : E[Y^*] = -1\}$. □

Now, we consider the case $\mathcal{Y} = L^\infty$, endowed with the norm topology.

**Theorem 1.28.** *Let $\mathcal{Y} = L^\infty$, endowed with the norm topology. A function $\rho : L^\infty \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous coherent risk measure if and only if there exists a set $M^* \subseteq \{Y^* \in ba(\Omega, \mathcal{F}, P)_+ : Y^*(\Omega) = 1\}$ such that*

$$\rho(Y) = \sup_{Y^* \in M^*} \langle -Y, Y^* \rangle . \tag{1.14}$$

*Moreover, $M^*$ can be chosen to be convex and weakly* closed.*

*Proof.* The results follow from Theorem 1.5 (b), (c) and (d) and since the function $Y \mapsto \sup_{Y^* \in M} \langle -Y, Y^* \rangle$ satisfies the properties of a coherent risk measure and is

lower semicontinuous.
In analogy to the proof of Theorem 1.25, we can show that

$$\rho^*(Y^*) = \mathcal{I}_{\mathcal{A}^*_\rho \cap \{Y^* \in \mathcal{Y}^* : Y^*(\Omega) = -1\}}(Y^*).$$

Since $\rho^*$ is convex and weakly* lower semicontinuous (Lemma A.4), the set $\mathcal{A}^*_\rho \cap \{Y^* \in \mathcal{Y}^* : Y^*(\Omega) = -1\}$ is convex and weakly* closed and $\rho$ can be represented with this set. □

Let $\rho$ be a coherent risk measure on $L^\infty$. In analogy to Theorem 1.20, we can give several conditions that ensure a representation of $\rho$ with respect to probability measures as in (1.13) instead of a representation with respect to finitely additive measures as in (1.14).

**Corollary 1.29.** *Let $\rho : L^\infty \to \mathrm{I\!R} \cup \{+\infty\}$ be a coherent risk measure. Then the following conditions are equivalent.*

*(i) $\rho$ can be represented by a set of probability measures $\mathcal{Q} \subseteq \widehat{\mathcal{Q}}$*

$$\rho(Y) = \sup_{Q \in \mathcal{Q}} E^Q[-Y].$$

*(ii) $\rho$ is weakly* lower semicontinuous.*

*(iii) $\rho$ satisfies the Fatou property.*

*(iv) The acceptance set $\mathcal{A}_\rho$ of $\rho$ is weakly* closed.*

*(v) $\rho$ is continuous from above.*

*Furthermore, if* $\operatorname{epi} \rho$ *is closed with respect to $P - a.s.$ convergent sequences, then $\rho$ is weakly* lower semicontinuous and the above properties are satisfied.*

*Proof.* The results follow immediately from Theorem 1.20. □

If one considers a set of acceptable positions $\mathcal{A} \subseteq \mathcal{Y}$ that is a convex cone, $\mathcal{Y}_+$-upward and satisfies $\inf\{t \in \mathrm{I\!R} : t\mathbf{1} \in \mathcal{A}\} \in \mathrm{I\!R}$, then the risk measure $\rho_\mathcal{A}$ defined via (1.11) is a coherent risk measure on $\mathcal{Y}$ (see Proposition 1.9, Theorem 1.5 (d)) with the acceptance set $\mathcal{A}_{\rho_\mathcal{A}} = \operatorname{rt} \mathcal{A}$ (Proposition 1.12). Another possibility to construct a coherent risk measure is to start with a set of probability measures $\mathcal{Q} \subseteq \widehat{\mathcal{Q}}$ such that $\{Z_Q : Q \in \mathcal{Q}\} \subseteq \mathcal{Y}^*$. Then, the function defined by

$$\rho(Y) := \sup_{Q \in \mathcal{Q}} E^Q[-Y]$$

is a coherent risk measure on $\mathcal{Y}$ with $\operatorname{dom} \rho^* = \overline{\operatorname{co}}^* \mathcal{Q}$ (see Theorem 1.25).
We give some examples to illustrate the variety in this class of risk measures. The examples are taken from [19].

**Example 1.30.** [[19], Example 4.9] Consider Example 1.21. If $\gamma$ satisfies $\gamma(Q) = 0$ for all $Q \in \mathcal{Q}$, then $\rho$ is a weakly* lower semicontinuous coherent risk measure on $L^\infty$ and takes the form

$$\rho(Y) = \sup_{Q \in \mathcal{Q}} E^Q[-Y].$$

**Example 1.31.** [[19], Example 4.8] Consider the worst-case risk measure $\rho_{max}$ on $L^\infty$ that measures the maximal loss. It is defined by

$$\rho_{max}(Y) := \text{ess. sup}(-Y).$$

The corresponding acceptance set $\mathcal{A}$ is given by the convex cone $L^\infty_+$. Thus, $\rho_{max}$ is a coherent risk measure. It is the most conservative coherent risk measure or, more general, for any monotone risk function $\rho$ that satisfies the translation property and $\rho(\mathbf{0}) = 0$ it holds

$$\forall Y \in L^\infty : \quad \rho(Y) \leq \rho_{max}(Y).$$

The risk measure $\rho_{max}$ can be represented in the form

$$\rho_{max}(Y) = \sup_{Q \in \widehat{\mathcal{Q}}} E^Q[-Y]$$

and is weakly* lower semicontinuous.

**Example 1.32.** [[19], Example 4.37, Section 4.4] Let $\mathcal{Q}_\alpha$ be the class of all $Q \in \widehat{\mathcal{Q}}$ whose density $dQ/dP$ is bounded by $1/\alpha$ for some fixed parameter $\alpha \in (0, 1]$. The corresponding coherent risk measure

$$AVaR_\alpha(Y) := \sup_{Q \in \mathcal{Q}_\alpha} E^Q[-Y]$$

is defined on $L^1$ and is called the Average Value at Risk at level $\alpha$. It can be written in terms of the Value at Risk

$$AVaR_\alpha(Y) = \frac{1}{\alpha} \int_0^\alpha VaR_\gamma(Y) d\gamma.$$

Sometimes, the Average Value at Risk is also called the Conditional Value at Risk ($\text{CVaR}_\alpha$) or the expected shortfall ($\text{ES}_\alpha$). The set $\mathcal{Q}_\alpha$ is equal to the maximal set $\mathcal{Q}_{max}$ of Theorem 1.25.
For $\alpha = 1$, one obtains $AVaR_1(Y) = E[-Y]$. For $Y \in L^\infty$, we have

$$AVaR_0(Y) := VaR_0(Y) := \lim_{\alpha \downarrow 0} AVaR_\alpha(Y) = \text{ess. sup}(-Y)$$

which is the worst-case risk measure on $L^\infty$ (Example 1.31).

**Example 1.33.** [[19], Example 4.38] Let $\mathcal{Q}$ be the class of all conditional distributions $P[\cdot|A]$ such that $A \in \mathcal{F}$ has $P(A) > \alpha$ for some fixed level $\alpha \in (0,1)$. The coherent risk measure induced by $\mathcal{Q}$ ,

$$WCE_\alpha(Y) := \sup\{E[-Y|A] : A \in \mathcal{F}, P(A) > \alpha\},$$

is called the worst conditional expectation at level $\alpha$. If the underlying probability space is atomless, the coherent risk measures $\mathrm{AVaR}_\alpha$ and $\mathrm{WCE}_\alpha$ coincides ([19], Corollary 4.62).

# Chapter 2

# Optimization Problems for Randomized Tests

In this section, we shall consider an optimization problem that arises from different problems in mathematical finance and test theory.
Let $\mathcal{Y}$ be a separated locally convex space with its topological dual space $\mathcal{Y}^*$ and $\rho : \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$. Let $\mathcal{X} = L^\infty$, endowed with the norm topology and $\mathcal{X}^* = ba(\Omega, \mathcal{F}, P)$, its topological dual space. Let $A : \mathcal{X} \to \mathcal{Y}$ be a linear and continuous operator and $b \in \mathcal{Y}$. We want to solve the problem

$$\min_{X \in \mathcal{X}_0} \rho(AX + b), \tag{2.1}$$

where the constraint set is

$$\mathcal{X}_0 = \{X \in \mathcal{X}_1 : \sup_{X^* \in C^*} \langle HX^*, X\rangle \leq c\} \tag{2.2}$$

with

$$\mathcal{X}_1 := \{X \in L^\infty : 0 \leq X \leq 1\} \subset \mathcal{X}.$$

The set $\mathcal{X}_1$ is called the set of randomized tests. $\mathcal{X}_0$ is a subset of $\mathcal{X}_1$ satisfying (2.2) with $c > 0$ and $C^* \subseteq L^1 \subset ba(\Omega, \mathcal{F}, P)$. Let $H$ be an element of $L^1$ such that for all $X^* \in C^*$ it holds $HX^* \in L^1$. We keep in mind that $\mathcal{Y}$ can be as in Chapter 1 the space $L^p$ for $p \in [1, \infty]$, endowed with the norm topology or the space $L^\infty$, endowed with the weak* or Mackey topology. In this chapter we may consider any separated locally convex space $\mathcal{Y}$.
The special choice of $\mathcal{X}_1$ being the set of randomized tests ensures that we can deduce a result about the structure of a solution to problem (2.1). In all the applications in Chapter 3 and 4 we will work with this set.
We now introduce a list of assumptions. In each of the following theorems and lemmata we will quote, which assumptions we use. For the main theorem, Theorem 2.9, we have to impose all of them.

**Assumption 2.1.** *We impose the following assumptions.*

*(A1)* $c > 0$.

*(A2) Let* $C^* \subseteq L^1$ *and* $H \in L^1$ *such that* $\{HX^* : X^* \in C^*\} \subseteq L^1$.

*(A3) Let* $\sup_{X^* \in C^*} \|HX^*\|_{L^1} < +\infty$ *and* $C^*$ *be compact.*

*(A4) The operator* $A : L^\infty \to \mathcal{Y}$ *is linear and continuous.*

*(A5) The functional* $\rho : \mathcal{Y} \to \mathrm{I\!R} \cup \{+\infty\}$ *is convex, lower semicontinuous, continuous in some* $AX_0 + b$ *with* $X_0 \in \mathcal{X}_0$ *and satisfies* $\rho(AX_0 + b) < +\infty$.

*(A6) The map* $X \mapsto \rho(AX + b)$ *is lower semicontinuous in the weak* topology on* $\mathcal{X}_0$.

*(A7) The map* $X \mapsto \langle Y^*, AX \rangle$ *is continuous in the weak* topology for all* $Y^* \in \mathcal{Y}^*$

**Remark 2.2.** Assumption (A6) is for example satisfied if the operator $A$ is continuous with respect to the weak* topology on $L^\infty$ and the topology used in $\mathcal{Y}$ and $\rho$ is lower semicontinuous.
Under the validity of (A4), condition (A7) is equivalent to $A^*Y^* \in L^1$ for all $Y^* \in \mathcal{Y}^*$, where $A^*$ denotes the adjoint operator of $A$ (see Definition 6.51 in [2]).

**Remark 2.3.** In Section 4.1.4, we will show that it is possible to weaken Assumption (A7) as follows

(A7') $A^*\widetilde{Y}^*$ admits a Hahn decomposition,

where $\widetilde{Y}^*$ is the solution of problem (2.4). Furthermore, in Remark 2.11 we discuss that the constant $c > 0$ can be replaced by a positive, continuous function $c(\cdot) : C^* \to \mathrm{I\!R}$ and thus Assumption (A1) can be generalized correspondingly. But unless otherwise stated, we shall work with Assumption 2.1 as above.

## 2.1 Motivation

Problem (2.1) arises in various applications. The two main cases are the problem of hedging in incomplete markets and the problem of testing compound hypotheses. We shall give a short motivation.

- **Testing Compound Hypotheses.**
  We want to discriminate a family $\mathcal{P}^*$ of probability measures (compound null hypothesis) against another family $\mathcal{Q}$ of probability measures (compound alternative hypothesis). This means, we look for a randomized test $\widetilde{\varphi}$ that

minimizes the probability of accepting $\mathcal{P}^*$ when it is false, while the probability of rejecting $\mathcal{P}^*$ when it is true should be less than a given acceptable significance level $\alpha \in (0,1)$. Thus, the problem, a special case of (2.1), is

$$\sup_{\varphi \in R_0} \inf_{Q \in \mathcal{Q}} E^Q[\varphi],$$

where $\mathcal{X}_0 = R_0 = \{\varphi \in \mathcal{X}_1 : \sup_{P^* \in \mathcal{P}^*} E^{P^*}[\varphi] \leq \alpha\}$. Thus, in this case we have $A\varphi = \varphi$, $b = 0$, $C^* = \mathcal{P}^*$, $H = \mathbf{1}$ and $c = \alpha$. The space $\mathcal{Y} = L^\infty$ is endowed with the Mackey topology, $\mathcal{Y}^* = L^1$ and the function $\rho : L^\infty \to \mathbb{R}$ is a coherent risk measure defined by $\rho(Y) := \sup_{Q \in \mathcal{Q}} E^Q[-Y]$. This problem will be considered in Section 3.1.

- **Hedging in Incomplete Markets.**
  We want to find an admissible strategy that minimizes the shortfall risk when hedging in incomplete markets. This dynamic problem can be split into a representation problem and a static problem. The latter problem is a special case of (2.1),
  $$\min_{\varphi \in R_0} \rho\left((\varphi - 1)H\right),$$
  where $\varphi$ is a randomized test. The constraint set $\mathcal{X}_0$ coincides with $R_0 = \{\varphi \in \mathcal{X}_1 : \sup_{P^* \in \mathcal{P}} E^{P^*}[\varphi H] \leq \widetilde{V}_0\}$. $H \in L^1_+$ is the payoff of a contingent claim and $\mathcal{P}$ is the set of equivalent martingale measures. Thus, this is problem (2.1) with $A\varphi = H\varphi$, $b = -H$, $C^* = \{Z_{P^*} : P^* \in \mathcal{P}\}$, $c = \widetilde{V}_0$ and $\mathcal{Y} = L^1$. $\rho$ is a risk measure that quantifies the risk of losses due to the shortfall. In Section 4.1, we consider the problem of hedging for different kinds of measures of risk when the set $\{Z_{P^*} : P^* \in \mathcal{P}\}$ is compact. In Section 4.1.1, we consider the most general case. In Sections 4.1.2, we consider a convex risk measure $\rho$ and in Section 4.1.3 a coherent risk measure $\rho$. In Section 4.1.4, we consider a robust version of the expectation of a loss function $l$ to quantify the risk of losses. The problem is solved for the case of Lipschitz continuous loss functions $l$. In the general case, we consider a modified problem
  $$\min_{\varphi \in R_0} \rho(\varphi).$$
  The function $\rho$ is a modification of the robust version of the expectation of a loss function $l$ and we have $A\varphi = \varphi$, $b = 0$, $C^* = \mathcal{P}$, $H = \mathbf{1}$, $c = \widetilde{V}_0$ and $\mathcal{Y} = L^\infty$, endowed with the norm topology.
  Analogously, the problem of hedging in general incomplete markets for different kind of risk measures is considered in Section 4.2.

The applications will be discussed in detail in Chapter 3 and 4. In this chapter, we work with the problem as general as possible since in the different applications, as motivated above, we have to vary the space $\mathcal{Y}$, the operator $A$, the set $C^*$, the

multiplicative element $H \in L^1$, the function $\rho$ and the constants $b, c \in \mathbb{R}$.
In this chapter, we will prove the existence of a solution to the primal problem (2.1), deduce the dual problem and verify the validity of strong duality. Then, we solve the inner problem of the dual problem and finally the whole problem.

## 2.2 Existence of a Solution to the Primal Problem

To show the existence of a solution to (2.1), we first prove the following lemma.

**Lemma 2.4.** *Let (A2) be satisfied. Then, the sets $\mathcal{X}_1$ and $\mathcal{X}_0$ are weakly* compact and convex.*

*Proof.* The unit sphere $B := \{X \in L^\infty : -1 \leq X \leq 1\}$ in $L^\infty$ is weakly* compact (Theorem V.4.2, [10]), since $L^1$ is a Banach space. It is sufficient to prove the closedness of $\mathcal{X}_1$ in the weak* topology. Then, the compactness of $\mathcal{X}_1$ in the weak* topology follows from Theorem V.4.3, [10], since $\mathcal{X}_1$ is a weakly* closed subset of the weakly* compact set $B$. Consider a net (see Section B.3 for definition) $\{X_\alpha\}_{\alpha \in D} \subseteq \mathcal{X}_1$ that converges to $X$ with respect to the weak* topology in $L^\infty$. This means, for all $X^* \in L^1$ it holds $E[X_\alpha X^*] \to E[XX^*]$. If there would exist $\Omega_1 \subseteq \Omega$ with $P(\Omega_1) > 0$ and $X(\omega) > 1$ for all $\omega \in \Omega_1$, then we can choose $\widehat{X}^* = 1_{\Omega_1}(\omega)$ in $L^1$ and obtain $E[X\widehat{X}^*] > P(\Omega_1)$. This is a contradiction to $E[X\widehat{X}^*] = \lim_\alpha E[X_\alpha \widehat{X}^*] \leq P(\Omega_1)$, which follows from $X_\alpha \leq 1$ for all $\alpha \in D$, since $X_\alpha \in \mathcal{X}_1$. Hence, $X(\omega) \leq 1$ for all $\omega \in \Omega$. Analogously, it can be shown that $X(\omega) \geq 0$ for all $\omega \in \Omega$. Hence, $\mathcal{X}_1$ is weakly* closed, hence, weakly* compact.
We show that $\mathcal{X}_0$ is weakly* closed. Consider a net $\{X_\alpha\}_{\alpha \in D} \subseteq \mathcal{X}_0$ that converges to $X$ with respect to the weak* topology in $L^\infty$. Since $\{HX^* : X^* \in C^*\} \subseteq L^1$ (Assumption (A2)), we obtain

$$\forall X^* \in C^* : \quad E[XHX^*] = \lim_\alpha E[X_\alpha HX^*] \leq c.$$

Hence, we can take the supremum on the left hand side and obtain that $\mathcal{X}_0$ is weakly* closed and, as a subset of a weakly* compact set, also weakly* compact.
The convexity of $\mathcal{X}_1$ and $\mathcal{X}_0$ is obvious. □

Now, we prove the existence of a solution to problem (2.1).

**Theorem 2.5.** *Let Assumption (A2), (A5) and (A6) be satisfied. There exists $\widetilde{X} \in \mathcal{X}_0$ solving the optimization problem (2.1) and $\rho(A\widetilde{X} + b)$ is finite. If $\rho$ is additionally strictly convex, then the difference of any two solutions has to be an element of* $\ker A := \{X \in \mathcal{X} : AX = 0\}$.

*Proof.* The constraint set $\mathcal{X}_0$ is weakly* compact, as proved in Lemma 2.4. Because we assumed $X \mapsto \rho(AX + b)$ to be lower semicontinuous in the weak* topology

on $\mathcal{X}_0$ (Assumption (A6)), there exists $\widetilde{X} \in \mathcal{X}_0$ solving (2.1) (cf. [46], 5.4(b)) and $\rho(A\widetilde{X} + b)$ is finite since $\rho$ is assumed to be finite in some $AX_0 + b$ with $X_0 \in R_0$ (Assumption (A5)). Thus, $\rho(A\widetilde{X} + b) \leq \rho(AX_0 + b) < +\infty$.
Let $\widetilde{X}_1$ be a solution. For any $X \in \mathcal{X}_0$ and for $\varepsilon \in (0,1)$ we define

$$X_\varepsilon = (1-\varepsilon)\widetilde{X}_1 + \varepsilon X.$$

If $\rho$ is strictly convex, we obtain

$$\rho(AX_\varepsilon + b) \leq (1-\varepsilon)\rho(A\widetilde{X}_1 + b) + \varepsilon\rho(AX + b).$$

The inequality is strict if $A\widetilde{X}_1 \neq AX$. Hence, for any two solutions $\widetilde{X}_1$ and $\widetilde{X}_2$ we have $A\widetilde{X}_1 = A\widetilde{X}_2$. This means, $A(\widetilde{X}_1 - \widetilde{X}_2) = 0$, hence $(\widetilde{X}_1 - \widetilde{X}_2) \in \ker A$. $\square$

## 2.3 The Dual Problem

We now want to deduce the dual problem of (2.1).
Since $\rho$ is lower semicontinuous, convex and proper, there exists a dual representation (biconjugation theorem, Theorem A.5) for $\rho$

$$\rho(Y) = \rho^{**}(Y) = \sup_{Y^* \in \mathcal{Y}^*} \{\langle Y^*, Y\rangle - \rho^*(Y^*)\}. \tag{2.3}$$

Equation (2.3) enables us to rewrite the primal problem (2.1) with value $p$

$$p = \min_{X \in \mathcal{X}_0} \{ \sup_{Y^* \in \mathcal{Y}^*} \{\langle Y^*, AX + b\rangle - \rho^*(Y^*)\}\}.$$

The dual problem to (2.1) with value $d$ is:

$$d = \sup_{Y^* \in \mathcal{Y}^*} \{ \inf_{X \in \mathcal{X}_0} \{\langle Y^*, AX + b\rangle - \rho^*(Y^*)\}\}. \tag{2.4}$$

The following strong duality theorem holds.

**Theorem 2.6.** *Let Assumptions (A2), (A4), (A5) and (A6) be satisfied. Strong duality holds, i.e., the values of the primal problem (2.1) and its dual problem (2.4) are equal: $p = d$ and there exists a solution $\widetilde{Y}^*$ of the dual problem (2.4).*
*Furthermore, $(\widetilde{X}, \widetilde{Y}^*)$ is a saddle point of the functional $(X, Y^*) \mapsto \langle Y^*, AX + b\rangle - \rho^*(Y^*)$ in $\mathcal{X}_0 \times \mathcal{Y}^*$, where $\widetilde{X}$ is the solution of (2.1). Thus,*

$$\min_{X \in \mathcal{X}_0} \{ \max_{Y^* \in \mathcal{Y}^*} \{\langle Y^*, AX + b\rangle - \rho^*(Y^*)\}\} = \max_{Y^* \in \mathcal{Y}^*} \{ \min_{X \in \mathcal{X}_0} \{\langle Y^*, AX + b\rangle - \rho^*(Y^*)\}\}.$$

*Proof.* Problem (2.1) can be rewritten as

$$p = \min_{X \in L^\infty} \{\rho(AX + b) + \mathcal{I}_{\mathcal{X}_0}(X)\} \tag{2.5}$$

We denote in (2.5): $f(X) := \mathcal{I}_{\mathcal{X}_0}(X)$ and $g(AX) := \rho(AX + b)$. The dual problem (see Theorem A.12) is

$$d = \sup_{Y^* \in \mathcal{Y}^*} \{-f^*(A^*Y^*) - g^*(-Y^*)\},$$

where $A^*$ is the adjoined operator of $A$ and $f^*, g^*$ are the conjugate functions of $f$ and $g$, respectively. The value $p$ of the primal problem is finite (Theorem 2.5). The function $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ is convex because of the convexity of $\mathcal{X}_0$ (see Example A.8). The function $g : \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$ is convex since $\rho$ is convex. Since $\rho$ is assumed to be continuous in some $AX_0 + b$ with $X_0 \in \mathcal{X}_0$, we have strong duality $p = d$ (Theorem A.12). To establish the dual problem, we calculate the conjugate functions $f^*$ and $g^*$.

$$\begin{aligned} f^*(A^*Y^*) &= \sup_{X \in \mathcal{X}} \{\langle A^*Y^*, X\rangle - f(X)\} = \sup_{X \in \mathcal{X}} \{\langle A^*Y^*, X\rangle - \mathcal{I}_{\mathcal{X}_0}(X)\} \\ &= \sup_{X \in \mathcal{X}_0} \{\langle Y^*, AX\rangle\}. \end{aligned}$$

The function $g$ is defined by $g(Y) = \rho(Y + b)$. Its conjugate function is [50, Theorem 2.3.1 (vi)]

$$g^*(Y^*) = \rho^*(Y^*) - \langle Y^*, b\rangle .$$

Then, the dual problem is

$$d = \sup_{\widehat{Y}^* \in \mathcal{Y}^*} \{- \sup_{X \in \mathcal{X}_0} \langle \widehat{Y}^*, AX\rangle - \rho^*(-\widehat{Y}^*) - \langle \widehat{Y}^*, b\rangle\}$$

We set $Y^* := -\widehat{Y}^*$ and obtain

$$d = \sup_{Y^* \in \mathcal{Y}^*} \{ \inf_{X \in \mathcal{X}_0} \{\langle Y^*, AX + b\rangle - \rho^*(Y^*)\}\}. \tag{2.6}$$

The existence of a solution $\widetilde{Y}^*$ to the dual problem (2.6) follows from the validity of strong duality (Theorem A.12). Let $\widetilde{X}$ be a solution to the primal problem (2.1) (Theorem 2.5). Since

$$p = \sup_{Y^* \in \mathcal{Y}^*} \{\langle Y^*, A\widetilde{X} + b\rangle - \rho^*(Y^*)\} \geq \langle \widetilde{Y}^*, A\widetilde{X} + b\rangle - \rho^*(\widetilde{Y}^*), \tag{2.7}$$

$$d = \inf_{X \in \mathcal{X}_0} \{\langle \widetilde{Y}^*, AX + b\rangle - \rho^*(\widetilde{Y}^*)\} \leq \langle \widetilde{Y}^*, A\widetilde{X} + b\rangle - \rho^*(\widetilde{Y}^*) \tag{2.8}$$

and because of strong duality we have

$$\langle \widetilde{Y}^*, A\widetilde{X} + b\rangle - \rho^*(\widetilde{Y}^*) \leq p = d \leq \langle \widetilde{Y}^*, A\widetilde{X} + b\rangle - \rho^*(\widetilde{Y}^*). \tag{2.9}$$

Hence, we have equality in (2.9) and also in (2.7) and (2.8). This means, the supremum in (2.7) and the infimum in (2.8) is attained. Thus,

$$\min_{X \in \mathcal{X}_0} \{ \max_{Y^* \in \mathcal{Y}^*} \{\langle Y^*, AX + b\rangle - \rho^*(Y^*)\}\} = \max_{Y^* \in \mathcal{Y}^*} \{ \min_{X \in \mathcal{X}_0} \{\langle Y^*, AX + b\rangle - \rho^*(Y^*)\}\}.$$

Thus, $(\widetilde{X}, \widetilde{Y}^*)$ is a saddle point of the function $(X, Y^*) \mapsto \langle Y^*, AX + b\rangle - \rho^*(Y^*)$. $\square$

This strong duality theorem motivates us to consider the inner problem of the dual problem.

## 2.4 The Inner Problem of the Dual Problem

Let us consider the inner problem of the dual problem (2.4) for an arbitrary, but fixed $Y^* \in \mathcal{Y}^*$:

$$\inf_{X \in \mathcal{X}_0} \{\langle Y^*, AX + b\rangle - \rho^*(Y^*)\} = \inf_{X \in \mathcal{X}_0} \{\langle Y^*, AX\rangle\} + \langle Y^*, b\rangle - \rho^*(Y^*).$$

Thus, the inner problem reduces to

$$\inf_{X \in \mathcal{X}_0} \langle Y^*, AX\rangle . \tag{2.10}$$

Denote by $\widetilde{p}(Y^*)$ the optimal value of (2.10). Rewriting the constraint set $\mathcal{X}_0$, problem (2.10) can be written as

$$\inf_{X \in \mathcal{X}_1} \langle Y^*, AX\rangle , \tag{2.11}$$

$$\forall X^* \in C^* : \quad \langle HX^*, X\rangle \leq c. \tag{2.12}$$

(2.11), (2.12) is an optimization problem on an infinite dimensional space with a linear objective function and a sublinear constraint or, formulated in another way, infinitely many linear constraints.

**Lemma 2.7.** *Let (A2) and (A7) be satisfied. There exists a solution $\widetilde{X}_{Y^*}$ to problem (2.11), (2.12) and $\widetilde{p}(Y^*)$ is finite.*

*Proof.* The assertion follows since $\mathcal{X}_0$ is weakly* compact (Lemma 2.4) and $X \mapsto \langle Y^*, AX\rangle$ is assumed to be continuous in the weak* topology for all $Y^* \in \mathcal{Y}^*$ (Assumption (A7)). □

Since $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, P)$ and $\mathcal{X}_1$ is the set of randomized test, we can give a result about the structure of a solution to (2.11), (2.12) and, using this, a result about the structure of a solution to (2.1). In the examples of Chapter 3 and 4 this will always be the case. Note that we do not need to specify the space $\mathcal{Y}$.
Let $\mathcal{B}$ denote the $\sigma$-algebra of all Borel sets on $C^*$. Let $\Lambda_+$ be the set of all finite measures on $(C^*, \mathcal{B})$. We assign to (2.11), (2.12) the following dual problem

$$\sup_{\lambda \in \Lambda_+} \{-E[(-A^*Y^* - H \int_{C^*} X^* d\lambda)^+] - c\lambda(C^*)\}. \tag{2.13}$$

Denote by $\widetilde{d}(Y^*)$ its optimal value. The following strong duality theorem holds.

