Analysis for Phase Field Models of Cahn-Hilliard Type

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Contents

Introduction

Since the last century there exists an active interest to model and analyze phase transitions mathematically. Phase transitions arise within the most diverse ranges of the daily life. We present some examples in order to give a first impression.

- The transition between solid, fluid and gaseous phases or in other words vaporizing/condensing (fluid $\leftrightarrow$ gaseous), melting/freezing (solid $\leftrightarrow$ fluid) and sublimation/resublimation (solid $\leftrightarrow$ gaseous);
- The transition between ferromagnetic and paramagnetic phases in magnetic materials at the *Curie-temperature*;
- The transition of some metals to superconductors at very low temperatures;
- The Bose-Einstein-condensate; a state in which the matter is cooled down, almost up to the lowest absolute temperature (0 K$= -273,15$ °C).

The well-known classical two-phase *Stefan problem*, a model problem for the analytic description of a phase transition, received especially much attention in the past.

$$\begin{aligned}
\kappa_i \partial_t \theta^i - d_i \Delta \theta^i &= 0, \quad && \text{in } \Omega^i(t), \\
B\theta^1 &= b, && \text{on } \partial\Omega^1(t), \\
\theta^i &= 0, && \text{on } \Gamma(t), \\
[d\partial_\nu \theta] &= \ell V, && \text{on } \Gamma(t), \\
\theta^i(0) &= \theta^i_0, && \text{in } \Omega^i_0, \\
\Gamma(0) &= \Gamma_0.
\end{aligned}$$

Here $\Omega \subset \mathbb{R}^n$ is a homogeneous material, consisting of two separated phases. The initial state of these two phases at $t = 0$ is given by Ω^1_0 and Ω^2_0, respectively, which are separated by a sharp interface Γ_0. It is assumed that Γ_0 does not intersect the boundary of Ω, to avoid so-called *contact angle problems* whose mathematical treatment is a challenging task. We denote by $\Gamma(t)$ the position of the moving interface at time t and $\Omega^1(t)$, $\Omega^2(t)$ denote the two phases, separated by $\Gamma(t)$. κ_i and d_i are the heat capacities and the heat conductivities of each phase, respectively. The quantity $[d\partial_\nu \theta] := d_2 \partial_\nu \theta^2 - d_1 \partial_\nu \theta^1$ represents the jump of the normal derivatives of θ^1 and θ^2 across the interface $\Gamma(t)$ and ℓ is the *latent heat*, which is needed for the phase transition. The normal velocity of $\Gamma(t)$ is denoted by V and B means Dirichlet or Neumann conditions on the boundary $\partial\Omega$.

The Stefan problem has been extensively studied by a number of authors during the last decades and it is still in the focus of mathematical analysts. In this model one assumes that the interface, which separates the two phases of the system, is infinitely thin. However, instead of such a sharp interface one observes *smeared* interfaces in experiments, which have a thickness of approximately $10^{-8}cm = 1$Å, the atomic radius. So, in the fifties of the last century, mathematicians started to derive models, which take into account a certain width of the interface between the phases. In these models one or more extra variables are introduced, to describe the state of the system, the so-called *order parameters*. An order parameter is a measure for the degree of order in a system with extremes -1 for total disorder and +1 for complete order. Otherwise the order parameter is assumed to take values between -1 and +1. Examples for such parameters are the mass density of the system under consideration (often assumed to be a conserved quantity) or the magnetic flux in ferromagnetism. But there are quite more possibilities to define order parameters.

Due to the large variety of such models we want to mention here two classical and very famous ones, namely

- the non-isothermal *Cahn-Hilliard* equation and,
- the *Penrose-Fife* model.

In contrast to the Penrose-Fife Model, the Cahn-Hilliard equation is based on the assumption that the absolute temperature θ of the system is far from zero and has only a small deviation from a fixed value θ^*. Then one introduces the relative temperature function $\tilde{\theta} := \theta - \theta^*$ and the nonlinearities in the differential operators may be approximated by linear terms, such that the quasilinear Penrose-Fife Model becomes a semilinear system.

In this thesis we will study the following models for phase transitions.

$$\begin{gathered} \partial_t \psi - \Delta\mu = 0, \quad \mu = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta, \\ \partial_t \left(b(\vartheta) + \lambda(\psi)\right) - \Delta\vartheta = 0, \end{gathered} \tag{0.1}$$

and

$$\begin{gathered} \partial_t \psi - \operatorname{div}(a\partial_t\psi) = \operatorname{div}(B\nabla\mu) \\ \mu - c\cdot\nabla\mu = \beta\partial_t\psi - \Delta\psi + \Phi'(\psi). \end{gathered} \tag{0.2}$$

In (0.1) the function ϑ is the reciprocal of the absolute temperature of the system, if one sets $b(s) = -1/s$. In this case we obtain the classical conserved Penrose-Fife equations which were proposed by Penrose & Fife in [32]. Conversely, if we set $b(s) = s$, the result is the classical non-isothermal Cahn-Hilliard equation, proposed by Cahn & Hilliard in [8]. The second model (0.2) was proposed by Gurtin [16] in order to model the action of forces that are associated with microscopic configurations of atoms which are not considered in the derivation of the classical Cahn-Hilliard equation. In this connection one often speaks of *microforces*. The equations (0.2) are a generalization of the classical Cahn-Hilliard equation with *constant* temperature and they are known as the Cahn-Hilliard-Gurtin equations.

Let us explain the equations in details. The function $b(\vartheta)$ is a contribution to the internal energy e. In fact, it holds that $e = b(\vartheta) + \lambda(\psi)$. It is possible to choose other functions than $b(s) = -1/s$ or $b(s) = s$ for b, provided that they satisfy certain assumptions, which are introduced below, in order to guarantee the mathematical well-posedness of the system. The nonlinearity Φ is the so called *physical potential* which characterizes the two different phases of the physical system. A prominent and often used example is the *double-well* potential

$$\Phi(s) = \Phi_0(s^2 - 1)^2,$$

with some positive constant $\Phi_0 > 0$. The two distinct minima of Φ correspond to each of the two phases. We remark here that there is no maximum principle for (0.1) available, since the equation for ψ is of fourth order, hence the interval $[-1, 1]$ is not an invariant set for (0.1), in general. Therefore, some authors use logarithmic physical potentials of the form

$$\Phi(s) = \frac{\theta}{2}((1+s)\log(1+s) + (1-s)\log(1-s)) - \frac{\theta_c}{2}s^2,$$

to ensure that the order parameter takes values between -1 and +1. For results on problem (0.1) with logarithmic potentials we refer to Abels & Wilke [1], Bonfoh [5] and the references cited therein. Next, the function λ represents the latent heat, which is crucial for appearance of a phase transition. Two examples are given by

$$\lambda(s) = \lambda_0(s - s_*) \quad \text{and} \quad \lambda(s) = \lambda_0(s^2 - s_*^2),$$

where $\lambda_0, s_* > 0$. The *chemical potential* μ is responsible for the mass transport inside the system and it is given by a variational derivative of an appropriate underlying energy functional. Last but not least, in (0.2), $B \in \mathbb{R}^{n\times n}$, $a, c \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ are free parameters with the constraint, that the $(n+1)\times(n+1)$-dimensional matrix

$$\begin{bmatrix} \beta & c^{\mathsf{T}} \\ a & B \end{bmatrix}$$

is positive semidefinite.

During the last years, many papers with different settings, were addressed to the global well-posedness and the qualitative behavior of the solutions of (0.1) and (0.2), as time tends to infinity. Due to the large variety of approaches and settings, we will only give a selection of papers, which represent the most important results, from our point of view. In the case of the Penrose-Fife equation Brokate & Sprekels [7] and Zheng [48] proved global well-posedness in an L_2-setting if the spatial dimension is equal to 1. For higher space dimensions this is still an open question. Sprekels & Zheng showed global well-posedness of the non-conserved equations (that is $\partial_t\psi = -\mu$) in higher space dimensions in [43], a similar result can be found in the article of Laurencot [25]. Concerning asymptotic behavior we refer to the articles of Kubo, Ito & Kenmochi [23], Shen & Zheng [40], Feireisl & Schimperna [14] and Rocca & Schimperna [38]. The last two authors studied well-posedness and qualitative behavior of solutions to the non-conserved Penrose-Fife equations. To be precise, they proved that each solution converges to a steady state, as time tends to infinity. Shen & Zheng [40] established the existence of attractors for the non-conserved equations, whereas Kubo, Ito & Kenmochi [23] studied the non-conserved as well as the conserved Penrose-Fife equations. Beside the proof of global well-posedness in the sense of weak solutions they also showed the existence of a global attractor.

In case of the Cahn-Hilliard equation global well-posedness has been shown by Hoffmann & Rybka [39], Elliott & Zheng [13], Racke & Zheng [37], Prüss, Racke & Zheng [34] and Prüss & Wilke [36]. The difference of these papers is the choice of the topology and the different boundary conditions. Hoffmann & Rybka [39] proved the existence of classical solutions with classical boundary conditions, while Elliott & Zheng [13] showed well-posedness in an L_2-setting, also with classical boundary conditions. Racke & Zheng [37] were the first, who considered the isothermal Cahn-Hilliard equation with a *dynamic boundary condition* of the form

$$\partial_t\psi - \sigma_s\Delta_\Gamma\psi + \gamma\partial_\nu\psi + \kappa(\psi - h) = 0,$$

where $\sigma_s, \gamma > 0$ and $\kappa \geq 0$. Such a condition has been proposed by Kenzler et. al. in [22]. The physical interpretation of this boundary condition is that the phase function ψ has the preferred value h at the boundary and the system is trying to approach this value by surface tension. The authors in [37] obtained global well-posedness of the system in an L_2-setting. Later, Prüss, Racke & Zheng [34] and Prüss & Wilke [36] extended the result of Racke & Zheng [37] to an L_p-setting and to the non-isothermal Cahn-Hilliard equation. In particular, in [34] and [36], the authors proved maximal L_p-regularity of the Cahn-Hilliard equation (isothermal / non-isothermal) with dynamic boundary conditions. Concerning the asymptotic behavior of solutions we refer to the references [2], [10], [39], [46] and [36]. There the authors show that every solution converges to a steady state as time tends to infinity by applying the so called *Lojasiewicz-Simon inequality* (see below).

The literature for the Cahn-Hilliard-Gurtin equations is not so vast, in contrast to the preceding two models. Results on existence and uniqueness can be found in the papers of Bonfoh & Miranville [6], Miranville [28], [29] and Miranville, Piétrus & Rakotoson [30]. In any of these papers the authors use a variational approach and some energy estimates to obtain global well-posedness in an L_2-setting, with some artificial periodic boundary conditions for a cuboid in $\mathbb{R}^3$. The qualitative behavior of solutions of the Cahn-Hilliard-Gurtin equation has been investigated in [6], [30] and [31]. In [6] and [30] the authors proved the existence of finite dimensional attractors, whereas Miranville & Rougirel [31] showed that each solution converges to a steady state, again with the help of the Lojasiewicz-Simon inequality. A basic restrictive assumption of

Miranville & Rougirel [31] is that the norms $|a|$, $|c|$ and $|B - I|$ have to be sufficiently small. This is needed for the proof of relative compactness. In this sense the Cahn-Hilliard-Gurtin equations are a small perturbation of the Cahn-Hilliard equation, if $\beta = 0$.

The purpose of this thesis is twofold. The first objective is to prove the global well-posedness of (0.1) and (0.2), subject to suitable and physically reasonable boundary and initial conditions in the strong sense of L_p. This will be done with the help of maximal regularity tools, which have recently been developed (cf. Chapter 1). The results on global well-posedness and maximal L_p-regularity for each model are completely new and we obtain optimal regularity results for each problem under consideration. The second part in the analysis of (0.1) and (0.2) is devoted to the study of the long-time behavior of the solutions. To be precise, we will show that each solution converges to a steady state as time tends to infinity, without a restriction on the initial value. In particular we show that for *any* initial value in an appropriate energy space, there exists a solution of the stationary problem such that the corresponding orbit converges to this steady state. Moreover, we are able to remove the smallness restrictions on a, c and $B - I$ in the Cahn-Hilliard-Gurtin equations, which were assumed by Miranville & Rougirel [31]. A convergence result for the conserved Penrose-Fife equations is not known to the author. The same holds for the non-isothermal Cahn-Hilliard equation. To prove convergence, we need to know that for each of the above models, there exists a strict Lyapunov functional $E : V \to \mathbb{R}$, defined on a suitable energy space V, which satisfies the *Lojasiewicz-Simon inequality* near some point φ in the ω-limit set of the solution. That is, there exist constants $\delta, C > 0,\ s \in (0, 1/2]$ such that for all $v \in V$ with $|v - \varphi|_V \le \delta$ there holds

$$|E(v) - E(\varphi)|^{1-s} \le C|E'(v)|_{V^*}, \tag{0.3}$$

where V^* is the topological dual space of V. In his famous work on semi-analytic and subanalytic sets [27], Lojasiewicz proved this inequality for analytic functions E in case $V \subset \mathbb{R}^n$. In the same paper he indicated that this inequality can be used to prove the convergence to steady states of solutions of the following gradient systems

$$\dot{u} + \nabla f(u) = 0.$$

Later, Simon [41] gave a proof of an infinite dimensional version of this inequality for analytic functionals E, defined on Hilbert spaces V. Recently, Jendoubi simplified Simon's rather complicated proof in [18] and he called the infinite dimensional version, the *Lojasiewicz-Simon* inequality. Since then it has been reproved in several articles; we refer to Chill [9] for a comprehensive study of this inequality in a functional analytic setting.

This thesis is structured as follows. In *Chapter 1* we explain some mathematical notations and function spaces and we introduce a joint functional calculus for two operators, which is due to Kalton & Weis [20] and which is an extension of the well-known Dunford calculus for closed operators. Furthermore we state a result on maximal L_p-regularity of parabolic problems, which is taken from Denk, Hieber & Prüss [11].

In *Chapter 2* we study the quasilinear equations (0.1) in case of classical boundary conditions. To be precise, we assume Neumann boundary conditions on μ, ψ and ϑ. The property of maximal L_p-regularity of a suitable linearized problem and the contraction mapping principle yield a unique local solution (ψ, ϑ) with optimal regularity

$$\psi \in H^1_p(0, T; L_p(\Omega)) \cap L_p(0, T; H^4_p(\Omega)),$$

and

$$\vartheta \in H^1_p(0, T; L_p(\Omega)) \cap L_p(0, T; H^2_p(\Omega)),$$

provided the nonlinearities are locally Lipschitz continuous. To establish global existence we first derive some higher order a priori estimates, by applying methods of semigroup theory, bootstrap arguments and the *Gagliardo-Nirenberg* interpolation inequality. On the basis of the results of Lieberman [26] and Ladyzhenskaya, Solonnikov & Uraltseva [24] we may then conclude that the solution ϑ of the quasilinear heat equation $(0.1)_2$ is Hölder continuous in time and space. As we will see, this is already sufficient for the global existence of (ψ, ϑ) in the optimal regularity class. For the proof of global well-posedness we need the following conditions on the nonlinear physical potential Φ.

$$\Phi \in C^{4-}(\mathbb{R}), \quad |\Phi'''(s)| \leq c_1(1+|s|^\gamma), \quad \text{for all } s \in \mathbb{R}, \tag{0.4}$$

$$\Phi(s) \geq -\frac{\eta}{2}s^2 - c_2, \quad \text{for all } s \in \mathbb{R}, \tag{0.5}$$

with some constants $c_i > 0$, $\eta < \lambda_1$, and λ_1 is the first nontrivial eigenvalue of the Neumann Laplacian. Furthermore we require $\gamma < 3$ if $n = 3$. We note that (0.5) is crucial, to obtain some energy estimates, which are needed in the proof of global well-posedness. For the latent heat λ we impose the growth condition

$$\lambda \in C^{4-}(\mathbb{R}), \quad |\lambda'(s)| \leq c_1(1+|s|),\ c_1 > 0, \quad \text{for all } s \in \mathbb{R} \text{ and } \lambda'', \lambda''' \in L_\infty(\mathbb{R}). \tag{0.6}$$

In the last section of Chapter 2 we investigate the long time behavior of the solution (ψ, ϑ). To this end the nonlinearities Φ, λ and b are assumed to be real analytic. We first show that the orbits $\psi(\mathbb{R}_+)$ and $\vartheta(\mathbb{R}_+)$ are relatively compact in a suitable energy space, to obtain useful properties of the ω-limit set $\omega(\psi, \vartheta)$. As it has already been pointed out, the Lojasiewicz-Simon inequality (0.3) will play a crucial role in the proof of convergence. The main result for the Penrose-Fife type reads as follows.

Theorem 0.0.1. *Let $p > (n+2)/2$, $p \geq 2$, $n \leq 3$, $J = [0,T]$, $\Omega \subset \mathbb{R}^n$ open, bounded with compact boundary $\Gamma = \partial\Omega \in C^4$ and assume that Φ and λ satisfy (0.4)-(0.6). Suppose furthermore that $b \in C^2(\mathbb{R})$ and that there exists $\sigma > 0$ such that we have the a priori bounds $b'(\vartheta(t,x)) \geq \sigma > 0$ for all $(t,x) \in \mathbb{R}_+ \times \Omega$ and $\vartheta \in L_\infty(\mathbb{R}_+ \times \Omega)$. Then there exists a unique solution (ψ, ϑ) of (0.1) with*

$$\psi \in H^1_p(J; L_p(\Omega)) \cap L_p(J; H^4_p(\Omega)),$$

and

$$\vartheta \in H^1_p(J; L_p(\Omega)) \cap L_p(J; H^2_p(\Omega)),$$

provided that the initial values (ψ_0, ϑ_0) satisfy the following conditions.

(i) $\psi_0 \in B^{4-4/p}_{pp}(\Omega)$;

(ii) $\vartheta_0 \in B^{2-2/p}_{pp}(\Omega)$;

(iii) $\partial_\nu \Delta\psi_0 = 0$, *if* $p > 5$;

(iv) $\partial_\nu \psi_0 = 0$, *if* $p > 5/3$;

(v) $\partial_\nu \vartheta_0 = 0$, *if* $p > 3$.

Moreover, if Φ, λ and b are real analytic, then the limits

$$\lim_{t\to\infty} \psi(t) =: \psi_\infty \quad \text{and} \quad \lim_{t\to\infty} \vartheta(t) =: \vartheta_\infty$$

exist in $H^1_2(\Omega)$ and $H^r_2(\Omega)$, $(r \in (0,1))$, respectively, and $(\psi_\infty, \vartheta_\infty)$ is a solution of the stationary problem.

At this point, we want to remark, that the bounds on $b'(\vartheta)$ and ϑ imply that the equation $(0.1)_2$ does not degenerate.

Chapter 3 is concerned with the non-isothermal Cahn-Hilliard equation (0.1), i.e. $b(s) = s$. This time we use a Neumann boundary condition for μ and a Robin boundary condition for ϑ. For the order parameter ψ we will take a *dynamic boundary condition* of the form

$$\partial_t\psi - \sigma_s\Delta_\Gamma\psi + \gamma\partial_\nu\psi + \kappa(\psi - h) = 0,$$

where $\sigma_s, \gamma > 0$ and $\kappa \geq 0$. Here we are interested in solutions with optimal regularity

$$\psi \in H^1_p(0,T; L_p(\Omega)) \cap L_p(0,T; H^4_p(\Omega)),$$

$$\vartheta \in H^1_p(0,T;L_p(\Omega)) \cap L_p(0,T;H^2_p(\Omega)),$$

and

$$\psi|_\Gamma \in H^1_p(0,T;W^{2-1/p}_p(\Gamma)) \cap L_p(0,T;W^{4-1/p}_p(\Gamma)).$$

It is a remarkable fact that we can reduce the non-isothermal Cahn-Hilliard equation to a problem for the function ψ. This is due to the fact, that this time the heat equation $(0.1)_2$ is linear. The structure of Chapter 3 is similar to that of Chapter 2. First we will use a result of Prüss, Racke & Zheng [34] to obtain maximal L_p-regularity of the linearized equations. Then the contraction mapping principle yields a unique local solution with optimal regularity. For the global existence we use an energy estimate on the solution ψ and again maximal L_p-regularity tools. Here we have to apply the Gagliardo-Nirenberg inequality several times and we have to deal with the additional traces, which are a result of the dynamic boundary condition. Having global existence of ψ, the same is true for ϑ by equation $(0.1)_2$. The proof of convergence of the solutions to a steady state follows the lines of Chapter 2. Let us state the main result for the non-isothermal Cahn-Hilliard equation.

Theorem 0.0.2. *Let $p \geq 2$, $n \leq 3$, $J = [0,T]$, $\Omega \subset \mathbb{R}^n$ open, bounded with compact boundary $\Gamma = \partial\Omega \in C^4$ and assume that Φ satisfies (0.4)-(0.5). Let furthermore $\lambda \in C^{4-}(\mathbb{R})$ and $\lambda', \lambda'', \lambda''' \in L_\infty(\mathbb{R})$. Then there exists a unique solution (ψ, ϑ) of (0.1) with $b(s) = s$ such that*

$$\psi \in H^1_p(J;L_p(\Omega)) \cap L_p(J;H^4_p(\Omega)),$$

$$\vartheta \in H^1_p(J;L_p(\Omega)) \cap L_p(J;H^2_p(\Omega)),$$

and

$$\psi|_\Gamma \in H^1_p(J;W^{2-1/p}_p(\Gamma)) \cap L_p(J;W^{4-1/p}_p(\Gamma)),$$

provided that the initial values (ψ_0, ϑ_0) satisfy the following conditions.

(i) $\psi_0 \in \{u \in B^{4-4/p}_{pp}(\Omega) : u|_\Gamma \in B^{4-3/p}_{pp}(\Gamma)\}$,

(ii) $\vartheta_0 \in B^{2-2/p}_{pp}(\Omega)$,

(iii) $\partial_\nu \Delta\psi_0 = \partial_\nu(\Phi'(\psi_0) - \lambda'(\psi_0)\vartheta_0)$, *if* $p > 5$,

(iv) $\alpha\vartheta_0 + \partial_\nu\vartheta_0 = 0$, *if* $p > 3$.

Moreover, if Φ and λ are real analytic and h is constant, then the limits

$$\lim_{t\to\infty} \psi(t) =: \psi_\infty \quad \text{and} \quad \lim_{t\to\infty} \vartheta(t) =: \vartheta_\infty$$

exist in $\{u \in H^1_2(\Omega) : u|_\Gamma \in H^1_2(\Gamma)\}$ and $L_2(\Omega)$, respectively, and $(\psi_\infty, \vartheta_\infty)$ is a solution of the stationary problem.

The results of this chapter are a joint work with Jan Prüß.

Finally in *Chapter 4* we will analyze the Cahn-Hilliard-Gurtin equations (0.2) with the Neumann boundary conditions $B\nabla\mu\cdot\nu = 0$ and $\partial_\nu\psi = 0$ on $\Gamma = \partial\Omega$. At this point we want to emphasize that in contrast to the Penrose-Fife Model or the Cahn-Hilliard equation, the chemical potential μ is *not* explicitly given. Instead, it is a solution of an elliptic problem, hence the Cahn-Hilliard-Gurtin equations form an elliptic-parabolic problem. This is the main difficulty in the analysis. Due to this fact the treatment of the linearized equations is more involved than in the two previous chapters. In a first step we will solve the full space problem in $\mathbb{R}^n$, without boundary condition. The second step is the analysis of the equations in the half space $\mathbb{R}^n_+$. Then, via a localization technique, transform of coordinates and perturbation, we obtain maximal L_p-regularity of the linear part for a bounded domain $\Omega \subset \mathbb{R}^n$ with compact boundary $\partial\Omega \in C^3$. To be precise we obtain for the linearized problem solutions of class

$$\psi \in H^1_p(0,T;H^1_p(\Omega)) \cap L_p(0,T;H^3_p(\Omega)),$$

and

$$\mu \in L_p(0,T; H_p^2(\Omega)).$$

The assumptions on the data are

(A) For $A := \beta B - \frac{1}{2}(a \otimes c + c \otimes a)$, there is a constant $\varepsilon > 0$, such that $(A\xi|\xi) \geq \varepsilon|\xi|^2$ for all $\xi \in \mathbb{R}^n$,

and $a, c \in C^1_{ub}(\overline{\Omega}; \mathbb{R}^n),\ B \in C^1_{ub}(\overline{\Omega}; \mathbb{R}^{n\times n})$ as well as,

$$\operatorname{div} a(x) = \operatorname{div} c(x) = 0, \quad \text{for all } x \in \Omega, \text{ and } a(x)\cdot\nu(x) = c(x)\cdot\nu(x) = 0, \quad \text{for all } x \in \Gamma, \tag{0.7}$$

$$B(x)\tau(x)\cdot\nu(x) = 0, \quad \text{for all } x \in \Gamma \text{ and all } \tau(x) \in T_x\Gamma, \tag{0.8}$$

where $T_x\Gamma$ denotes the tangential space in a point $x \in \Gamma$ on $\Gamma = \partial\Omega$. Condition (A) ensures that the Cahn-Hilliard-Gurtin problem equations form an elliptic-parabolic problem. Assumption (0.7) is useful for the proof of dissipativity of certain differential operators and (0.8) is used in the half space case for a symmetry argument.

The remaining part of Chapter 4 is similar to Chapters 2 and 3. With the same technique as in Chapter 3 we are able to establish global existence under the assumption

(H) There exists a constant $\varepsilon > 0$ such that

$$\beta z_0^2 + (a + c|z_1)z_0 + (Bz_1|z_1) \geq \varepsilon(z_0^2 + |z_1|^2),$$

for all $(z_0, z_1) \in \mathbb{R} \times \mathbb{R}^n$.

This condition is needed for some crucial energy estimates to obtain global well-posedness of the system. In the Appendix of Chapter 4, we show that (H) already implies (A).

Finally, in the last section, the Lojasiewicz-Simon inequality is the tool which leads to the convergence theorem. This time, we have the following result.

Theorem 0.0.3. *Let $p \geq 2$, $n \leq 3$, $J = [0,T]$, $\Omega \subset \mathbb{R}^n$ open, bounded with compact boundary $\Gamma = \partial\Omega \in C^3$ and assume that Φ satisfies* (0.4)-(0.5). *Suppose furthermore that the data (β, a, c, B) satisfy (H),* (0.7), (0.8) *and let $a, c \in C^1_{ub}(\overline{\Omega}; \mathbb{R}^n),\ B \in C^1_{ub}(\overline{\Omega}; \mathbb{R}^{n\times n})$. Then there exists a unique solution (ψ, ϑ) of* (0.2) *with*

$$\psi \in H_p^1(J; H_p^1(\Omega)) \cap L_p(J; H_p^3(\Omega)),$$

and

$$\mu \in L_p(J; H_p^2(\Omega)).$$

provided that the initial value ψ_0 satisfies the following conditions.