**Theorem 2.8.** *Let Assumptions (A1) - (A4) and (A7) be satisfied. Then, strong duality holds true for problems (2.11), (2.12) and (2.13), i.e.,*

$$\forall Y^* \in \mathcal{Y}^* : \quad \widetilde{d}(Y^*) = \widetilde{p}(Y^*).$$

*Moreover, for each* $Y^* \in \mathcal{Y}^*$ *there exists a solution* $\widetilde{\lambda}_{Y^*} \in \Lambda_+$ *to problem (2.13). The optimal randomized test* $\widetilde{X}_{Y^*}$ *of problem (2.11), (2.12) has the following structure:*

$$\widetilde{X}_{Y^*}(\omega) = \begin{cases} 1 & : \quad -A^*Y^* > H \int\limits_{C^*} X^* d\widetilde{\lambda}_{Y^*} \\ 0 & : \quad -A^*Y^* < H \int\limits_{C^*} X^* d\widetilde{\lambda}_{Y^*} \end{cases} \qquad P-a.s. \tag{2.14}$$

*and*

$$E[HX^*\widetilde{X}_{Y^*}] = c \qquad \widetilde{\lambda}_{Y^*} - a.s. \tag{2.15}$$

*Proof.* Let $\mathcal{L}$ be the linear space of all continuous functions $l : C^* \to \mathbb{R}$ on the compact set $C^*$ (Assumption (A3)) with pointwise addition, multiplication with real numbers and pointwise partial order $l_1 \leq l_2 \Leftrightarrow l_2 - l_1 \in \mathcal{L}_+ := \{l \in \mathcal{L} : \forall X^* \in C^* : l(X^*) \geq 0\}$. We endow $\mathcal{L}$ with the norm $\|l\|_{\mathcal{L}} = \sup_{X^* \in C^*} |l(X^*)|$, which ensures that $\mathcal{L}$ is a Banach space ([10], Section IV.6).
We define a linear and continuous operator $B : (L^\infty, \|\cdot\|_{L^\infty}) \to (\mathcal{L}, \|\cdot\|_{\mathcal{L}})$ by $(BX)(X^*) := -\langle HX^*, X\rangle$ for $X^* \in C^*$. Assumption (A3) ensures that $B$ is bounded and thus continuous. We define the functions $\mathbf{1}, \mathbf{0} \in \mathcal{L}$ by

$$\forall X^* \in C^* : \mathbf{1}(X^*) = 1 \in \mathbb{R},\ \mathbf{0}(X^*) = 0 \in \mathbb{R}.$$

The constraint (2.12) can be rewritten as

$$c\mathbf{1} + BX \geq \mathbf{0} \Leftrightarrow BX \in \mathcal{L}_+ - c\mathbf{1}.$$

Then, we can write problem (2.11), (2.12) equivalently as

$$\widetilde{p}(Y^*) = \min_{X \in L^\infty} \{\langle Y^*, AX\rangle + \mathcal{I}_{\mathcal{X}_1}(X) + \mathcal{I}_{\mathcal{L}_+ - c\mathbf{1}}(BX)\}, \tag{2.16}$$

where Lemma 2.7 ensures that the minimum in (2.16) is attained. Let $\Lambda$ be the space of finite signed measures on $(C^*, \mathcal{B})$, regarded as the dual space of $\mathcal{L}$ with the bilinear form $\langle l, \lambda\rangle = \int_{C^*} l d\lambda$ for $l \in \mathcal{L}$ and $\lambda \in \Lambda$ (see [2], Corollary 13.15). We want to establish the dual problem of (2.16) as in Theorem A.12

$$\widetilde{d}(Y^*) = \sup_{\lambda \in \Lambda} \{-f^*(B^*\lambda) - g^*(-\lambda)\}, \tag{2.17}$$

where $f(X) := \langle Y^*, AX\rangle + \mathcal{I}_{\mathcal{X}_1}(X)$ and $g(BX) := \mathcal{I}_{\mathcal{L}_+ - c\mathbf{1}}(BX)$. The conjugate function of $g$ is

$$\begin{aligned} g^*(\lambda) &= \sup_{\widetilde{l} \in \mathcal{L}} \{\langle \widetilde{l}, \lambda\rangle - \mathcal{I}_{\mathcal{L}_+ - c\mathbf{1}}(\widetilde{l})\} = \sup_{\widetilde{l} \in \mathcal{L}_+ - c\mathbf{1}} \langle \widetilde{l}, \lambda\rangle = \sup_{l \in \mathcal{L}_+} \langle l - c\mathbf{1}, \lambda\rangle \\ &= \sup_{l \in \mathcal{L}_+} \langle l, \lambda\rangle - c \int\limits_{C^*} d\lambda = \mathcal{I}_{\mathcal{L}_+^*}(\lambda) - c\lambda(C^*), \end{aligned}$$

where $\mathcal{L}_+^* := \{\lambda \in \Lambda : \forall l \in \mathcal{L}_+ : \langle l, \lambda \rangle \leq 0\}$ is the negative dual cone of $\mathcal{L}_+$. The last equality follows from Example A.11 since $\mathcal{L}_+$ is a cone containing $\mathbf{0} \in \mathcal{L}$. To determine the conjugate function of $f$ at $B^*\lambda$, i.e.,

$$f^*(B^*\lambda) \;=\; \sup_{X \in L^\infty} \Big\{ \langle B^*\lambda, X\rangle - \langle Y^*, AX\rangle - \mathcal{I}_{\mathcal{X}_1}(X) \Big\},$$

we have to calculate $\langle B^*\lambda, X\rangle$, where $B^* : \Lambda \to ba(\Omega, \mathcal{F}, P)$ is the adjoined operator of $B$. By definition of $B^*$, the equation $\langle B^*\lambda, X\rangle = \langle \lambda, BX\rangle$ has to be satisfied for all $X \in L^\infty, \lambda \in \Lambda$ (see [2], Definition 6.51). Thus,

$$\forall X \in L^\infty, \forall\, \lambda \in \Lambda :\ \langle B^*\lambda, X\rangle = \int_{C^*} -\langle HX^*, X\rangle\, d\lambda = -\int_{C^*} E[HX^*X] d\lambda$$

The last equality holds true, since $\{HX^* : X^* \in C^*\} \subseteq L^1$ (Assumption (A2)). Furthermore, we have $A^*Y^* \in L^1$ for all $Y^* \in \mathcal{Y}^*$ (Assumption (A7), Remark 2.2). Hence the conjugate function of $f$ at $B^*\lambda$ is

$$f^*(B^*\lambda) = \sup_{X \in \mathcal{X}_1} \{- \int_{C^*} E[HX^*X] d\lambda - E[A^*Y^*X]\}.$$

The dual problem (2.17) becomes

$$\begin{aligned} \widetilde{d}(Y^*) \;&=\; \sup_{\lambda \in \Lambda} \Big\{ - \sup_{X \in \mathcal{X}_1} \{- \int_{C^*} E[HX^*X] d\lambda - E[A^*Y^*X]\} - \mathcal{I}_{-\mathcal{L}_+^*}(\lambda) - c\lambda(C^*) \Big\}, \\ &=\; \sup_{\lambda \in -\mathcal{L}_+^*} \Big\{ - \sup_{X \in \mathcal{X}_1} \{- \int_{C^*} E[HX^*X] d\lambda - E[A^*Y^*X]\} - c\lambda(C^*) \Big\}, \qquad (2.18) \end{aligned}$$

where $-\mathcal{L}_+^* = \{\lambda \in \Lambda : \forall l \in \mathcal{L}_+ : \langle l, \lambda\rangle \geq 0\}$. It holds $-\mathcal{L}_+^* = \Lambda_+ := \{\lambda \in \Lambda : \forall M^* \in \mathcal{B} : \lambda(M^*) \geq 0\}$. This is the set of finite measures on $(C^*, \mathcal{B})$. To prove this, we take $\lambda \in -\mathcal{L}_+^*$ and suppose that $\lambda \notin \Lambda_+$, i.e., there exists a set $\overline{M}^* \in \mathcal{B}$ with $\lambda(\overline{M}^*) < 0$. Define $\bar{l}$ by $\bar{l}(X^*) := 1_{\overline{M}^*}(X^*) \in L_+$. Then, $\langle \bar{l}, \lambda \rangle = \lambda(\overline{M}^*) < 0$, which is a contradiction to $\lambda \in -\mathcal{L}_+^*$. Thus, $-\mathcal{L}_+^* \subseteq \Lambda_+$. Vice versa, take $\lambda \in \Lambda_+$. Then for all $l \in \mathcal{L}_+$ it holds $l(X^*) \geq 0$ for all $X^* \in C^*$. Thus, $\langle l, \lambda\rangle \geq \langle \mathbf{0}, \lambda\rangle = 0$ for all $l \in \mathcal{L}_+$. This means, $\lambda \in -\mathcal{L}_+^*$. Thus, we can rewrite (2.18) and obtain

$$\widetilde{d}(Y^*) = \sup_{\lambda \in \Lambda_+} \Big\{ - \sup_{X \in \mathcal{X}_1} \{- \int_{C^*} E[HX^*X] d\lambda - E[A^*Y^*X]\} - c\lambda(C^*) \Big\}. \qquad (2.19)$$

The spaces $(\Omega, \mathcal{F}, P)$ and $(C^*, \mathcal{B}, \lambda)$ for $\lambda \in \Lambda_+$ are positive, finite measure spaces, and thus also $\sigma$-finite. Furthermore, the function $f(\omega, X^*) = H(\omega)X^*(\omega)X(\omega)$ is measurable for all $X \in \mathcal{X}_1$ and it holds that for all $\lambda \in \Lambda_+$ and for all $X \in \mathcal{X}_1$

$$\int_{C^*} \int_{\Omega} |HX^*X| dP d\lambda \overset{\|X\|_{L^\infty} \leq 1}{\leq} \sup_{X^* \in C^*} \|HX^*\|_{L^1} \lambda(C^*) \overset{(A3)}{<} +\infty.$$

Thus, we can apply Tonelli's Theorem (Theorem B.22) and obtain that the order of integration can be changed, i.e., for all $\lambda \in \Lambda_+$ and for all $X \in \mathcal{X}_1$

$$\int\limits_{C^*}\int\limits_{\Omega} HX^*XdPd\lambda = \int\limits_{\Omega}\int\limits_{C^*} HX^*Xd\lambda dP < +\infty. \tag{2.20}$$

Since in (2.19) only elements $\lambda \in \Lambda_+$ and $X \in \mathcal{X}_1$ have to be considered, we can change the order of integration and obtain

$$\widetilde{d}(Y^*) = \sup_{\lambda\in\Lambda_+}\Big\{-\sup_{X\in\mathcal{X}_1} E[X(-A^*Y^* - H\int\limits_{C^*} X^*d\lambda)] - c\lambda(C^*)\Big\} \tag{2.21}$$

We can apply Tonelli's Theorem (Theorem B.22) also for the function $g(\omega, X^*) = |H(\omega)X^*(\omega)X(\omega)|$. Thus, we can choose in the equation corresponding to (2.20), i.e., in

$$\forall\lambda \in \Lambda_+, \forall X \in \mathcal{X}_1 : \int\limits_{C^*}\int\limits_{\Omega} |HX^*X|dPd\lambda = \int\limits_{\Omega}\int\limits_{C^*} |HX^*X|d\lambda dP < +\infty,$$

$X = \mathbf{1} \in \mathcal{X}_1$ and obtain that $H\int_{C^*} X^*d\lambda \in L^1$ for all $\lambda \in \Lambda_+$. Together with $A^*Y^* \in L^1$ for all $Y^* \in \mathcal{Y}^*$ (Assumption (A7), Remark 2.2), we obtain $-A^*Y^* - H\int_{C^*} X^*d\lambda \in L^1$ for all $\lambda \in \Lambda_+, Y^* \in \mathcal{Y}^*$. Since $X \in \mathcal{X}_1$ is a randomized test, it follows that the supremum over all $X \in \mathcal{X}_1$ in (2.21) is attained by an $\overline{X} \in \mathcal{X}_1$ satisfying

$$\overline{X}(\omega) = \begin{cases} 1 & : \quad \omega \in \{\omega \in \Omega : -(A^*Y^*)(\omega) > (H\int\limits_{C^*} X^*d\lambda)(\omega)\} \\ 0 & : \quad \omega \in \{\omega \in \Omega : -(A^*Y^*)(\omega) < (H\int\limits_{C^*} X^*d\lambda)(\omega)\} \end{cases} \qquad P-a.s. \tag{2.22}$$

In the following, we shall use the simpler notation as in (2.14). If we denote for the moment $-A^*Y^* - H\int_{C^*} X^*d\lambda =: \nu_\lambda \in L^1$ and with $\nu_\lambda^+$ the positive part, the value of the dual problem is

$$\widetilde{d}(Y^*) = \sup_{\lambda\in\Lambda_+}\{-E[\nu_\lambda^+] - c\lambda(C^*)\}.$$

This is equation (2.13). Strong duality holds if $f$ and $g$ are convex, $g$ is continuous in some $BX_0$ with $X_0 \in \operatorname{dom} f$ and $\widetilde{p}(Y^*)$ is finite (see Theorem A.12). The existence of a primal solution ensures the finiteness of $\widetilde{p}(Y^*)$ (Lemma 2.7). The function $f$ is convex since $\mathcal{X}_1$ is a convex set and $g$ is convex since the set $\mathcal{L}_+ - c\mathbf{1}$ is convex (see Example A.8). The function $g$ is continuous in some $BX_0$ with $X_0 \in \operatorname{dom} f$ if $BX_0 \in \operatorname{int}(\mathcal{L}_+ - c\mathbf{1})$. If we take $X_0 \equiv 0$, then $X_0 \in \operatorname{dom} f$ since $X_0 \in \mathcal{X}_1$ and we see that $BX_0 = \mathbf{0} \in \operatorname{int}(\mathcal{L}_+ - c\mathbf{1})$ since $\operatorname{int}\mathcal{L}_+ \neq \emptyset$ (Lemma B.12) and $c > 0$ (Assumption (A1)). Hence, we have strong duality.

To indicate the dependence from the selected $Y^* \in \mathcal{Y}^*$, we use the notation $\widetilde{X}_{Y^*}$ and $\widetilde{\lambda}_{Y^*}$ for the primal and dual solution, respectively. The existence of a solution $\widetilde{X}_{Y^*} \in \mathcal{X}_0$ of the primal problem follows from Lemma 2.7. Now, with strong duality the existence of a dual solution $\widetilde{\lambda}_{Y^*} \in \Lambda_+$ follows and the values of the primal and dual objective function at $\widetilde{X}_{Y^*}$, respectively $\widetilde{\lambda}_{Y^*}$, coincide (see Theorem A.12). This leads to a necessary and sufficient optimality condition. Let us consider the primal objective function. Note that $A^*Y^* \in L^1$ for all $Y^* \in \mathcal{Y}^*$ ((A7), Remark 2.2).

$$\begin{aligned}\langle Y^*, AX\rangle &= \langle A^*Y^*, X\rangle = E[X(A^*Y^* + H\int_{C^*} X^* d\lambda)] - E[XH\int_{C^*} X^* d\lambda] \\ &= E[X\nu_\lambda^-] - E[X\nu_\lambda^+] - E[XH\int_{C^*} X^* d\lambda].\end{aligned}$$

We subtract from the primal objective function the dual objective function. Because of strong duality, the difference has to be zero at $\widetilde{X}_{Y^*}$, respectively $\widetilde{\lambda}_{Y^*}$:

$$E[\nu^+_{\widetilde{\lambda}_{Y^*}}(1 - \widetilde{X}_{Y^*})] + E[\nu^-_{\widetilde{\lambda}_{Y^*}}\widetilde{X}_{Y^*}] + \int\limits_{C^*} (c - E[HX^*\widetilde{X}_{Y^*}])d\widetilde{\lambda}_{Y^*} = 0.$$

The sum of these three nonnegative integrals is zero if and only if $\widetilde{X}_{Y^*} \in \mathcal{X}_0$ satisfies condition (2.14) and (2.15) of Theorem 2.8. □

## 2.5 Result about the Structure of a Solution

Now, it is possible to get a result about the solution to the original problem (2.1).

**Theorem 2.9.** *Let Assumption 2.1 be satisfied. Then, there exists a pair $(\widetilde{Y}^*, \widetilde{\lambda}) \in \mathcal{Y}^* \times \Lambda_+$ solving*

$$\max_{Y^* \in \mathcal{Y}^*, \lambda \in \Lambda_+} \{\langle Y^*, b\rangle - \rho^*(Y^*) - E[(-A^*Y^* - H\int\limits_{C^*} X^* d\lambda)^+] - c\lambda(C^*)\}. \tag{2.23}$$

*The solution to (2.1) is*

$$\widetilde{X}(\omega) = \begin{cases} 1 & : \quad -A^*\widetilde{Y}^* > H\int\limits_{C^*} X^* d\widetilde{\lambda} \\ 0 & : \quad -A^*\widetilde{Y}^* < H\int\limits_{C^*} X^* d\widetilde{\lambda} \end{cases} \qquad P - a.s. \tag{2.24}$$

*with*

$$E[HX^*\widetilde{X}] = c \qquad \widetilde{\lambda} - a.s. \tag{2.25}$$

*and $(\widetilde{X}, \widetilde{Y}^*)$ is the saddle point of the functional $(X, Y^*) \mapsto \langle Y^*, AX + b\rangle - \rho^*(Y^*)$ in $\mathcal{X}_0 \times \mathcal{Y}^*$ as described in Theorem 2.6.*

**Remark 2.10.** It follows that there exists a $[0,1]$-valued random variable $\delta$ such that $\widetilde{X}$ as in Theorem 2.9 satisfies

$$\widetilde{X}(\omega) = 1_{\{-A^*\widetilde{Y}^* > H \int\limits_{C^*} X^* d\widetilde{\lambda}\}}(\omega) + \delta(\omega) 1_{\{-A^*\widetilde{Y}^* = H \int\limits_{C^*} X^* d\widetilde{\lambda}\}}(\omega).$$

$\delta$ has to be chosen such that $\widetilde{X}$ satisfies (2.25).

*Proof of Theorem 2.9.* Consider the dual problem of (2.1) given in (2.4), where Theorem 2.6 ensures that the supremum with respect to $Y^* \in \mathcal{Y}^*$ and the infimum with respect to $X \in \mathcal{X}_0$ are attained. We obtain by the validity of strong duality for the inner problem (Theorem 2.8):

$$\begin{aligned} &\max_{Y^* \in \mathcal{Y}^*} \min_{X \in \mathcal{X}_0} \{\langle Y^*, AX + b\rangle - \rho^*(Y^*)\} = \max_{Y^* \in \mathcal{Y}^*} \{\widetilde{p}(Y^*) + \langle Y^*, b\rangle - \rho^*(Y^*)\} \\ &= \max_{Y^* \in \mathcal{Y}^*} \{\widetilde{d}(Y^*) + \langle Y^*, b\rangle - \rho^*(Y^*)\} \\ &= \max_{Y^* \in \mathcal{Y}^*, \lambda \in \Lambda_+} \{\langle Y^*, b\rangle - \rho^*(Y^*) - E[(-A^*Y^* - H \int\limits_{C^*} X^* d\lambda)^+] - c\lambda(C^*)\}. \end{aligned}$$

With Theorem 2.6 it follows that $\widetilde{Y}^*$ attains the maximum with respect to $Y^* \in \mathcal{Y}^*$. Theorem 2.8 shows the existence of a $\widetilde{\lambda} = \widetilde{\lambda}_{\widetilde{Y}^*}$ that attains the maximum with respect to $\lambda \in \Lambda_+$. Thus, there exists a pair $(\widetilde{Y}^*, \widetilde{\lambda})$ solving (2.23). The application of Theorem 2.8 with $Y^* = \widetilde{Y}^*$ leads to the results. □

**Remark 2.11.** The theory works analogously if we replace the constant $c > 0$ by a positive, continuous function $c(\cdot) : C^* \to \mathbb{R}$, this means $c(\cdot) \in \mathcal{L}$, with $c(X^*) > 0$ for all $X^* \in C^*$. Then the constraint set in problem (2.1) becomes

$$\mathcal{X}_0 = \{X \in \mathcal{X}_1 : \forall X^* \in C^* : \langle HX^*, X\rangle \leq c(X^*)\}.$$

Thus, Assumption (A1) can be modified as follows.

(A1') $c(\cdot) \in \mathcal{L}$ with $c(X^*) > 0$ for all $X^* \in C^*$.

It is easy to show that $\mathcal{X}_0$ remains convex and weakly* compact under this modification. The results are similar. In Theorem 2.8, (2.15) has to be replaced by

$$E[HX^*\widetilde{X}_{Y^*}] = c(X^*) \qquad \widetilde{\lambda}_{Y^*} - a.s.$$

and equation (2.13) turns into

$$\widetilde{d}(Y^*) = \sup_{\lambda \in \Lambda_+} \{-E[(-A^*Y^* - H \int\limits_{C^*} X^* d\lambda)^+] - \int\limits_{C^*} c(X^*) d\lambda\}.$$

In Theorem 2.9, (2.25) has to be replaced by

$$E[HX^*\widetilde{X}] = c(X^*) \qquad \widetilde{\lambda} - a.s.$$

and problem (2.23) becomes

$$\max_{Y^*\in\mathcal{Y}^*,\lambda\in\Lambda_+}\{\langle Y^*,b\rangle-\rho^*(Y^*)-E[(-A^*Y^*-H\int\limits_{C^*}X^*d\lambda)^+]-\int\limits_{C^*}c(X^*)d\lambda\}.$$

We shall use this modification in Section 3.2.

In the following chapters, we will show that the optimization problem considered in Chapter 2 arises in a naturally way from the problem of hedging contingent claims as well from testing hypotheses.

# Chapter 3

# Test Theory

In this chapter, we shall study the classical problem of testing hypotheses. In Section 3.1, we consider the most general case of testing a compound null hypothesis, consisting of a family of probability measures against another family of probability measures, the compound alternative hypothesis. In Section 3.2, we formulate a more abstract test problem by discriminating not only families of probability measures but families of measures and by using a positive, continuous function on the parameter set of the null hypothesis instead of a constant significance level $\alpha \in (0, 1)$.

## 3.1 Testing of Compound Hypotheses

Let $(\Omega, \mathcal{F})$ be a measurable space. The general problem in test theory is to discriminate a family $\mathcal{P}^*$ of probability measures (compound null hypothesis) against another family $\mathcal{Q}$ of probability measures (compound alternative hypothesis). Suppose that $P$ is another probability measure such that all $P^* \in \mathcal{P}^*$ and $Q \in \mathcal{Q}$ are absolutely continuous with respect to $P$. Recall that the Radon-Nikodym derivative $dQ/dP$ of a probability measure $Q$ is denoted by $Z_Q$. Let the set $Z_{\mathcal{P}^*} := \{Z_{P^*} : P^* \in \mathcal{P}^*\}$ as a subset of $L^1$ be compact.

Let $R$ denote the set of all randomized tests, i.e., the set of all random variables $\varphi : \Omega \to [0, 1]$. We want to minimize the probability of accepting $\mathcal{P}^*$ when it is false (probability of type-II-error), while the probability of rejecting $\mathcal{P}^*$ when it is true (probability of type-I-error) should be less than a given acceptable significance level $\alpha \in (0, 1)$. In other words, we look for a randomized test $\widetilde{\varphi}$ that maximizes the smallest power $\inf_{Q\in\mathcal{Q}} E^Q[\varphi]$ over all randomized tests $\varphi$ of size less or equal to a significance level $\alpha$: $\sup_{P^*\in\mathcal{P}^*} E^{P^*}[\varphi] \le \alpha$. This means, we look for $\widetilde{\varphi}$ solving

$$\sup_{\varphi\in R_0} \inf_{Q\in\mathcal{Q}} E^Q[\varphi], \tag{3.1}$$

where

$$R_0 = \{\varphi \in R : \sup_{P^*\in\mathcal{P}^*} E^{P^*}[\varphi] \le \alpha\}.$$

The optimal randomized test $\widetilde{\varphi}$ can be interpreted as follows. If the outcome $\omega \in \Omega$ is observed, then the hypothesis $\mathcal{P}^*$ is rejected with probability $\widetilde{\varphi}(\omega)$.
Problem (3.1) can be identified as a special case of the optimization problem (2.1). Thus, we can deduce the optimal randomized test $\widetilde{\varphi}$ by applying the theory deduced in Chapter 2. Note that in problem (3.1) the set $\mathcal{Q}$ can be replaced by $\overline{\text{co}}\mathcal{Q}$ without altering the optimal value or the solution $\widetilde{\varphi}$. The set $\overline{\text{co}}\mathcal{Q}$ denotes the closure of the convex hull of the densities $Z_Q$ of $\mathcal{Q}$ with respect to the norm topology in $L^1$. This means, the problem of testing the compound null hypothesis $\mathcal{P}^*$ against the compound alternative hypothesis $\overline{\text{co}}\mathcal{Q}$ is equivalent to the problem of testing $\mathcal{P}^*$ against $\mathcal{Q}$. This result follows from Theorem 1.25 and Remark 1.27 as we shall see in the proof of Theorem 3.1.
Let us denote the $\sigma$-algebra of all Borel sets of $Z_{\mathcal{P}^*}$ with $\mathcal{B}$ and the set of finite measures on $(Z_{\mathcal{P}^*}, \mathcal{B})$ with $\Lambda_+$. We give a short survey about the procedure deduced in Chapter 2, adapted to the setting of problem (3.1).

(i) Prove the existence of a solution $\widetilde{\varphi}$ to the primal problem (3.1) (Theorem 2.5)

$$-p = \max_{\varphi \in R_0} \inf_{Q \in \mathcal{Q}} E^Q[\varphi].$$

(ii) Prove the validity of strong duality $p = d$ (Theorem 2.6) between the primal problem (3.1) and its Fenchel dual problem

$$-d = \inf_{Q \in \overline{\text{co}}\mathcal{Q}} \sup_{\varphi \in R_0} E^Q[\varphi]. \tag{3.2}$$

We obtain the existence of a dual solution $\widetilde{Q} \in \overline{\text{co}}\mathcal{Q}$ and can show that the problem is a saddle point problem

$$\max_{\varphi \in R_0} \min_{Q \in \overline{\text{co}}\mathcal{Q}} E^Q[\varphi] = \min_{Q \in \overline{\text{co}}\mathcal{Q}} \max_{\varphi \in R_0} E^Q[\varphi].$$

(iii) Consider the inner problem of the dual problem (3.2) for an arbitrary $Q \in \overline{\text{co}}\mathcal{Q}$:

$$p^i(Q) := \sup_{\varphi \in R_0} E^Q[\varphi]. \tag{3.3}$$

Prove the existence of a solution $\widetilde{\varphi}_Q$ to (3.3) (Lemma 2.7). Prove the validity of strong duality $p^i(Q) = d^i(Q)$ between (3.3) and its Fenchel dual problem

$$d^i(Q) = \inf_{\lambda \in \Lambda_+} \Big\{ \int_\Omega [Z_Q - \int_{\mathcal{P}^*} Z_{P^*} d\lambda]^+ dP + \alpha\lambda(Z_{\mathcal{P}^*}) \Big\}.$$

Deduce the necessary and sufficient structure of a solution to the inner problem $\widetilde{\varphi}_Q$ (Theorem 2.8). Note that $p^i(Q)$ coincides with $-\widetilde{p}(-Y^*)$ in the notation of Chapter 2.

(iv) Apply Theorem 2.6 and 2.8 to the primal problem (3.1) and deduce the necessary and sufficient structure of a solution $\widetilde{\varphi}$ to (3.1) (Theorem 2.9).

The result is as follows.

**Theorem 3.1** (generalized Neyman-Pearson lemma)**.** *Let $\mathcal{P}^*$, $\mathcal{Q}$ be families of probability measure such that all $P^* \in \mathcal{P}^*$ and all $Q \in \mathcal{Q}$ are absolutely continuous with respect to a probability measure $P$ and let $Z_{\mathcal{P}^*}$ be a compact set. Let $R$ be the set of all randomized tests and $\alpha \in (0,1)$. Then, there exists a solution $\widetilde{\varphi}$ to (3.1). Furthermore, there exists a pair $(\widetilde{Q}, \widetilde{\lambda}) \in \overline{\mathrm{co}}\mathcal{Q} \times \Lambda_+$ solving*

$$\min_{Q \in \overline{\mathrm{co}}\mathcal{Q}, \lambda \in \Lambda_+} \Big\{ E[(Z_Q - \int_{\mathcal{P}^*} Z_{P^*} d\lambda)^+] + \alpha\lambda(Z_{\mathcal{P}^*}) \Big\}. \tag{3.4}$$

*It holds:*

- *The optimal randomized test of (3.1) has the following structure:*

$$\widetilde{\varphi} = \begin{cases} 1 & : \quad \widetilde{Z}_Q > \int_{\mathcal{P}^*} Z_{P^*} d\widetilde{\lambda} \\ 0 & : \quad \widetilde{Z}_Q < \int_{\mathcal{P}^*} Z_{P^*} d\widetilde{\lambda} \end{cases} \qquad P - a.s. \tag{3.5}$$

  *with*

$$E^{P^*}[\widetilde{\varphi}] = \alpha \qquad \widetilde{\lambda} - a.s. \tag{3.6}$$

- *$(\widetilde{\varphi}, \widetilde{Q})$ is a saddle point of the functional $(\varphi, Q) \mapsto E^Q[\varphi]$ in $R_0 \times \overline{\mathrm{co}}\mathcal{Q}$.*

*Proof.* Problem (3.1) can be identified, up to the sign, with problem (2.1) by setting $\mathcal{X} = L^\infty$, endowed with the norm topology, $\mathcal{X}^* = ba(\Omega, \mathcal{F}, P)$. The space $\mathcal{Y} = L^\infty$ is endowed with the Mackey topology with respect to the dual pair $(L^\infty, L^1)$. This is the finest locally convex Hausdorff topology which still preserves the topological dual $L^1$ (see Definition B.4). Thus, $\mathcal{Y}^* = L^1$. The operator $A : (L^\infty, \|\cdot\|_{L^\infty}) \to (L^\infty, \text{Mackey topology})$ is the identical operator $A\varphi = \varphi$, $b = 0$, $C^* = Z_{\mathcal{P}^*}$, $H = \mathbf{1}$, $c = \alpha$, $\mathcal{X}_1 = R$ and $\mathcal{X}_0 = R_0$. The function $\rho : L^\infty \to \mathbb{R}$ is defined by $\rho(Y) := \sup_{Q \in \mathcal{Q}} E^Q[-Y]$. Let us verify condition (A1)-(A7) of Assumption 2.1:

(A1): $c = \alpha > 0$.