(i) $\psi_0 \in B_{pp}^{3-2/p}(\Omega)$,

(ii) $\partial_\nu \psi_0 = 0$, *if* $p > 3/2$.

Moreover, if Φ is real analytic, then the limit

$$\lim_{t\to\infty} \psi(t) =: \psi_\infty$$

exists in $H_2^1(\Omega)$ and ψ_∞ is a solution of the stationary problem.

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Chapter 1

Mathematical Preliminaries

1.1 Some notation, Function spaces, Laplace- and Fourier transform

In this section we fix some notations used throughout the thesis and recall some basic definitions.

By $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ we denote the sets of natural numbers, integers, real and complex numbers, respectively. Let further $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$. Furthermore $(\cdot|\cdot)$ means the scalar product in $\mathbb{C}^n$.

The symbol $\mathcal{B}(X, Y)$ means the space of all bounded, linear operators from X to Y and we write for short $\mathcal{B}(X) = \mathcal{B}(X, X)$. If A is a linear operator in some Banach space X then $D(A), R(A), N(A)$ stand for domain, range and null space of A, respectively, while $\rho(A), \sigma(A)$ designate resolvent set and spectrum of A. For a closed operator A we denote by D_A the domain of A equipped with the graph norm.

In what follows, let X be a Banach space. For $\Omega \subset \mathbb{R}^n$ open or closed, $C(\Omega; X)$ and $C_{ub}(\Omega; X)$ stand for the continuous resp. bounded and uniformly continuous functions $f : \Omega \to X$. Furthermore, if $\Omega \subset \mathbb{R}^n$ is open and $k \in \mathbb{N}$, $C^k(\overline{\Omega}; X)$ $(C^k_{ub}(\overline{\Omega}; X))$ means the space of all functions $f : \overline{\Omega} \to X$ for which the partial derivatives $\partial^\alpha f$ exist on Ω and can be continuously extended to a function belonging to $C(\overline{\Omega}; X)$ $(C_{ub}(\overline{\Omega}; X))$, for $0 \le |\alpha| \le k$. As usual $C^{(k+1)-}(\Omega; X)$ is the space of all functions in $C^k(\Omega; X)$ whose k^{th} derivative is locally Lipschitz continuous. Lastly, by $C_0^\infty(\Omega)$ we denote the space of all infinitely times continuously differentiable functions $f : \Omega \to X$, having compact support in Ω, that is, the set

$$\operatorname{supp} f := \overline{\{y \in \Omega : f(y) \neq 0\}} \subset \Omega$$

is compact.

If $\Omega \subset \mathbb{R}^n$ is a Lebesgue measurable set and $1 \le p < \infty$, then $L_p(\Omega; X)$ denotes the space of all (equivalence classes of) Bochner-measurable functions $f : \Omega \to X$ such that

$$|f|_p := \left(\int_\Omega |f(y)|_X^p \, dy \right)^{1/p} < \infty.$$

$L_p(\Omega; X)$ is a Banach space when normed by $|\cdot|_p$. Similarly, $L_\infty(\Omega; X)$ stands for the space of (equivalence classes of) Bochner-measurable essentially bounded functions $f : \Omega \to X$, with norm

$$|f|_\infty := \operatorname*{ess\,sup}_{y \in \Omega} |f(y)|.$$

With this norm, $L_\infty(\Omega; X)$ is a Banach space. For the special case that $X = L_q(G)$, with $G \subset \mathbb{R}^n$ Lebesgue measurable, we write the norm of f in $L_p(\Omega; X)$ for short as $|f|_{p,q}$ for all $1 \le p, q \le \infty$. From time to time we will also use the notation $(\cdot|\cdot)_2$ for the inner product in $L_2(\Omega)$.

For $\Omega \subset \mathbb{R}^n$ open, $H_p^m(\Omega; X)$, $m \in \mathbb{N}$ denotes the classical Sobolev space, that is, the space of all functions $f : \Omega \to X$ having distributional derivatives $\partial^\alpha f \in L_p(\Omega; X)$ of order $0 \le |\alpha| \le m$. The norm in $H_p^m(\Omega; X)$ is given by

$$|f|_{H_p^m(\Omega;X)} = \left(\sum_{|\alpha|\le m} |\partial^\alpha f|_p^p \right)^{1/p} \quad \text{for } 1 \le p < \infty,$$

and

$$|f|_{H_\infty^m(\Omega;X)} = \max_{|\alpha|\le m} |\partial^\alpha f|_\infty, \quad \text{for } p = \infty.$$

Further, we define the Bessel potential spaces $H_p^{sm}(\Omega; X)$, by means of complex interpolation, i.e.

$$H_p^{sm}(\Omega; X) = [L_p(\Omega; X); H_p^m(\Omega; X)]_s\ , \quad \text{for } s \in (0, 1).$$

We will frequently also use the Besov spaces $B_{pp}^{sm}(\Omega; X)$ which can be defined via real interpolation, i.e.

$$B_{pp}^{sm}(\Omega; X) = (L_p(\Omega; X); H_p^m(\Omega; X))_{s,p}\ , \quad \text{for } s \in (0, 1).$$

Recall that $B_{pp}^s(\Omega; X) = W_p^s(\Omega; X)$, provided that $s \notin \mathbb{N}$, where $W_p^s(\Omega; X)$ denotes the Sobolev-Slobodeckij space. For a definition of this space we refer to Triebel [44]. In case $J = [0, T] \subset \mathbb{R}$ is an interval, we denote by ${}_0H_p^s(J; X)$ the space of all functions $f : J \to X$ in $H_p^s(J; X)$, such that $f|_{t=0} = 0$, whenever the trace at $t = 0$ exists.

If not indicated otherwise, $f * g$ means the convolution, defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau)\ d\tau, \quad t \ge 0,$$

for two functions, supported on the half line $\mathbb{R}_+$.

Let $f \in L_{1,loc}(\mathbb{R}_+; X)$ be of subexponential growth, i.e. $\int_0^\infty e^{-\omega t}|f(t)|\ dt < \infty$ with some $\omega \in \mathbb{R}$. Then the *Laplace transform* of f is defined by

$$(\mathcal{L}f)(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\ dt, \quad \operatorname{Re}\lambda \ge \omega.$$

If $f \in C_0^\infty(\mathbb{R}^n; X)$, then the *Fourier transform* of f is given by

$$(\mathcal{F}f)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(x|\xi)} f(x)\ dx, \quad \xi \in \mathbb{R}^n.$$

1.2 Sectorial operators, $\mathcal{H}^\infty$-calculus, $\mathcal{R}$-boundedness

We begin with the definition of *sectorial operators*.

Definition 1.2.1. *Let X be a complex Banach space and A a closed linear operator in X. Then A is called* sectorial*, if $\overline{D(A)} = X$, $\overline{R(A)} = X$, $N(A) = \{0\}$, $(-\infty, 0) \in \rho(A)$ and*

$$\sup_{t>0} t|(t + A)^{-1}| \le M,$$

for some constant $M > 0$.

The class of these operators will be denoted by $\mathcal{S}(X)$. Let furthermore

$$\Sigma_\phi := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \phi\}.$$

Then one may apply a Neumann series argument, to conclude that $\rho(-A) \supset \Sigma_\phi$ for some $\phi > 0$ and

$$\sup_{\lambda\in\Sigma_\phi} |\lambda(\lambda + A)^{-1}| < \infty,$$

provided that $A \in \mathcal{S}(X)$. Therefore it makes sense to define the *spectral angle* of $A \in \mathcal{S}(X)$ by

$$\phi_A := \inf\{\phi : \rho(-A) \supset \Sigma_{\pi-\phi},\ \sup_{\lambda\in\Sigma_{\pi-\phi}} |\lambda(\lambda+A)^{-1}| < \infty\}.$$

Now we turn our attention to the $\mathcal{H}^\infty$-calculus. Let $\phi \in (0,\pi]$ and define the space of holomorphic functions on Σ_ϕ by $\mathcal{H}(\Sigma_\phi) := \{f : \Sigma_\phi \to \mathbb{C} \text{ holomorphic}\}$. Furthermore, we define the space

$$\mathcal{H}^\infty(\Sigma_\phi) := \{f : \Sigma_\phi \to \mathbb{C} \text{ holomorphic and bounded}\}.$$

The space $\mathcal{H}^\infty(\Sigma_\phi)$ is a Banach algebra, when equipped with the norm

$$|f|_\infty^\phi = \sup_{\lambda\in\Sigma_\phi} |f(\lambda)|.$$

In addition, we define $\mathcal{H}_0(\Sigma_\phi) := \bigcup_{\alpha,\beta<0} \mathcal{H}_{\alpha,\beta}(\Sigma_\phi)$, where $\mathcal{H}_{\alpha,\beta}(\Sigma_\phi) := \{f \in \mathcal{H}(\Sigma_\phi) : |f|_{\alpha,\beta}^\infty < \infty\}$, with

$$|f|_{\alpha,\beta}^\infty := \sup_{|\lambda|\le 1} |\lambda^\alpha f(\lambda)| + \sup_{|\lambda|\ge 1} |\lambda^{-\beta} f(\lambda)|.$$

Suppose that $A \in \mathcal{S}(X)$ and let $\phi \in (\phi_A,\pi)$. Select any $\varphi \in (\phi,\pi)$ and denote by Γ_φ the contour, defined by $\Gamma_\varphi(t) = -te^{i\varphi}$ if $t \le 0$ and $\Gamma_\varphi(t) = te^{-i\varphi}$ if $t \ge 0$. Then the *Dunford integral*

$$f(A) = \frac{1}{2\pi i}\int_{\Gamma_\varphi} f(\lambda)(\lambda - A)^{-1}\, d\lambda, \quad f \in \mathcal{H}_0(\Sigma_\phi),$$

converges in $\mathcal{B}(X)$ and does not depend on the choice of φ. Moreover, via the mapping $\Phi_A(f) = f(A)$, it defines a functional calculus $\Phi_A : \mathcal{H}_0(\Sigma_\phi) \to \mathcal{B}(X)$.

Definition 1.2.2. *A sectorial operator A in X admits a bounded $\mathcal{H}^\infty$-calculus if there are $\phi > \phi_A$ and a constant $K_\phi < \infty$ such that*

$$|f(A)| \le K_\phi |f|_\infty^\phi,$$

for all $f \in \mathcal{H}_0(\Sigma_\phi)$.

The class of these operators will be denoted by $\mathcal{H}^\infty(X)$. If $A \in \mathcal{H}^\infty(X)$, then the functional calculus for A on $\mathcal{H}_0(\Sigma_\phi)$ extends uniquely to $\mathcal{H}^\infty(\Sigma_\phi)$, by approximation.

We consider next operators with bounded imaginary powers. This subclass has been introduced by Prüss & Sohr [35]. First, note that for any $A \in \mathcal{S}(X)$, one can define complex powers A^z of A, where $z \in \mathbb{C}$ is arbitrary.

Definition 1.2.3. *A sectorial operator A in X is said to admit bounded imaginary powers, if $A^{is} \in \mathcal{B}(X)$ for each $s \in \mathbb{R}$ and there exists a constant $C > 0$ such that $|A^{is}| \le C$ for $|s| \le 1$.*

The class of such operators will be denoted by $\mathcal{BIP}(X)$ and we define the *power angle* of A by

$$\theta_A := \limsup_{|s|\to\infty} \frac{1}{|s|} \log |A^{is}|.$$

Since for each $s \in \mathbb{R}$, the function $f_s(z) = z^{is}$ belongs to $\mathcal{H}^\infty(\Sigma_\phi)$, $\phi \in (0,\pi)$, we have the inclusions

$$\mathcal{H}^\infty(X) \subset \mathcal{BIP}(X) \subset \mathcal{S}(X).$$

We come now to $\mathcal{R}$-sectorial operators. First we will define the notion of $\mathcal{R}$-boundedness.

Definition 1.2.4. *Let X and Y be Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{B}(X,Y)$ is called $\mathcal{R}$-bounded, if there is a constant $C > 0$ and $p \in [1,\infty)$, such that for each $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ and for all independent, symmetric, $\{-1,1\}$-valued random variables ε_j on a probability space $(\Omega,\mathcal{M},\mu)$ the inequality*

$$|\sum_{j=1}^N \varepsilon_j T_j x_j|_{L_p(\Omega;Y)} \le C|\sum_{j=1}^N \varepsilon_j x_j|_{L_p(\Omega;X)},$$

is valid. The smallest of such constants $C > 0$ is called $\mathcal{R}$-bound of $\mathcal{T}$, which is denoted by $\mathcal{R}(\mathcal{T})$.

It follows from Kahane's inequality, that the definition of $\mathcal{R}$-boundedness is independent of $p \in [1, \infty)$, see [12, Remark 3.2 (2)]. Now we are in a position to define $\mathcal{R}$-sectorial operators.

Definition 1.2.5. *Let X be a complex Banach space and assume that A is a sectorial operator in X. The A is called $\mathcal{R}$-sectorial, if the set*

$$\{t(t+a)^{-1} : t > 0\}$$

is $\mathcal{R}$-bounded. The $\mathcal{R}$-angle $\phi_A^{\mathcal{R}}$ of A is defined by means of

$$\phi_A^{\mathcal{R}} := \inf\{\theta \in (0, \pi) : \mathcal{R}_A(\pi - \theta) < \infty\},$$

where

$$\mathcal{R}_A(\theta) := \mathcal{R}(\lambda(\lambda + a)^{-1} : |\arg \lambda| \leq \theta\}.$$

The class of $\mathcal{R}$-sectorial operators in X is denoted by $\mathcal{RS}(X)$ and if the Banach space X *is of class* $\mathcal{HT}$, that is, the Hilbert transform

$$(Hf)(t) := \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\varepsilon \leq |s| \leq 1/\varepsilon} f(t-s) \, \frac{ds}{s}$$

acts as a bounded operator in $L_p(\mathbb{R}; X)$ for some $p \in (1, \infty)$, we have the inclusion

$$\mathcal{BIP}(X) \subset \mathcal{RS}(X), \quad \phi_A^R \leq \theta_A.$$

We close this section with the definition of an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus.

Definition 1.2.6. *Let X be a complex Banach space and suppose that $A \in \mathcal{H}^\infty(X)$. The operator A is said to admit an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus if the set*

$$\{f(A) : f \in \mathcal{H}^\infty(\Sigma_\phi), |f|_\infty^\phi \leq 1\}$$

is $\mathcal{R}$-bounded for some $\phi > 0$. The $\mathcal{RH}^\infty$-angle $\phi_A^{R\infty}$ of A is defined as the infimum of the $\mathcal{R}$-bounds, w.r.t. such angles ϕ.

The class of such operators is denoted by $\mathcal{RH}^\infty(X)$.

1.3 Joint functional calculus, Sums of closed operators

In this section, we state a result, which is due to Kalton & Weis [20] and is called *operator valued $\mathcal{H}^\infty$ functional calculus.*

Theorem 1.3.1. *Let X be a Banach space, $A \in \mathcal{H}^\infty(X)$, $F \in \mathcal{H}^\infty(\Sigma_\phi; \mathcal{B}(X))$ such that*

$$F(\lambda)(\mu - A)^{-1} = (\mu - A)^{-1}F(\lambda), \quad \mu \in \rho(A), \ \lambda \in \Sigma_\phi.$$

Suppose in addition, that $\phi > \phi_A^\infty$ and $\mathcal{R}(F(\Sigma_\phi)) < \infty$. Then $F(A) \in \mathcal{B}(X)$ and $|F(A)|_{\mathcal{B}(X)} \leq C_A \mathcal{R}(F(\Sigma_\phi))$, where C_A denotes a constant, only depending on A.

It is remarkable, that a conclusion of this theorem is a version of the well-known *Dore-Venni Theorem.*

Corollary 1.3.2. *Suppose $A \in \mathcal{H}^\infty(X)$ and $B \in \mathcal{RS}(X)$ are commuting operators, such that $\phi_A^\infty + \phi_B^R < \pi$. Then $A+B$ with domain $D(A+B) = D(A) \cap D(B)$ is closed, $A+B \in \mathcal{S}(X)$, with $\phi_{A+B} \leq \max\{\phi_A^\infty, \phi_B^R\}$ and*

$$|Ax| + |Bx| \leq C|(A+B)x|, \quad x \in D(A) \cap D(B), \tag{1.1}$$

for some constant $C > 0$. In particular, if A or B is invertible, then $A+B$ is invertible as well.

Example: Let $1 < p < \infty$, $J = [0,T]$, $X = L_p(J \times \mathbb{R}^n) \in \mathcal{HT}$ and $(t,x) \in J \times \mathbb{R}^n$. Set $B = \partial_t$, with domain $D(B) =_0 H_p^1(\mathbb{R}_+; L_p(\mathbb{R}^n))$ and define A as the natural extension of $-\Delta_x$ in $L_p(\mathbb{R}^n)$, with $D(-\Delta_x) = H_p^2(\mathbb{R}^n)$ to X, that is, $D(A) = L_p(J; H_p^2(\mathbb{R}^n))$ and $Af = -\Delta_x f$ for each $f \in D(A)$. Then A and B are commuting operators and $A, B \in \mathcal{H}^\infty(X)$ with $\mathcal{H}^\infty$-angles $\phi_A^\infty = 0$ and $\phi_B^\infty = \pi/2$. Since B is invertible and $\mathcal{H}^\infty(X) \subset \mathcal{RS}(X)$ it follows that $A + B$ is invertible and the estimate (1.1) is valid for each $x \in D(A) \cap D(B)$. In other words, the parabolic problem

$$\begin{aligned} \partial_t u - \Delta_x u &= f, \\ u(0) &= 0, \end{aligned}$$

has a unique solution $u \in {}_0H_p^1(J; L_p(\mathbb{R}^n)) \cap L_p(J; H_p^2(\mathbb{R}^n))$, for each $f \in L_p(J \times \mathbb{R}^n)$ and the estimate

$$|\partial_t u|_X + |\Delta_x u|_X \leq C|f|_X,$$

for some constant $C > 0$ is valid.

The following result is known as the *mixed derivative theorem* and is due to Sobolevskii [42].

Proposition 1.3.3. *Suppose A, B are sectorial operators in a Banach space X, commuting in the resolvent sense. Assume that their spectral angles satisfy the parabolicity condition $\phi_A + \phi_B < \pi$. Further suppose that the pair (A, B) is coercively positive, i.e. $A + \mu B$ with natural domain $D(A) \cap D(B)$ is closed for each $\mu > 0$ and there is a constant $M > 0$ such that*

$$|Ax| + \mu|Bx| \leq M|Ax + \mu Bx|, \quad x \in D(A) \cap D(B),\ \mu > 0.$$

Then there exists a constant $C > 0$ such that

$$|A^\alpha B^{1-\alpha} x| \leq C|Ax + Bx|,$$

for all $x \in D(A) \cap D(B)$ and $\alpha \in [0, 1]$.

1.4 Model problems, Maximal L_p-regularity

In this paragraph we collect some known results on the solvability and regularity of second order problems on the halfline, which occur in a natural way after a transformation of an arbitrary domain $\Omega \subset \mathbb{R}^n$ to a halfspace $\mathbb{R}_+^n$. At the end of this section we state a result for maximal L_p-regularity of parabolic partial differential equations with inhomogeneous boundary conditions.

We start with the following problem with a Dirichlet boundary condition

$$\begin{aligned} -u''(y) + F^2 u(y) &= f(y), \quad y > 0, \\ u(0) &= \phi, \end{aligned} \tag{1.2}$$

in $L_p(\mathbb{R}_+; X)$, where X is a Banach space. The following result is due to Prüss [33].

Theorem 1.4.1. *Suppose X is a Banach space of class $\mathcal{HT}$, $p \in (1, \infty)$. Let $F \in \mathcal{BIP}(X)$ be invertible with power angle $\theta_F < \pi/2$, and let D_F^j denote the domain $D(F^j)$, equipped with its graph norm, $j = 1, 2$.*

Then (1.2) has a unique solution $u \in H_p^2(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; D_F^2)$ if and only if the following conditions are satisfied.

(i) $f \in L_p(\mathbb{R}_+; X)$;

(ii) $\phi \in D_F(2 - 1/p, p)$.

In this case we have in addition $u \in H_p^1(\mathbb{R}_+; D_F^1)$.

There is a corresponding result for the abstract second order problem with a Robin condition

$$\begin{aligned} -u''(y) + F^2 u(y) &= f(y), \quad y > 0, \\ -u'(0) + Du(0) &= \psi, \end{aligned} \tag{1.3}$$

in $L_p(\mathbb{R}_+; X)$

Theorem 1.4.2. *Suppose X is a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$. Let $F \in \mathcal{BIP}(X)$ be invertible with power angle $\theta_F < \pi/2$, and let D_F^j denote the domain $D(F^j)$, equipped with its graph norm, $j = 1, 2$. Suppose that D is sectorial in X, belongs to $\mathcal{BIP}(\overline{R(D)})$, commutes with F and is such that $\theta_F + \theta_D < \pi$.*

Then (1.3) has a unique solution $u \in H_p^2(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; D_F^2)$ if and only if the following conditions are satisfied.

(i) $f \in L_p(\mathbb{R}_+; X)$;

(ii) $\psi \in D_F(1 - 1/p, p)$.

In this case we have in addition $u \in H_p^1(\mathbb{R}_+; D_F^1)$.

We turn our attention now to problems of the form

$$\begin{aligned} \partial_t u(t,x) + \mathcal{A}(t,x,D)u(t,x) &= f(t,x), \quad t \in J, x \in \Omega, \\ \mathcal{B}_j(t,x,D)u(t,x) &= g_j(t,x), \quad t \in J,\ x \in \Gamma,\ j = 1,\dots,m, \\ u(0,x) &= u_0(x), \quad x \in \Omega, \end{aligned} \tag{1.4}$$

where $J = [0,T]$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with compact boundary $\Gamma = \partial\Omega \in C^{2m},\ m \in \mathbb{N}$. The partial differential operator $\mathcal{A}(t,x,D)$ has order $2m,\ m \in \mathbb{N}$, and the boundary operators $\mathcal{B}_j(t,x,D)$ are of order $m_j < 2m,\ j = 1,\dots,m,\ m_j \in \mathbb{N}$. To be precise, let E be a Banach space and let

$$\begin{aligned} \mathcal{A}(t,x,D) &= \sum_{|\alpha| \le 2m} a_\alpha(t,x) D^\alpha, \\ \mathcal{B}_j(t,x,D) &= \sum_{|\beta| \le m_j} b_{j\beta}(t,x) D^\beta, \end{aligned}$$

where a_α and $b_{j\beta}$ are variable coefficients with values in $\mathcal{B}(E)$ and $D^\alpha = (-i)^{|\alpha|}\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$. The principal parts $\mathcal{A}_\#(t,x,D)$, $\mathcal{B}_{j,\#}(t,x,D)$ of the operators $\mathcal{A}(t,x,D)$, $\mathcal{B}_j(t,x,D)$ and the coefficients of $\mathcal{A}(t,x,D)$ and $\mathcal{B}(t,x,D)$ should satisfy the following conditions:

(E) *(Ellipticity of the principal part) For all $t \in J,\ x \in \bar{\Omega}$ and $\zeta \in \mathbb{R}^n$ with $|\zeta| = 1$ it holds that*

$$\sigma(\mathcal{A}_\#(t,x,\zeta)) \subset \mathbb{C}_+,$$

*i.e. $\mathcal{A}(t,x,D)$ is **normal elliptic**.*

(LS) (*Lopatinskii-Shapiro condition) For all $t \in J,\ x \in \Gamma$, $\zeta \in \mathbb{R}^n$ with $(\zeta, \nu(x)) = 0$, $\lambda \in \bar{\mathbb{C}}_+$ with $|\lambda| + |\zeta| \neq 0$ and all $h \in E^m$ the system of ordinary differential equations*

$$\begin{aligned} \lambda v(y) + \mathcal{A}_\#(t,x,\zeta + i\nu(x)\partial_y)v(y) &= 0, \quad y > 0 \\ \mathcal{B}_{j,\#}(t,x,\zeta + i\nu(x)\partial_y)v(0) &= h_j, \quad j = 1,\dots,m \end{aligned}$$

admits a unique solution $v \in C_0(\mathbb{R}_+; E)$.

(A) *There are $r_k, s_k \ge p$ with $\frac{1}{s_k} + \frac{n}{2mr_k} < 1 - \frac{k}{2m}$, such that*

$$\begin{aligned} &a_\alpha \in L_{s_k}(J_0; (L_{r_k} + L_\infty)(\Omega; \mathcal{B}(E))), \quad |\alpha| = k < 2m, \\ &a_\alpha \in C(J_0 \times \bar{\Omega}; \mathcal{B}(E)), \quad |\alpha| = 2m. \end{aligned}$$

(B) *There are $s_{jk}, r_{jk} \ge p$ with $\frac{1}{s_{jk}} + \frac{n-1}{2mr_{jk}} < \kappa_j + \frac{m_j - k}{2m}$, such that*

$$b_{j\beta} \in W_{s_{jk}}^{\kappa_j}(J_0; L_{r_{jk}}(\Gamma; \mathcal{B}(E))) \cap L_{s_{jk}}(J_0; W_{r_{jk}}^{2m\kappa_j}(\Gamma; \mathcal{B}(E))),\ |\beta| = k \le m_j.$$

For the data f, g_j, u_0 we suppose the following conditions:

(D)

(i) $f \in L_p(J_0 \times \Omega; E) =: X$,

(ii) $g_j \in W_p^{\kappa_j}(J_0; L_p(\Gamma; E)) \cap L_p(J_0; W_p^{2m\kappa_j}(\Gamma; E)) =: Y_j$,*with* $\kappa_j := \frac{2m-m_j-1/p}{2m}$,

(iii) $u_0 \in B_{pp}^{2m(1-1/p)}(\Omega; E) =: X_p$,

(iv) *If* $\kappa_j > 1/p$, *Then* $\mathcal{B}_j(x, D)u_0(x) = h_j(0, x)$, *for all* $x \in \Gamma$.

The next result is due to Denk, Hieber & Prüss [11, Theorem 2.1].