(A2): For the Radon-Nikodym derivative of $P^* \in \mathcal{P}^*$ with respect to $P$ it holds $Z_{P^*} \in L^1$. Since $H = \mathbf{1} \in L^1$, we have $\{HX^* : X^* \in C^*\} = \{Z_{P^*}\mathbf{1} : P^* \in \mathcal{P}^*\} \subseteq L^1$.

(A3): Since $\mathcal{P}^*$ is a set of probability measures, it follows that $\sup_{P^* \in \mathcal{P}^*} \|Z_{P^*}\mathbf{1}\|_{L^1} = 1 < +\infty$. The set $Z_{\mathcal{P}^*} = \{Z_{P^*} : P^* \in \mathcal{P}^*\}$ is compact by assumption.

(A4): The operator $A : (L^\infty, \|\cdot\|_{L^\infty}) \to (L^\infty, \text{Mackey topology})$, defined by $A\varphi := \varphi$, is linear and continuous, since every sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq L^\infty$ converging in the norm topology in $L^\infty$, converges also in the weaker Mackey topology on $L^\infty$ (see Example B.5 and [2], Lemma 2.47-4.).

(A5): The function $\rho$ as defined above can be interpreted as a coherent risk measure (cf. Theorem 1.25). $\rho$ is lower semicontinuous in the weak* topology, since it admits by definition a dual representation with a set of probability measures $\mathcal{Q}$ (see Corollary 1.29 (i), (ii)). Furthermore, $\rho$ is convex. Since $\mathcal{Q}$ is a set of probability measures, we have $\sup_{Q\in\mathcal{Q}} \|Z_Q\|_{L^1} = 1$. Hence, $\rho$ is finite for all $Y \in L^\infty$:

$$\rho(Y) = \sup_{Q\in\mathcal{Q}} E^Q[-Y] \leq \sup_{Q\in\mathcal{Q}} |\langle Y, Z_Q\rangle| \leq \|Y\|_{L^\infty} \sup_{Q\in\mathcal{Q}} \|Z_Q\|_{L^1} = \|Y\|_{L^\infty} < +\infty.$$

Thus, we can apply Corollary B.11 and obtain that $\rho$ is continuous with respect to the Mackey topology in every $Y \in \mathcal{Y}$ and thus trivially lower semicontinuous with respect to the Mackey topology. Hence, Assumption (A5) is satisfied.

(A6): The map $X \mapsto \rho(AX+b)$ coincides with $\rho(X)$, since $A$ is the identical operator and $b = 0$. It is already shown that $\rho$ is lower semicontinuous in the weak* topology.

(A7): The map $X \mapsto \langle Y^*, AX\rangle = E[Y^*X]$ is continuous in the weak* topology of $L^\infty$ for all $Y^* \in \mathcal{Y}^*$, since $\mathcal{Y}^* = L^1$.

Thus, Assumption 2.1 is satisfied and we can apply the theory deduced in Chapter 2. The existence of a solution $\widetilde{\varphi}$ to (3.1) follows from Theorem 2.5. Since $\rho$ is a coherent risk measure, Theorem 1.25 and Remark 1.27 yield $\operatorname{dom}\rho^* = -\overline{\operatorname{co}}\mathcal{Q}$. Thus, the solution $\widetilde{Q}$ to the dual problem (3.2) is in general attained in $\overline{\operatorname{co}}\mathcal{Q}$. The identical operator $A$ is self-adjoint. Hence, $A^*Y^* = Y^*$. Note that $Z_Q = -Y^* \in -\operatorname{dom}\rho^*$. By an application of Theorem 2.9, we obtain the stated results. □

**Remark 3.2.** It follows that there exists a $[0,1]$-valued random variable $\delta$ such that $\widetilde{\varphi}$ as in Theorem 3.1 satisfies

$$\widetilde{\varphi}(\omega) = 1_{\{\widetilde{Z}_Q > \int_{\mathcal{P}^*} Z_{P^*}d\widetilde{\lambda}\}}(\omega) + \delta(\omega)1_{\{\widetilde{Z}_Q = \int_{\mathcal{P}^*} Z_{P^*}d\widetilde{\lambda}\}}(\omega).$$

$\delta$ has to be chosen such that $\widetilde{\varphi}$ satisfies (3.6).

The case of testing a compound hypothesis against a simple alternative hypothesis has been considered in a variety of papers. The problem of testing a compound hypothesis against a compound alternative hypothesis has been studied for instance by Cvitanić and Karatzas [7]. Since Cvitanić and Karatzas [7] seem to present the up to now most general result in this topic, we want to give a short overview over the differences between [7] and Theorem 3.1 in terms of the obtained results and the methods used to solve the problem. In [7] the enlargement

$$\mathcal{D} := \{D \in L^1 : D \geq 0, \forall \varphi \in R_0 : E[\varphi D] \leq \alpha\} \supseteq Z_{\mathcal{P}^*}$$

of the convex hull of the densities of $\mathcal{P}^*$ is introduced. The set $\mathcal{D}$ is convex, bounded in $L^1$ and closed under $P - a.s.$ convergence. Furthermore, it is assumed in [7], that the set of densities of $\mathcal{Q}$ is convex and closed under $P - a.s.$ convergence. The basic observation in [7] is

$$\forall Q \in \mathcal{Q}, \forall D \in \mathcal{D}, \forall z > 0, \forall \varphi \in R_0 : \quad E^Q[\varphi] \leq E[(Z_Q - zD)^+] + \alpha z. \tag{3.7}$$

Then, the existence of a quadruple $(\widehat{Q}, \widehat{D}, \widehat{z}, \widehat{\varphi}) \in (\mathcal{Q} \times \mathcal{D} \times (0, \infty) \times R_0)$ that satisfies equality in (3.7) is shown and the structure of the optimal randomized test $\widehat{\varphi}$ is deduced:

$$\widehat{\varphi} = 1_{\{\widehat{z}\widehat{D} < \widehat{Z}_Q\}} + \delta 1_{\{\widehat{z}\widehat{D} = \widehat{Z}_Q\}}, \tag{3.8}$$

where $\delta$ is a suitable random variable and $(\widehat{Q}, \widehat{D}, \widehat{z})$ is a solution of

$$\inf_{z>0, (Q,D) \in (\mathcal{Q}, \mathcal{D})} \{\alpha z + E[(Z_Q - zD)^+]\}. \tag{3.9}$$

With the method deduced in this thesis, it is not necessary to introduce the enlarged set $\mathcal{D}$ and to impose the above assumption to $\mathcal{Q}$. Let us study the relationship between Theorem 3.1 and the results of [7]. With Tonelli's Theorem (Theorem B.22) it is easy to show that $k \int_{\mathcal{P}^*} Z_{P^*} d\lambda \in \mathcal{D}$ for all $\lambda \in \Lambda_+$, where $k = \lambda(Z_{\mathcal{P}^*})^{-1}$ if $\lambda(Z_{\mathcal{P}^*}) \neq 0$ and zero if $\lambda(Z_{\mathcal{P}^*}) = 0$. The case $\lambda(Z_{\mathcal{P}^*}) = 0$ implies $\lambda(B) = 0$ for all $B \in \mathcal{B}$ and thus $\int_{\mathcal{P}^*} Z_{P^*} d\lambda = 0$.
If we consider in (3.7) only elements $k \int_{\mathcal{P}^*} Z_{P^*} d\lambda \in \mathcal{D}$, then inequality (3.7) coincides with weak duality between the primal and dual objective function of $p^i(Q)$ and $d^i(Q)$ (cf. Theorem A.12) and reduces to

$$\forall Q \in \overline{\text{co}}\mathcal{Q}, \forall \lambda \in \Lambda_+, \forall \varphi \in R_0 : \quad E^Q[\varphi] \leq E[(Z_Q - \int_{\mathcal{P}^*} Z_{P^*} d\lambda)^+] + \alpha\lambda(Z_{\mathcal{P}^*}). \tag{3.10}$$

Problem (3.9) reduces to (3.4). To summarize the methods, Cvitanić and Karatzas [7] proved the existence of a primal and a dual solution that satisfy equality in (3.7). In order to do this, stronger assumptions had to be imposed. In our method, the validity of strong duality, hence equality in (3.10) was shown directly by Fenchel duality (cf. step (ii) and (iii) of the procedure, Theorem A.12). Then, the existence of a dual solution follows. Both methods lead to a result about the structure of a solution. But now it is possible to show the impact of the original set $\mathcal{P}^*$ to the sets that define the solution $\widehat{\varphi}$ in Cvitanić and Karatzas [7] (see 3.8):

$$\widehat{z}\widehat{D} = \int_{\mathcal{P}^*} Z_{P^*} d\widetilde{\lambda}, \tag{3.11}$$

where $(\widetilde{Q}, \widetilde{\lambda})$ is the optimal pair in (3.4). This means, $\widehat{z} = \widetilde{\lambda}(Z_{\mathcal{P}^*})$ and

$$\widehat{D} = k \int_{\mathcal{P}^*} Z_{P^*} d\widetilde{\lambda}, \tag{3.12}$$

where $k = \widetilde{\lambda}(Z_{\mathcal{P}^*})^{-1}$ if $\widetilde{\lambda}(Z_{\mathcal{P}^*}) \neq 0$ and zero if $\widetilde{\lambda}(Z_{\mathcal{P}^*}) = 0$. Let us summarize the improvements of our method. With Theorem 3.1 it is now possible to give a result about the structure of the solution $\widetilde{\varphi}$ in terms of the original sets $\mathcal{P}^*$ and $\mathcal{Q}$. It is not necessary to embed $\mathcal{P}^*$ into the larger set $\mathcal{D}$, to impose the assumption that $\{Z_Q : Q \in \mathcal{Q}\}$ is convex and closed under $P - a.s.$ convergence and to impose $\mathcal{P}^* \cap \mathcal{Q} = \emptyset$, but we have to impose that $Z_{\mathcal{P}^*}$ is compact. The assumption that the elements of $\mathcal{P}^*$ and $\mathcal{Q}$ are probability measures that are absolutely continuous to another probability measure $P$ can be weakened as we shall see in the next section.

## 3.2 The Generalized Test Problem

Instead of probability measures $P^*$ and $Q$ as considered in the test problem in Section 3.1, we consider more general subsets $\mathcal{H}$ and $\mathcal{G}$ of $L^1$ and instead of a constant $\alpha \in (0, 1)$, we consider a positive, continuous function $\alpha(g)$.
We want to solve the following optimization problem

$$\sup_{\varphi \in R} \inf_{h \in \mathcal{H}} E[\varphi h], \tag{3.13}$$

subject to

$$\sup_{g \in \mathcal{G}} E[\varphi g] \leq \alpha(g). \tag{3.14}$$

This is no longer a test problem in the classical sense, but the structure of the problem is similar to the problem of testing compound hypotheses. Problem (3.13), (3.14), the so called generalized test problem, arises for example from the problem of hedging in incomplete markets (see Remark 4.8). This problem will be studied in detail in Chapter 4.
Let us denote the constraint set with $R_0 := \{\varphi \in R : \varphi \text{ satisfies } (3.14)\}$. We consider the measurable space $(\mathcal{G}, \mathcal{B})$, where $\mathcal{B}$ is a $\sigma$-algebra of all Borel sets of $\mathcal{G}$ and denote the set of finite measures on $(\mathcal{G}, \mathcal{B})$ with $\Lambda_+$.
Taking Remark 2.11 into account, we obtain the following theorem with the help of the results from Chapter 2.

**Theorem 3.3.** *Let $\mathcal{H}, \mathcal{G} \subseteq L^1(\Omega, \mathcal{F}, P)$ with $\sup_{h \in \mathcal{H}} \|h\|_{L^1} < +\infty$, $\sup_{g \in \mathcal{G}} \|g\|_{L^1} < +\infty$ and $\mathcal{G}$ compact. Let $\alpha(g)$ be a continuous function on the measurable space $(\mathcal{G}, \mathcal{B})$ with $\alpha(g) > 0$ for all $g \in \mathcal{G}$. Let $R$ be the set of randomized tests. Then, there exists a solution $\widetilde{\varphi}$ to (3.13), (3.14). Furthermore, there exists a pair $(\widetilde{h}, \widetilde{\lambda}) \in \overline{\text{co}}\mathcal{H} \times \Lambda_+$ solving*

$$\min_{h \in \overline{\text{co}}\mathcal{H}, \lambda \in \Lambda_+} \Big\{ E[(h - \int_{\mathcal{G}} g d\lambda)^+] + \int_{\mathcal{G}} \alpha(g) d\lambda \Big\}, \tag{3.15}$$

*where $\Lambda_+$ is the set of finite measures on $(\mathcal{G}, \mathcal{B})$. We obtain:*

- *The optimal randomized test of (3.13), (3.14) has the following structure:*

$$\widetilde{\varphi} = \begin{cases} 1 & : \quad \widetilde{h} > \int_{\mathcal{G}} g d\widetilde{\lambda} \\ 0 & : \quad \widetilde{h} < \int_{\mathcal{G}} g d\widetilde{\lambda} \end{cases} \qquad P - a.s. \tag{3.16}$$

*with*

$$E[\widetilde{\varphi} g] = \alpha(g) \qquad \widetilde{\lambda} - a.s. \tag{3.17}$$

- *$(\widetilde{\varphi}, \widetilde{h})$ is a saddle point of the functional $(\varphi, h) \mapsto E[\varphi h]$ in $R_0 \times \overline{\mathrm{co}}\mathcal{H}$.*

*Proof of Theorem 3.3.* The proof is similar to the proof of Theorem 3.1. Let the space $\mathcal{X} = L^\infty$ be endowed with the norm topology, $\mathcal{X}^* = ba(\Omega, \mathcal{F}, P)$. The space $\mathcal{Y} = L^\infty$ is endowed with the Mackey topology with respect to the dual pair $(L^\infty, L^1)$ (see Section B.1). Thus, $\mathcal{Y}^* = L^1$. Let $\mathcal{X}_1 = R$, $\mathcal{X}_0 = R_0$, $A\varphi = \varphi$, $b = 0$ and $H = \mathbf{1}$. In contrast to the proof of Theorem 3.1 we have $C^* = \mathcal{G}$ and $\alpha(g)$ is the positive, continuous function $c(\cdot)$ in Remark 2.11. The function $\rho : L^\infty \to \mathbb{R}$ is defined by $\rho(X) := \sup_{h \in \mathcal{H}} E[-Xh]$. Then, problem (2.1) turns, up to the sign, into (3.13), (3.14). We verify condition (A1)-(A7) of Assumption 2.1:

(A1'): Replacing $c$ by the positive, continuous function $\alpha(g)$ as discussed in Remark 2.11, one can see that Assumption (A1') is satisfied.

(A2): $\{HX^* : X^* \in C^*\} = \mathcal{G} \subseteq L^1$.

(A3): $\sup_{g \in \mathcal{G}} \|g\mathbf{1}\|_{L^1} < +\infty$ and $\mathcal{G}$ compact as assumed in Theorem 3.3.

(A4): The operator $A : (L^\infty, \|\cdot\|_{L^\infty}) \to (L^\infty, \text{Mackey topology})$ defined by $A\varphi := \varphi$ is linear and continuous, since every sequence $\{\varphi_n\}_{n \in \mathbb{N}} \subseteq L^\infty$ converging in the norm topology in $L^\infty$, converges also in the weaker Mackey topology on $L^\infty$ (see [2], Lemma 2.47-4.).

(A5): The function $\rho$ as defined above is lower semicontinuous in the weak* topology, since it admits by definition a dual representation with $\mathcal{H} \subseteq L^1$ and is convex (see Theorem 1.6 (i), (ii)). Furthermore, since $\sup_{h \in \mathcal{H}} \|h\|_{L^1} < +\infty$ by assumption, $\rho$ is finite and hence continuous with respect to the Mackey topology (Corollary B.11).

(A6): The map $X \mapsto \rho(AX + b)$ coincides with $\rho(X)$ since $A$ is the identical operator and $b = 0$. It is already shown that $\rho$ is lower semicontinuous in the weak* topology.

(A7): The map $X \mapsto \langle Y^*, AX \rangle = E[Y^* X]$ is continuous in the weak* topology of $L^\infty$ for all $Y^* \in \mathcal{Y}^*$, since $\mathcal{Y}^* = L^1$.

Thus, Assumption 2.1 is satisfied and we can apply the theory deduced in Chapter 2. The existence of a solution $\widetilde{\varphi}$ follows from Theorem 2.5. Example A.10 yields $\operatorname{dom}\rho^* = -\overline{\operatorname{co}}\mathcal{H}$. Thus, the dual solution $\widetilde{h}$ is in general attained in $\overline{\operatorname{co}}\mathcal{H}$. The application of Theorem 2.9 under the modifications mentioned in Remark 2.11 yield the stated results. $\square$

This kind of generalized test problem was studied for the case of a simple alternative hypothesis ($\mathcal{H}$ being a singleton) and a positive, bounded and measurable function $\alpha$ in Witting [47], Section 2.5.1. For this case it was shown with Lagrange duality that the structure (3.16), (3.17) of a test is sufficient for optimality. Furthermore, it was shown in [47] that for a finite set $\mathcal{G}$ the conditions (3.16), (3.17) are sufficient and necessary for optimality. In Rudloff [35] we could show that in the general case of a infinite set $\mathcal{G}$ and a positive, constant function $\alpha$, the structure (3.16), (3.17) of $\widetilde{\varphi}$ is necessary and sufficient for optimality.
In Theorem 3.3, we show that a generalization of these results is even possible for the case where both, the hypothesis $\mathcal{G}$ and the alternative hypothesis $\mathcal{H}$, are compound hypothesis and $\alpha$ is a positive, continuous function.

**Remark 3.4.** The reason why in [47], Section 2.5.1, the function $\alpha$ is assumed to be a measurable bounded function, whereas in Theorem 3.3 the function $\alpha$ is assumed to be continuous, is that in [47] Lagrange duality is done with a dual space to the space of all measurable bounded functions that is not the topological dual space with respect to the supremum norm. In [47] the space of finite $\sigma$-additive signed measures is used as a dual space (see Example 1.63 in [47]) and a weak duality result is obtained ([47], Section 2.5.1). To apply Fenchel duality (Theorem A.12), we have to work with the topological dual space with respect to the supremum norm, i.e., with the space of bounded, finitely additive set functions (see [10], Theorem IV.5.1). Since Tonelli's Theorem (Theorem B.22) does not hold for finitely additive set functions, we can not work with this space and have to impose the assumption that the set $C^* = \mathcal{G}$ is compact. Then, it is possible to work in the proof of Theorem 2.8 with the space of continuous functions on a compact set and its norm dual of finite $\sigma$-additive signed measures (see [2], Corollary 13.15). Thus, we can apply Tonelli's Theorem and obtain a strong duality result.

# Chapter 4

# Hedging in Complete and Incomplete Markets

The problem of pricing and hedging a contingent claim with payoff $H$ is well understood in the context of arbitrage-free option pricing in complete markets (see Black and Scholes [4], Merton [30]). In this situation, a perfect hedge is always possible, i.e., there exists a dynamic strategy such that trading in the underlying assets replicates the payoff of the contingent claim. Then, the price of the contingent claim turns out to be the expectation of $H$ with respect to the equivalent martingale measure which is unique. However, the possibility of a perfect hedge is restricted to the complete market and thus, to certain models and restrictive assumptions. In more realistic models the market will be incomplete, i.e., a perfect hedge as in the Black-Scholes-Merton model is not possible and the equivalent martingale measure is not unique any longer. Thus, a contingent claim bears an intrinsic risk that cannot be hedged away completely. Therefore, we are faced with the problem of searching strategies which reduce the risk of the resulting shortfall as much as possible.
One can still stay on the safe side using a superhedging strategy (see [13] for a survey). Then, the replicating portfolio at final time $T$ is in any case larger than the payoff of the contingent claim. But from a practical point of view, the cost of superhedging is often too high (see for instance [21]). For this reason, we consider the possibility of investing less capital than the superhedging price of the liability. This leads to a shortfall, the risk of which, measured by a suitable risk measure, should be minimized.
A similar problem arises when hedging in complete markets and the investor is unwilling or unable to pay the unique arbitrage free price of a contingent claim and wants to invest a sum less than this price. The aim is to find a hedging strategy that minimizes the losses due to the difference between the claim and the hedging portfolio at time $T$, measured by a suitable risk measure. This is a special case of the above mentioned problem since we have only to deal with an unique equivalent martingale measure. In Section 4.1, we shall consider this problem and the problem

of hedging in special incomplete markets. In Section 4.2, we consider the general incomplete market.
We study the hedging problem using different types of risk measures. First, we give a general result and then deduce in the following subsections the corresponding results for different kinds of risk measures and compare the obtained results with the recent literature.
In our setting, the discounted price process of the $d$ underlying assets is described as an $\mathbb{R}^d$-valued semimartingale $S = (S_t)_{t\in[0,T]}$ on a complete probability space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and $\mathcal{F} = \mathcal{F}_T$. A semimartingale is the sum of a continuous local martingale and a finite-variation process that is right-continuous with left-hand limits (for details regarding the notation and the filtration we refer to [25]). Let $\mathcal{P}$ denote the set of equivalent martingale measures with respect to $P$. Since we assume the absence of arbitrage opportunities, it holds $\mathcal{P} \neq \emptyset$.
Recall that $\widehat{\mathcal{Q}}$ denotes the set of all probability measures on $(\Omega, \mathcal{F})$ absolutely continuous with respect to $P$. For $Q \in \widehat{\mathcal{Q}}$ we denote the expectation with respect to $Q$ by $E^Q$ and the Radon-Nikodym derivative $dQ/dP$ by $Z_Q$. Let us denote $Z_{\mathcal{P}} := \{Z_{P^*} : P^* \in \mathcal{P}\}$.
A self-financing strategy is given by an initial capital $V_0 \geq 0$ and a predictable process $\xi$ such that the resulting value process

$$V_t = V_0 + \int_0^t \xi_s dS_s, \quad t \in [0,T],$$

is well defined. Such a strategy $(V_0, \xi)$ is called admissible if the corresponding value process $V$ satisfies $V_t \geq 0$ for all $t \in [0,T]$.
Consider a contingent claim. Its payoff is given by an $\mathcal{F}_T$ -measurable, nonnegative random variable $H \in L^1$. We assume

$$U_0 = \sup_{P^*\in\mathcal{P}} E^{P^*}[H] < +\infty. \tag{4.1}$$

The above equation is the dual characterization of the superhedging price $U_0$, the smallest amount $V_0$ such that there exists an admissible strategy $(V_0, \xi)$ with value process $V_t$ satisfying $V_T \geq H$ (see [13] for an overview over this topic). In the complete case, where the equivalent martingale measure $P^*$ is unique, $U_0 = E^{P^*}[H]$ is the unique arbitrage-free price of the contingent claim.
Since superhedging can be quite expensive in the incomplete market (see [21] for the general semimartingale case), we search for the best hedge an investor can achieve with a smaller amount $\widetilde{V}_0 < U_0$. In other words, we look for an admissible strategy $(V_0, \xi)$ with $0 < V_0 \leq \widetilde{V}_0$ that minimizes the risk of losses due to the shortfall $\{\omega : V_T(\omega) < H(\omega)\}$, this means we want to minimize the risk of $-(H - V_T)^+$. The risk will be measured by a suitable risk measure $\rho$. Thus, we consider the dynamic optimization problem of finding an admissible strategy that minimizes

$$\min_{(V_0,\xi)} \rho\left(-(H - V_T)^+\right) \tag{4.2}$$

under the capital constraint of investing less capital than the superhedging price

$$0 < V_0 \le \widetilde{V}_0 < U_0. \tag{4.3}$$

The dynamic optimization problem (4.2), (4.3) can be split into the following two problems:

1. Static optimization problem: Find an optimal modified claim $\widetilde{\varphi}H$, where $\widetilde{\varphi}$ is a randomized test solving

$$\min_{\varphi\in R_0} \rho\left((\varphi-1)H\right), \tag{4.4}$$

$$R_0 = \{\varphi:\ \Omega \to [0,1],\ \mathcal{F}_T-\text{measurable},\ \sup_{P^*\in\mathcal{P}} E^{P^*}[\varphi H] \le \widetilde{V}_0\}. \tag{4.5}$$

2. Representation problem: Find a superhedging strategy for the modified claim $\widetilde{\varphi}H$.

The representation problem can be solved by the optional decomposition theorem of Föllmer and Kabanov [14] (see Appendix, Theorem C.3). The idea of splitting the dynamic optimization problem in this way was introduced by Föllmer and Leukert [16], minimizing the probability of a shortfall. It was used for the expectation of a loss function as risk measure in [17], for coherent risk measures in Nakano [31, 32], Rudloff [36] and for convex risk measures in Rudloff [38] analogously. The only property of $\rho$ that is needed in the proof is monotonicity.

**Theorem 4.1.** *Let $\rho : L^1 \to \mathbb{R}\cup\{+\infty\}$ be a monotone function and let $\widetilde{\varphi}$ be a solution of the minimization problem (4.4) and $(\widetilde{V}_0, \widetilde{\xi})$ be the admissible strategy, where $\widetilde{\xi}$ is determined by the optional decomposition of the claim $\widetilde{\varphi}H$. Then the strategy $(\widetilde{V}_0, \widetilde{\xi})$ solves the optimization problem (4.2), (4.3) and it holds*

$$\min_{(V_0,\xi)} \rho(-(H-V_T)^+) = \min_{\varphi\in R_0} \rho\left((\varphi-1)H\right). \tag{4.6}$$

To prove the theorem, we first review the optional decomposition theorem (Theorem C.3) in our setting (see also [16], [17]). Therefore, we consider the modified claim $\widetilde{\varphi}H$, where $\widetilde{\varphi}$ is the solution of (4.4) and define $\widetilde{U}$ as a right-continuous version of the process

$$\widetilde{U}_t = \text{ess.}\sup_{P^*\in\mathcal{P}} E^{P^*}[\widetilde{\varphi}H|\mathcal{F}_t].$$

For the definition of the essential supremum, see Section C. The process $\widetilde{U}$ is a $\mathcal{P}$-supermartingale, i.e., a supermartingale with respect to any equivalent martingale measure $P^* \in \mathcal{P}$ (see [16], [17]). By the optional decomposition theorem (Theorem C.3) there exists an admissible strategy $(\widetilde{U}_0, \widetilde{\xi})$ and an increasing optional process $\widetilde{C}$ with $\widetilde{C}_0 = 0$ such that

$$\widetilde{U}_t = \widetilde{U}_0 + \int_0^t \widetilde{\xi}_s dS_s - \widetilde{C}_t.$$

One obtains that $\widetilde{U}_0 = \sup_{P^* \in \mathcal{P}} E^{P^*}[\widetilde{\varphi} H]$ is the superhedging price and $\widetilde{\xi}$ the superhedging strategy of the modified claim $\widetilde{\varphi} H$.

**Remark 4.2.** In the complete case where the equivalent martingale measure is unique, $(\widetilde{U}_0, \widetilde{\xi})$ is simply the replicating strategy for the modified claim $\widetilde{\varphi} H$. Thus, $\widetilde{U}_0 = E^{P^*}[\widetilde{\varphi} H]$ is the unique arbitrage-free price of the contingent claim $\widetilde{\varphi} H$.

**Remark 4.3.** In the incomplete market, when a risk measure $\rho$ is used that allows the construction of $\widetilde{\varphi}$ via the Neyman-Pearson lemma directly (cf. [16] and some special cases of [17]), one can see that $\widetilde{U}_0 = \widetilde{V}_0$ since the optimal test $\widetilde{\varphi}$ attains the bound $\widetilde{V}_0$ in (4.5). In Theorem 4.9, equation (4.13) shows (except in the case where the dual solution takes only the value zero (see Remark 4.15 for the convex hedging case)) that in the general case the bound $\widetilde{V}_0$ is as well attained by the optimal test.