Theorem 1.4.3. *Let* $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ *be a bounded domain, with compact boundary* $\Gamma = \partial\Omega \in C^{2m}$. *Assume that* $E \in \mathcal{HT}$ *and suppose that the conditions **(E)**, **(LS)**, **(A)** and **(B)** are satisfied. Then* (1.4) *has a unique solution*

$$u \in H_p^1(J_0; L_p(\Omega; E)) \cap L_p(J_0; H_p^{2m}(\Omega; E)),$$

if and only if the data f, g_j *and* u_0 *satisfy the conditions in **(D)**. Furthermore, the inequality*

$$|\partial_t u|_X + |D^{2m}u|_X \leq M(|f|_X + |u_0|_{X_p} + \sum_{j=1}^{m} |g_j|_{Y_j})$$

holds for some constant $M > 0$.

Chapter 2

Conserved Penrose-Fife Type Models

2.1 Derivation of the Model

Here we are interested in the conserved Penrose-Fife type equations

$$\begin{aligned} \partial_t\psi = \Delta\mu, \quad \mu = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta, \quad t \in J,\ x \in \Omega, \\ \partial_t\left(b(\vartheta) + \lambda(\psi)\right) - \Delta\vartheta = 0, \quad t \in J,\ x \in \Omega, \end{aligned} \tag{2.1}$$

For the case $b(s) = -1/s$ we obtain the conserved Penrose-Fife equations, for which we will give a short derivation. In this context, we will follow the lines of Alt & Pawlow [3]. We start with the free energy functional

$$F(\psi, \vartheta) = \int_\Omega \left(\frac{1}{2}|\nabla\psi|^2 + \Phi(\psi) + \log\vartheta - \lambda(\psi)\vartheta\right)\, dx.$$

By definition, the chemical potential μ is given by the variational derivative of F with respect to ψ, i.e.

$$\mu = \frac{\delta F}{\delta\psi} = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta.$$

According to [32, (2.8)] the internal energy e of the system under consideration is given by the variational derivative of F with respect to ϑ, i.e.

$$e = -\frac{\delta F}{\delta\vartheta} = -\frac{1}{\vartheta} + \lambda(\psi).$$

To obtain kinetic equations we assume that the order parameter ψ and the internal energy e are conserved quantities. The according conservation laws are given by

$$\partial_t\psi + \operatorname{div} j = 0, \quad \partial_t e + \operatorname{div} q = 0,$$

with the boundary conditions $(j|\nu) = (q|\nu) = 0$, where ν is the outer unit normal on $\partial\Omega$. Here q is the heat flux, which in this paper is assumed to be given by the Fourier law $q = -\nabla\vartheta$ and j denotes the phase flux of the order parameter ψ which is assumed to be of the form $j = -\nabla\mu$, which is a constitutive and well accepted law. Since $(j|\nu) = (q|\nu) = 0$ we obtain from the constitutive laws the boundary conditions $\partial_\nu\mu = 0$ and $\partial_\nu\vartheta = 0$ for the chemical potential μ and the inverse temperature ϑ, respectively. Since (2.1) is of fourth order with respect to the function ψ we need an additional boundary condition. An appropriate and classical one from a variational point of view is $\partial_\nu \psi = 0$. Finally, this yields the initial-boundary value problem

$$\begin{aligned} \partial_t \psi - \Delta\mu = f_1, \quad \mu = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta, \quad & t \in J,\ x \in \Omega, \\ \partial_t \left(b(\vartheta) + \lambda(\psi)\right) - \Delta\vartheta = f_2, \quad & t \in J,\ x \in \Omega, \\ \partial_\nu \mu = g_1, \quad & t \in J,\ x \in \partial\Omega, \\ \partial_\nu \psi = g_2, \quad & t \in J,\ x \in \partial\Omega, \\ \partial_\nu \vartheta = g_3, \quad & t \in J,\ x \in \partial\Omega, \\ \psi(0) = \psi_0,\ \vartheta(0) = \vartheta_0, \quad & t = 0,\ x \in \Omega, \end{aligned} \tag{2.2}$$

The functions $f_j, g_j, \psi_0, \vartheta_0, \Phi, \lambda$ and b are given. In the following sections we will prove well-posedness of (2.2) for solutions in the optimal regularity class

$$\psi \in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^4(\Omega)),$$

$$\vartheta \in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega)),$$

where $J = [0, T]$ and $\Omega \subset \mathbb{R}^n$ is open and bounded, with compact boundary $\Gamma = \partial\Omega \in C^4$.

2.2 The Linear Problem

In this section we deal with a linearized version of (2.2).

$$\begin{aligned} \partial_t u + \Delta^2 u + \Delta(\eta_1 v) = f_1, \quad & t \in J,\ x \in \Omega, \\ \partial_t v - a_0 \Delta v + \eta_2 \partial_t u = f_2, \quad & t \in J,\ x \in \Omega, \\ \partial_\nu \Delta u + \partial_\nu(\eta_1 v) = g_1, \quad & t \in J,\ x \in \partial\Omega, \\ \partial_\nu u = g_2, \quad & t \in J,\ x \in \partial\Omega, \\ \partial_\nu v = g_3, \quad & t \in J,\ x \in \partial\Omega, \\ u(0) = u_0,\ v(0) = v_0, \quad & t = 0,\ x \in \Omega. \end{aligned} \tag{2.3}$$

Here $\eta_1 = \eta_1(x), \eta_2 = \eta_2(x), a_0 = a_0(x)$ are given functions such that

$$\eta_1 \in H_p^2(\Omega) \cap W_\infty^1(\Omega),\ \eta_2 \in H_p^1(\Omega) \cap L_\infty(\Omega) \quad \text{and} \quad a_0 \in C(\overline{\Omega}). \tag{2.4}$$

We assume furthermore that $a_0(x) \geq \sigma > 0$ for all $x \in \overline{\Omega}$ and some constant $\sigma > 0$. Hence equation $(2.3)_2$ does not degenerate. We are interested in solutions

$$u \in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^4(\Omega)) =: Z^1$$

and

$$v \in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega)) =: Z^2$$

of (2.3). By the well-known trace theorems (cf. [4, Theorem 4.10.2])

$$Z^1 \hookrightarrow C(J; B_{pp}^{4-4/p}(\Omega)) \quad \text{and} \quad Z^2 \hookrightarrow C(J; B_{pp}^{2-2/p}(\Omega)), \tag{2.5}$$

we necessarily have $u_0 \in B_{pp}^{4-4/p}(\Omega) =: X_p^1$, $v_0 \in B_{pp}^{2-2/p}(\Omega) =: X_p^2$ and the compatibility conditions

$$\partial_\nu \Delta u_0 + \partial_\nu(\eta_1 v_0) = g_1|_{t=0}, \quad \partial_\nu u_0 = g_2|_{t=0}, \quad \text{as well as} \quad \partial_\nu v_0 = g_3|_{t=0},$$

should be satisfied, whenever $p > 5$, $p > 5/3$ and $p > 3$, respectively (cf. Theorem 1.4.3). For the forthcoming calculations we need the following assumption.

To solve (2.3) we will assume in the sequel that $p > (n+2)/2,\ p \geq 2,\ n \in \mathbb{N}$, wherefore we have the embeddings

$$B_{pp}^{4-4/p}(\Omega) \hookrightarrow H_p^2(\Omega) \cap C^1(\overline{\Omega}) \quad \text{and} \quad B_{pp}^{2-2/p}(\Omega) \hookrightarrow H_p^1(\Omega) \cap C(\overline{\Omega}) \tag{2.6}$$

at our disposal.

Suppose that the function $u \in Z^1$ in (2.3) is already known. Then in a first step we will solve the linear heat equation

$$\partial_t v - a_0 \Delta v = f_2 - \eta_2 \partial_t u, \tag{2.7}$$

subject to the boundary and initial conditions $\partial_\nu v = g_3$ and $v(0) = v_0$. By the properties of the function a_0 we may apply Theorem 1.4.3 to obtain a unique solution $v \in Z^2$ of (2.7), provided that $f_2 \in L_p(J_0 \times \Omega)$, $v_0 \in B_{pp}^{2-2/p}(\Omega)$,

$$g_3 \in W_p^{1/2-1/2p}(J; L_p(\Gamma)) \cap L_p(J; W_p^{1-1/p}(\Gamma)) =: Y_3,$$

and the compatibility condition $\partial_\nu v_0 = g_3|_{t=0}$ if $p > 3$ is valid. The solution may then be represented by the variation of parameters formula

$$v(t) = v_1(t) - \int_0^t e^{-A(t-s)} \eta_2 \partial_t u(s)\, ds, \tag{2.8}$$

where A denotes the L_p-realization of the differential operator $\mathcal{A}(x) = -a_0(x)\Delta_N$, Δ_N means the Neumann-Laplacian and e^{-At} stands for the bounded analytic semigroup, which is generated by $-A$ in $L_p(\Omega)$. Furthermore the function $v_1 \in Z^2$ solves the linear problem

$$\partial_t v_1 - a_0 \Delta v_1 = f_2, \quad \partial_\nu v_1 = g_3, \quad v_1(0) = v_0.$$

We fix a function $w^* \in Z^1$ such that $w^*|_{t=0} = u_0$ and make use of (2.8) and the fact that $(u - w^*)|_{t=0} = 0$ to obtain

$$\begin{aligned} v(t) &= v_1(t) + v_2(t) - (\partial_t + A)^{-1} \eta_2 \partial_t (u - w^*) \\ &= v_1(t) + v_2(t) - \partial_t^{1/2} (\partial_t + A)^{-1} \partial_t^{1/2} \eta_2 (u - w^*), \end{aligned}$$

with $v_2(t) := -\int_0^t e^{-A(t-s)} \eta_2 \partial_t w^*$. Set $v^* = v_1 + v_2 \in Z^2$ and

$$F(u) = -\partial_t^{1/2} (\partial_t + A)^{-1} \partial_t^{1/2} \eta_2 (u - w^*).$$

Then we may reduce (2.3) to the problem

$$\begin{aligned} \partial_t u + \Delta^2 u &= \Delta G(u) + f_1, & t \in J,\ x \in \Omega, \\ \partial_\nu \Delta u &= \partial_\nu G(u) + g_1, & t \in J,\ x \in \partial\Omega, \\ \partial_\nu u &= g_2 & t \in J,\ x \in \partial\Omega, \\ u(0) &= u_0, & t = 0,\ x \in \Omega, \end{aligned} \tag{2.9}$$

where $G(u) := -\eta_1(F(u) + v^*)$. For a given $T \in (0, T_0]$ we set

$${}_0Z^1 := \{u \in Z^1(T) : u|_{t=0} = 0\}$$

and

$$E_0 := X(T) \times Y_1(T) \times Y_2(T), \qquad {}_0E_0 := \{(f, g, h) \in E_0 : g|_{t=0} = h|_{t=0} = 0\}$$

where $X(T) := L_p(J \times \Omega)$,

$$Y_1(T) := W_p^{1/4-1/4p}(J; L_p(\Gamma)) \cap L_p(J; W_p^{1-1/p}(\Gamma)),$$

and

$$Y_2(T) := W_p^{3/4-1/4p}(J; L_p(\Gamma)) \cap L_p(J; W_p^{3-1/p}(\Gamma)).$$

The spaces Z^1 and E_0 are endowed with the canonical norms $|\cdot|_1$ and $|\cdot|_0$, respectively. Let $B = -\Delta_\Gamma$ be the Laplace-Beltrami operator on Γ and denote by $e^{-B^2 t}$ the analytic semigroup, generated by $-B^2$. By Theorem 1.4.3 there exists a unique global solution $u^* \in Z^1$ of the linear problem

$$\begin{aligned} \partial_t u^* + \Delta^2 u^* &= f_1, \quad t \in J,\ x \in \Omega, \\ \partial_\nu \Delta u^* &= g_1 - e^{-B^2 t} g_0, \quad t \in J,\ x \in \partial\Omega, \\ \partial_\nu u^* &= g_2 \quad t \in J,\ x \in \partial\Omega, \\ u^*(0) &= u_0, \quad t = 0,\ x \in \Omega, \end{aligned}$$

provided that $f_1 \in X(T_0)$, $g_j \in Y_j(T_0)$, $j = 1, 2$, and $u_0 \in X_p^1$. Here $g_0 = 0$ if $p < 5$ and $g_0 = g_1|_{t=0} - \partial_\nu \Delta u_0$, if $p > 5$. We will now apply the contraction mapping principle to solve (2.9). For that purpose we define a linear operator $L : Z^1 \to E_0$ by

$$Lw = \begin{bmatrix} \partial_t w + \Delta^2 w \\ \partial_\nu \Delta w \\ \partial_\nu w \end{bmatrix}.$$

Considering L as an operator from ${}_0Z^1$ to ${}_0E_0$ we obtain from Theorem 1.4.3 that L is bounded and bijective, hence an isomorphism. The open mapping theorem then implies that L is invertible with bounded inverse L^{-1}. Next we define a mapping $\tilde{G} : Z^1 \times {}_0Z^1 \to {}_0E_0$ by

$$\tilde{G}(u^*, w) = \begin{bmatrix} \Delta G(u^* + w) \\ \partial_\nu G(u^* + w) - \tilde{g}_0 \\ 0 \end{bmatrix},$$

where $\tilde{g}_0 = 0$ if $p < 5$ and $\tilde{g}_0 = e^{-B^2 t}[\partial_\nu G(u^*)|_{t=0}]$, if $p > 5$. It is not difficult to see that $u := u^* + w$ is a solution of (2.9) if and only if

$$Lw = \tilde{G}(u^*, w) \quad \text{or equivalently} \quad w = L^{-1}\tilde{G}(u^*, w).$$