*Proof of Theorem 4.1.* Let $(V_0, \xi)$ with $V_0 \leq \widetilde{V}_0$ be an admissible strategy. We define the corresponding success ratio $\varphi = \varphi_{(V_0,\xi)}$ as

$$\varphi_{(V_0,\xi)} := 1_{\{V_T \geq H\}} + \frac{V_T}{H} 1_{\{V_T < H\}}.$$

Thus, $-(H - V_T)^+ = (\varphi - 1)H$. Since $V_t$ is a $\mathcal{P}$-supermartingale and $\varphi H \leq V_T$:

$$\forall P^* \in \mathcal{P}: \quad E^{P^*}[\varphi H] \leq E^{P^*}[V_T] \leq V_0 \leq \widetilde{V}_0,$$

hence, $\varphi \in R_0$. Thus,

$$\rho(-(H - V_T)^+) = \rho((\varphi - 1)H) \geq \rho((\widetilde{\varphi} - 1)H), \tag{4.7}$$

where $\widetilde{\varphi}$ is the solution to the static optimization problem (4.4). Inequality (4.7) is especially satisfied for the success ratio of the admissible strategy $(\overline{V}_0, \widetilde{\xi})$, where $\widetilde{\xi}$ is the superhedging strategy for the modified claim $\widetilde{\varphi} H$, determined by the optional decomposition theorem (Theorem C.3) and $\overline{V}_0 \in [\widetilde{U}_0, \widetilde{V}_0]$, where $\widetilde{U}_0 = \sup_{P^* \in \mathcal{P}} E^{P^*}[\widetilde{\varphi} H]$ is the superhedging price of the modified claim $\widetilde{\varphi} H$. Thus,

$$\rho((\varphi_{(\overline{V}_0,\widetilde{\xi})} - 1)H) \geq \rho((\widetilde{\varphi} - 1)H). \tag{4.8}$$

To show the revers inequality, let us consider $\varphi_{(\overline{V}_0,\widetilde{\xi})} H = \min(\widetilde{V}_T, H)$, where $\widetilde{V}_T = \overline{V}_0 + \int_0^T \widetilde{\xi}_s dS_s$. It holds

$$\begin{aligned}
\widetilde{V}_T &= \overline{V}_0 + \int_0^T \widetilde{\xi}_s dS_s = \overline{V}_0 + \widetilde{U}_T + \widetilde{C}_T - \widetilde{U}_0 \\
&= \overline{V}_0 + \operatorname*{ess.\,sup}_{P^* \in \mathcal{P}} E^{P^*}[\widetilde{\varphi} H | \mathcal{F}_T] + \widetilde{C}_T - \widetilde{U}_0 = \widetilde{\varphi} H + \widetilde{C}_T + \overline{V}_0 - \widetilde{U}_0 \\
&\geq \widetilde{\varphi} H.
\end{aligned}$$

Thus, $\varphi_{(\overline{V}_0,\widetilde{\xi})}H \geq \widetilde{\varphi}H$. Since $\rho$ is monotone, we obtain

$$\rho((\varphi_{(\overline{V}_0,\widetilde{\xi})} - 1)H) \leq \rho((\widetilde{\varphi} - 1)H).$$

Together with (4.8), we see that $\varphi_{(\overline{V}_0,\widetilde{\xi})}$ attains the minimum of the static optimization problem (4.4). Due to (4.7), we now have

$$\min_{(V_0,\xi)} \rho(-(H - V_T)^+) \geq \rho(-(H - \widetilde{V}_T)^+).$$

Hence, $(\overline{V}_0, \widetilde{\xi})$ with $\overline{V}_0 \in [\widetilde{U}_0, \widetilde{V}_0]$ is the strategy that attains the minimum in the dynamic optimization problem (4.2), (4.3) and it holds

$$\min_{(V_0,\xi)} \rho(-(H - V_T)^+) = \min_{\varphi \in R_0} \rho\left((\varphi - 1)H\right).$$

$\square$

**Remark 4.4.** As mentioned in Remark 4.3, relationship (4.13) of Theorem 4.9 means that $\widetilde{U}_0 = \widetilde{V}_0$. Thus, the optimal strategy is $(\widetilde{V}_0, \widetilde{\xi})$.

The static optimization problem (4.4) can be identified as the optimization problem considered in (2.1) by defining that $H \in L^1_+$ is the payoff of the contingent claim, $A\varphi = H\varphi$, $b = -H$, $c = \widetilde{V}_0$, $C^* = Z_{\mathcal{P}}$, $\mathcal{X} = \mathcal{Y}^* = L^\infty$, $\mathcal{X}^* = ba(\Omega, \mathcal{F}, P)$, $\mathcal{Y} = L^1$, $\mathcal{X}_1 = R$, $\mathcal{X}_0 = R_0$ and $\rho$ is a suitable risk measure.
The problem (4.4) has been studied using different types of risk measures. Föllmer and Leukert [16] used the so called quantile hedging to determine a portfolio strategy which minimizes the probability of loss. This idea leads to partial hedges. In this approach, losses could be very substantial, even if they occur with a very small probability. Therefore, Föllmer and Leukert [17] proposed to use the expectation of a loss function as risk measure instead. Nakano [31, 32] and Rudloff [36] used coherent risk measures and Rudloff [37, 38] convex risk measures to quantify the shortfall risk. We want to study the hedging problem using a risk measure as general as possible.
Since in Theorem 2.8 and 2.9 we need the assumption that $C^* = Z_{\mathcal{P}}$ is a compact set, we shall divide the following considerations into two cases. In Section 4.1 we shall assume that $Z_{\mathcal{P}}$ is compact, which includes the important case of complete markets. In Section 4.2 we solve the general case of incomplete markets. Since in this general setting we can no longer apply Theorem 2.8 and 2.9, we shall solve the inner problem of the dual problem with a duality method presented in [27].

## 4.1 Hedging in Complete and Special Incomplete Markets

In this section, we consider the problem of hedging in complete markets, i.e., the set $\mathcal{P} = \{P^*\}$ is a singleton, when the investor is unwilling or unable to pay the unique arbitrage free price of a contingent claim and wants to invest a sum less than this price. Since the results also hold true in the more general case of an incomplete market with $Z_{\mathcal{P}} := \{Z_{P^*} : P^* \in \mathcal{P}\}$ compact, we work in this more general setting. We want to study the hedging problem using a risk measure as general as possible. In Theorem 4.1 we needed to assume that $\rho$ is monotone. To solve the problem by application of the results of Chapter 2, we additionally need to assume that $\rho$ is convex, lower semicontinuous and continuous in some $(\varphi_0 - 1)H$ with $\varphi_0 \in R_0$. First, we shall give the result for this most general case. Then, in the next subsections we shall add more properties to $\rho$ which leads to different types of risk measures and we analyze the influence on the results.

### 4.1.1 The General Case

First, we consider a risk function as general as possible. We impose the following assumption.

**Assumption 4.5.** *Let $\rho : L^1 \to \mathbb{R} \cup \{+\infty\}$ be a monotone, convex, lower semicontinuous function, that is continuous and finite in some $(\varphi_0 - 1)H$ with $\varphi_0 \in R_0$ and satisfies $\rho(0) < +\infty$.*

**Remark 4.6.** Especially, if $\rho$ is a lower semicontinuous convex function with $\rho(Y) < +\infty$ for all $Y \in L^1$, then $\rho$ is continuous for all $Y \in L^1$ since $L^1$ is a Banach space ([11], Corollary I.2.5).
In general, a lower semicontinuous convex function $\rho : L^1 \to \mathbb{R} \cup \{+\infty\}$ is continuous in some $Y$ if $Y$ is an interior point of the domain of $\rho$ (see [11], Corollary I.2.5).

Let us consider the measurable space $(Z_{\mathcal{P}}, \mathcal{B})$, where $\mathcal{B}$ is the $\sigma$-algebra of all Borel sets on $Z_{\mathcal{P}}$. We denote by $\Lambda_+$ the set of finite measures on $(Z_{\mathcal{P}}, \mathcal{B})$.

**Remark 4.7.** We review the procedure deduced in Chapter 2 to solve the static optimization problem (4.4), where $\rho$ is a function satisfying Assumption 4.5:

(i) Prove the existence of a solution $\widetilde{\varphi}$ to the primal problem (4.4) (Theorem 2.5)

$$p = \min_{\varphi \in R_0} \rho\left((\varphi - 1)H\right) = \min_{\varphi \in R_0} \{ \sup_{Y^* \in L^\infty_+} \{E[(1-\varphi)HY^*] - \rho^*(-Y^*)\}\}.$$

(ii) Deduce the dual problem to (4.4) by Fenchel duality:

$$d = \sup_{Y^* \in L^\infty_+} \{ \inf_{\varphi \in R_0} \{E[(1-\varphi)HY^*] - \rho^*(-Y^*)\}\} \tag{4.9}$$

and prove the validity of strong duality $p = d$ (Theorem 2.6). We obtain the existence of a dual solution and can show that the problem is a saddle point problem.

(iii) Consider the inner problem of the dual problem (4.9) for an arbitrary $Y^* \in L^\infty_+$:

$$p^i(Y^*) := \max_{\varphi \in R_0} E[\varphi H Y^*]. \tag{4.10}$$

Prove the existence of a solution $\widetilde{\varphi}_{Y^*}$ to (4.10) (Lemma 2.7). Deduce the dual problem by Fenchel duality:

$$d^i(Y^*) = \inf_{\lambda \in \Lambda_+} \Big\{ \int_\Omega [HY^* - H \int_{\mathcal{P}} Z_{P^*} d\lambda]^+ dP + \widetilde{V}_0 \lambda(Z_{\mathcal{P}}) \Big\}.$$

Prove the validity of strong duality $p^i(Y^*) = d^i(Y^*)$ and deduce the necessary and sufficient structure of a solution $\widetilde{\varphi}_{Y^*}$ to the inner problem (4.10) (Theorem 2.8). Note that $p^i(Y^*) = -\widetilde{p}(-Y^*)$ in the notation of Chapter 2.

(iv) Apply Theorem 2.6 and 2.8 to the primal problem (4.4) and deduce the necessary and sufficient structure of a solution $\widetilde{\varphi}$ to (4.4) (Theorem 2.9).

**Remark 4.8.** Problem (4.10) can be identified as a problem of test theory. Let us define the measures $O$ and $O^* = O^*(P^*)$ by $\frac{dO}{dP} = HY^*$ and $\frac{dO^*}{dP^*} = H$ for $P^* \in \mathcal{P}$. Problem (4.10) turns into

$$\max_{\varphi \in R} E^O[\varphi]$$

subject to

$$\forall P^* \in \mathcal{P}: \quad E^{O^*}[\varphi] \le \widetilde{V}_0 =: \alpha.$$

This is equivalent of looking for an optimal test $\widetilde{\varphi}_{Y^*}$ when testing the compound hypothesis $H_0 = \{O^*(P^*) : P^* \in \mathcal{P}\}$, parameterized by the class of equivalent martingale measures, against the simple alternative $H_1 = \{O\}$ in a generalized sense. In the generalized test problem (see Section 3.2), $O$ and $O^*$ are not necessarily probability measures, but measures and the significance level $\alpha$ is generalized to be a positive continuous function $\alpha(P^*)$.

We now give the main results by applying the procedure described in Remark 4.7 to problem (4.4).

**Theorem 4.9** (Solution to the Generalized Hedging Problem)**.** *Let $\rho$ be as in Assumption 4.5 and let $Z_{\mathcal{P}}$ be compact. Then, there exists a solution $\widetilde{\varphi}$ to (4.4). If $\rho$ is strictly convex, then any two solutions coincide $P-a.s.$ on $\{H > 0\}$. There exists a pair $(\widetilde{Y}^*, \widetilde{\lambda})$ solving*

$$\max_{Y^* \in L^\infty_+, \lambda \in \Lambda_+} \Big\{ E[HY^* \wedge H \int_{\mathcal{P}} Z_{P^*} d\lambda] - \widetilde{V}_0 \lambda(Z_{\mathcal{P}}) - \rho^*(-Y^*) \Big\}, \tag{4.11}$$

*where $x \wedge y = min(x, y)$. It follows that:*

- *The solution of the static optimization problem (4.4) is*

$$\widetilde{\varphi} = \begin{cases} 1 & : \quad H\widetilde{Y}^* > H \int_{\mathcal{P}} Z_{P^*} d\widetilde{\lambda} \\ 0 & : \quad H\widetilde{Y}^* < H \int_{\mathcal{P}} Z_{P^*} d\widetilde{\lambda} \end{cases} \qquad P - a.s. \tag{4.12}$$

  *with*

$$E^{P^*}[\widetilde{\varphi} H] = \widetilde{V}_0 \qquad \widetilde{\lambda} - a.s. \tag{4.13}$$

- $(\widetilde{\varphi}, \widetilde{Y}^*)$ *is a saddle point of the functional* $(\varphi, Y^*) \mapsto E[(1-\varphi)HY^*] - \rho^*(-Y^*)$ *in* $R_0 \times L^\infty_+$.

- $(\widetilde{V}_0, \widetilde{\xi})$ *solves the dynamic hedging problem (4.2), (4.3), where* $\widetilde{\xi}$ *is the superhedging strategy of the modified claim* $\widetilde{\varphi} H$*, obtained by the optional decomposition theorem (Theorem C.3).*

**Remark 4.10.** It follows that there exists a $[0, 1]$-valued random variable $\delta$ such that $\widetilde{\varphi}$ as in Theorem 4.9 satisfies

$$\widetilde{\varphi}(\omega) = 1_{\{H\widetilde{Y}^* > H \int_{\mathcal{P}} Z_{P^*} d\widetilde{\lambda}\}}(\omega) + \delta(\omega) 1_{\{H\widetilde{Y}^* = H \int_{\mathcal{P}} Z_{P^*} d\widetilde{\lambda}\}}(\omega).$$

$\delta$ has to be chosen such that $\widetilde{\varphi}$ satisfies (4.13).

**Remark 4.11.** Theorem 4.9 gives a result about the structure of a solution to the hedging problem for every risk measure satisfying Assumption 4.5. Note that we do not need a translation property for $\rho$ to obtain this result.

*Proof of Theorem 4.9* 1) We can apply the theory of Chapter 2 by setting $H$ equal to the payoff of the contingent claim in $L^1_+$ , $A\varphi = H\varphi$, $b = -H$, $C^* = Z_{\mathcal{P}}$ and $c = \widetilde{V}_0$. We have $\mathcal{X} = L^\infty$, endowed with the norm topology, $\mathcal{X}^* = ba(\Omega, \mathcal{F}, P)$, $\mathcal{Y} = L^1$, $\mathcal{Y}^* = L^\infty$. The function $\rho$ is as in Assumption 4.5 and $\mathcal{X}_1$ is the set of randomized tests and coincides with $R = \{\varphi : \ \Omega \to [0, 1], \ \mathcal{F}_T - \text{measurable}\}$. Hence, $\mathcal{X}_0 = R_0$. Then, the static optimization problem (4.4) can be identified as a special case of problem (2.1).

2) First, we verify that condition (A1)-(A7) of Assumption 2.1 are satisfied:

(A1): $c = \widetilde{V}_0 > 0$ (see (4.3)).

(A2): $H \in L^1_+, C^* = Z_{\mathcal{P}} \subseteq L^1$ and $\{HX^* : X^* \in C^*\} = \{HZ_{P^*} : P^* \in \mathcal{P}\} \subseteq L^1$, since we assumed in (4.1) the superhedging price of $H$ to be finite.

(A3): (4.1) also ensures that $\sup_{X^* \in C^*} \|HX^*\|_{L^1} < +\infty$. The set $Z_{\mathcal{P}}$ is assumed to be compact.

(A4): The operator $A : L^\infty \to L^1$, defined by $A\varphi := H\varphi$, is linear and continuous.

(A5): Since $\rho$ is as in Assumption 4.5, it satisfies the condition (A5).

(A6): We prove that the function $f : L^\infty \to \mathbb{R} \cup \{+\infty\}$, defined by $f(\varphi) := \rho((\varphi - 1)H)$, is lower semicontinuous in the weak* topology. Because of Assumption 4.5, $\rho$ admits a dual representation (see Theorem 1.5 (b))

$$\rho(Y) = \sup_{Y^* \in L^\infty_+} \{E[-YY^*] - \rho^*(-Y^*)\}.$$

Thus,

$$\begin{aligned} f(\varphi) &= \sup_{Y^* \in L^\infty_+} \{E[(1-\varphi)HY^*] - \rho^*(-Y^*)\} \\ &= \sup_{Y^* \in L^\infty_+} \{E[HY^*] + E[\varphi H(-Y^*)] - \rho^*(-Y^*)\}. \end{aligned}$$

The function $\varphi \mapsto E[\varphi H(-Y^*)] + E[HY^*] - \rho^*(-Y^*)$ is weakly* continuous for all $Y^* \in L^\infty$ since $H(-Y^*) \in L^1$. Since $f(\varphi)$ is the pointwise supremum of weakly* continuous functions, $f$ is weakly* lower semicontinuous (Lemma 2.38, [2]).

(A7): The map $\varphi \mapsto \langle Y^*, A\varphi\rangle = E[Y^* H\varphi]$ is continuous in the weak* topology for all $Y^* \in L^\infty$, since $HY^* \in L^1$ for all $Y^* \in L^\infty$.

Thus, all conditions in Assumption 2.1 are satisfied.

3) The existence of a solution $\widetilde{\varphi}$ to (4.4) follows from Theorem 2.5. If $\rho$ is additionally strictly convex, then any two solutions coincides $P-a.s.$ on $\{H > 0\}$. By definition of the adjoined operator $A^*$ of $A$ (see Definition 6.51 in [2]), the equation $\langle A\varphi, Y^*\rangle = \langle \varphi, A^*Y^*\rangle$ has to be satisfied for all $\varphi \in L^\infty, Y^* \in L^\infty$. Since from the validity of (A7) we obtain $A^*Y^* \in L^1$ for all $Y^* \in L^\infty$ (cf. Remark 2.2), it holds

$$\forall \varphi \in L^\infty, \forall Y^* \in L^\infty : \quad \int_\Omega H\varphi Y^* dP = \int_\Omega \varphi A^* Y^* dP. \tag{4.14}$$

Suppose $A^*Y^* < HY^*$ on $\Omega_1 \subseteq \Omega$ with $P(\Omega_1) > 0$. Define $\varphi(\omega) = 1_{\Omega_1}(\omega)$. This $\varphi \in L^\infty$ violates (4.14). The case $A^*Y^* > HY^*$ on $\Omega_2 \subseteq \Omega$ with $P(\Omega_2) > 0$ is analogous. We conclude $A^*Y^* = HY^* = AY^*$, i.e., the operator $A$ is self-adjoined.
In our setting, the optimization problem (2.23) becomes (4.11). Note that, since $\rho$ is monotone, it is convenient to work with $-Y^* \in \operatorname{dom}\rho^*$ (cf. Theorem 1.5), whereas in Chapter 2 we work with $\overline{Y}^* \in \operatorname{dom}\rho^*$. By applying Theorem 2.9, we obtain the existence of an optimal pair $(\widetilde{Y}^*, \widetilde{\lambda})$ solving (4.11) and the structure (4.12), (4.13) of an optimal randomized test $\widetilde{\varphi}$. Furthermore, $(\widetilde{\varphi}, \widetilde{Y}^*)$ is a saddle point of the functional $(\varphi, Y^*) \mapsto E[(1-\varphi)HY^*] - \rho^*(-Y^*)$.

4) Equation (4.13) and Theorem 4.1 show, that $(\widetilde{V}_0, \widetilde{\xi})$ solves the dynamic hedging problem (4.2), (4.3), where $\widetilde{\xi}$ is the superhedging strategy of the modified claim $\widetilde{\varphi}H$ obtained by the optional decomposition theorem (Theorem C.3).

□

If $\rho$ satisfies additionally to Assumption 4.5 the translation property and $\rho(0) = 0$, it forms a convex risk measure (see Section 1.2). If it is additionally to this positively homogeneous, it is a coherent risk measure (Section 1.3). The translation property is a natural assumption for a risk measure used as a risk adjusted capital requirement, but is not necessary for the proof of Theorem 4.9. Furthermore, there are risk measures, that do not necessarily have this property, e.g. the expectation of a loss function. This risk measure was used in the context of hedging in [17].
In the following subsections, we shall analyze the hedging problem (4.2) and therefore the corresponding static optimization problem (4.4) using different important risk measures to quantify the shortfall risk. In Section 4.1.2, we use convex risk measures and in Section 4.1.3 coherent risk measures. These risk measures will be special cases of functions satisfying Assumption 4.5. We shall analyze the influence of different additional properties of these risk measures on the results of Theorem 4.9. Furthermore, we shall compare these results with results that can be found in the recent literature using these special risk measures when hedging in incomplete markets. We shall show that Theorem 4.9 is widely applicable and that the obtained results improve previous results in the case $Z_{\mathcal{P}}$ compact.
In Section 4.1.4, we shall consider the hedging problem when the risk is measured by a robust version of the expectation of a loss function. For Lipschitz continuous loss functions the problem can be solved by an application of Theorem 4.9. We show that the linear case is related to the coherent hedging problem and can be solved analogously. We compare our results with the literature. The case of a general loss function turns out to fit not exactly to the setting of Theorem 4.9. We show which assumptions can be weakened and give proposals how the problem could be solved in general.
We start with the case of convex risk measures.

### 4.1.2 Convex Hedging

In this section we consider the problem of hedging when the attitude towards losses is modelled by a convex risk measure. This problem was studied in Rudloff [37, 38].

**Assumption 4.12.** *Let $\rho : L^1 \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex risk measure that is continuous and finite in some $(\varphi_0 - 1)H$ with $\varphi_0 \in R_0$.*

**Remark 4.13.** Note that, if $\rho(Y) < +\infty$ for all $Y \in L^1$, a lower semicontinuous convex risk measure turns out to be continuous (see Remark 4.6). Finite valued convex risk measures are discussed in [18], [19], where also examples can be found.

A convex risk measure $\rho$ is lower semicontinuous if and only if its acceptance set $\mathcal{A}_\rho$ is closed (Proposition 1.8 (vi)).

Convex risk measures have been studied in Section 1.2 and are by definition convex, monotone, satisfy the translation property and $\rho(\mathbf{0}) = 0$. Lower semicontinuous convex risk measures on $L^1$ admit the following dual representation (Theorem 1.16)

$$\rho(Y) = \sup_{Q \in \mathcal{Q}} \{E^Q[-Y] - \sup_{\widetilde{Y} \in \mathcal{A}_\rho} E^Q[-\widetilde{Y}]\}, \tag{4.15}$$

where $\mathcal{Q} := \{Q \in \widehat{\mathcal{Q}} : Z_Q \in L^\infty\}$ is the set of all probability measures $Q$, absolutely continuous to $P$ and with densities in $L^\infty$ and $\mathcal{A}_\rho$ is the acceptance set of $\rho$.
The dynamic convex hedging problem consists in finding an admissible strategy solving

$$\min_{(V_0, \xi)} \rho\left(-(H - V_T)^+\right), \quad 0 < V_0 \leq \widetilde{V}_0 < U_0. \tag{4.16}$$

With Theorem 4.1, it follows that the corresponding static optimization problem is

$$\min_{\varphi \in R_0} \rho\left((\varphi - 1)H\right) = \min_{\varphi \in R_0} \{\sup_{Q \in \mathcal{Q}} \{E^Q[(1-\varphi)H] - \sup_{Y \in \mathcal{A}_\rho} E^Q(-Y)\}\}, \tag{4.17}$$

where $R_0$ is as in (4.5). By applying Theorem 4.9 and Theorem 1.16, we obtain the following result:

**Corollary 4.14** (Convex Hedging)**.** *Let $\rho$ be a convex risk measure satisfying Assumption 4.12 and let $Z_\mathcal{P}$ be compact. Then, there exists a solution $\widetilde{\varphi}$ to (4.17). Furthermore, there exists a pair $(\widetilde{Q}, \widetilde{\lambda}) \in \mathcal{Q} \times \Lambda_+$ solving*

$$\max_{Q \in \mathcal{Q}, \lambda \in \Lambda_+} \left\{E[HZ_Q \wedge H \int_\mathcal{P} Z_{P^*} d\lambda] - \widetilde{V}_0 \lambda(Z_\mathcal{P}) - \sup_{Y \in \mathcal{A}_\rho} E^Q(-Y)\right\}. \tag{4.18}$$

*It follows that:*

- *The solution of the static optimization problem (4.17) is*

$$\widetilde{\varphi} = \begin{cases} 1 & : \quad H\widetilde{Z}_Q > H \int_\mathcal{P} Z_{P^*} d\widetilde{\lambda} \\ 0 & : \quad H\widetilde{Z}_Q < H \int_\mathcal{P} Z_{P^*} d\widetilde{\lambda} \end{cases} \qquad P - a.s.$$

  *with*

$$E^{P^*}[\widetilde{\varphi} H] = \widetilde{V}_0 \qquad \widetilde{\lambda} - a.s. \tag{4.19}$$

- *$(\widetilde{\varphi}, \widetilde{Q})$ is a saddle point of the functional $(\varphi, Q) \mapsto E^Q[(1-\varphi)H] - \sup_{Y \in \mathcal{A}_\rho} E^Q(-Y)$ in $R_0 \times \mathcal{Q}$.*

- *$(\widetilde{V}_0, \widetilde{\xi})$ solves the dynamic convex hedging problem (4.16), where $\widetilde{\xi}$ is the superhedging strategy of the modified claim $\widetilde{\varphi} H$, obtained by the optional decomposition theorem (Theorem C.3).*

It is not longer possible to show the essential uniqueness of a solution $\widetilde{\varphi}$ to (4.17) on $\{H > 0\}$ since a convex risk measure cannot be strictly convex. The translation property of $\rho$ and $\rho(\mathbf{0}) = 0$ imply the linearity of $\rho$ on the one dimensional subspace $L(\mathbf{1})$ spanned by the random variable $\mathbf{1}$ (see Proposition 1.14). This means that for convex risk measures one can only show the existence, but not the essential uniqueness of a solution.

*Proof of Corollary 4.14.* Since $\rho$ satisfies Assumption 4.12, we can apply Theorem 4.9. Together with the dual representation of convex risk measures (see Theorem 1.16) we obtain the stated results. □

**Remark 4.15.** From equation (4.19) it follows (except in the case where $\widetilde{\lambda}$ is the zero-measure, i.e., $\widetilde{\lambda}(B) = 0$ for all $B \in \mathcal{B}$) that $\widetilde{U}_0 := \sup_{P^* \in \mathcal{P}} E^{P^*}[\widetilde{\varphi}H] = \widetilde{V}_0$ (see Remark 4.3 and 4.4). This ensures, that $\widetilde{V}_0$ is the minimal amount of capital that is necessary to solve together with $\widetilde{\xi}$ the dynamic problem (4.16).
Let us check if the case $\widetilde{\lambda}(B) = 0$ for all $B \in \mathcal{B}$ can be excluded. If $\widetilde{\lambda}$ is the zero-measure, the optimal randomized test is

$$\widetilde{\varphi} = \begin{cases} 1 & : \quad H\widetilde{Z}_Q > 0 \\ \delta & : \quad H\widetilde{Z}_Q = 0 \end{cases} \qquad P - a.s.,$$

where $\delta$ is a $[0,1]$-valued random variable such that (if possible) $\widetilde{\varphi} \in R_0$, for instance $\delta = \mathbf{0}$. Equation (4.19) has no longer an impact on $\widetilde{\varphi}$ since $\widetilde{\lambda}$ takes only the value zero. Then, the optimal value of the static optimization problem (4.17) becomes (see Theorem 1.5 (a), (c))

$$p = \rho(\widetilde{\varphi}H - H) = E^{\widetilde{Q}}[H] - E^{\widetilde{Q}}[\widetilde{\varphi}H] - \rho^*(-\widetilde{Z}_Q) = -\rho^*(-\widetilde{Z}_Q) \leq 0.$$

From $\rho$ monotone and $\widetilde{\varphi}H - H \leq 0$, we obtain $\rho(\widetilde{\varphi}H - H) \geq 0$ and thus $\rho(\widetilde{\varphi}H - H) = 0$. This means, the risk of the difference between the modified claim $\widetilde{\varphi}H$ and $H$ is zero. In some special cases we can exclude that $\widetilde{\lambda}$ takes only value zero. If $\widetilde{Q}$ is a probability measure equivalent to $P$, then $\widetilde{\lambda}(B) = 0$ for all $B \in \mathcal{B}$ implies $\widetilde{\varphi} \notin R_0$. Thus, in this case $\widetilde{\lambda}(B) = 0$ for all $B \in \mathcal{B}$ is not possible.

### 4.1.3 Coherent Hedging

In this section, we consider the hedging problem when the risk of losses due to the shortfall is measured by a coherent risk measure. This problem was studied in Nakano [31, 32] and Rudloff [36]. Coherent risk measures are convex risk measures that are additionally positively homogeneous. In this section, we deduce the main results for the case $Z_{\mathcal{P}}$ compact and show the differences between the method used in [31, 32] and our method to solve the problem. We show that our results give more information about the structure of a solution. A comparison of the results in the general incomplete market can be found in Section 4.2.2.

**Assumption 4.16.** *Let $\rho : L^1 \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous coherent risk measure that is continuous and finite in some $(\varphi_0 - 1)H$ with $\varphi_0 \in R_0$.*

To assume that $\rho$ is continuous in some $(\varphi_0-1)H$ with $\varphi_0 \in R_0$ is not very restrictive. If we take for instance a finite valued lower semicontinuous coherent risk measure $\rho$ as considered for example in [19], then $\rho$ is continuous. (see Remark 4.6).
The dynamic coherent hedging problem is to find an admissible strategy solving

$$\min_{(V_0,\xi)} \rho\left(-\left(H - V_T\right)^+\right), \quad 0 < V_0 \leq \widetilde{V}_0 < U_0. \tag{4.20}$$

With Theorem 4.1 and the dual representation of a lower semicontinuous coherent risk measure (Theorem 1.25) it follows that the corresponding static optimization problem, the primal problem, is

$$\min_{\varphi \in R_0} \rho\left((\varphi - 1)H\right) = \min_{\varphi \in R_0} \{\sup_{Q \in \mathcal{Q}} E^Q[(1-\varphi)H]\}, \tag{4.21}$$

where

$$R_0 = \{\varphi : \ \Omega \to [0,1], \ \mathcal{F}_T - \text{measurable}, \ \sup_{P^* \in \mathcal{P}} E^{P^*}[\varphi H] \leq \widetilde{V}_0\} \tag{4.22}$$

and $\mathcal{Q}$, the maximal representing set, is a convex and weakly* closed subset of $\{Q \in \widehat{\mathcal{Q}} : Z_Q \in L^\infty\}$ determined by the dual representation of $\rho$ (Theorem 1.25). The dual problem of (4.21) is (see Remark 4.7, (ii))

$$d = \max_{Q \in \mathcal{Q}} \{\min_{\varphi \in R_0} E^Q[(1-\varphi)H]\}. \tag{4.23}$$

The inner problem of (4.23) for a fixed $Q \in \mathcal{Q}$ is (see Remark 4.7, (iii))

$$p^i(Q) := \max_{\varphi \in R_0} E^Q[\varphi H]. \tag{4.24}$$

Its dual problem (see Remark 4.7, (iii)), deduced via Fenchel duality, is

$$d^i(Q) = \inf_{\lambda \in \Lambda_+} \Big\{ \int_\Omega [HZ_Q - H \int_{\mathcal{P}} Z_{P^*} d\lambda]^+ dP + \widetilde{V}_0 \lambda(Z_{\mathcal{P}}) \Big\}. \tag{4.25}$$

**Corollary 4.17** (Coherent Hedging)**.** *Let $\rho$ be a coherent risk measure satisfying Assumption 4.16 and let $Z_{\mathcal{P}}$ be compact. Then, there exists a solution $\widetilde{\varphi}$ to (4.21). Furthermore, there exists a pair $(\widetilde{Q}, \widetilde{\lambda}) \in \mathcal{Q} \times \Lambda_+$ solving*

$$\max_{Q \in \mathcal{Q}, \lambda \in \Lambda_+} \Big\{ E[HZ_Q \wedge H \int_{\mathcal{P}} Z_{P^*} d\lambda] - \widetilde{V}_0 \lambda(Z_{\mathcal{P}}) \Big\}. \tag{4.26}$$

*It follows that:*

- *The solution of the static optimization problem (4.21) is*

$$\widetilde{\varphi} = \begin{cases} 1 & : \quad H\widetilde{Z}_Q > H\int_{\mathcal{P}} Z_{P^*} d\widetilde{\lambda} \\ 0 & : \quad H\widetilde{Z}_Q < H\int_{\mathcal{P}} Z_{P^*} d\widetilde{\lambda} \end{cases} \qquad P - a.s. \tag{4.27}$$

*with*

$$E^{P^*}[\widetilde{\varphi}H] = \widetilde{V}_0 \qquad \widetilde{\lambda} - a.s. \tag{4.28}$$

- $(\widetilde{\varphi}, \widetilde{Q})$ *is a saddle point of the functional* $(\varphi, Q) \mapsto E^Q[(1-\varphi)H]$ *in* $R_0 \times \mathcal{Q}$.

- $(\widetilde{V}_0, \widetilde{\xi})$ *solves the dynamic coherent hedging problem (4.20), where* $\widetilde{\xi}$ *is the superhedging strategy of the modified claim* $\widetilde{\varphi}H$*, obtained by the optional decomposition theorem (Theorem C.3).*

*Proof.* Since coherent risk measures are also convex risk measures, the results follow from Corollary 4.14 and the dual representation of coherent risk measures (see Theorem 1.25). □

The problem (4.20) of hedging with coherent risk measures was studied by Nakano [31, 32]. In [32], the decomposition of the dynamic problem and the existence of a solution to the static problem was shown. In [31] a similar result as in Corollary 4.17 was obtained. We now want to make the differences in the methods clear that are used in the proofs and show in which way Corollary 4.17 is an improvement of Theorem 4.11 in [31] for the case $Z_{\mathcal{P}}$ compact. Nakano [31] followed the method of Cvitanić and Karatzas [7] (see Section 3.1) to show that the solution of the static optimization problem is a Neyman-Pearson test. In Nakano [31] it is necessary to introduce the enlarged sets

$$\mathcal{Z} = \{Z \in L^\infty_+ \mid E[Z] \le 1, \forall X \in L^1_+ : E[XZ] \le \rho(-X)\} \supseteq \{Z_Q : Q \in \mathcal{Q}\}$$

and

$$\mathcal{D} = \{D \in L^1_+ \mid E[D] \le 1, E[DH] \le U_0, \forall \varphi \in R_0 : E[D\varphi H] \le \widetilde{V}_0\} \supseteq Z_{\mathcal{P}},$$

where $\mathcal{Z}$ is closed under $P - a.s.$ convergence and convex and $\mathcal{D}$ is bounded in $L^1$, convex and closed under $P-a.s.$ convergence. These enlarged sets were introduced to ensure the existence of a quadruple $(\widehat{Z}, \widehat{D}, \widehat{z}, \widehat{\varphi}) \in (\mathcal{Z} \times \mathcal{D} \times (0,\infty) \times R_0)$ that yield equality in

$$\forall Z \in \mathcal{Z}, \forall D \in \mathcal{D}, \forall z > 0, \forall \varphi \in R_0 : \quad E[Z(H - \varphi H)] \ge E[H(Z \wedge zD)] - \widetilde{V}_0 z \tag{4.29}$$

In Theorem 4.11 in [31] the typical 0-1-structure of an optimal randomized test $\widehat{\varphi}$ is deduced, but with respect to elements from the larger sets $\mathcal{Z}$ and $\mathcal{D}$:

$$\widehat{\varphi} = 1_{\{\widehat{z}\widehat{D} < \widehat{Z}\}} + \delta 1_{\{\widehat{z}\widehat{D} = \widehat{Z}\}},$$

where $(\widehat{z}, \widehat{Z}, \widehat{D})$ attain the supremum of

$$\sup_{z\geq 0, Z\in\mathcal{Z}, D\in\mathcal{D}} \{E[H(Z\wedge zD)] - \widetilde{V}_0 z\}.$$

We can show that inequality (4.29) corresponds to the validity of weak duality between the inner problem (4.24) and its dual problem (4.25) that is automatically satisfied (cf. Theorem A.12). That is

$$\forall Q\in\mathcal{Q}, \forall\lambda\in\Lambda_+, \forall\varphi\in R_0: \quad E^Q[(1-\varphi)H] \geq E[HZ_Q\wedge H\int_{\mathcal{P}} Z_{P^*}d\lambda] - \widetilde{V}_0\lambda(Z_{\mathcal{P}}).$$

With our method, it is not necessary to consider the larger sets $\mathcal{Z}$ and $\mathcal{D}$. We prove the validity of strong duality via Fenchel duality directly (step (ii) and (iii), Remark 4.7). The existence of a dual solution follows from the validity of strong duality (cf. Theorem A.12). This makes it possible to deduce the 0-1-structure of $\widetilde{\varphi}$ with respect to elements from the original sets $\mathcal{Q}$ and $\mathcal{P}$. In contrast to this, Nakano [31] proved the existence of a solution to the dual problem. Therefore, it was necessary to consider the larger sets $\mathcal{Z}$ and $\mathcal{D}$. The application of Corollary 4.17 shows that there is a one-to-one relationship between the optimal elements $\widehat{Z}$, $\widehat{D}$ and $\widehat{z}$ of [31] and elements of $\mathcal{Q}$ and $\mathcal{P}$:

$$\widehat{Z} = \begin{cases} \widetilde{Z}_Q & : \{H>0\} \\ 0 & : \{H=0\} \end{cases},$$

$$\widehat{D} = \begin{cases} k\int_{\mathcal{P}} Z_{P^*}d\widetilde{\lambda} & : \{H>0\} \\ 0 & : \{H=0\} \end{cases},$$

$$\widehat{z} = \widetilde{\lambda}(Z_{\mathcal{P}}),$$

where $(\widetilde{Q}, \widetilde{\lambda})$ is the optimal pair in (4.26) and $k = \widetilde{\lambda}(Z_{\mathcal{P}})^{-1}$ if $\widetilde{\lambda}(Z_{\mathcal{P}}) \neq 0$ and zero if $\widetilde{\lambda}(Z_{\mathcal{P}}) = 0$. It holds $\widetilde{\varphi} = \widehat{\varphi}$. Thus, the direct application of convex duality gives more detailed information about the structure of the optimal randomized test $\widetilde{\varphi}$. Another difference to [31] is that we consider coherent risk measures that can also attain the value $+\infty$. Furthermore, we now can show in equation (4.28) of Corollary 4.17 that the upper bound of the constraint in (4.22) is attained (except in the pathological case where $\widetilde{\lambda}$ takes only the value zero (see Remark 4.15)). Then, $\widetilde{U}_0 = \sup_{P^*\in\mathcal{P}} E^{P^*}[\widetilde{\varphi}H] = \widetilde{V}_0$. It follows, that $\widetilde{V}_0$, the upper capital boundary, is the minimal required capital that is necessary for the optimal hedge and thus, $(\widetilde{V}_0, \widetilde{\xi})$ solves the optimization problem (4.20) (see Proof of Theorem 4.1 and Remark 4.4). This was not possible to deduce from the analogous result $E[\widehat{\varphi}H\widehat{D}] = \widetilde{V}_0$, $\widehat{D}\in\mathcal{D}$ in Nakano [31]. A comparison of the results in the general incomplete market can be found in Section 4.2.2.

### 4.1.4 Robust Efficient Hedging

In the concept of efficient hedging the expectation of a loss function $l$ is used as the risk measure in problem (4.2). This problem was introduced by Föllmer and Leukert [17] (see also [19]).

**Assumption 4.18.** *Let $l : \mathbb{R} \to \mathbb{R}$ be a nondecreasing convex function with $l(x) = 0$ for all $x \leq 0$.*

The function $l$ is continuous since it is a convex and finite valued function on $\mathbb{R}$ ([11], Corollary I.2.3). Let $L^0 = L^0(\Omega, \mathcal{F}, P)$ be the space of $P - a.s.$ finite random variables and $L^0_+ := \{Y \in L^0 : Y \geq \mathbf{0} \ P - a.s.\}$. We define $L : L^1 \to L^0_+$ by

$$L(Y)(\omega) := l(Y(\omega)).$$

The function $L$ maps into $L^0_+$ since $l$ is continuous and maps into $\mathbb{R}_+$. We consider the dynamic efficient hedging problem derived from (4.2) with the risk measure $\rho_0(Y) = E[L(-Y)]$ for $Y \in L^1$. This means, we look for an admissible strategy that is a solution of

$$\min_{(V_0,\xi)} E[L((H - V_T)^+)], \quad 0 < V_0 \leq \widetilde{V}_0 < U_0.$$

We want to generalize this problem and consider a robust version (see Remark 8.12 in [19]) of the expectation of a loss function defined as

$$\rho_1(Y) = \sup_{Q \in \mathcal{Q}} E^Q[L(-Y)], \quad Y \in L^1, \tag{4.30}$$

where $\mathcal{Q} \subseteq \widehat{\mathcal{Q}}$ is a set of probability measures absolutely continuous with respect to $P$. By passing from a single probability measure $P$ to a whole set $\mathcal{Q}$ of probability measures one can take into account an uncertainty regarding the underlying model. This can be the case if for instance the underlying asset price process is modelled via an jump-diffusion process and there is uncertainty regarding the jump intensities (see [26] for several examples).
In the following, we will study the robust efficient hedging problem using the risk measure $\rho_1$. The dynamic problem is to find an admissible strategy solving

$$\min_{(V_0,\xi)} \sup_{Q \in \mathcal{Q}} E^Q[L((H - V_T)^+)], \quad 0 < V_0 \leq \widetilde{V}_0 < U_0. \tag{4.31}$$

**Remark 4.19.** A special case of (4.31) is the problem of quantile hedging. In this case, the probability of losses due to the shortfall has to be minimized. We obtain this problem from (4.31) by setting $\mathcal{Q} = \{P\}$ and using the non-convex loss function $l(x) = 1_{(0,\infty)}(x)$. This problem was solved in [16].

We impose the following assumption on $l, \mathcal{Q}$ and $H$.

**Assumption 4.20.** $\sup_{Q\in\mathcal{Q}} E^Q[L(H)] < +\infty$.

Let us analyze the properties of $\rho_1$.

**Proposition 4.21.** *Under Assumption 4.18, the function $\rho_1 : L^1 \to \mathbb{R}\cup\{+\infty\}$ is monotone, convex, lower semicontinuous and satisfies $\rho_1(\boldsymbol{0}) = 0$.*

*Proof.* $\rho_1$ is monotone and convex since $l$ is nondecreasing and convex. $\rho_1$ satisfies $\rho_1(\mathbf{0}) = 0$ since $l(0) = 0$. To prove the lower semicontinuity of $\rho_1$, we prove that $\operatorname{epi}\rho_1$ is closed. Take a sequence $(Y_n, r_n) \in \operatorname{epi}\rho_1$ for all $n \in \mathbb{N}$ with $Y_n \to Y$ in $L^1$ and $r_n \to r$. Thus, for all $n \in \mathbb{N}$ it holds $\rho_1(Y_n) \le r_n$. Since $Y_n \to Y$ in $L^1$, there is a subsequence $Y_{n_k}$ converging $P - a.s.$ to $Y$ (see Theorem 10.38 and 10.39 in [2]). The sequence $L(-Y_{n_k})$ converges $P - a.s.$ to $L(-Y)$ since $l$ is continuous. $L(-Y_{n_k})$ is for all $k \in \mathbb{N}$ a nonnegative random variable due to Assumption 4.18. Thus, we can apply Fatou's Lemma (Lemma B.21) and obtain

$$\forall Q \in \mathcal{Q}: \quad E^Q[L(-Y)] = E^Q[\liminf_{k\to\infty} L(-Y_{n_k})] \le \liminf_{k\to\infty} E^Q[L(-Y_{n_k})]. \tag{4.32}$$

Since $(Y_{n_k}, r_{n_k}) \in \operatorname{epi}\rho_1$ for all $k \in \mathbb{N}$, we have

$$\forall Q \in \mathcal{Q}: \quad E^Q[L(-Y_{n_k})] \le \sup_{\widehat{Q}\in\mathcal{Q}} E^{\widehat{Q}}[L(-Y_{n_k})] \le r_{n_k}.$$

Together with (4.32) we obtain

$$\forall Q \in \mathcal{Q}: \quad E^Q[L(-Y)] \le r.$$

Hence, $\rho_1(Y) = \sup_{Q\in\mathcal{Q}} E^Q[L(-Y)] \le r$. This means, $(Y, r) \in \operatorname{epi}\rho_1$ and thus $\rho_1$ is lower semicontinuous in $L^1$. $\square$

Since $\rho_1$ is monotone, we can apply Theorem 4.1 and obtain the static optimization problem that corresponds to the dynamic problem (4.31)

$$\min_{\varphi\in R_0} \sup_{Q\in\mathcal{Q}} E^Q[L((1-\varphi)H)], \tag{4.33}$$

where $R_0$ is as in (4.5)

$$R_0 = \{\varphi :\ \Omega \to [0,1],\ \mathcal{F}_T - \text{measurable},\ \sup_{P^*\in\mathcal{P}} E^{P^*}[\varphi H] \le \widetilde{V}_0\}.$$

By Proposition 4.21, $\rho_1$ is monotone, convex, lower semicontinuous and satisfies $\rho_1(\mathbf{0}) = 0$. To apply Theorem 4.9, $\rho_1$ has to satisfy Assumption 4.5. Thus, $\rho_1$ has to be continuous and finite in some $(\varphi_0 - 1)H$ with $\varphi_0 \in R_0$.
Since $\rho_1$ is a convex and lower semicontinuous functional on $L^1$, it is continuous in the interior of its effective domain ([11], Corollary I.2.5). The points, where $\rho_1$ takes

finite values, depend on the choice of the loss function $l$. Let us consider the simple example of $l(x) = x^2$ and $\mathcal{Q} = \{P\}$. Let us ignore for the moment the condition $l(x) = 0$ for $x \leq 0$, which does not have an impact on the optimization problem (4.31) since we work only with nonnegative values. Then, the effective domain of the function $\rho_1(Y) = E[Y^2]$ consists of all elements of $L^2$. Since the interior of $L^2$ as a linear subspace of $L^1$ is empty, there does not exist a point $Y \in L^1$ such that $\rho_1(Y) = E[Y^2]$ is continuous. Thus, in general, we can not expect to find an inner point of the domain of $\rho_1$ and thus, a point where $\rho_1$ is continuous.
In the following, we shall consider several special cases, where the continuity of $\rho_1$ in some $(\varphi_0 - 1)H$ with $\varphi_0 \in R_0$ can be verified and thus, Theorem 4.9 can be applied to solve the problem.

### The Special Case of Lipschitz Continuous Loss Functions

In this section, we consider a special case, i.e., we impose stronger assumptions to solve problem (4.31). These assumptions are for instance satisfied if the loss function is Lipschitz continuous and the set $\mathcal{Q}$ of measures satisfies a certain condition. In addition to Assumption 4.20, we shall impose the following in this section.

**Assumption 4.22.** *Let* $\{Z_Q : Q \in \mathcal{Q}\} \subseteq L^\infty$ *with* $\sup_{Q\in\mathcal{Q}} \|Z_Q\|_{L^\infty} < +\infty$.

Let $\varepsilon > 0$. We denote by $U_\varepsilon(H) := \{Y \in L^1 : \|Y - H\|_{L^1} \leq \varepsilon\}$ the $\varepsilon-$neighborhood of $H \in L^1$.

**Assumption 4.23.** *Let $l$ be such that there exists an $\varepsilon-$neighborhood $U_\varepsilon(H)$ of $H$ with*

$$\forall Y \in U_\varepsilon(H) : \quad L(Y) - L(H) \in L^1.$$

**Remark 4.24.** Assumption 4.23 is for instance satisfied if $l$ is Lipschitz continuous, i.e., there exists a constant $c \in \mathbb{R}$ such that for all $x, y \in \mathbb{R}$

$$|l(x) - l(y)| \leq c|x - y|.$$

Then, it follows that $L$ is Lipschitz continuous and maps into $L^1$ since for all $Y_1, Y_2 \in L^1$ we have

$$\begin{aligned} \|L(Y_1) - L(Y_2)\|_{L^1} &= \int_\Omega |L(Y_1)(\omega) - L(Y_2)(\omega)|dP = \int_\Omega |l(Y_1(\omega)) - l(Y_2(\omega))|dP \\ &\leq \int_\Omega c|Y_1(\omega) - Y_2(\omega)|dP = c\|Y_1 - Y_2\|_{L^1}. \end{aligned}$$

**Remark 4.25.** If $l$ is Lipschitz continuous and Assumption 4.22 is satisfied, then Assumption 4.20 holds since

$$\sup_{Q\in\mathcal{Q}} E^Q[L(H)] \leq \|L(H)\|_{L^1} \sup_{Q\in\mathcal{Q}} \|Z_Q\|_{L^\infty} \leq c\|H\|_{L^1} \sup_{Q\in\mathcal{Q}} \|Z_Q\|_{L^\infty} < +\infty.$$

**Proposition 4.26.** *Let Assumption 4.20, 4.22 and 4.23 be satisfied. Then, $\rho_1$ is continuous and finite in $-H = (\varphi_0 - 1)H$ with $\varphi_0 = \mathbf{0} \in R_0$.*

*Proof.* For all $-Y \in U_\varepsilon(H)$ we have $L(-Y) - L(H) \in L^1$ due to Assumption 4.23. Together with Assumption 4.22 and 4.20, we obtain that for all $-Y \in U_\varepsilon(H)$

$$\begin{aligned}\rho_1(Y) &= \sup_{Q\in\mathcal{Q}} E^Q[L(-Y)] \leq \sup_{Q\in\mathcal{Q}} E^Q[L(-Y) - L(H)] + \sup_{Q\in\mathcal{Q}} E^Q[L(H)] \\ &\leq \|L(-Y) - L(H)\|_{L^1} \sup_{Q\in\mathcal{Q}} \|Z_Q\|_{L^\infty} + \sup_{Q\in\mathcal{Q}} E^Q[L(H)] < +\infty.\end{aligned}$$

Since $-Y \in U_\varepsilon(H)$ if and only if $Y \in U_\varepsilon(-H)$, we obtain that for all $Y \in U_\varepsilon(-H)$ the convex function $\rho_1$ is bounded above by a finite constant. Thus, by Lemma I.2.1, [11], $\rho_1$ is continuous in $-H = (\varphi_0 - 1)H$ with $\varphi_0 = \mathbf{0} \in R_0$. □

This means, if Assumption 4.18, 4.20, 4.22 and 4.23 are satisfied (for instance if we work with a Lipschitz continuous loss functions $l$ and a finite set $\mathcal{Q}$ with $\{Z_Q : Q \in \mathcal{Q}\} \subseteq L^\infty$), we can apply Theorem 4.9 to deduce a result about the structure of a solution $\widetilde{\varphi}$ to (4.33). Since $\rho_1 : L^1 \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, convex, proper and monotone (Proposition 4.21), it has the following dual representation (see Theorem 1.5 (b))

$$\rho_1(Y) = \sup_{Y^* \in L^\infty_+} \{E[-YY^*] - \rho_1^*(-Y^*)\}.$$

In the following theorem we shall work with this dual representation instead of the representation (4.30) of $\rho_1$. The application of Theorem 4.9 yields the following result.

**Corollary 4.27** (Robust Efficient Hedging)**.** *Let Assumption 4.18, 4.20, 4.22 and 4.23 be satisfied and let $Z_{\mathcal{P}}$ be compact. Then, there exists a solution $\widetilde{\varphi}$ to (4.33). If $\rho_1$ is strictly convex, then any two solutions coincide $P - a.s.$ on $\{H > 0\}$. Furthermore, there exists a pair $(\widetilde{Y}^*, \widetilde{\lambda}) \in L^\infty_+ \times \Lambda_+$ solving*

$$\max_{Y^* \in L^\infty_+, \lambda \in \Lambda_+} \Big\{E[HY^* \wedge H \int_{\mathcal{P}} Z_{P^*} d\lambda] - \widetilde{V}_0 \lambda(Z_{\mathcal{P}}) - \rho_1^*(-Y^*)\Big\}. \tag{4.34}$$

*Let $(\widetilde{Y}^*, \widetilde{\lambda})$ be the optimal pair in (4.34). It follows that:*

- *The solution of the static optimization problem (4.33) is*

$$\widetilde{\varphi} = \begin{cases} 1 & : \quad H\widetilde{Y}^* > H \int_{\mathcal{P}} Z_{P^*} d\widetilde{\lambda} \\ 0 & : \quad H\widetilde{Y}^* < H \int_{\mathcal{P}} Z_{P^*} d\widetilde{\lambda} \end{cases} \qquad P - a.s.$$

  *with*

$$E^{P^*}[\widetilde{\varphi} H] = \widetilde{V}_0 \qquad \widetilde{\lambda} - a.s.$$

- $(\widetilde{\varphi}, \widetilde{Y}^*)$ *is a saddle point of the functional* $(\varphi, Y^*) \mapsto E[(1-\varphi)HY^*] - \rho_1^*(-Y^*)$ *in* $R_0 \times L_+^\infty$.
- $(\widetilde{V}_0, \widetilde{\xi})$ *solves the dynamic robust efficient hedging problem (4.31), where* $\widetilde{\xi}$ *is the superhedging strategy of the modified claim* $\widetilde{\varphi}H$, *obtained by the optional decomposition theorem (Theorem C.3).*

*Proof.* The Assumptions 4.18, 4.20, 4.22 and 4.23 ensure that Assumption 4.5 is satisfied (see Proposition 4.21 and 4.26). Thus, we can apply Theorem 4.9 and the stated results follow. □

**Remark 4.28.** The function $\rho_1$ is strictly convex if for instance $l(x)$ is strictly convex and $\mathcal{Q}$ has only finitely many elements.

### The Linear Case

Let us impose Assumption 4.22 for this section. Since the "linear" loss function $l(x) = x^+$ is Lipschitz continuous, we can apply Corollary 4.27 (see Remark 4.24 and 4.25). But, in the linear case we can even go a step further.
Problem (4.33) with $l(x) = x^+$ is equivalent to problem (4.33) with $l(x) = x$ since we work only with nonnegative values. Thus, the static optimization problem in the linear case is

$$\min_{\varphi \in R_0} \sup_{Q \in \mathcal{Q}} E^Q[(1-\varphi)H], \tag{4.35}$$

where $R_0$ is as in (4.5) and the risk measure that is used is

$$\rho_2(Y) = \sup_{Q \in \mathcal{Q}} E^Q[-Y],$$

defined on $L^1$. Since we impose Assumption 4.22, the risk measure $\rho_2$ is a coherent risk measure on $L^1$ (cf. Section 1.3) that is finite valued, thus continuous ([11], Corollary I.2.5). The maximal representing set of $\rho_2$ is $\mathcal{Q}_{\max} = \overline{\text{co}}^* \mathcal{Q}$, the weak* closure of the convex hull of the densities of $\mathcal{Q}$ (see Theorem 1.25). Then, the 0-1-structure of $\widetilde{\varphi}$ can be deduced with respect to a $\widetilde{Q} \in \overline{\text{co}}^* \mathcal{Q}$ instead of $\widetilde{Y}^* \in L_+^\infty$ as in Corollary 4.27. Thus, we obtain by an application of Corollary 4.17 with the maximal representing set $\overline{\text{co}}^* \mathcal{Q}$ of $\rho_2$ the following Corollary.

**Corollary 4.29** (Robust Efficient Hedging with linear loss function)**.** *Let Assumption 4.22 be satisfied and let* $Z_{\mathcal{P}}$ *be compact. Then, there exists a solution* $\widetilde{\varphi}$ *to (4.35). Furthermore, there exists a pair* $(\widetilde{Q}, \widetilde{\lambda}) \in \overline{\text{co}}^* \mathcal{Q} \times \Lambda_+$ *solving*

$$\max_{Q \in \overline{\text{co}}^* \mathcal{Q}, \lambda \in \Lambda_+} \Big\{ E[HZ_Q \wedge H \int_{\mathcal{P}} Z_{P^*} d\lambda] - \widetilde{V}_0 \lambda(Z_{\mathcal{P}}) \Big\}. \tag{4.36}$$

*Let* $(\widetilde{Q}, \widetilde{\lambda})$ *be the optimal pair in (4.36). It follows that:*

- *The solution of the static optimization problem (4.35) is*

$$\widetilde{\varphi} = \begin{cases} 1 & : \quad H\widetilde{Z}_Q > H \int_{\mathcal{P}} Z_{P^*} d\widetilde{\lambda} \\ 0 & : \quad H\widetilde{Z}_Q < H \int_{\mathcal{P}} Z_{P^*} d\widetilde{\lambda} \end{cases} \qquad P-a.s.$$

  *with*

$$E^{P^*}[\widetilde{\varphi}H] = \widetilde{V}_0 \qquad \widetilde{\lambda}-a.s.$$

- $(\widetilde{\varphi}, \widetilde{Q})$ *is a saddle point of the functional* $(\varphi, Q) \mapsto E^Q[(1-\varphi)H]$ *in* $R_0 \times \overline{\mathrm{co}}^* \mathcal{Q}$.

- $(\widetilde{V}_0, \widetilde{\xi})$ *solves the dynamic problem (4.31) in the linear case, where* $\widetilde{\xi}$ *is the superhedging strategy of the modified claim* $\widetilde{\varphi}H$*, obtained by the optional decomposition theorem (Theorem C.3).*

Corollary 4.29 is a generalization of Proposition 4.1 in Föllmer and Leukert [17] and a generalization of Theorem 1.19 in Xu [48] for the case $Z_{\mathcal{P}}$ compact. In [17] and [48] the set $\mathcal{Q} = \{P\}$ is a singleton. In [17], the problem is solved in the complete financial market, i.e., $\mathcal{P} = \{P^*\}$ and in [48] the problem is solved in the incomplete financial market. Furthermore, in [48] the optimal strategy is computed in three complete market cases. In analogy to Nakano [31] (see Section 4.1.3), Xu [48] enlarged the set $\mathcal{P}$ that contains the equivalent martingale measures and deduced the 0-1-structure of the optimal randomized test with respect to an element from the enlarged set. With our method this is not necessary, we work directly with the set $\mathcal{P}$. Furthermore, we do not need to impose the assumption that the discounted asset price $S$ is locally bounded as used in [48]. A comparison of the results in the general incomplete market can be found in Section 4.2.3.
With our method it is possible to solve the problem not only in the case $\mathcal{Q} = \{P\}$, but also for more general sets $\mathcal{Q}$ satisfying Assumption 4.22 and even for more general loss functions satisfying Assumption 4.23.

**Prospect: The General Case**

This section should be understood as a discussion and as a prospect of further research. The problem of robust efficient hedging does, in general, not satisfy Assumption 4.5. We shall show which results are affected and we give proposals how the problem could be solved.
Let us consider a general loss function $l$ and problem (4.33) in the context of Chapter 2. All conditions of Assumption 2.1 are satisfied except the continuity of $\rho_1$ in some $(\varphi_0 - 1)H$ with $\varphi_0 \in R_0$ as postulated in (A5). The lack of this condition has an effect on the validity of strong duality in Theorem 2.6. Thus, the equality between the values of the primal problem (4.33) and its Fenchel dual problem is no longer ensured.

This motivates us to use the special structure of the problem and to define a modified risk measure on the space $L^\infty$ that might satisfy the required assumptions. First, we define $\widetilde{L} : L^\infty \to L^0_+$ by

$$\widetilde{L}(Y) := L(H - YH)$$

Consider the function $\rho : L^\infty \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\rho(Y) := \begin{cases} \sup_{Q \in \mathcal{Q}} E^Q[\widetilde{L}(Y)] & : \quad Y \in L^\infty_+ \\ +\infty & : \quad Y \notin L^\infty_+ \end{cases} . \tag{4.37}$$

Then, problem (4.33) is equivalent to the static optimization problem

$$\min_{\varphi \in R_0} \rho(\varphi) \tag{4.38}$$

We shall deduce several properties of $\rho$.