Consider a ball $\mathbb{B}_R \subset {}_0Z^1$ and define a mapping $\mathcal{T} : \mathbb{B}_R \to {}_0Z^1$ by $\mathcal{T}w = L^{-1}\tilde{G}(u^*, w)$. In order to apply the contraction mapping principle we have to show that $\mathcal{T}$ is a self mapping, i.e. $\mathcal{T}\mathbb{B}_R \subset \mathbb{B}_R$ and that $\mathcal{T}$ defines a strict contraction on $\mathbb{B}_R$, i.e. there exists a constant $\kappa < 1$, such that

$$|\mathcal{T}w - \mathcal{T}\bar{w}|_1 \le \kappa |w - \bar{w}|_1,$$

for all $w, \bar{w} \in \mathbb{B}_R$. Firstly we show that the operator $\Delta\partial_t^{1/2}(\partial_t + A)^{-1}\partial_t^{1/2}$ is of lower order compared to $(\partial_t + \Delta^2)u$, $u \in Z^1$. By the mixed-derivative theorem we obtain

$$u \in Z^1 \hookrightarrow H_p^{3/4}(J; H_p^1(\Omega)) \hookrightarrow H_p^s(J; H_p^1(\Omega)),$$

for every $s \in (0, 3/4)$. Moreover by (2.6) it holds that $\eta_2 \in H_p^1(\Omega)$, hence $\eta := \eta_2(u - w) \in {}_0H_p^{3/4}(J; H_p^1(\Omega))$. To see this, we compute

$$\begin{aligned} |\nabla(\eta_2(u - w))|_{L_p(\Omega;\mathbb{R}^n)} &\le |\eta_2 \nabla(u - w)|_{L_p(\Omega;\mathbb{R}^n)} + |(u - w)\nabla\eta_2|_{L_p(\Omega;\mathbb{R}^n)} \\ &\le |\eta_2|_{L_\infty(\Omega)}|u - v|_{H_p^1(\Omega)} + |u - w|_{L_\infty(\Omega;\mathbb{R}^n)}|\eta_2|_{H_p^1(\Omega)} < \infty. \end{aligned}$$

The regularity w.r.t. the variable t is clear, since η_2 does not depend on t. It follows that

$$(\partial_t + A)^{-1}\partial_t^{1/2}\eta \in {}_0H_p^{s+1/2}(J; H_p^1(\Omega)) \cap {}_0H_p^{s-1/2}(J; H_p^3(\Omega)) \hookrightarrow {}_0H_p^{s+\theta-1/2}(J; H_p^{3-2\theta}(\Omega)),$$

for each $\theta \in (0, 1)$ and $s \in (1/2, 3/4)$. Thus, it holds that

$$\partial_t^{1/2}(\partial_t + A)^{-1}\partial_t^{1/2} : {}_0H_p^s(J; H_p^1(\Omega)) \to {}_0H_p^{s+\theta-1}(J; H_p^{3-2\theta}(\Omega)), \tag{2.10}$$

for all $s \in (1/2, 3/4)$ and $\theta \in (1 - s, 1)$. In particular, this shows that $\Delta\partial_t^{1/2}(\partial_t + A)^{-1}\partial_t^{1/2}$ is of lower order, if e.g. $\theta = 1/2$.

Since L has the property of maximal L_p-regularity we obtain the estimate

$$|\mathcal{T}w|_1 = |L^{-1}\tilde{G}(u^*, w)|_1 \leq M\left(|\Delta G(w+u^*)|_{X(T)} + |\partial_\nu G(w+u^*)|_{Y_1(T)} + |g_0|_{Y_1(T)}\right), \tag{2.11}$$

with a constant $M > 0$ which does not depend on T, since the time trace of w at $t = 0$ vanishes whenever it exists. For convenience we will use the notation $\tilde{w} = w + u^*$. Making use of (2.6) we estimate as follows

$$\begin{aligned}|\Delta G(w+u^*)|_{X(T)} &\leq |\Delta(\eta_1 v^*)|_{X(T)} + |\Delta(\eta_1 F(\tilde{w}))|_{X(T)} \\ &\leq |\Delta(\eta_1 v^*)|_{X(T)} + |F(\tilde{w})\Delta\eta_1|_{X(T)} + 2|\nabla\eta_1 \cdot \nabla F(\tilde{w})|_{X(T)} + |\eta_1 \Delta F(\tilde{w})|_{X(T)} \\ &\leq |\Delta(\eta_1 v^*)|_{X(T)} + |\Delta\eta_1|_{L_p(\Omega)}|F(\tilde{w})|_{L_p(J;L_\infty(\Omega))} \\ &\qquad + 2|\nabla\eta_1|_{L_\infty(\Omega)}|\nabla F(\tilde{w})|_{X(T)} + |\eta_1|_{L_\infty(\Omega)}|\Delta F(\tilde{w})|_{X(T)}.\end{aligned}$$

Note that by (2.6) it holds that $\eta_1 v^* \in L_p(J; H_p^2(\Omega))$ and therefore $|\Delta(\eta_1 v^*)|_{X(T)} \to 0$ as $T \to 0$, since $\eta_1 v^*$ is a fixed function. The same holds for the function $|g_0|_{Y_1(T)}$. Choose $\theta = 1/2$ in (2.10). Then we have the embedding

$$H_p^2(\Omega) \hookrightarrow L_\infty(\Omega), \quad p > (n+2)/2,$$

at our disposal and by Hölder's inequality and (2.10) we obtain

$$|F(\tilde{w})|_{L_p(J;L_\infty(\Omega))} \leq T^{1/r'p}|F(\tilde{w})|_{L_{rp}(J;H_p^2(\Omega))} \leq CT^{1/r'p}|\eta_2(\tilde{w} - w^*)|_{H_p^s(J;H_p^1(\Omega))},$$

where $1/r + 1/r' = 1$ and $r' > 0$ is sufficiently large. Now, for an arbitrarily small $\varepsilon > 0$, the embedding

$${}_0W_p^{s+\varepsilon}(J; L_p(\Omega)) \hookrightarrow {}_0H_p^s(J; L_p(\Omega)) \tag{2.12}$$

is valid (cf. Triebel [44] or Kalton et al. [19]). Therefore, it holds that

$$\begin{aligned}|\eta_2(\tilde{w} - w^*)|_{H_p^s(J;H_p^1(\Omega))} &\leq C|\eta_2(\tilde{w} - w^*)|_{H_p^s(J;L_p(\Omega))} + |\nabla(\eta_2(\tilde{w} - w^*))|_{H_p^s(J;L_p(\Omega))} \\ &\leq C|\eta_2|_{L_\infty(\Omega)}|\tilde{w} - w^*|_{W_p^{s+\varepsilon}(J;L_p(\Omega))} \\ &\qquad + C|\eta_2|_{L_\infty(\Omega)}|\nabla(\tilde{w} - w^*)|_{W_p^{s+\varepsilon}(J;L_p(\Omega))} + C|\nabla\eta_2|_{L_p(\Omega)}|\tilde{w} - w^*|_{W_p^{s+\varepsilon}(J;L_\infty(\Omega))},\end{aligned}$$

where $C > 0$ is independent of T, since $(\tilde{w} - w^*)|_{t=0} = 0$. By the mixed derivative theorem we obtain the embeddings

$${}_0Z^1(T) \hookrightarrow {}_0H_p^\theta(J; H_p^{4(1-\theta)}(\Omega)) \hookrightarrow {}_0W_p^{s+\varepsilon}(J; L_\infty(\Omega)) \quad \text{if} \quad s \in \left(\frac{1}{2}, \frac{n+4}{2(n+2)}\right) \tag{2.13}$$

since we assume $p > (n+2)/2$. Here $\varepsilon > 0$ has to be sufficiently small. This yields

$$|F(\tilde{w})|_{L_p(J;L_\infty(\Omega))} \leq CT^{1/r'p}|\tilde{w} - w^*|_{Z^1(T)} \leq CT^{1/r'p}(R + |u^*|_{Z^1(T)} + |w^*|_{Z^1(T)}).$$

One more time we make use of (2.10) with $\theta = 1/2$ to obtain

$$|\nabla F(\tilde{w})|_{X(T)} + |\Delta F(\tilde{w})|_{X(T)} \leq CT^{1/r'p}|\eta_2(\tilde{w} - w^*)|_{H_p^s(J;H_p^1(\Omega))},$$

and then as before this implies the estimate

$$|\nabla F(\tilde{w})|_{X(T)} + |\Delta F(\tilde{w})|_{X(T)} \leq CT^{1/r'p}(R + |u^*|_{Z^1(T)} + |w^*|_{Z^1(T)}).$$

All together there exists a constant $C > 0$ and a function $\kappa = \kappa(T)$ with $\kappa(T) \to 0$ as $T \to 0$ such that

$$|\Delta G(\tilde{w})|_{X(T)} \leq \kappa(T)(1 + R). \tag{2.14}$$

Now we turn to the estimate for $\partial_\nu G$ in $Y_1(T)$. Trace theory yields the embedding

$$H_p^{1/2}(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega)) \hookrightarrow Y_1(T).$$

Hence, by the above calculations it suffices to estimate the term $G(w+u^*)$ in $H_p^{1/2}(J;L_p(\Omega))$ for each $w \in \mathbb{B}_R$. Then we have again by Hölder's inequality

$$
\begin{aligned}
|G(w+u^*)|_{H_p^{1/2}(J;L_p(\Omega))} &\leq |\eta_1|_{L_\infty(\Omega)}|F(\tilde w)+v^*|_{H_p^{1/2}(J;L_p(\Omega))}\\
&\leq |\eta_1|_{L_\infty(\Omega)}\Big(|F(\tilde w)|_{H_p^{1/2}(J;L_p(\Omega))}+|v^*|_{H_p^{1/2}(J;L_p(\Omega))}\Big)\\
&\leq C|\eta_1|_{L_\infty(\Omega)}\Big(|\eta_2(\tilde w-w^*)|_{H_p^{1/2}(J;L_p(\Omega))}+|v^*|_{H_p^{1/2}(J;L_p(\Omega))}\Big)\\
&\leq C|\eta_1|_{L_\infty(\Omega)}\Big(|\eta_2|_{L_\infty(\Omega)}|\tilde w-w^*|_{H_p^{1/2}(J;L_p(\Omega))}+|v^*|_{H_p^{1/2}(J;L_p(\Omega))}\Big) \quad (2.15)\\
&\leq C|\eta_1|_{L_\infty(\Omega)}\Big(T^{1/r'p}|\eta_2|_{L_\infty(\Omega)}|\tilde w-w^*|_{H_{rp}^{1/2}(J;L_p(\Omega))}+|v^*|_{H_p^{1/2}(J;L_p(\Omega))}\Big)\\
&\leq \Big(T^{1/r'p}|\tilde w-w^*|_1+|v^*|_{H_p^{1/2}(J;L_p(\Omega))}\Big)\\
&\leq \kappa(T)\,(1+R)\,,
\end{aligned}
$$

with a function $\kappa=\kappa(T)$, such that $\kappa(T)\to 0$ as $T\to 0$ and $r'>0$ has to be sufficiently large. Here we used the fact that

$$\partial_t^{1/2}(\partial_t+A)^{-1}\partial_t^{1/2} : {}_0H_p^{1/2}(J;L_p(\Omega)) \to {}_0H_p^{1/2}(J;L_p(\Omega)),$$

is a bounded linear operator and $|v^*|_{H_p^{1/2}(J;L_p(\Omega))}\to 0$ as $T\to 0$, since v^* is a fixed function. Combining (2.14) and (2.15) with (2.11) we obtain the self mapping property of $\mathcal{T}$, provided that T is sufficiently small. For the contraction mapping property we use again maximal L_p-regularity to obtain

$$
\begin{aligned}
&|\mathcal{T}w-\mathcal{T}\bar w|_1\\
&\quad\leq M\left(|\Delta(G(w+u^*)-G(\bar w+u^*))|_{X(T)}+|\partial_\nu(G(w+u^*)-G(\bar w+u^*))|_{Y_1(T)}\right). \qquad (2.16)
\end{aligned}
$$

Using the same methods as above and setting $w_1=w+u^*$, $w_2=\bar w+u^*$ we may estimate the first term on the right side of the latter inequality in the following way.

$$
\begin{aligned}
|\Delta(G(w+u^*)-G(\bar w+u^*))|_{X(T)} &\leq |(F(w_1)-F(w_2))\Delta\eta_1|_{X(T)}\\
&\qquad + 2|\nabla\eta_1\cdot\nabla(F(w_1)-F(w_2))|_{X(T)}\\
&\qquad\qquad + |\eta_1\Delta(F(w_1)-F(w_2))|_{X(T)}\\
&\leq CT^{1/r'p}|w_1-w_2|_1 = CT^{1/r'p}|w-\bar w|_1.
\end{aligned}
$$

Trace theory implies the estimate

$$|\partial_\nu(G(w+u^*)-G(\bar w+u^*))|_{Y_1(T)} \leq |G(w+u^*)-G(\bar w+u^*)|_{H_p^{1/2}(J;L_p(\Omega))\cap L_p(J;H_p^2(\Omega))}.$$

The same computations which lead to (2.15) yield

$$
\begin{aligned}
|G(w+u^*)-G(\bar w+u^*)|_{H_p^{1/2}(J;L_p(\Omega))} &\leq |\eta_1|_{L_\infty(\Omega)}|F(w_1)-F(w_2)|_{H_p^{1/2}(J;L_p(\Omega))}\\
&\leq CT^{1/r'p}|w_1-w_2|_1 = CT^{1/r'p}|w-\bar w|_1,
\end{aligned}
$$

whence we see that there exists a function $\kappa=\kappa(T)$ with $\kappa(T)\to 0$ as $T\to 0$ such that

$$|\mathcal{T}w-\mathcal{T}\bar w|_1 \leq \kappa(T)|w-\bar w|_1.$$

Choosing $T>0$ small enough we obtain the desired estimate. Finally the contraction mapping principle yields a unique fixed point $\tilde u\in\mathbb{B}_R$ of $\mathcal{T}$ or equivalently $\tilde u+u^*\in Z^1(T)$ is the unique local solution of (2.9). Then $v\in Z^2$ defined by (2.8), with u replaced by $\tilde u$, is the unique (local) solution of (2.7), hence the pair $(\tilde u,v)\in Z^1\times Z^2$ solves (2.3). Due to the linearity of (2.3), the invariance w.r.t time shifts and the property of maximal regularity the local solution exists globally in time. We summarize these considerations in

Theorem 2.2.1. *Let $n \in \mathbb{N}$, $p > (n+2)/2$, $p \geq 2$ and $p \neq 3, 5$. Suppose $\Omega \subset \mathbb{R}^n$ is bounded open with compact boundary $\Gamma = \partial\Omega \in C^4$ and let $J = [0, T]$. Assume that (2.4) holds and that there exists $\sigma > 0$ such that $a_0(x) \geq \sigma > 0$ for all $x \in \overline{\Omega}$.*

Then there exists a unique solution (u, v) of (2.3) such that

$$u \in H^1_p(J; L_p(\Omega)) \cap L_p(J; H^4_p(\Omega)) = Z^1$$

and

$$v \in H^1_p(J; L_p(\Omega)) \cap L_p(J; H^2_p(\Omega)) = Z^2,$$

if and only if the data are subject to the following conditions.

(i) $f_1, f_2 \in L_p(J \times \Omega) = X$,

(ii) $g_1 \in W_p^{1/4-1/4p}(J; L_p(\Gamma)) \cap L_p(J; W_p^{1-1/p}(\Gamma)) = Y_1$,

(iii) $g_2 \in W_p^{3/4-1/4p}(J; L_p(\Gamma)) \cap L_p(J; W_p^{3-1/p}(\Gamma)) = Y_2$,

(iv) $g_3 \in W_p^{1/2-1/2p}(J; L_p(\Gamma)) \cap L_p(J; W_p^{1-1/p}(\Gamma)) = Y_3$,

(v) $u_0 \in B_{pp}^{4-4/p}(\Omega) = X^1_p$

(vi) $v_0 \in B_{pp}^{2-2/p}(\Omega) = X^2_p$,

(vii) $\partial_\nu \Delta u_0 + \partial_\nu(\eta_1 v_0) = g_1|_{t=0}$, *if* $p > 5$, $\quad \partial_\nu u_0 = g_2|_{t=0}$, *if* $p > 5/3$,

(viii) $\partial_\nu v_0 = g_3|_{t=0}$, *if* $p > 3$.

2.3 Local Well-Posedness

In this section we will use the following setting. For $T_0 > 0$, to be fixed later, and a given $T \in (0, T_0]$ we define

$$\mathbb{E}_1 := Z^1(T) \times Z^2(T), \qquad {}_0\mathbb{E}_1 := \{(u, v) \in \mathbb{E}_1 : (u, v)|_{t=0} = 0\}$$

and

$$\mathbb{E}_0 := X(T) \times X(T) \times Y_1(T) \times Y_2(T) \times Y_3(T),$$

as well as

$${}_0\mathbb{E}_0 := \{(f_1, f_2, g_1, g_2, g_3) \in \mathbb{E}_0 : g_1|_{t=0} = g_2|_{t=0} = g_3|_{t=0} = 0\},$$

with canonical norms $|\cdot|_1$ and $|\cdot|_0$, respectively. The aim of this section is to find a local solution $(\psi, \vartheta) \in \mathbb{E}_1$ of the quasilinear system (2.2). Therefore we will again apply Banach's fixed point theorem. For this purpose let $f_1, f_2 \in X(T_0)$, $g_j \in Y_j(T_0)$, $j = 1, 2$, $\psi_0 \in X^1_p$ and $\vartheta_0 \in X^2_p$ be given such that the compatibility conditions

$$\partial_\nu \Delta\psi_0 - \partial_\nu \Phi'(\psi_0) + \partial_\nu(\lambda'(\psi_0)\vartheta_0) = -g_1|_{t=0},\ \partial_\nu\psi_0 = g_2|_{t=0} \quad \text{and} \quad \partial_\nu\vartheta_0 = g_3|_{t=0},$$

hold whenever $p > 5$, $p > 5/3$ and $p > 3$, respectively. We set $a_0(x) = 1/b'(\vartheta_0(x))$, $\eta_1(x) = \lambda'(\psi_0(x))$ and $\eta_2(x) = a_0(x)\eta_1(x)$ with the assumption

(H1) $b \in C^2(\mathbb{R})$ and there is a constant $\sigma > 0$ such that $b'(\vartheta_0(x)) \geq \sigma > 0$ for all $x \in \overline{\Omega}$.

Note that it follows from (2.6) that the conditions in (2.4) are satisfied, provided $p > (n+2)/2$ and there exists $\sigma > 0$ such that $a_0(x) \geq \sigma > 0$ for all $x \in \overline{\Omega}$, by (H1). Thanks to Theorem 2.2.1 we may define a pair of functions $(u^*, v^*) \in \mathbb{E}_1$ as the solution of the problem

$$\begin{aligned}
\partial_t u^* + \Delta^2 u^* + \Delta(\eta_1 v^*) &= f_1, \quad t \in J,\ x \in \Omega,\\
\partial_t v^* - a_0\Delta v^* + \eta_2\partial_t u^* &= a_0 f_2, \quad t \in J,\ x \in \Omega,\\
\partial_\nu\Delta u^* + \partial_\nu(\eta_1 v^*) &= -g_1 - e^{-B^2 t} g_0, \quad t \in J,\ x \in \partial\Omega,\\
\partial_\nu u^* &= g_2, \quad t \in J,\ x \in \partial\Omega,\\
\partial_\nu v^* &= g_3, \quad t \in J,\ x \in \partial\Omega,\\
u^*(0) = \psi_0,\ v^*(0) &= \vartheta_0, \quad t = 0,\ x \in \Omega,
\end{aligned} \tag{2.17}$$

where $B = -\Delta_\Gamma$ is the Laplace-Beltrami operator on Γ and $e^{-B^2 t}$ is the analytic semigroup which is generated by $-B^2$. Furthermore $g_0 = 0$ if $p < 5$ and $g_0 = -g_1|_{t=0} - (\partial_\nu \Delta\psi_0 + \partial_\nu(\eta_1\vartheta_0))$ if $p > 5$. In the sequel we will need the following regularity assumption on the nonlinearities λ and Φ.

(H2) The functions λ, Φ belong to $C^{4-}(\mathbb{R})$.

Define a linear operator $\mathbb{L} : \mathbb{E}_1 \to \mathbb{E}_0$ by

$$\mathbb{L}(u,v) = \begin{bmatrix} \partial_t u + \Delta^2 u + \eta_1 \Delta v \\ \partial_t v - a_0 \Delta v + \eta_2 \partial_t u \\ \partial_\nu \Delta u + \partial_\nu(\eta_1 v) \\ \partial_\nu u \\ \partial_\nu v \end{bmatrix}.$$

Consider $\mathbb{L}$ as an operator from ${}_0\mathbb{E}_1$ to ${}_0\mathbb{E}_0$. Then, by Theorem 2.2.1, the operator $\mathbb{L} : {}_0\mathbb{E}_1 \to {}_0\mathbb{E}_0$ is bounded and bijective, hence an isomorphism with bounded inverse $\mathbb{L}^{-1}$. For all $(u,v) \in \mathbb{E}_1$ we set

$$G_1(u,v) = (\lambda'(\psi_0) - \lambda'(u))v + \Phi'(u),$$
$$G_2(u,v) = (a_0\lambda'(\psi_0) - a(v)\lambda'(u))\partial_t u - (a_0 - a(v))\Delta v - (a_0 - a(v)) f_2,$$

where $a(v(t,x)) = 1/b'(v(t,x))$ and $a_0 = a(\vartheta_0)$. Lastly we define a nonlinear mapping $G : \mathbb{E}_1 \times {}_0\mathbb{E}_1 \to {}_0\mathbb{E}_0$ by

$$G((u^*,v^*);(u,v)) = \begin{bmatrix} \Delta G_1(u+u^*, v+v^*) \\ G_2(u+u^*, v+v^*) \\ \partial_\nu G_1(u+u^*, v+v^*) - \tilde{g}_0 \\ 0 \\ 0 \end{bmatrix},$$

where $\tilde{g}_0 = 0$ if $p < 5$ and $\tilde{g}_0 = e^{-B^2 t}\partial_\nu G_1(\psi_0, \vartheta_0)$ if $p > 5$. Then it is easy to see that $\psi = u + u^*$ and $\vartheta = v + v^*$ is a solution of (2.2) if and only if

$$\mathbb{L}(u,v) = G((u^*,v^*);(u,v))$$

or equivalently

$$(u,v) = \mathbb{L}^{-1} G((u^*,v^*);(u,v)).$$

In order to apply the contraction mapping principle we consider a ball $\mathbb{B}_R = \mathbb{B}_R^1 \times \mathbb{B}_R^2 \subset {}_0\mathbb{E}_1$, where $R \in (0,1]$. Furthermore we define a mapping $\mathcal{T} : \mathbb{B}_R \to {}_0\mathbb{E}_1$ by $\mathcal{T}(u,v) = \mathbb{L}^{-1}G((u^*,v^*);(u,v))$. We shall prove that $\mathcal{T}\mathbb{B}_R \subset \mathbb{B}_R$ and that $\mathcal{T}$ defines a strict contraction on $\mathbb{B}_R$. Therefore we define the shifted ball $\mathbb{B}_R(u^*,v^*) = \mathbb{B}_R^1(u^*) \times \mathbb{B}_R^2(v^*) \subset \mathbb{E}_1$ by

$$\mathbb{B}_R(u^*,v^*) = \{(u,v) \in \mathbb{E}_1 : (u,v) = (\tilde{u},\tilde{v}) + (u^*,v^*),\ (\tilde{u},\tilde{v}) \in \mathbb{B}_R\}.$$

To ensure that the mapping G_2 is well defined, we choose $T_0 > 0$ and $R > 0$ sufficiently small. This yields that all functions $v \in \mathbb{B}_R^2(v^*)$ have only a small deviation from the initial value ϑ_0. To see this, write

$$|\vartheta_0(x) - v(t,x)| \le |\vartheta_0(x) - v^*(t,x)| + |v^*(t,x) - v(t,x)| \le \mu(T) + R,$$

for all functions $v \in \mathbb{B}_R^2(v^*)$, where $\mu = \mu(T)$ is defined by

$$\mu(T) = \max_{(t,x)\in[0,T]\times\Omega} |v^*(t,x) - \vartheta_0(x)|.$$

Observe that $\mu(T) \to 0$ as $T \to 0$, by the continuity of v^* and ϑ_0. This in turn implies that $b'(v(t,x)) \ge \sigma/2 > 0$ for sufficiently small $T_0 > 0$, $R > 0$ and all $v \in \mathbb{B}_R^2(v^*)$. Moreover, for all $v, \bar{v} \in \mathbb{B}_R^2(v^*)$ we obtain the estimates

$$|a(\vartheta_0(x)) - a(v(t,x))| \le C|\vartheta_0(x) - v(t,x)| \tag{2.18}$$

and
$$|a(\bar{v}(t,x)) - a(v(t,x))| \le C|\bar{v}(t,x) - v(t,x)|, \tag{2.19}$$
valid for all $(t,x) \in [0,T] \times \Omega$, with some constant $C > 0$, since b' is locally Lipschitz continuous by (H1).

The next proposition provides all the facts to show the desired properties of the operator $\mathcal{T}$.

Proposition 2.3.1. *Let $n \in \mathbb{N}$ and $p > (n+2)/2$, $p \ge 2$. Furthermore assume that (H1), (H2) hold and let $J = [0,T] \subset [0,T_0]$. Then there exists a constant $C > 0$, independent of T, and functions $\mu_j = \mu_j(T)$ with $\mu_j(T) \to 0$ as $T \to 0$, such that for all $(u,v), (\bar{u},\bar{v}) \in \mathbb{B}_R(u^*,v^*)$ the following statements hold.*

(i) $|\Delta\Phi'(u) - \Delta\Phi'(\bar{u})|_{X(T)} \le (\mu_1(T) + R)|(u,v) - (\bar{u},\bar{v})|_1$,

(ii) $|\Delta((\lambda'(\psi_0) - \lambda'(u))v) - \Delta((\lambda'(\psi_0) - \lambda'(\bar{u}))\bar{v})|_{X(T)} \le C(\mu_2(T) + R)|(u,v) - (\bar{u},\bar{v})|_1$,

(iii) $|(a_0\lambda'(\psi_0) - a(v)\lambda'(u))\partial_t u - (a_0\lambda'(\psi_0) - a(\bar{v})\lambda'(\bar{u}))\partial_t \bar{u}|_{X(T)} \le C(\mu_3(T) + R)|(u,v) - (\bar{u},\bar{v})|_1$,

(iv) $|(a_0 - a(v))\Delta v - (a_0 - a(\bar{v}))\Delta\bar{v}|_{X(T)} \le C(\mu_4(T) + R)|(u,v) - (\bar{u},\bar{v})|_1$,

(v) $|(a(v) - a(\bar{v}))f_2|_{X(T)} \le C\mu_5(T)|(u,v) - (\bar{u},\bar{v})|_1$,

(vi) $|\partial_\nu\Phi'(u) - \partial_\nu\Phi'(\bar{u})|_{Y_1(T)} \le (\mu_6(T) + R)|(u,v) - (\bar{u},\bar{v})|_1$,

(vii) $|\partial_\nu((\lambda'(\psi_0) - \lambda'(u))v) - \partial_\nu((\lambda'(\psi_0) - \lambda'(\bar{u}))\bar{v})|_{Y_1(T)} \le C(\mu_7(T) + R)|(u,v) - (\bar{u},\bar{v})|_1$.