**Proposition 4.30.** *Suppose Assumption 4.18 and 4.20 hold. Then, the function $\rho$ defined in (4.37) is monotone, convex and proper with* $\operatorname{dom}\rho = L^\infty_+$*. Furthermore, $\rho$ is lower semicontinuous and there exists a $\varphi_0 \in R_0$, such that $\rho$ is continuous and finite in $\varphi_0$.*

*Proof.* $\rho$ is monotone and convex, since $l$ is nondecreasing and convex (Assumption 4.18). Take $Y \in L^\infty_+$. Then, since $l$ is nondecreasing, $H \in L^1_+$ (see page 50) and because of Assumption 4.20, $\rho(Y) = \sup_{Q \in \mathcal{Q}} E^Q[L(H - YH)] \leq \sup_{Q \in \mathcal{Q}} E^Q[L(H)] < +\infty$. For $Y \notin L^\infty_+$, we have $\rho(Y) = +\infty$. Thus, $\operatorname{dom}\rho = L^\infty_+$ and $\rho$ is proper.
To show the lower semicontinuity of $\rho$, we shall show that the epigraph $\operatorname{epi}\rho$ is closed. Take a sequence $(Y_n, r_n) \in \operatorname{epi}\rho$ for all $n \in \mathbb{N}$ with $Y_n \to Y$ in the norm topology of $L^\infty$ and $r_n \to r$. Thus, for all $n \in \mathbb{N}$ we have $Y_n \in L^\infty_+$ with $\rho(Y_n) \leq r_n$. Then

$$\forall n \in \mathbb{N}, \forall Q \in \mathcal{Q} : \quad E^Q[\widetilde{L}(Y_n)] \leq \sup_{\widehat{Q} \in \mathcal{Q}} E^{\widehat{Q}}[\widetilde{L}(Y_n)] \leq r_n. \tag{4.39}$$

Take $Q \in \mathcal{Q}$. Since from $Y_n \to Y$ in the norm topology of $L^\infty$ it follows $Y_n \to Y$ $P-a.s.$ (see Section 4.3 in [12]), we obtain that $\widetilde{L}(Y_n)$ converges $P-a.s.$ to $\widetilde{L}(Y)$. Since for all $Y \in L^\infty_+$ it holds $0 \leq \widetilde{L}(Y) \leq L(H)$ because of Assumption 4.18 and $H \in L^1_+$. It follows that $|\widetilde{L}(Y_n)|$ is dominated by $L(H)$. Because of Assumption 4.20, $L(H)$ is integrable with respect to $Q \in \mathcal{Q}$. Thus, we can apply Corollary B.20 and obtain together with (4.39)

$$\forall Q \in \mathcal{Q} : \quad E^Q[\widetilde{L}(Y)] = \lim_{n\to\infty} E^Q[\widetilde{L}(Y_n)] \leq r.$$

Since $L^\infty_+$ is a closed set, $Y \in L^\infty_+$. It follows that $\rho(Y) = \sup_{Q \in \mathcal{Q}} E^Q[\widetilde{L}(Y)] \leq r$. Thus, $(Y, r) \in \operatorname{epi}\rho$, hence $\rho$ is lower semicontinuous.

Since $\rho$ is lower semicontinuous and convex and because of $L^\infty$ endowed with the norm topology is a Banach space, $\rho$ is continuous in the interior of its domain (see [11], Corollary 2.5). This means, $\rho$ is continuous in $\operatorname{int}(L^\infty_+) \neq \emptyset$ (see Lemma B.13). Take $\varphi_0 \in \operatorname{int}(L^\infty_+)$ with $\varphi_0 \equiv c$, $c \in (0,1)$ such that $cU_0 \leq \widetilde{V}_0$. Such a constant $c$ always exists. Then $\sup_{P^* \in \mathcal{P}} E^{P^*}[\varphi_0 H] \leq cU_0 \leq \widetilde{V}_0$. Thus, $\varphi_0 \in R_0$ and $\rho$ is continuous and finite in $\varphi_0$. □

**Proposition 4.31.** *Suppose Assumption 4.18 and 4.20 hold. Then, the function $\rho$ defined in (4.37) is lower semicontinuous with respect to the weak* topology.*

*Proof.* The proof is similar to the proof of the lower semicontinuity of $\rho$ in Proposition 4.30. We show that $\operatorname{epi}\rho$ is closed with respect to $P-a.s.$ convergent sequences. Take a sequence $\{(Y_n, r_n)\}_{n\in\mathbb{N}} \subset \operatorname{epi}\rho$ with $Y_n \to Y$ $P-a.s$ and $r_n \to r$. Thus, for all $n \in \mathbb{N}$ it holds $Y_n \in L^\infty_+$ with $\rho(Y_n) \leq r_n$. Then

$$\forall n \in \mathbb{N}, \forall Q \in \mathcal{Q}: \quad E^Q[\widetilde{L}(Y_n)] \leq \sup_{\widehat{Q}\in\mathcal{Q}} E^{\widehat{Q}}[\widetilde{L}(Y_n)] \leq r_n. \tag{4.40}$$

Since $|\widetilde{L}(Y_n)|$ is dominated by the $Q-$integrable function $L(H)$ for all $n \in \mathbb{N}$, we can apply Corollary B.20 and obtain together with (4.40)

$$\forall Q \in \mathcal{Q}: \quad E^Q[\widetilde{L}(Y)] = \lim_{n\to\infty} E^Q[\widetilde{L}(Y_n)] \leq r.$$

Since $L^\infty_+$ is closed with respect to $P - a.s.$ convergent sequences, $Y \in L^\infty_+$. Hence,

$$\rho(Y) = \sup_{Q\in\mathcal{Q}} E^Q[\widetilde{L}(Y)] \leq r.$$

Thus, $\operatorname{epi}\rho$ is closed with respect to $P-a.s.$ convergent sequences. Since $\rho$ is convex, we can apply Theorem 1.7 and obtain that $\rho$ is lower semicontinuous with respect to the weak* topology. □

Since we work with the modified problem (4.38), we cannot apply Theorem 4.9 and have to work directly with the results of Chapter 2. The modified problem (4.38) turns out to be a special case of optimization problem (2.1) by setting $A\varphi = \varphi$, $b = 0$, $C^* = Z_{\mathcal{P}}$, $c = \widetilde{V}_0$ and $\mathcal{X} = \mathcal{Y} = L^\infty$, endowed with the norm topology. Hence, $\mathcal{X}^* = \mathcal{Y}^* = ba(\Omega, \mathcal{F}, P)$. $H \in L^1_+$ is the payoff of the contingent claim. It holds $\mathcal{X}_1 = R$ and $\mathcal{X}_0 = R_0$. We check (A1)-(A7) of Assumption 2.1:

(A1): We have $c = \widetilde{V}_0 > 0$ (see (4.3)).

(A2): It holds $H \in L^1_+, C^* = Z_{\mathcal{P}} \subseteq L^1$ and $\{HX^* : X^* \in C^*\} = \{HP^* : Z_{P^*} \in \mathcal{P}\} \subseteq L^1$, since we assumed in (4.1) the superhedging price of $H$ to be finite.

(A3): Inequality (4.1) also ensures that $\sup_{X^*\in C^*} \|HX^*\|_{L^1} < +\infty$.

(A4): The operator $A : L^\infty \to L^\infty$, defined by $A\varphi := \varphi$, is linear and continuous.

(A5): Proposition 4.30 ensures the conditions for $\rho$.

(A6): The map $\varphi \mapsto \rho(A\varphi + b)$ coincides with $\rho(\varphi)$ and is lower semicontinuous in the weak* topology as proved in Proposition 4.31.

Note that Assumption (A7) is not satisfied, in general: The map $\varphi \mapsto \langle Y^*, A\varphi\rangle = \langle Y^*, \varphi\rangle$ is, in general, not continuous in the weak* topology for all $Y^* \in \operatorname{dom}\rho^*$ since $\operatorname{dom}\rho^* \subseteq ba(\Omega, \mathcal{F}, P)$.

**Remark 4.32.** One could think of endowing $\mathcal{Y} = L^\infty$ with the weak* topology since we already proved that $\rho$ is weakly* lower semicontinuous (Proposition 4.31). Then, Assumption (A7) is satisfied since $\operatorname{dom}\rho^* \subseteq \mathcal{Y}^* = L^1$. But in this case, we can not ensure the continuity of $\rho$ in $\varphi_0 \in R_0$ as postulated in (A5) since $\operatorname{int} L^\infty_+ = \emptyset$ with respect to the weak* topology (Lemma B.14, cf. proof of Proposition 4.30).

It would be sufficient for the application of Theorem 2.9 to postulate that $\widetilde{Y}^*$, the solution to the dual problem, is an element of $L^1$ and thus $\varphi \mapsto \langle \widetilde{Y}^*, \varphi\rangle$ is continuous in the weak* topology. But, in general, this condition is not satisfied.
In the following, we show which results in the theorems of Chapter 2 do not longer hold since Assumption (A7) is not satisfied and we shall give proposals how the problem could be solved in spite of this. Assumption (A7) has no impact on Theorem 2.5 and 2.6.
Since $\rho$ is lower semicontinuous, convex and monotone with $\rho(\mathbf{0}) < +\infty$ (Proposition 4.30), it has a dual representation with respect to elements of $\mathcal{Y}^*_+ = ba(\Omega, \mathcal{F}, P)_+$ (see Theorem 1.5 (b)). Since $\rho$ is weakly* lower semicontinuous (Proposition 4.31), it is sufficient to consider elements of $L^1_+$ in the dual representation (see Theorem 1.6)

$$\rho(Y) = \sup_{Y^* \in ba(\Omega,\mathcal{F},P)_+} \{\langle Y, -Y^*\rangle - \rho^*(-Y^*)\} = \sup_{Y^* \in L^1_+} \{E[-YY^*] - \rho^*(-Y^*)\}.$$

Then, the primal problem (4.38) can be written as

$$\min_{\varphi \in R_0} \{ \sup_{Y^* \in L^1_+} \{E[-\varphi Y^*] - \rho^*(-Y^*)\}\},$$

where Theorem 2.5 ensures the existence of a primal solution $\widetilde{\varphi}$. The continuity of $\rho$ in some $\varphi_0 \in R_0$ (Proposition 4.30) ensures strong duality between the primal and its dual problem (Theorem 2.6) with respect to $\mathcal{Y}^* = ba(\Omega, \mathcal{F}, P)$, i.e.,

$$\min_{\varphi \in R_0} \{ \sup_{Y^* \in L^1_+} \{E[-\varphi Y^*] - \rho^*(-Y^*)\}\} = \sup_{Y^* \in ba(\Omega,\mathcal{F},P)_+} \{ \inf_{\varphi \in R_0} \{\langle \varphi, -Y^*\rangle - \rho^*(-Y^*)\}\}.$$

Strong duality also ensures the existence of a dual solution $\widetilde{Y}^* \in ba(\Omega, \mathcal{F}, P)_+$ and with equation (2.6) of Theorem 2.6 we obtain

$$\min_{\varphi \in R_0} \{ \sup_{Y^* \in L^1_+} \{E[-\varphi Y^*] - \rho^*(-Y^*)\}\} = \max_{Y^* \in ba(\Omega,\mathcal{F},P)_+} \{ \min_{\varphi \in R_0} \{\langle \varphi, -Y^*\rangle - \rho^*(-Y^*)\}\}. \tag{4.41}$$

In Theorem 2.8 we considered the inner problem of the dual problem for every $Y^* \in \mathcal{Y}^*$, but it is sufficient to consider the inner problem just for $\widetilde{Y}^* \in ba(\Omega, \mathcal{F}, P)_+$, i.e., the problem

$$\max_{\varphi \in R_0} \langle \varphi, \widetilde{Y}^* \rangle, \tag{4.42}$$

where (2.6) of Theorem 2.6 ensures the existence of a solution $\widetilde{\varphi}$. The dual problem of (4.42) is

$$\inf_{\lambda \in \Lambda_+} \Big\{ \sup_{\varphi \in R} \langle \widetilde{Y}^* - H \int\limits_{\mathcal{P}} Z_{P^*} d\lambda, \varphi \rangle - c\lambda(Z_{\mathcal{P}}) \Big\}, \tag{4.43}$$

where $\Lambda_+$ is the set of all finite measures on $(Z_{\mathcal{P}}, \mathcal{B})$ and $\mathcal{B}$ is a $\sigma$-algebra of all Borel sets on $Z_{\mathcal{P}}$. Assumption (A7) does not have an impact on the validity of strong duality between (4.42) and (4.43) as in the proof of Theorem 2.8

$$\max_{\varphi \in R_0} \langle \varphi, \widetilde{Y}^* \rangle = \min_{\lambda \in \Lambda_+} \Big\{ \sup_{\varphi \in R} \langle \widetilde{Y}^* - H \int\limits_{\mathcal{P}} Z_{P^*} d\lambda, \varphi \rangle - c\lambda(Z_{\mathcal{P}}) \Big\}. \tag{4.44}$$

The validity of strong duality also ensures the existence of a dual solution $\widetilde{\lambda} \in \Lambda_+$ (Theorem A.12).

Assumption (A2) and (A3) ensure that $H \int_{\mathcal{P}} Z_{P^*} d\lambda \in L^1$ and thus, the signed measure with density $H \int_{\mathcal{P}} Z_{P^*} d\lambda$ admits a Hahn decomposition for $\lambda \in \Lambda_+$. For simplicity, we write $Y$ admits a Hahn decomposition instead of the measure with density $Y$ admits a Hahn decomposition. With Assumption (A7) we wanted to ensure that $\widetilde{Y}^*$ is an element of $L^1$ and thus, the whole term $\widetilde{Y}^* - H \int_{\mathcal{P}} Z_{P^*} d\lambda$ admits a Hahn decomposition which could be used in (4.44) (as it was in (2.22) in the proof of Theorem 2.8) and would lead to a result about the structure of a solution $\widetilde{\varphi}$ (see Theorem 2.9).

This makes clear that it is possible to weaken Assumption (A7) as proposed in Remark 2.3:

(A7') $A^* \widetilde{Y}^*$ admits a Hahn decomposition.

At this point, we give several proposals of further research that could lead to a possibility to solve the modified problem (4.38) and thus, also the original problem (4.33) or that work directly with problem (4.33).

- **Does $\widetilde{Y}^*$ admit a Hahn decomposition?** $\widetilde{Y}^*$ is an element of $ba(\Omega, \mathcal{F}, P)_+$, a nonnegative, finitely additive set functions on $(\Omega, \mathcal{F})$ with bounded variation, absolutely continuous to $P$ (see [49], Chapter IV, 9, Example 5). In [5] it was shown, that a bounded finitely additive real-valued measure $\widetilde{Y}^*$ admits a Hahn decomposition if and only if it attains its norm on the unit ball of $L^\infty$. This is equivalent to the condition, that $\widetilde{Y}^*$ attains its bounds (see [40]). Schmidt [43], Lemma 2.1, showed that it is sufficient to show that the upper bound is

attained.
In (4.42) we see that $\widetilde{Y}^*$ attains its supremum over the set $R_0$ (follows from (2.6) of Theorem 2.6). If it also attains its supremum over the set $R$, and thus over the unit ball in $L^\infty$ (since $\widetilde{Y}^* \in ba(\Omega, \mathcal{F}, P)_+$), $\widetilde{Y}^*$ would admit a Hahn decomposition ([5], Theorem 1) and we could solve the problem analogously to Theorem 2.9.

- **Approximation.** If $\widetilde{Y}^*$ does not admit a Hahn decomposition, $\widetilde{Y}^*$ could be approximated by a sequence $\widetilde{Y}_n^* \in ba(\Omega, \mathcal{F}, P)$, where $\widetilde{Y}_n^*$ admits a Hahn decomposition for all $n \in \mathbb{N}$ or, if possible, even with a sequence $\widetilde{Y}_n^* \in L^1$. For every $\widetilde{Y}_n^* \in L^1$, there exists a solution $\widetilde{\varphi}_{\widetilde{Y}_n^*}$ to

$$\max_{\varphi \in R_0} \langle \varphi, \widetilde{Y}_n^* \rangle,$$

since $R_0$ is weakly* compact and the map $\varphi \mapsto \langle \varphi, \widetilde{Y}_n^* \rangle$ is weakly* continuous for $\widetilde{Y}_n^* \in L^1$. Then, one has to analyze the behavior of the sequence $\widetilde{\varphi}_{\widetilde{Y}_n^*}$ and to check if it converges to the solution $\widetilde{\varphi}$.
Furthermore, another possible approximation can be discussed. From (4.41) and Proposition 4.31 it follows that

$$\begin{aligned} \rho(\widetilde{\varphi}) &= \langle \widetilde{\varphi}, -\widetilde{Y}^* \rangle - \rho^*(-\widetilde{Y}^*) = \max_{Y^* \in ba(\Omega, \mathcal{F}, P)_+} \{\langle \widetilde{\varphi}, -Y^* \rangle - \rho^*(-Y^*)\} \\ &= \sup_{Y^* \in L^1_+} \{E[-\widetilde{\varphi} Y^*] - \rho^*(-Y^*)\}. \end{aligned}$$

Hence, there exists a maximizing sequence $Y_n^* \in L^1_+$ such that $E[-\widetilde{\varphi} Y_n^*] - \rho^*(-Y_n^*)$ converges to $\langle \widetilde{\varphi}, -\widetilde{Y}^* \rangle - \rho^*(-\widetilde{Y}^*)$. Again, for every $Y_n^* \in L^1$, there exists a solution $\widetilde{\varphi}_{Y_n^*}$ to

$$\max_{\varphi \in R_0} \langle \varphi, Y_n^* \rangle$$

and one could analyze the behavior of the sequence $\widetilde{\varphi}_{Y_n^*}$

- **Is a weaker condition for strong duality satisfied?** Consider problem (4.33) and check if a weaker condition than the continuity in $(\varphi_0 - 1)H$ with $\varphi_0 \in R_0$ leads to a strong duality result (see [50], Theorem 2.7.1 for a list of conditions that lead to strong duality).

It seems to be worthwhile to do further research in this direction since for special cases there already exist results in the literature. In Föllmer and Leukert [17] the efficient hedging problem was considered. This is a special case of problem (4.31) with $\mathcal{Q} = \{P\}$, a singleton. For this case, the decomposition of the dynamic problem into the static problem and the representation problem (cf. Theorem 4.1) was

already proved and the existence of a solution to the static problem was shown. Furthermore, a solutions for the linear case $l(x) = x$ in the complete market $\mathcal{P} = \{P^*\}$ was deduced by an application of the Neyman-Pearson lemma. The linear case $l(x) = x$ in the incomplete market was solved by Xu [48]. These special cases can be solved with the method deduced in this thesis as well (see also Section 4.2.3). It is even possible to solve the problem for the more general case of Lipschitz continuous loss functions and with a more general set $\mathcal{Q}$ satisfying Assumption 4.22. Interesting for further research are more general loss functions, for instance the function $l(x) = (x^+)^p, p \geq 1$.
Kirch [26] considered the general robust efficient hedging problem (4.31) with the following assumptions concerning the loss function $l$. In [26], $l(x, \omega)$ was assumed to be strictly convex, increasing, continuous differentiable on $(0, H)$ and bounded for all $x \geq 0$. It is then possible to express the solution in terms of the inverse of the derivative of the utility function $u := -\widetilde{l}$. The problem was solved by enlarging the sets $\mathcal{Q}$ and $\mathcal{P}$ by passing to the closed convex hull of the densities of $\mathcal{Q}$ in $L^1$ and to the closure of the densities of $\mathcal{P}$ in $L^0$. The solution to the problem could be reduced to a solution to a simple problem (fixed $Q \in \overline{\text{co}}\mathcal{Q}$ and fixed $P^*$ in the closure of the densities of $\mathcal{P}$). In some cases only an approximation of the solution by a sequence of simple problems was possible. These results motivate further research in this area using the method deduced in this thesis.

## 4.2 Hedging in Incomplete Markets

In this section, we study the problem of hedging in incomplete markets in the general case, i.e., we only assume that the set of equivalent martingale measures satisfies $\mathcal{P} \neq \emptyset$ due to absence of arbitrage opportunities. We do no longer impose compactness of $Z_{\mathcal{P}}$. Let us consider problem (4.2), (4.3), i.e., the dynamic optimization problem of finding an admissible strategy that solves

$$\min_{(\widetilde{V}_0, \xi)} \rho\left(-\left(H - V_T\right)^+\right), \quad 0 < \widetilde{V}_0 < U_0. \tag{4.45}$$

We summarize the assumptions of this section.

**Assumption 4.33.**

- $\mathcal{P} \neq \emptyset$.
- *The payoff of the contingent claim satisfies* $H \in L^1_+$.
- *The superhedging price of* $H$ *is finite, i.e.,* $U_0 = \sup_{P^* \in \mathcal{P}} E^{P^*}[H] < +\infty$.

**Assumption 4.34.** *The risk measure* $\rho : L^1 \to \mathbb{R} \cup \{+\infty\}$ *is a monotone, convex, lower semicontinuous function that is continuous and finite in some* $(\varphi_0 - 1)H$ *with* $\varphi_0 \in R_0$ *and satisfies* $\rho(0) < +\infty$.

We can apply all theorems of Chapter 2 that do not need Assumption (A3) (compactness of $Z_{\mathcal{P}}$). Since $\rho$ is monotone, we can apply Theorem 4.1 and obtain that the corresponding static optimization problem is

$$\min_{\varphi \in R_0} \rho\left((\varphi - 1)H\right), \tag{4.46}$$

$$R_0 = \{\varphi : \ \Omega \to [0,1], \ \mathcal{F}_T - \text{measurable}, \ \sup_{P^* \in \mathcal{P}} E^{P^*}[\varphi H] \le \widetilde{V}_0\}.$$

Theorem 2.5 ensures the existence of a solution $\widetilde{\varphi}$ to (4.46) and the dual representation of $\rho$ (Theorem 1.5 (b)) enables us to rewrite (4.46) as follows

$$p = \min_{\varphi \in R_0} \rho\left((\varphi - 1)H\right) = \min_{\varphi \in R_0} \{ \sup_{Y^* \in L_+^\infty} \{E[(1-\varphi)HY^*] - \rho^*(-Y^*)\}\}.$$

Theorem 2.6 ensures strong duality between (4.46) and its Fenchel dual problem

$$\sup_{Y^* \in L_+^\infty} \{ \inf_{\varphi \in R_0} \{E[(1-\varphi)HY^*] - \rho^*(-Y^*)\}\} \tag{4.47}$$

and ensures the existence of a saddle point, i.e.,

$$\min_{\varphi \in R_0} \{ \max_{Y^* \in L_+^\infty} \{E[(1-\varphi)HY^*] - \rho^*(-Y^*)\}\} = \max_{Y^* \in L_+^\infty} \{ \min_{\varphi \in R_0} \{E[(1-\varphi)HY^*] - \rho^*(-Y^*)\}\}.$$

It is no longer possible to solve the inner problem of the dual problem with the theory deduced in Chapter 2 (Theorem 2.8) since $Z_{\mathcal{P}}$ is not assumed to be compact. In this section, we want to solve the inner problem with the method in Xu [48] that is based on a duality approach deduced by Kramkov and Schachermayer [27]. Let us consider the inner problem of the dual problem for a fixed $Y^* \in L_+^\infty$

$$\min_{\varphi \in R_0} E[(1-\varphi)HY^*]. \tag{4.48}$$

The existence of a solution $\widetilde{\varphi}_{Y^*}$ to (4.48) follows from Lemma 2.7. Problem (4.48) is the static problem to the dynamic problem of finding an admissible strategy that minimizes

$$\min_{(\widetilde{V}_0, \xi)} E[(H - V_T)^+ Y^*], \quad 0 < \widetilde{V}_0 < U_0. \tag{4.49}$$

This problem was solved in [48] for the case $Y^* = \mathbf{1}$. We want to adopt the method to our case. Therefore, we introduce the set of admissible, self-financing value processes $V$ starting at initial capital $x > 0$

$$\mathcal{V}(x) := \{V : V_t = x + \int_0^t \xi_s dS_s \ge 0, \quad t \in [0,T]\}$$

and the set of contingent claims super-replicable by some admissible self-financing strategies with initial capital $x$

$$\mathcal{C}(x) := \{g \in L^0(\Omega, \mathcal{F}, P) : 0 \le g \le V_T \text{ for some } V \in \mathcal{V}(x)\}.$$

We consider the state dependent utility function $U : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$

$$U(x,\omega) := H(\omega)Y^*(\omega) - (H(\omega) - x)^+ Y^*(\omega) = (H(\omega) \wedge x)Y^*(\omega) \tag{4.50}$$

and the primal problem for $x > 0$

$$\begin{aligned} u(x) &= \sup_{V \in \mathcal{V}(x)} E[U(V_T(\omega),\omega)] \\ &= \sup_{g \in \mathcal{C}(x)} E[U(g(\omega),\omega)] = \sup_{g \in \mathcal{C}(x)} E[(H \wedge g)Y^*]. \end{aligned} \tag{4.51}$$

If necessary, we use the notation $u_{Y^*}(x)$ to emphasize the dependence from the selected $Y^* \in L^\infty_+$. Note that the problem $u(\widetilde{V}_0)$ is equivalent to problem (4.49) in the sense that if $\widetilde{g} \in \mathcal{C}(\widetilde{V}_0)$ is a solution to (4.51) for $x = \widetilde{V}_0$, then the admissible self-financing superhedging strategy $(\widetilde{V}_0, \widetilde{\xi})$ of $\widetilde{g}$ solves (4.49), where $\widetilde{\xi}$ is obtained by the optional decomposition theorem (Theorem C.3). Furthermore, it holds (with (4.6)) that

$$-u(\widetilde{V}_0) + E[HY^*] = \min_{(\widetilde{V}_0,\xi)} E[(H - V_T)^+ \, Y^*] = \min_{\varphi \in R_0} E[(1-\varphi)HY^*]. \tag{4.52}$$

As in [48] and [27], we define the following set of processes $Y$

$$\mathcal{Y}(y) := \{Y \geq 0 : Y_0 = y \text{ and } VY \text{ is a } P\text{-supermartingale for any } V \in \mathcal{V}(1)\}$$

and the set $\mathcal{D}(y)$ of random variables $h$ by

$$\mathcal{D}(y) := \{h \in L^0(\Omega, \mathcal{F}, P) : 0 \leq h \leq Y_T \text{ for some } Y \in \mathcal{Y}(y)\}.$$

The dual relation between $\mathcal{C}(1)$ and $\mathcal{D}(1)$ (or equivalently between $\mathcal{V}(1)$ and $\mathcal{Y}(1)$) is for instance shown in [27], Proposition 3.1.
Let us consider the function $W : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ defined by

$$W(y,\omega) := \sup_{x \geq 0}\{U(x,\omega) - xy\}$$

for $y \geq 0$. It holds $W(y,\omega) = (-U + \mathcal{I}_{\mathbb{R}_+})^*(-y,\omega)$ for each $\omega \in \Omega$ (see Definition A.3 for the definition of the conjugate function). With (4.50) and because of $W(0,\omega) \geq U(0,\omega) = 0$ we obtain

$$W(y,\omega) = (Y^*(\omega) - y)^+ H(\omega). \tag{4.53}$$

We assign to (4.51) the following dual problem

$$\begin{aligned} w(y) &= \inf_{Y \in \mathcal{Y}(y)} E[W(Y_T(\omega),\omega)] \\ &= \inf_{h \in \mathcal{D}(y)} E[W(h(\omega),\omega)] = \inf_{h \in \mathcal{D}(y)} E[(Y^* - h)^+ H]. \end{aligned} \tag{4.54}$$

The utility function $U(\cdot,\omega)$ and the value function $u$ are concave, continuous and increasing. The functions $W(\cdot,\omega)$ and $w$ are convex, continuous and decreasing. For the definition of the subdifferential of a convex function we refer to Definition A.13 and for that of a concave function we refer to Definition A.14. For fixed $\omega \in \Omega$ we consider the function $U : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $U(g) := U(g(\omega),\omega)$ for $g \in \mathcal{C}(x)$. The subdifferential $\partial U(g)$ is then understood for each $\omega \in \Omega$ in the sense that $h \in \partial U(g)\ P - a.s.$ Analogously we define the function $W(h) : \mathbb{R}_+ \to \mathbb{R}_+$ for $h \in \mathcal{D}(y)$ and the subdifferential $\partial W(h)$. The following duality theorem holds true.

**Theorem 4.35.** *Let Assumptions 4.33 be satisfied. Then, it holds:*
*(i) For $x > 0$ and $y > 0$ an optimal solution $\widetilde{g}(x) \in \mathcal{C}(x)$ to (4.51) exists and an optimal solution $\widetilde{h}(y) \in \mathcal{D}(y)$ to (4.54) exists.*
*(ii) The value functions $u$ and $w$ satisfy the following relationship*

$$\begin{aligned} w(y) &= \sup_{x>0}\{u(x) - xy\} \quad \text{for any } y > 0 \text{ and} \\ u(x) &= \inf_{y>0}\{w(y) + xy\} \quad \text{for any } x > 0. \end{aligned} \tag{4.55}$$

*(iii) Let $x > 0$ and $y > 0$ such that $y \in \partial u(x)$. Then, $E[\widetilde{g}\widetilde{h}] = xy$ and $\widetilde{h} \in \partial U(\widetilde{g})\ P - a.s.$, or equivalently, $\widetilde{g} \in -\partial W(\widetilde{h})\ P - a.s.$ if and only if $\widetilde{g}$ solves (4.51) and $\widetilde{h}$ solves (4.54).*

*Proof.* The assumptions of Theorem C.7 are satisfied since $\mathcal{P} \neq \emptyset$, $U(\cdot,\omega)$ is continuous, increasing and concave for any fixed $\omega$ and $U(0,\omega) = 0$. Furthermore, the right-hand derivative satisfies $U^r(0,\omega) \geq 0$ (cf. Remark C.8) and $U^r(\infty,\omega) = \lim_{x\to\infty} U^r(x,\omega) = 0$. Since $Y^* \in L^\infty_+$ and $H \in L^1_+$, is holds $U(x,\omega) \leq H(\omega)Y^*(\omega)\ P-a.s.$ for all $x \geq 0$ and $HY^* \in L^1$ since $E[HY^*] \leq \|Y^*\|_{L^\infty}\|H\|_{L^1} < +\infty$. Then, the assertion of the theorem follows from Theorem C.7. □

**Remark 4.36.** Note that the relationship in Theorem 4.35 (ii) means that $w(y) = (-u + \mathcal{I}_{\mathbb{R}_{>0}})^*(-y)$ and $u(x) = -(w + \mathcal{I}_{\mathbb{R}_{>0}})^*(-x)$.
Let us consider the condition $x > 0$ and $y > 0$ such that $y \in \partial u(x)$ in Theorem 4.35 (iii). It holds (cf. [48])

$$y \in \partial u(x) \Leftrightarrow u(x) = w(y) + xy,$$

which means that the infimum in (4.55) is attained.
For $x > 0$, we have $\partial u(x) \neq \emptyset$ since $u$ is continuous in the interior of its effective domain (see [11], Corollary I.2.3) and for all $y \in \partial u(x)$ it holds $y \geq 0$.

The structure of a primal solution with respect to a dual solution can be deduced as follows.

**Theorem 4.37.** *Let Assumption 4.33 be satisfied. Let $x > 0$ and $y > 0$ such that $y \in \partial u(x)$. Let $\widetilde{h}(y) \in \mathcal{D}(y)$ be an optimal solution to (4.54). Then, there is an optimal solution $\widetilde{g}(x)$ to (4.51) such that*

$$\widetilde{g} = (1_{\{0 \leq \widetilde{h} < Y^*\}} + \delta 1_{\{\widetilde{h} = Y^*\}})H$$

*and*

$$E[\widetilde{g}\widetilde{h}] = xy,$$

*where $\delta$ is a $[0,1]$-valued random variable.*

*Proof.* Let $x > 0$ and $y > 0$ such that $y \in \partial u(x)$. From Theorem 4.35 the existence of an optimal solution $\widetilde{g}(x) \in \mathcal{C}(x)$ to (4.51) and $\widetilde{h}(y) \in \mathcal{D}(y)$ to (4.54) follows. Furthermore, it holds $E[\widetilde{g}\widetilde{h}] = xy$ and $\widetilde{g} \in -\partial W(\widetilde{h})$ $P-a.s.$ It holds (cf. [48], page 8) that

$$\widetilde{g} \in -\partial W(\widetilde{h}) \Leftrightarrow W(\widetilde{h}) = U(\widetilde{g}) - \widetilde{g}\widetilde{h}.$$

With (4.50) and (4.53) this becomes

$$\widetilde{g} \in -\partial W(\widetilde{h}) \Leftrightarrow (Y^* - \widetilde{h})^+ H = (H \wedge \widetilde{g})Y^* - \widetilde{g}\widetilde{h}.$$

It follows

$$-\partial W(\widetilde{h}) = \begin{cases} 0 & \text{if } \widetilde{h} > Y^* \\ H & \text{if } 0 < \widetilde{h} < Y^* \\ [H, \infty) & \text{if } \widetilde{h} = 0 \\ [0, H] & \text{if } \widetilde{h} = Y^*. \end{cases}$$

Thus, $\widetilde{g} = (1_{\{0 \leq \widetilde{h} < Y^*\}} + \delta 1_{\{\widetilde{h} = Y^*\}})H \in -\partial W(\widetilde{h})$ $P-a.s.$ is an optimal solution to (4.51), where $\delta$ is an $[0,1]$-valued random variable such that $E[\widetilde{g}\widetilde{h}] = xy$ is satisfied. □

To emphasis the dependence of the value function $u$ and the solutions $\widetilde{g}$ and $\widetilde{h}$ from the selected $Y^* \in L^\infty_+$, we use the notation $u_{Y^*}, \widetilde{g}_{Y^*}$ and $\widetilde{h}_{Y^*}$.
Let the initial capital be $x = \widetilde{V}_0$. We conclude that the optimal solution $\widetilde{g}_{Y^*}(\widetilde{V}_0)$ to (4.51) can be written as $\widetilde{g}_{Y^*} = \widetilde{\varphi}_{Y^*}H$, where $\widetilde{\varphi}_{Y^*} = 1_{\{0 \leq \widetilde{h} < Y^*\}} + \delta 1_{\{\widetilde{h} = Y^*\}} \in R_0$ is the solution to (4.48).
Now, we are ready to go back to the static optimization problem (4.46) and to deduce a result about the structure of its solution.

**Theorem 4.38** (Solution to the Generalized Hedging Problem)**.** *Let Assumption 4.33 and 4.34 be satisfied. There exists a solution $\widetilde{\varphi}$ to problem (4.46) and a solution $\widetilde{Y}^* \in L^\infty_+$ to problem (4.47).*

*Let* $\widetilde{y} > 0$ *such that* $\widetilde{y} \in \partial u_{\widetilde{Y}^*}(\widetilde{V}_0)$. *Then, there exists an* $\widetilde{h} \in \mathcal{D}(\widetilde{y})$ *such that the triple* $(\widetilde{Y}^*, \widetilde{y}, \widetilde{h}) \in (L^\infty_+ \times \mathbb{R}_{>0} \times \mathcal{D}(y))$ *solves*

$$\max_{Y^* \in L^\infty_+, y>0, h \in \mathcal{D}(y)} \{E[(Y^* \wedge h)H] - \widetilde{V}_0 y - \rho^*(-Y^*)\}. \tag{4.56}$$

*It follows that:*

- *The solution to (4.46) is*

$$\widetilde{\varphi} = \begin{cases} 1 & : \quad 0 \leq \widetilde{h} < \widetilde{Y}^* \\ 0 & : \quad \widetilde{h} > \widetilde{Y}^* \end{cases} \qquad P - a.s.$$

  *with*

$$E[\widetilde{\varphi} H \widetilde{h}] = \widetilde{V}_0 \widetilde{y}.$$

- $(\widetilde{\varphi}, \widetilde{Y}^*)$ *is the saddle point of the functional* $(\varphi, Y^*) \mapsto E[(1-\varphi)HY^*] - \rho^*(-Y^*)$ *in* $R_0 \times L^\infty_+$.

- $(\widetilde{V}_0, \widetilde{\xi})$ *solves the dynamic hedging problem (4.45), where* $\widetilde{\xi}$ *is the superhedging strategy of the modified claim* $\widetilde{\varphi} H$*, obtained by the optional decomposition theorem (Theorem C.3).*

*Proof.* Theorem 2.5 ensures the existence of a solution $\widetilde{\varphi}$ to (4.46). Consider the dual problem of (4.46) given in (4.47), where Theorem 2.6 ensures that the supremum with respect to $Y^* \in L^\infty_+$ and the infimum with respect to $\varphi \in R_0$ are attained. We obtain

$$\begin{aligned}
\max_{Y^* \in L^\infty_+} \min_{\varphi \in R_0} \{E[(1-\varphi)HY^*] - \rho^*(-Y^*)\} &\overset{(4.52)}{=} \max_{Y^* \in L^\infty_+} \{-u_{Y^*}(\widetilde{V}_0) + E[HY^*] - \rho^*(-Y^*)\} \\
&\overset{(4.55)}{=} \max_{Y^* \in L^\infty_+} \{-\min_{y>0} \{w(y) + \widetilde{V}_0 y\} + E[HY^*] - \rho^*(-Y^*)\} \\
&\overset{(4.54)}{=} \max_{Y^* \in L^\infty_+} \{-\min_{y>0} \{\min_{h \in \mathcal{D}(y)} E[(Y^* - h)^+ H] + \widetilde{V}_0 y\} + E[HY^*] - \rho^*(-Y^*)\} \\
&= \max_{Y^* \in L^\infty_+, y>0, h \in \mathcal{D}(y)} \{E[(Y^* \wedge h)H] - \widetilde{V}_0 y - \rho^*(-Y^*)\}.
\end{aligned}$$

With Theorem 2.6 it follows that $\widetilde{Y}^*$ attains the maximum with respect to $Y^* \in L^\infty_+$. Remark 4.36 shows that the condition $\widetilde{y} > 0$ such that $\widetilde{y} \in \partial u_{\widetilde{Y}^*}(\widetilde{V}_0)$ ensures that $\widetilde{y}$ attains the above infimum with respect to $y > 0$. Theorem 4.35 shows that $\widetilde{h} := \widetilde{h}_{\widetilde{Y}^*}(\widetilde{y}) \in \mathcal{D}(\widetilde{y})$ attains the above infimum with respect to $h \in \mathcal{D}(\widetilde{y})$. Thus, there exists a triple $(\widetilde{Y}^*, \widetilde{y}, \widetilde{h}) \in (L^\infty_+ \times \mathbb{R}_{>0} \times \mathcal{D}(y))$ solving (4.56). The application of Theorem 4.37 with $Y^* = \widetilde{Y}^*$ leads to the result about the structure of $\widetilde{\varphi}$. It follows that $(\widetilde{\varphi}, \widetilde{Y}^*)$ is the saddle point described in Theorem 2.6. Theorem 4.1 shows that $(\widetilde{V}_0, \widetilde{\xi})$ solves the dynamic hedging problem (4.45), where $\widetilde{\xi}$ is the superhedging strategy of the modified claim $\widetilde{\varphi} H$ obtained by the optional decomposition theorem (Theorem C.3). $\square$

If we compare Theorem 4.38 (general incomplete market) with Theorem 4.9 (complete and special incomplete markets), we see that both lead to a structural result of the solution $\widetilde{\varphi}$ to the static optimization problem (4.4). If $Z_{\mathcal{P}}$ is compact, the 0-1-structure of an optimal randomized test $\widetilde{\varphi}$ can be deduced with elements from $\mathcal{P}$ and elements from the representing set of the risk measure ($L^\infty_+$ or $\mathcal{Q}$, this depends on the choice of the risk measure). In the general case, this is not possible any longer. The 0-1-structure of an optimal randomized test $\widetilde{\varphi}$ can be deduced as well with elements from the representing set $L^\infty_+$ (respectively $\mathcal{Q}$), but no longer with elements from $\mathcal{P}$. We have to pass to a larger set $\mathcal{D}(y)$ which is a subset of $L^0_+$. Thus, in the case where $Z_{\mathcal{P}}$ is compact, we can deduce a more detailed result about the structure of $\widetilde{\varphi}$.
When special risk measures $\rho$ as in Section 4.1 are considered, the results are analogously to Theorem 4.38. A special choice of $\rho$ has an impact on the optimization problem (4.56) regarding the set, the solution $\widetilde{Y}^*$ is attained in, and on the representation of the conjugate function $\rho^*$ of $\rho$.

### 4.2.1 Convex Hedging

If $\rho : L^1 \to \mathbb{R} \cup \{+\infty\}$ satisfies additionally to Assumption 4.34 the translation property, it forms a convex risk measure as in Assumption 4.12. Then, $\rho$ admits the dual representation (see Theorem 1.16)

$$\rho(Y) = \sup_{Q \in \mathcal{Q}} \{E^Q[-Y] - \sup_{\widetilde{Y} \in \mathcal{A}_\rho} E^Q[-\widetilde{Y}]\}, \tag{4.57}$$

where $\mathcal{Q} := \{Q \in \widehat{\mathcal{Q}} : Z_Q \in L^\infty\}$ is the set of all probability measures $Q$, absolutely continuous to $P$ and with densities in $L^\infty$ and $\mathcal{A}_\rho$ is the acceptance set of $\rho$. If we consider problem (4.46) with a convex risk measure $\rho$ satisfying Assumption 4.12, its Fenchel dual problem (see Theorem 2.6) is

$$\sup_{Q \in \mathcal{Q}} \inf_{\varphi \in R_0} \{E^Q[(1-\varphi)H] - \sup_{Y \in \mathcal{A}_\rho} E^Q[-Y]\}. \tag{4.58}$$

Then, Theorem 4.38 and the dual representation (4.57) of $\rho$ lead to the following corollary.

**Corollary 4.39** (Convex Hedging)**.** *Let Assumption 4.33 be satisfied and let $\rho$ be a convex risk measure satisfying Assumption 4.12. There exists a solution $\widetilde{\varphi}$ to problem (4.46) and a solution $\widetilde{Q} \in \mathcal{Q}$ to (4.58). Let $\widetilde{y} > 0$ such that $\widetilde{y} \in \partial u_{\widetilde{Q}}(\widetilde{V}_0)$. Then, there exists an $\widetilde{h} \in \mathcal{D}(\widetilde{y})$ such that the triple $(\widetilde{Q}, \widetilde{y}, \widetilde{h}) \in (\mathcal{Q} \times \mathbb{R}_{>0} \times \mathcal{D}(y))$ solves*

$$\max_{Q \in \mathcal{Q}, y>0, h \in \mathcal{D}(y)} \{E[(Z_Q \wedge h)H] - \widetilde{V}_0 y - \sup_{Y \in \mathcal{A}_\rho} E^Q[-Y]\}.$$

*It follows that:*

- *The solution to (4.46) is*

$$\widetilde{\varphi} = \begin{cases} 1 & : \quad 0 \leq \widetilde{h} < \widetilde{Z}_Q \\ 0 & : \quad \widetilde{h} > \widetilde{Z}_Q \end{cases} \qquad P - a.s.$$

  *with*

$$E[\widetilde{\varphi} H \widetilde{h}] = \widetilde{V}_0 \widetilde{y}.$$

- $(\widetilde{\varphi}, \widetilde{Q})$ *is the saddle point of the functional* $(\varphi, Q) \mapsto E^Q[(1-\varphi)H]$ $- \sup_{Y \in \mathcal{A}_\rho} E^Q[-Y]$ *in* $R_0 \times \mathcal{Q}$.

- $(\widetilde{V}_0, \widetilde{\xi})$ *solves the dynamic hedging problem (4.45), where* $\widetilde{\xi}$ *is the superhedging strategy of the modified claim* $\widetilde{\varphi} H$, *obtained by the optional decomposition theorem (Theorem C.3).*

### 4.2.2 Coherent Hedging

Let us consider a coherent risk measure $\rho : L^1 \to \mathbb{R} \cup \{+\infty\}$ satisfying Assumption 4.16. Then, the Fenchel dual problem of problem (4.46) is (see Theorem 2.6)

$$\sup_{Q \in \mathcal{Q}} \inf_{\varphi \in R_0} E^Q[(1-\varphi)H], \tag{4.59}$$

where $\mathcal{Q}$, the maximal representing set of $\rho$, is a convex and weakly* closed subset of $\{Q \in \widehat{\mathcal{Q}} : Z_Q \in L^\infty\}$ (see Theorem 1.25). Theorem 4.38 and the dual representation (Theorem 1.25) of $\rho$ lead to the following corollary.

**Corollary 4.40** (Coherent Hedging)**.** *Let Assumption 4.33 be satisfied and let* $\rho$ *be a coherent risk measure satisfying Assumption 4.16. There exists a solution* $\widetilde{\varphi}$ *to problem (4.46) and a solution* $\widetilde{Q} \in \mathcal{Q}$ *to problem (4.59). Let* $\widetilde{y} > 0$ *such that* $\widetilde{y} \in \partial u_{\widetilde{Q}}(\widetilde{V}_0)$. *Then, there exists an* $\widetilde{h} \in \mathcal{D}(\widetilde{y})$ *such that the triple* $(\widetilde{Q}, \widetilde{y}, \widetilde{h}) \in (\mathcal{Q} \times \mathbb{R}_{>0} \times \mathcal{D}(y))$ *solves*

$$\max_{Q \in \mathcal{Q}, y > 0, h \in \mathcal{D}(y)} \{E[(Z_Q \wedge h)H] - \widetilde{V}_0 y\}.$$

*It follows that:*

- *The solution to (4.46) is*

$$\widetilde{\varphi} = \begin{cases} 1 & : \quad 0 \leq \widetilde{h} < \widetilde{Z}_Q \\ 0 & : \quad \widetilde{h} > \widetilde{Z}_Q \end{cases} \qquad P - a.s.$$

  *with*

$$E[\widetilde{\varphi} H \widetilde{h}] = \widetilde{V}_0 \widetilde{y}.$$

- $(\widetilde{\varphi}, \widetilde{Q})$ *is the saddle point of the functional* $(\varphi, Q) \mapsto E^Q[(1-\varphi)H]$ *in* $R_0 \times \mathcal{Q}$.
- $(\widetilde{V}_0, \widetilde{\xi})$ *solves the dynamic hedging problem (4.45), where* $\widetilde{\xi}$ *is the superhedging strategy of the modified claim* $\widetilde{\varphi}H$*, obtained by the optional decomposition theorem (Theorem C.3).*

As in Section 4.1.3, we can compare Corollary 4.40 with the results of Nakano [32]. Corollary 4.40 shows that the typical 0-1-structure of an optimal randomized test $\widetilde{\varphi}$ is deduced with respect to elements from the sets $\mathcal{Q}$ and $\mathcal{D}(\widetilde{y})$. Thus, with our method it is not necessary to consider the enlarged set $\mathcal{Z}$ that contains the set $\{Z_Q : Q \in \mathcal{Q}\}$ as in [32]. But in contrast to the complete case (Corollary 4.17) considered in Section 4.1, it is no longer possible to deduce the structure of $\widetilde{\varphi}$ directly with elements from $Z_{\mathcal{P}}$.

### 4.2.3 Robust Efficient Hedging

Let us consider a Lipschitz continuous loss function $l$ satisfying Assumption 4.18. Let the risk measure in the problem of hedging in incomplete markets be the robust version of the expectation of the loss function (see Section 4.1.4)

$$\rho_1(Y) = \sup_{Q \in \mathcal{Q}} E^Q[L(-Y)], \quad Y \in L^1,$$

where $L : L^1 \to L^0_+$ is as in Section 4.1.4 defined by $L(Y)(\omega) := l(Y(\omega))$. The probability measures $\mathcal{Q} \subseteq \widehat{\mathcal{Q}}$ take into account an uncertainty regarding the underlying model and satisfy Assumption 4.22. We can show that the risk measure $\rho_1$ satisfies Assumption 4.34 (see Proposition 4.21 and 4.26) and has the dual representation

$$\rho_1(Y) = \sup_{Y^* \in L^\infty_+} \{E[-YY^*] - \rho_1^*(-Y^*)\}.$$

Thus, this fits exactly into the setting of Theorem 4.38 and we can solve the problem by an application of this theorem. Analogously, we can treat the more general case where the loss function $l$ and the set of probability measures $\mathcal{Q}$ satisfy Assumption 4.18, 4.20, 4.22 and 4.23.
In the case of a linear loss function, we can go a step further and deduce the structure of the solution $\widetilde{\varphi}$ with respect to elements from $\mathcal{Q}$. We consider the hedging problem (4.46) with the risk measure

$$\rho_2(Y) = \sup_{Q \in \mathcal{Q}} E^Q[-Y], \tag{4.60}$$

which is a continuous coherent risk measure on $L^1$ with the maximal representing set $\mathcal{Q}_{\max} = \overline{\text{co}}^* \mathcal{Q}$ (see Section 4.1.4). Its Fenchel dual problem is

$$\sup_{Q \in \overline{\text{co}}^* \mathcal{Q}} \inf_{\varphi \in R_0} E^Q[(1-\varphi)H]. \tag{4.61}$$

An application of Corollary 4.40 with the maximal representing set $\overline{\mathrm{co}}^*\mathcal{Q}$ yields the following result.

**Corollary 4.41** (Robust Efficient Hedging with linear loss function)**.** *Let the risk measure $\rho$ be as in (4.60) and let Assumption 4.22 and 4.33 be satisfied. There exists a solution $\widetilde{\varphi}$ to problem (4.46) and a solution $\widetilde{Q} \in \overline{\mathrm{co}}^*\mathcal{Q}$ to its dual problem (4.61). Let $\widetilde{y} > 0$ such that $\widetilde{y} \in \partial u_{\widetilde{Q}}(\widetilde{V}_0)$. Then, there exists an $\widetilde{h} \in \mathcal{D}(\widetilde{y})$ such that the triple $(\widetilde{Q}, \widetilde{y}, \widetilde{h}) \in (\overline{\mathrm{co}}^*\mathcal{Q} \times \mathbb{R}_{>0} \times \mathcal{D}(y))$ solves*

$$\max_{Q \in \overline{\mathrm{co}}^*\mathcal{Q}, y>0, h \in \mathcal{D}(y)} \{E[(Z_Q \wedge h)H] - \widetilde{V}_0 y\}.$$

*It follows that:*

- *The solution to (4.46) is*

$$\widetilde{\varphi} = \begin{cases} 1 & : \quad 0 \leq \widetilde{h} < \widetilde{Z}_Q \\ 0 & : \quad \widetilde{h} > \widetilde{Z}_Q \end{cases} \qquad P - a.s.$$

  *with*

$$E[\widetilde{\varphi} H \widetilde{h}] = \widetilde{V}_0 \widetilde{y}.$$

- *$(\widetilde{\varphi}, \widetilde{Q})$ is the saddle point of the functional $(\varphi, Q) \mapsto E^Q[(1-\varphi)H]$ in $R_0 \times \overline{\mathrm{co}}^*\mathcal{Q}$.*

- *$(\widetilde{V}_0, \widetilde{\xi})$ solves the dynamic hedging problem (4.45), where $\widetilde{\xi}$ is the superhedging strategy of the modified claim $\widetilde{\varphi}H$, obtained by the optional decomposition theorem (Theorem C.3).*

With Theorem 4.38, the robust efficient hedging problem can be solved when the loss function $l$ and the set of probability measures $\mathcal{Q}$ satisfy Assumption 4.18, 4.20, 4.22 and 4.23. Corollary 4.41 treats the special case of a linear loss function. These results generalize Proposition 4.1 in Föllmer and Leukert [17] and Theorem 1.19 in Xu [48]. In [17] and [48] the set $\mathcal{Q} = \{P\}$ is a singleton and a linear loss function is considered. In [17], the problem is solved in the complete financial market, i.e., $\mathcal{P} = \{P^*\}$ and in [48] the problem is solved in the incomplete financial market. With our method it is possible to solve the problem not only in the case $\mathcal{Q} = \{P\}$, but also for more general sets $\mathcal{Q}$ satisfying Assumption 4.22 and even for more general loss functions.

**Example 4.42.** For a risk measure $\rho$ (regardless if it is a convex or coherent risk measure or as general as in Assumption 4.5), the structure of a solution $\widetilde{\varphi}$ to the hedging problem (4.2), (4.3) is not given explicitly (it depends on the dual solutions $(\widetilde{Y}^*, \widetilde{\lambda})$ in the case where $Z_{\mathcal{P}}$ is compact (see Theorem 4.9), respectively on the dual solutions $(\widetilde{Y}^*, \widetilde{y}, \widetilde{h})$ in the general incomplete market (see Theorem 4.38)). We give

a very simple example that is connected to different kinds of risk measures and a special case, where the problem can be solved explicitly.
Let us consider the problem of minimizing the risk of losses $-(H - V_T)^+$ due to the shortfall where the risk is measured by the coherent risk measure $\rho : L^1 \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\rho(X) = E^Q[-X].$$

This means, the representing set in the dual representation of the coherent risk measure (see Theorem 1.25) is a singleton, $\mathcal{Q} = \{Q\}$ with $Z_Q \in L^\infty$. Thus we look for an admissible strategy $(V_0, \widetilde{\xi})$ that minimizes

$$\rho(-(H - V_T)^+) = E^Q[(H - V_T)^+] \tag{4.62}$$

under the constraint

$$0 < V_0 \leq \widetilde{V}_0, \tag{4.63}$$

where $\widetilde{V}_0$ is a given capital constraint that is strictly less than the superhedging price $U_0$ of $H$. Theorem 4.1 shows that the corresponding static optimization problem is

$$\max_{\varphi \in R} E^Q[\varphi H] \tag{4.64}$$

under the constraint

$$\forall P^* \in \mathcal{P}: \quad E^{P^*}[\varphi H] \leq \widetilde{V}_0. \tag{4.65}$$

The same optimization problem with $HZ_Q = Z_P$ arises in [16], Section 4, where the problem of quantile hedging in the incomplete case is considered. The risk measure used there is just the probability of the shortfall.
In [17], the expectation of a loss function is used as a risk measure. In Section 4, the problem of minimizing the expected shortfall is considered. This means, the linear loss function $l(x) = x$ is used. This leads to the optimization problem (4.64), (4.65) with $Q = P$.
Corollary 4.17 and 4.40 make it possible to solve these problems not only in the complete market. We consider two cases. First, let $Z_{\mathcal{P}}$ be compact. Under the assumption $\widetilde{V}_0 > 0$, the following conditions are necessary and sufficient for the optimality of $\widetilde{\varphi}$ with respect to the optimization problem (4.64), (4.65) and give a result about the structure of the solution (see Corollary 4.17):

$$\widetilde{\varphi} = \begin{cases} 1 & : \quad HZ_Q > H \int_{\mathcal{P}} Z_{P^*} d\widetilde{\lambda} \\ 0 & : \quad HZ_Q < H \int_{\mathcal{P}} Z_{P^*} d\widetilde{\lambda} \end{cases} \qquad P - a.s.$$

with

$$E^{P^*}[\widetilde{\varphi} H] = \widetilde{V}_0 \qquad \widetilde{\lambda} - a.s.,$$

where $\widetilde{\lambda}$, a finite measure on $\mathcal{P}$, is the solution of the dual problem of (4.64), (4.65), i.e., a solution to

$$\inf_{\lambda\in\Lambda_+}\Big\{E[(HZ_Q - H\int_{\mathcal{P}} Z_{P^*}d\lambda)^+] + \widetilde{V}_0\lambda(Z_{\mathcal{P}})\Big\}.$$

In the general incomplete market, we can apply Corollary 4.40 and obtain that the structure of the optimal solution $\widetilde{\varphi}$ of (4.64), (4.65) is the following

$$\widetilde{\varphi} = \begin{cases} 1 & : \quad 0 \le \widetilde{h} < Z_Q \\ 0 & : \quad \widetilde{h} > Z_Q \end{cases} \qquad P-a.s.$$

with

$$E[\widetilde{\varphi}H\widetilde{h}] = \widetilde{V}_0\widetilde{y},$$

where $\widetilde{y} \in \partial u_Q(\widetilde{V}_0)$ is assumed to satisfy $\widetilde{y} > 0$ and $\widetilde{h} \in \mathcal{D}(\widetilde{y})$ solves

$$\inf_{h\in\mathcal{D}(\widetilde{y})} E[(Z_Q - h)^+H].$$

In both cases, the dynamic coherent hedging problem (4.62), (4.63) can be solved by the optional decomposition theorem (Theorem C.3). The solution is $(\widetilde{V}_0, \widetilde{\xi})$, where $\widetilde{\xi}$ is the superhedging strategy of the corresponding modified claim $\widetilde{\varphi}H$ (see Theorem 4.1).
If additionally $\mathcal{P} = \{P^*\}$ is a singleton as in [32], Proposition 4.1, i.e., we work in a complete financial market, but with capital constraint $\widetilde{V}_0 < U_0 = E^{P^*}[H]$ and can apply Corollary 4.17. Then, the static problem can be solved explicitly. The optimal solution is

$$\widetilde{\varphi}(\omega) = 1_{\{Z_Q > \widetilde{a}Z_{P^*}\}}(\omega) + \delta 1_{\{Z_Q = \widetilde{a}Z_{P^*}\}}(\omega),$$

where

$$\widetilde{a} = \inf\{a \mid E^{P^*}[H1_{\{Z_Q > aZ_{P^*}\}}] \le \widetilde{V}_0\}$$

and

$$\delta = \begin{cases} \dfrac{\widetilde{V}_0 - E^{P^*}[H1_{\{Z_Q > \widetilde{a}Z_{P^*}\}}]}{E^{P^*}[H1_{\{Z_Q = \widetilde{a}Z_{P^*}\}}]} & : \quad P^*(\{Z_Q = \widetilde{a}Z_{P^*}\} \cap \{H > 0\}) > 0 \\ c \in [0,1] \text{ arbitrarily} & : \quad P^*(\{Z_Q = \widetilde{a}Z_{P^*}\} \cap \{H > 0\}) = 0. \end{cases}$$

When $Q$ is equal to $P$ this coincides with Proposition 4.1 in [17].
If there would be no capital constraint in the complete case, the optimal randomized test of the static problem would be $\widetilde{\varphi} = 1$ on $\{H > 0\}$. That means $\widetilde{\varphi}H = H$. Thus, the optimal strategy of problem (4.62), (4.63) would be exactly the replicating strategy $(E^{P^*}[H], \widetilde{\xi})$ of the claim $H$.

# Appendix

## A Results from Convex Analysis

For the convenience of the reader, we collect some important definitions and results from Convex Analysis that are used in this thesis. Most of the results can be found in [11] or [50], if not, we give the proofs.

**Definition A.1** ([2], Definition 5.52)**.** *A topology on a linear vector space $\mathcal{X}$ is called* **locally convex** *if every neighborhood of zero contains a convex neighborhood of zero.*

**Definition A.2** ([2], Definition 2.5)**.** *A topology on $\mathcal{X}$ is called* **separated** *or* **Hausdorff** *if any two distinct points can be separated by two disjoint neighborhoods of the points. That is, for each pair $X_1, X_2 \in \mathcal{X}$ with $X_1 \neq X_2$ there exist neighborhoods $U(X_1)$ and $U(X_2)$ such that $U(X_1) \cap U(X_2) = \emptyset$.*

We call a linear vector space $\mathcal{X}$ with a locally convex Hausdorff topology a **separated locally convex space**. In the following, let $\mathcal{X}$ and $\mathcal{Y}$ be separated locally convex spaces and $\mathcal{X}^*, \mathcal{Y}^*$ their topological dual spaces. This means, $\mathcal{X}^*$ is the vector space of all continuous linear functionals on $\mathcal{X}$. Let $\langle X, X^* \rangle$ denote the value of the continuous linear functional $X^*$ at $X$. For $\mathcal{Y}^*$ the notation is analogously.

**Definition A.3** ([50], Section 2.3)**.** *Let $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$. The function $f^* : \mathcal{X}^* \to \mathbb{R} \cup \{\pm\infty\}$*

$$f^*(X^*) := \sup_{X \in \mathcal{X}} \{\langle X, X^* \rangle - f(X)\}$$

*is called the* **conjugate** *or Fenchel conjugate of $f$. The* **biconjugate** *$f^{**} : \mathcal{X} \to \mathbb{R} \cup \{\pm\infty\}$ of $f^*$ is defined as*

$$f^{**}(X) := \sup_{X^* \in \mathcal{X}^*} \{\langle X, X^* \rangle - f^*(X^*)\}.$$

We recall that a function $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ is called **proper** if $\operatorname{dom} f \neq \emptyset$.

**Theorem A.4** ([50], Theorem 2.3.1, Corollary 2.3.2)**.** *Let $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a function. Then, $f^*$ is convex and weakly* lower semicontinuous. It holds, $f^*$ is proper if and only if $f$ is proper.*

**Theorem A.5** (biconjugation theorem, [50], Theorem 2.3.3)**.** *Let $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be convex, proper and lower semicontinuous. Then,*

$$f = f^{**}.$$

**Example A.6.** [[50], Section 2.3] Let $\emptyset \neq M \subseteq \mathcal{X}$. The **support function** $\delta_M : \mathcal{X}^* \to \mathbb{R} \cup \{+\infty\}$ of $M$ is defined as

$$\delta_M(X^*) := \sup_{X \in M} \langle X, X^* \rangle .$$

Let $\emptyset \neq M^* \subseteq \mathcal{X}^*$. The support function $\delta_{M^*} : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ of $M^*$ is defined similarly as

$$\delta_{M^*}(X) := \sup_{X^* \in M^*} \langle X, X^* \rangle .$$

The support function $\delta_M$ is weakly* lower semicontinuous, positively homogeneous and subadditive and $\delta_{M^*}$ is lower semicontinuous, positively homogeneous and subadditive.

**Example A.7.** [[50], Section 2.3] Let $\emptyset \neq M \subseteq \mathcal{X}$. The support function of $M$ coincides with the support function on the closed convex hull of $M$

$$\delta_M(X^*) = \sup_{X \in M} \langle X, X^* \rangle = \delta_{\overline{\text{co}}M}(X^*) = \sup_{X \in \overline{\text{co}}M} \langle X, X^* \rangle .$$

**Example A.8.** [[50], Section 2.1] Let $M \subseteq \mathcal{X}$. The indicator function $\mathcal{I}_M : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$

$$\mathcal{I}_M(X) = \begin{cases} 0 & : \quad X \in M \\ +\infty & : \quad X \notin M. \end{cases}$$

is convex if and only if the set $M$ is convex.

**Example A.9.** [[11], Example I.4.3] Let $\emptyset \neq M \subseteq \mathcal{X}$. Then,

$$\mathcal{I}_M^{**}(X) = \mathcal{I}_{\overline{\text{co}}M}(X).$$

**Example A.10.** Let $\emptyset \neq M \subseteq \mathcal{X}$. The conjugate of the support function of $M$ is an indicator function of the closed convex hull of $M$

$$\delta_M^*(X) = \mathcal{I}_{\overline{\text{co}}M}(X).$$

*Proof.* Consider the function $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ defined by $f(X) := \mathcal{I}_M(X)$. By definition of the conjugate, it holds

$$f^*(X^*) = \sup_{X \in \mathcal{X}} \{\langle X, X^* \rangle - \mathcal{I}_M(X)\} = \sup_{X \in M} \langle X, X^* \rangle = \delta_M(X^*).$$

By Example A.9, we obtain

$$f^{**}(X) = \mathcal{I}_M^{**}(X) = \delta_M^*(X) = \mathcal{I}_{\overline{\text{co}}M}(X).$$

□

We recall that the **negative dual cone** $K^*$ of a cone $K \subseteq \mathcal{X}$ is defined by $K^* = \{X^* \in \mathcal{X}^* : \forall X \in K : \langle X, X^* \rangle \leq 0\}$.

**Example A.11.** Let $K \subseteq \mathcal{X}$ be a cone containing $\mathbf{0} \in \mathcal{X}$. Then, the support function of $K$ is the indicator function of the negative dual cone $K^*$ of $K$

$$\delta_K(X^*) = \sup_{X \in K} \langle X, X^* \rangle = \mathcal{I}_{K^*}(X^*).$$

*Proof.* First, take $X^* \in K^*$. Since $0 \in K$, $\sup_{X \in K} \langle X, X^* \rangle = 0$ for all $X^* \in K^*$. Now, take $X^* \notin K^*$. This means, there exists $\overline{X} \in K$, such that $\langle \overline{X}, X^* \rangle > 0$. Consider a sequence $t_n > 0$ for all $n \in \mathbb{N}$ with $t_n \to +\infty$. Since $K$ is a cone, $t_n \overline{X} \in K$ for all $n \in \mathbb{N}$. We obtain

$$\forall X^* \notin K^* : \quad \langle t_n \overline{X}, X^* \rangle = t_n \langle \overline{X}, X^* \rangle \to +\infty.$$

Thus, $\sup_{X \in K} \langle X, X^* \rangle = +\infty$ for all $X^* \notin K^*$. □

The following theorem is a special case of the fundamental duality formula, Theorem III.4.1 in [11].

**Theorem A.12** (Fenchel's duality theorem)**.** *Let $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$, $g : \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$ and $A : \mathcal{X} \to \mathcal{Y}$ be a linear and continuous operator with the adjoint operator $A^* : \mathcal{Y}^* \to \mathcal{X}^*$. Let $p, d \in \mathbb{R} \cup \{\pm\infty\}$ be the values of the primal and the dual optimization problem,*

$$\begin{aligned} p &= \inf_{X \in \mathcal{X}} \{f(X) + g(AX)\} \\ d &= \sup_{Y^* \in \mathcal{Y}^*} \{-f^*(A^*Y^*) - g^*(-Y^*)\}, \end{aligned}$$

*respectively. Then,* **weak duality** *holds true, i.e., $d \leq p$.*
*If $f$ and $g$ are convex, the value $p$ of the primal problem is finite and there exists a point $X_0 \in \operatorname{dom} f$ such that $g$ is continuous and finite in $AX_0 \in \mathcal{Y}$, then* **strong duality** *holds true, i.e., $d = p$ and there exists a $\widetilde{Y}^* \in \mathcal{Y}^*$ that attains the supremum in the dual problem.*

*Proof.* The first part of the theorem (weak duality) follows from Proposition III.1.1 in [11]. The results about strong duality follow from Theorem III.4.1 and Remark III.4.2 in [11]. □

**Definition A.13** ([11], Section I.5)**.** *Let $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$. An element $X^* \in \mathcal{X}^*$ is called a* **subgradient** *of the function $f$ at $X$ if*

$$f(X) + f^*(X^*) = \langle X, X^* \rangle.$$

*The set of all subgradients of the function $f$ at $X$ is denoted by $\partial f(X)$ and is called the* **subdifferential** *of $f$ at $X$.*

If $\partial f(X) \neq \emptyset$, then $f = f^{**}$ and $\partial f(X) = \partial f^{**}(X)$ ([11], Section I.5). This justifies why only proper convex functions are considered when discussing subdifferentiability. A function $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ is called **concave** if $-f$ is convex. For a concave function $f$, the subdifferential is defined as the negative of the subdifferential of the convex function $-f$.

**Definition A.14** ([33], Section 30). *Let $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be concave.*

$$\partial f(X) := -\partial(-f)(X).$$

# B Results from Functional Analysis

## B.1 Barrelledness, Weak* and Mackey Topology

In this section, we give the definitions of the weak* and the Mackey topology on $\mathcal{X}^*$. Furthermore, we shall present the theory that leads to an important result that is used in this thesis: A weakly* lower semicontinuous, convex and finite valued functional on $L^\infty$ is continuous with respect to the Mackey topology.

**Definition B.1** ([23], Definition 15). *Let $\mathcal{X}$ be a separated locally convex space and $\mathcal{X}^*$ its topological dual space. The coarsest locally convex Hausdorff topology on $\mathcal{X}$ for which the map $X \mapsto \langle X, X^* \rangle$ is continuous for each $X^* \in \mathcal{X}^*$, is called the* **weak topology** *$\sigma(\mathcal{X}, \mathcal{X}^*)$ on $\mathcal{X}$.*
*The coarsest locally convex Hausdorff topology on $\mathcal{X}^*$ for which the map $X^* \mapsto \langle X, X^* \rangle$ is continuous for each $X \in \mathcal{X}$, is called the* **weak* topology** *$\sigma(\mathcal{X}^*, \mathcal{X})$ on $\mathcal{X}^*$.*

**Definition B.2** ([2], Definition 4.66). *A* **dual pair** *is a pair $(\mathcal{X}, \mathcal{X}^*)$ of vector spaces together with a function $(X, X^*) \mapsto \langle X, X^* \rangle$, from $\mathcal{X} \times \mathcal{X}^*$ into $\mathbb{R}$ such that $\langle X, X^* \rangle$ is a bilinear form that satisfies the following. If $\langle X, X^* \rangle = 0$ for each $X^* \in \mathcal{X}^*$, then $X = 0$ and if $\langle X, X^* \rangle = 0$ for each $X \in \mathcal{X}$, then $X^* = 0$.*

**Definition B.3** ([2], Definition 5.85). *A locally convex Hausdorff topology $\tau$ on $\mathcal{X}$ is* **consistent** *with the dual pair $(\mathcal{X}, \mathcal{X}^*)$ if $(\mathcal{X}, \tau)^* = \mathcal{X}^*$. Consistent topologies on $\mathcal{X}^*$ are defined analogously.*

**Definition B.4** (Mackey-Arens, [23], Theorem I.11, Definition I.17). *The* **Mackey topology** *is the finest locally convex Hausdorff topology on $\mathcal{X}^*$ consistent with the dual pair $(\mathcal{X}, \mathcal{X}^*)$.*

**Example B.5.** Let $\mathcal{X}^* = L^\infty(\Omega, \mathcal{F}, P)$. The Mackey topology with respect to the dual pair $(L^\infty, L^1)$ is finer than the weak* topology on $L^\infty$. If $L^\infty$ is endowed with the weak* topology or the Mackey topology, its topological dual space can be identified with $L^1$. The norm topology on $L^\infty$ is finer than the Mackey topology with respect to the dual pair $(L^\infty, L^1)$. If $L^\infty$ is endowed with the norm topology, its topological dual space can be identified with $ba(\Omega, \mathcal{F}, P)$.

**Definition B.6** ([23], Definition I.2, II.1 and II.2). *A set $A \subseteq \mathcal{X}$ is said to be* **circled** *if $tA \subseteq A$ for every $t \in \mathbb{R}$ with $|t| \leq 1$ and it is called* **absorbing** *if for each $X \in \mathcal{X}$ there is an $\alpha > 0$ such that $X \in tA$ for all $t \in \mathbb{R}$ with $|t| \geq \alpha$.*
*A locally convex space $(\mathcal{X}, \tau)$ is called a* **barrelled space** *if each closed, circled, convex and absorbing subset of $\mathcal{X}$ is a neighborhood of zero.*

**Example B.7.** Every Banach space is a barrelled space (see [23], Corollary I.1). Thus, the space $L^p, p \in [0, \infty]$, endowed with the norm topology, is a barrelled space.

**Theorem B.8** ([23], Corollary II.2, II.4). *A locally convex space $(\mathcal{X}, \tau)$ is barrelled if and only if $\tau$ is the Mackey topology.*

**Example B.9.** The space $L^\infty$, endowed with the Mackey topology with respect to the dual pair $(L^\infty, L^1)$ is a barrelled space.

**Theorem B.10** ([11], Corollary I.2.5). *Let $\mathcal{X}$ be a barrelled space. Then, every lower semicontinuous, convex function $\rho : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ is continuous over the interior of its effective domain.*

Now, it is possible to prove the following corollary.

**Corollary B.11.** *A weakly* lower semicontinuous, convex and finite valued functional on $L^\infty$ is continuous with respect to the Mackey topology.*

*Proof.* A weakly* lower semicontinuous functional of $L^\infty$ is also lower semicontinuous in the finer Mackey topology with respect to the dual pair $(L^\infty, L^1)$ ([2], Lemma 2.48-2.). Since $L^\infty$, endowed with the Mackey topology is a barrelled space (Theorem B.8), we can apply Theorem B.10 and obtain the stated result. □

## B.2 Ordering Cones and their Interior

In this subsection, we shall consider several important cones that induce an order relation in the corresponding space and answer the question if these cones have an empty (or not empty) interior. The existence of interior points is important when dealing with the question if the indicator function of the cone is continuous in at least one point.
First, let us consider the Banach space $\mathcal{L}$ of continuous functions $l : C^* \to \mathbb{R}$ on a compact set $C^*$, endowed with the supremum norm $\|l\|_{\mathcal{L}} = \sup_{X^* \in C^*} |l(X^*)|$. Let $\mathcal{L}_+ := \{l \in \mathcal{L} : \forall X^* \in C^* : l(X^*) \geq 0\}$ be the cone generating the pointwise partial order on $\mathcal{L}$ defined by $l_1 \leq l_2$ if and only if $l_2 - l_1 \in \mathcal{L}_+$.

**Lemma B.12.** *It holds* $\operatorname{int} \mathcal{L}_+ \neq \emptyset$.

*Proof.* Consider $l_0(X^*) := \mathbf{1}(X^*) = 1$ for all $X^* \in C^*$. Thus, $l_0 \in \mathcal{L}_+$. $l_0$ is an interior point of $\mathcal{L}_+$ if and only if there exists an $\varepsilon-$neighborhood $U_\varepsilon(l_0) := \{l \in \mathcal{L} : \|l - l_0\|_{\mathcal{L}} < \varepsilon\}$ of $l_0$ with $\varepsilon > 0$ such that $U_\varepsilon(l_0) \subseteq \mathcal{L}_+$. For all $l \in U_\varepsilon(l_0)$ it holds $\sup_{X^* \in C^*} |l(X^*) - 1| < \varepsilon$, i.e., for all $X^* \in C^*$ we have $1 - \varepsilon < l(X^*) < 1 + \varepsilon$. Thus, for $0 < \varepsilon \leq 1$ it holds that $U_\varepsilon(l_0) \subseteq \mathcal{L}_+$. □

Let us consider the space $L^\infty = L^\infty(\Omega, \mathcal{F}, P)$, endowed with the norm $\|X\|_{L^\infty} = \inf\{c \geq 0 : P[|X| > c] = 0\}$ and its ordering cone $L^\infty_+ := \{X \in L^\infty : X \geq \mathbf{0}\ P-a.s.\}$.

**Lemma B.13.** *It holds* $\operatorname{int} L^\infty_+ \neq \emptyset$ *with respect to the norm topology.*

*Proof.* Consider $X_0(\omega) := 1\ P - a.s.$ Thus, $X_0 \in L^\infty_+$. Define $U_\varepsilon(X_0) := \{X \in L^\infty : \|X - X_0\|_{L^\infty} < \varepsilon\}$. Thus, for all $X \in U_\varepsilon(X_0)$ it holds $|X(\omega) - 1| < \varepsilon\ P - a.s.$, i.e., $1 - \varepsilon < X(\omega) < 1 + \varepsilon\ P - a.s.$ Thus, $U_\varepsilon(X_0) \subseteq L^\infty_+$ for $0 < \varepsilon \leq 1$. □

Now, let us consider $L^\infty$, endowed with the weak* topology and $L^\infty_+$ as above.

**Lemma B.14.** *It holds* $\operatorname{int} L^\infty_+ = \emptyset$ *with respect to the weak* topology.*

*Proof.* Consider $A_n \in \mathcal{F}$ with $P(A_n) > 0$ for all $n \in \mathbb{N}$ and with $P(A_n) \to 0$. Take an arbitrary $X \in L^\infty_+$. Then, the sequence $X - 2\|X\|_{L^\infty}\mathcal{I}_{A_n}$ converges with respect to the weak* topology to $X$, since

$$\forall Y \in L^1 : \quad E[(X - 2\|X\|_{L^\infty}\mathcal{I}_{A_n})Y] = E[XY] - 2\|X\|_{L^\infty}E[\mathcal{I}_{A_n}Y] \to E[XY].$$

This means, to every $X \in L^\infty_+$ there exists a sequence converging with respect to the weak* topology to $X$, but

$$\forall n \in \mathbb{N} : \quad X - 2\|X\|_{L^\infty}\mathcal{I}_{A_n} \notin L^\infty_+.$$

Thus, every $\varepsilon-$neighborhood $U_\varepsilon(X)$ of $X \in L^\infty_+$ with respect to the weak* topology contains elements that are not in $L^\infty_+$. Thus, $\operatorname{int} L^\infty_+ = \emptyset$ with respect to weak* topology. Note that the sequence $\{X - 2\|X\|_{L^\infty}\mathcal{I}_{A_n}\}_{n\in\mathbb{N}}$ does not converge with respect to the norm topology on $L^\infty$. □

For the sake of completeness let us consider $L^p = L^p(\Omega, \mathcal{F}, P), p \in [1, \infty)$, endowed with the strong topology, generated by the norm $\|X\|^p_{L^p} = E[|X|^p]$ and the ordering cone $L^p_+ := \{X \in L^p : X \geq \mathbf{0}\ P - a.s.\}$. It is well-known that $\operatorname{int} L^p_+ = \emptyset$.

## B.3 Nets

In this section we recall the definition of a net (see for instance [2]).
A sequence in $\mathcal{X}$ is a function from the natural numbers $\mathbb{N}$ into $\mathcal{X}$. A net is a direct generalization of the notion of a sequence. Instead of the natural numbers, the index set can be more general. The key issue is that the index set has a sense of direction. A **direction** $\succeq$ on a (not necessarily infinite) set $D$ is a reflexive transitive binary

relation with the property that each pair has an upper bound. That is, for each pair $\alpha, \beta \in D$ there exists some $\gamma \in D$ satisfying $\gamma \succeq \alpha$ and $\gamma \succeq \beta$. A **directed set** is any set $D$, equipped with a direction $\succeq$.

**Definition B.15** (Definition 2.8, [2]). *A* **net** *in a set $\mathcal{X}$ is a function $X : D \to \mathcal{X}$, where $D$ is a directed set. The directed set is called the* **index set** *of the net and the members of $D$ are indexes.*

In particular, sequences are nets. A net $\{X_\alpha\}$ in a topological spaces **converges** to some point $X$, if for each neighborhood $V$ of $X$, there is some index $\alpha_0$, depending on $V$, such that $X_\alpha \in V$ for all $\alpha \geq \alpha_0$. We say $X$ is the **limit** of the net and write $X_\alpha \to X$. In separated topological vector spaces limits are unique.

**Theorem B.16** (Theorem 2.9, [2]). *A topological space is separated if and only if every net converges to at most one point.*

Whenever possible, it is desirable to replace nets with sequences. One case, that allows this (see [2], Section 2.9), is the case of a first countable topology (each point has a countable neighborhood base). This class of spaces includes all metric spaces, hence all Banach spaces. Thus, in the Banach space $L^p(\Omega, \mathcal{F}, P), p \in [1, \infty]$ with the norm topology, it is sufficient to work with sequences. Note that the space $L^\infty(\Omega, \mathcal{F}, P)$, endowed with the weak* topology is not first countable due to the following results.

**Lemma B.17** ([29], Corollary 2.3.12). *A separated topological vector space is metrizable if and only if it is first countable.*

**Lemma B.18** ([29], Proposition 2.6.12). *Let $\mathcal{X}$ be a Banach space. Then, the weak* topology on the dual $\mathcal{X}^*$ is metrizable if and only if $\mathcal{X}$ is finite dimensional.*

Thus, we have to work with nets if the space $L^\infty(\Omega, \mathcal{F}, P)$ is endowed with the weak* topology.

## B.4 Auxiliary Results about Integration

**Theorem B.19** (Lebesgue's dominated convergence theorem, [10], Theorem IV.10.10). *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. If $Y_n$ is a sequence of $\mu-$integrable functions which converges $\mu - a.s.$ to $Y$ and if $Z$ is a $\mu-$integrable function such that $|Y_n(\omega)| \leq Z(\omega)$ $\mu - a.s.$ for all $n \in \mathbb{N}$, then $Y$ is $\mu-$integrable and*

$$\int_\Omega Y(\omega) d\mu = \lim_{n \to \infty} \int_\Omega Y_n(\omega) d\mu.$$

In this thesis we need the following version of Lebesgue's dominated convergence theorem.

**Corollary B.20.** *Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $Y^* \in L^1$. We define a $\sigma-$additive set function $Q = Q(Y^*)$, absolutely continuous with respect to $P$, by the Radon-Nikodym derivative $\frac{dQ}{dP} = Y^*$. If $Y_n$ is a sequence of measurable functions which converges $P - a.s.$ to $Y$, and if $Z$ is a $Q-$integrable function such that $|Y_n(\omega)| \leq Z(\omega)$ $P - a.s.$ for all $n \in \mathbb{N}$. Then, $Y$ is $Q-$integrable and*

$$E^Q[Y] = \lim_{n\to\infty} E^Q[Y_n].$$

*If there exists a constant $c \in \mathbb{R}$, such that $|Y_n(\omega)| \leq c$ $P - a.s.$ for all $n \in \mathbb{N}$, then*

$$\forall Y^* \in L^1 : \quad E[YY^*] = \lim_{n\to\infty} E[Y_n Y^*].$$

*Proof.* Because of $Q$ absolutely continuous with respect to $P$, $Y_n$ converges also $Q - a.s.$ to $Y$ and we have $|Y_n(\omega)| \leq Z(\omega)$ $Q - a.s.$ for all $n \in \mathbb{N}$ and thus, $Y_n$ is also $Q-$integrable. Hence, we can apply Theorem B.19. If additionally $|Y_n(\omega)| \leq Z(\omega) = c$ $P - a.s.$ for all $n \in \mathbb{N}$, then $Z$ is integrable with respect to $Q(Y^*)$ for all $Y^* \in L^1$ and we obtain the stated results. □

**Theorem B.21** (Lemma von Fatou, [10], Theorem III.6.19)**.** *Let $(\Omega, \mathcal{F}, \mu)$ be a positive measure space. If $\{Y_n\}_{n\in\mathbb{N}}$ is a sequence of nonnegative measurable, but not necessarily integrable, functions, then*

$$\int_\Omega \liminf_{n\to\infty} Y_n(\omega) d\mu \leq \liminf_{n\to\infty} \int_\Omega Y_n(\omega) d\mu.$$

**Theorem B.22** (Tonelli, [10], Corollary III.11.15)**.** *Let $(R, \Sigma_R, \nu) = (S, \Sigma_S, \mu) \times (T, \Sigma_T, \lambda)$ be the product of two positive, $\sigma$-finite measure spaces. Let $f : S \times T \to \mathbb{R}$ be measurable with respect to the product $\sigma$-algebra $\Sigma_R$ and let*

$$\int\limits_S \Big\{ \int\limits_T |f(s,t)| d\lambda(dt) \Big\} \mu(ds) < +\infty.$$

*Then $f$ is $\nu$-integrable and*

$$\int\limits_S \Big\{ \int\limits_T f(s,t) d\lambda(dt) \Big\} \mu(ds) = \int\limits_T \Big\{ \int\limits_S f(s,t) \mu(ds) \Big\} d\lambda(dt) < +\infty.$$

# C Results from Stochastic Finance

We give the definition of the essential supremum of a family of random variables and review the optional decomposition theorem of Föllmer and Kabanov [14].

**Theorem C.1** (Theorem A.32, [19])**.** *Let $\Phi$ be any set of random variables on $(\Omega, \mathcal{F}, P)$.*

(i) *There exists a random variable* $\varphi^*$ *such that*

$$\forall \varphi \in \Phi : \quad \varphi^* \geq \varphi \; P - a.s. \tag{C.1}$$

*Moreover,* $\varphi^*$ *is* $P - a.s.$ *unique in the following sense: Any other random variable* $\psi$ *with property (C.1)satisfies* $\psi \geq \varphi^* \; P - a.s.$

(ii) *Suppose that* $\Phi$ *is directed upwards, i.e., for* $\varphi, \widetilde{\varphi} \in \Phi$ *there exists* $\psi \in \Phi$ *with* $\psi \geq \varphi \vee \widetilde{\varphi}$*. Then, there exists an increasing sequence* $\varphi_1 \leq \varphi_2 \leq ...$ *in* $\Phi$*, such that* $\varphi^* = \lim_{n\to\infty} \varphi_n \; P - a.s.$

**Definition C.2** (Definition A.33, [19])**.** *The random variable* $\varphi^*$ *in Theorem C.1 is called the* **essential supremum** *of* $\Phi$ *with respect to* $P$*, and we write*

$$\text{ess. sup}\, \Phi = \text{ess.} \sup_{\varphi \in \Phi} \varphi := \varphi^*.$$

**Theorem C.3** (optional decomposition theorem, [14], Theorem 1)**.** *Let* $S$ *be an* $\mathbb{R}^d$*-valued right-continuous semimartingale on a complete probability space* $(\Omega, \mathcal{F}, P)$ *with filtration* $(\mathcal{F}_t)_{t\in[0,T]}$*. Let* $\mathcal{P} \neq \emptyset$ *denote the set of equivalent measures with respect to* $P$ *such that* $S$ *is a local martingale with respect to* $P^* \in \mathcal{P}$*. Let* $U_t$ *be a right-continuous process which is a local supermartingale with respect to any* $P^* \in \mathcal{P}$*. Then there exists an increasing, right-continuous optional (that means adapted) process* $C$ *with* $C_0 = 0$ *and a predictable* $\xi$ *such that*

$$U_t = U_0 + \int_0^t \xi_s dS_s - C_t.$$

The following theorem is a duality result of Xu [48], based on a result for expected utility maximization problems due to Kramkov and Schachermayer [27]. First, we review the assumptions of this theory. Let the discounted asset price process be a semimartingale $S = (S_t)_{t\in[0,T]}$ on a complete probability space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)_{t\in[0,T]}$ that satisfies the usual conditions.

**Assumption C.4.** *Let the set of equivalent martingale measures* $\mathcal{P}$ *satisfy* $\mathcal{P} \neq \emptyset$*.*

**Assumption C.5.** *Let* $U(x,\omega) : (\mathbb{R}_+ \times \Omega) \to \mathbb{R}_+$ *be a utility function that satisfies* $U(\cdot,\omega)$ *is continuous, increasing and concave for any fixed* $\omega$ *and* $U(0,\omega) = 0$*. The right-hand derivative satisfies* $U^r(0,\omega) > 0$ *and* $U^r(\infty,\omega) = \lim_{x\to\infty} U^r(x,\omega) = 0$ *for all* $\omega \in \Omega$*.*

**Assumption C.6.** *Let* $U$ *satisfy* $U(x,\omega) \leq Z(\omega) \; P - a.s.$ *for all* $x \geq 0$ *and let* $Z \in L^1(\Omega, \mathcal{F}, P)$*.*

We introduce the set of admissible, self-financing value processes $V$ starting at initial capital $x > 0$

$$\mathcal{V}(x) := \{V : V_t = x + \int_0^t \xi_s dS_s \geq 0, \quad t \in [0,T]\}.$$

Let us denote the set of contingent claims super-replicable by some admissible self-financing strategies with initial capital $x$ by

$$\mathcal{C}(x) := \{g \in L^0(\Omega, \mathcal{F}, P) : 0 \leq g \leq V_T \text{ for some } V \in \mathcal{V}(x)\}.$$

We consider the optimization problem of maximizing the expected utility at time $T$

$$u(x) = \sup_{V \in \mathcal{V}(x)} E[U(V_T(\omega), \omega)] = \sup_{g \in \mathcal{C}(x)} E[U(g(\omega), \omega)]. \tag{C.2}$$

Let us define the set of processes $Y$ by

$$\mathcal{Y}(y) := \{Y \geq 0 : Y_0 = y \text{ and } VY \text{ is a } P\text{-supermartingale for any } V \in \mathcal{V}(1)\}$$

and the set of random variables $h$ by

$$\mathcal{D}(y) := \{h \in L^0(\Omega, \mathcal{F}, P) : 0 \leq h \leq Y_T \text{ for some } Y \in \mathcal{Y}(y)\}.$$

Let us consider the function $W(y, \omega) : (\mathbb{R}_+ \times \Omega) \to \mathbb{R}_+$ defined for $y \geq 0$ by

$$W(y, \omega) := \sup_{x \geq 0} \{U(x, \omega) - xy\}.$$

It holds $W(y, \omega) = (-U + \mathcal{I}_{\mathbb{R}_+})^*(-y, \omega)$. We assign a dual problem to (C.2) by

$$w(y) = \inf_{Y \in \mathcal{Y}(y)} E[W(Y_T(\omega), \omega)] = \inf_{h \in \mathcal{D}(y)} E[W(h(\omega), \omega)]. \tag{C.3}$$

The following duality theorem holds true.

**Theorem C.7** ([48], Theorem 1.9 and 1.13)**.** *Let Assumptions C.4, C.5 and C.6 be satisfied. Then, it holds:*
*(i) For $x > 0$ and $y > 0$ an optimal solution $\widetilde{g}(x) \in \mathcal{C}(x)$ to (C.2) exists and an optimal solution $\widetilde{h}(y) \in \mathcal{D}(y)$ to (C.3) exists.*
*(ii) The value functions $u$ and $w$ satisfy the following relationship*

$$\begin{aligned} w(y) &= \sup_{x>0} \{u(x) - xy\} \quad \textit{for any } y > 0 \textit{ and} \\ u(x) &= \inf_{y>0} \{w(y) + xy\} \quad \textit{for any } x > 0. \end{aligned}$$

*(iii) Let $x > 0$ and $y > 0$ such that $y \in \partial u(x)$. Then, $E[\widetilde{g}\widetilde{h}] = xy$ and $\widetilde{h} \in \partial U(\widetilde{g})$ $P-a.s.$, or equivalently, $\widetilde{g} \in -\partial W(\widetilde{h})$ $P-a.s.$ if and only if $\widetilde{g}$ solves (C.2) and $\widetilde{h}$ solves (C.3).*

**Remark C.8.** It is easy to verify that the results of Theorem C.7 remains true if we substitute the condition $U^r(0, \omega) > 0$ in Assumption C.5 by $U^r(0, \omega) \geq 0$ (see proofs of Theorem 1.9 and 1.13 in [48]).

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