The proof is given in the Appendix.

It is now easy to verify the self-mapping property of $\mathcal{T}$. Let $(u,v) \in \mathbb{B}_R$. By Proposition 2.3.1 there exists a function $\mu = \mu(T)$ with $\mu(T) \to 0$ as $T \to 0$ such that

$$\begin{aligned}
|\mathcal{T}(u,v)|_1 &= |\mathbb{L}^{-1}G((u^*,v^*),(u,v))|_1 \le |\mathbb{L}^{-1}||G((u^*,v^*),(u,v))|_0 \\
&\le C(|G((u^*,v^*),(u,v)) - G((u^*,v^*),(0,0))|_0 + |G((u^*,v^*),(0,0))|_0) \\
&\le C(|G_1(u+u^*,v+v^*) - G_1(u^*,v^*)|_{X(T)} + |G_2(u+u^*,v+v^*) - G_2(u^*,v^*)|_{X(T)} \\
&\qquad + |\partial_\nu G_1(u+u^*,v+v^*) - \partial_\nu G_1(u^*,v^*)|_{Y_1(T)} + |G((u^*,v^*),(0,0))|_0) \\
&\le C(\mu(T) + R)|(u,v)|_1 + |G((u^*,v^*),(0,0))|_0 \\
&\le C(\mu(T) + R)R + |G((u^*,v^*),(0,0))|_0.
\end{aligned}$$

Hence we see that $\mathcal{T}\mathbb{B}_R \subset \mathbb{B}_R$ if T and R are sufficiently small, since $G((u^*,v^*),(0,0))$ is a fixed function. Furthermore for all $(u,v), (\bar{u},\bar{v}) \in \mathbb{B}_R$ we have

$$\begin{aligned}
|\mathcal{T}(u,v) - \mathcal{T}(\bar{u},\bar{v})|_1 &= |\mathbb{L}^{-1}(G((u^*,v^*),(u,v)) - G((u^*,v^*),(\bar{u},\bar{v})))|_1 \\
&\le |\mathbb{L}^{-1}||G((u^*,v^*),(u,v)) - G((u^*,v^*),(\bar{u},\bar{v}))|_0 \\
&\le C(|G_1(u+u^*,v+v^*) - G_1(\bar{u}+u^*,\bar{v}+v^*)|_{X(T)} \\
&\qquad + |\partial_\nu G_1(u+u^*,v+v^*) - \partial_\nu G_1(\bar{u}+u^*,\bar{v}+v^*)|_{Y_1(T)} \\
&\qquad\qquad + |G_2(u+u^*,v+v^*) - G_2(\bar{u}+u^*,\bar{v}+v^*)|_{X(T)}) \\
&\le C(\mu(T) + R)|(u,v) - (\bar{u},\bar{v})|_1.
\end{aligned}$$

Thus $\mathcal{T}$ is a strict contraction on $\mathbb{B}_R$, if T and R are again small enough. Therefore we may apply the contraction mapping principle to obtain a unique fixed point $(\tilde{u},\tilde{v}) \in \mathbb{B}_R$ of $\mathcal{T}$. In other words the functions $(\psi,\vartheta) = (\tilde{u}+u^*,\tilde{v}+v^*) \in \mathbb{E}_1$ are the unique local solutions of (2.2). We summarize the preceding calculations in

Theorem 2.3.2. *Let $n \in \mathbb{N}$, $p > (n+2)/2$, $p \ge 2$, $p \ne 3,5$. Assume furthermore that (H1),(H2) hold. Then there exists an interval $J = [0,T] \subset [0,T_0]$ and a unique solution (ψ,ϑ) of (2.2) on J, with*
$$\psi \in H_p^1(J;L_p(\Omega)) \cap L_p(J;H_p^4(\Omega)) = Z^1(T)$$

and

$$\vartheta \in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega)) = Z^2(T),$$

provided the data are subject to the following conditions.

(i) $f_1, f_2 \in L_p(J_0 \times \Omega) = X$,

(ii) $g_1 \in W_p^{1/4-1/4p}(J_0; L_p(\Gamma)) \cap L_p(J_0; W_p^{1-1/p}(\Gamma)) = Y_1$,

(iii) $g_2 \in W_p^{3/4-1/4p}(J_0; L_p(\Gamma)) \cap L_p(J_0; W_p^{3-1/p}(\Gamma)) = Y_2$,

(iv) $g_3 \in W_p^{1/2-1/2p}(J_0; L_p(\Gamma)) \cap L_p(J_0; W_p^{1-1/p}(\Gamma)) = Y_3$,

(v) $\psi_0 \in B_{pp}^{4-4/p}(\Omega) = X_p^1$

(vi) $\vartheta_0 \in B_{pp}^{2-2/p}(\Omega) = X_p^2$,

(vii) $\partial_\nu \Delta\psi_0 - \partial_\nu \Phi'(\psi_0) + \partial_\nu(\lambda'(\psi_0)\vartheta_0) = -g_1|_{t=0}$, *if* $p > 5$,

(viii) $\partial_\nu \psi_0 = g_2|_{t=0}$, *if* $p > 5/3$,

(ix) $\partial_\nu \vartheta_0 = g_3|_{t=0}$, *if* $p > 3$.

The solution depends continuously on the given data and if the data are independent of t*, the map* $(\psi_0, \vartheta_0) \mapsto (\psi, \vartheta)$ *defines a local semiflow on the natural phase manifold*

$$\mathcal{M}_p := \{(\psi_0, \vartheta_0) \in X_p^1 \times X_p^2 : \psi_0 \text{ and } \vartheta_0 \text{ satisfy } (vii) - (ix)\}.$$

2.4 Global Well-Posedness

In this section we will investigate the global existence of the solution to the conserved Penrose-Fife type system

$$\begin{aligned}
\partial_t\psi - \Delta\mu = 0, \quad \mu = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta, &\quad t > 0,\ x \in \Omega,\\
\partial_t\left(b(\vartheta) + \lambda(\psi)\right) - \Delta\vartheta = 0, &\quad t > 0,\ x \in \Omega,\\
\partial_\nu \mu = 0, &\quad t > 0,\ x \in \partial\Omega,\\
\partial_\nu \psi = 0, &\quad t > 0,\ x \in \partial\Omega,\\
\partial_\nu \vartheta = 0, &\quad t > 0,\ x \in \partial\Omega,\\
\psi(0) = \psi_0,\ \vartheta(0) = \vartheta_0, &\quad t = 0,\ x \in \Omega,
\end{aligned} \tag{2.20}$$

with respect to time if the spatial dimension is less or equal to 3. Note that the boundary conditions are equivalent to $\partial_\nu\vartheta = \partial_\nu\psi = \partial_\nu\Delta\psi = 0$. Assuming that (H1) and (H2) hold, a successive application of Theorem 2.3.2 yields a maximal interval of existence $J_{\max} = [0, T_{\max})$ for the solution $(\psi, \vartheta) \in Z^1 \times Z^2$ of (2.20). In the sequel we will make use of the following assumptions.

(H3) There exist some constants $c_j > 0$, $\eta, \gamma > 0$ such that

$$\Phi(s) \geq -\frac{\eta}{2}s^2 - c_1,$$

$$|\Phi'''(s)| \leq c_2(1 + |s|^\gamma),$$

for all $s \in \mathbb{R}$, where $\eta < \lambda_1$ with λ_1 being the smallest nontrivial eigenvalue of the negative Laplacian on Ω with Neumann boundary conditions and $\gamma < 3$ if $n = 3$.

(H4) There is a constant $c > 0$ such that $|\lambda'(s)| \leq c(1 + |s|)$ for all $s \in \mathbb{R}$ and $\lambda'', \lambda''' \in L_\infty(\mathbb{R})$.

(H5) There is a constant $\kappa > 0$ such that

$$b'(\vartheta(t, x)) \geq \kappa > 0$$

for all $(t, x) \in J_{\max} \times \Omega$ and let $\vartheta \in L_\infty(J_{\max} \times \Omega)$.

Remark: Condition (H3) is certainly fulfilled, if Φ is a polynomial of degree $2m$, $m \in \mathbb{N}$, $m < 3$.

We prove global well-posedness by contradiction. For this purpose, assume that $T_{\max} < \infty$. We multiply $\partial_t \psi = \Delta\mu$ by μ and integrate by parts to the result

$$\frac{d}{dt}\left(\frac{1}{2}|\nabla\psi|_2^2 + \int_\Omega \Phi(\psi)\ dx\right) + |\nabla\mu|_2^2 - \int_\Omega \lambda'(\psi)\vartheta\partial_t\psi\ dx = 0. \tag{2.21}$$

Next we multiply $(2.20)_2$ by ϑ and integrate by parts. This yields

$$\int_\Omega \vartheta b'(\vartheta)\partial_t\vartheta\ dx + |\nabla\vartheta|_2^2 + \int_\Omega \lambda'(\psi)\vartheta\partial_t\psi\ dx = 0. \tag{2.22}$$

Set $\beta'(s) = sb'(s)$ and add (2.21) to (2.22) to obtain the equation

$$\frac{d}{dt}\Big(\frac{1}{2}|\nabla\psi|_2^2 + \int_\Omega \Phi(\psi)\ dx + \int_\Omega \beta(\vartheta)\ dx\Big) + |\nabla\mu|_2^2 + |\nabla\vartheta|_2^2 = 0. \tag{2.23}$$

Integrating (2.23) with respect to t, we obtain

$$E(\psi(t), \vartheta(t)) + |\nabla\mu|_{2,2}^2 + |\nabla\vartheta|_{2,2}^2 = E(\psi_0, \vartheta_0), \tag{2.24}$$

for the functional

$$E(u, v) := \frac{1}{2}|\nabla u|_2^2 + \int_\Omega \Phi(u)\ dx + \int_\Omega \beta(v)\ dx.$$

It follows from (H3) and the Poincaré-Wirtinger inequality that

$$\begin{aligned}\frac{\varepsilon}{2}\int_\Omega |\nabla\psi(t)|^2\ dx &+ \frac{1-\varepsilon}{2}\int_\Omega |\nabla\psi(t)|^2\ dx + \int_\Omega \Phi(\psi(t))\ dx \\ &\geq \frac{\varepsilon}{2}\int_\Omega |\nabla\psi(t)|^2\ dx + \frac{(1-\varepsilon)\lambda_1 - \eta}{2}|\psi(t)|_2^2 - c_1|\Omega| - \frac{\lambda_1}{2|\Omega|}\left(\int_\Omega \psi_0\ dx\right),\end{aligned}$$

since by equation $\partial_t\psi = \Delta\mu$ and the boundary condition $\partial_\nu\mu = 0$, it holds that

$$\int_\Omega \psi(t, x)\ dx \equiv \int_\Omega \psi_0(x)\ dx.$$

Hence for a sufficiently small $\varepsilon > 0$ we obtain the a priori estimates

$$\psi \in L_\infty(J_{\max}; H_2^1(\Omega)) \quad \text{and} \quad |\nabla\mu|, |\nabla\vartheta| \in L_2(J_{\max}; L_2(\Omega)), \tag{2.25}$$

since $\beta(\vartheta(t, x))$ is uniformly bounded on $J_{\max} \times \Omega$, by (H5). However, things are more involved for higher order estimates. Here we have the following result.

Proposition 2.4.1. *Let $n \leq 3$, $p > (n+2)/2$, $p \geq 2$ and let (ψ, ϑ) be the maximal continued solution of (2.20) with initial value $\psi_0 \in X_p^1$ and $\vartheta_0 \in X_p^2$.*

Then $\psi \in L_\infty(J_{\max} \times \Omega)$ and $\vartheta \in H_2^1(J_{\max}; L_2(\Omega)) \cap L_\infty(J_{\max}; H_2^1(\Omega))$. Moreover, it holds that $\partial_t\psi \in L_r(J_{\max} \times \Omega)$, where $r := \min\{p, 2(n+4)/n\}$.

Proof. The proof is given in the Appendix.

□

Define the new function $u = b(\vartheta)$. Then u satisfies the differential equation in divergence form

$$\partial_t u - \operatorname{div}(a(t, x)\nabla u) = f, \tag{2.26}$$

subject to the boundary and initial conditions $\partial_\nu u = 0$ and $u(0) = b(\vartheta_0)$, where $a(t, x) := 1/b'(\vartheta(t, x))$ and $f := -\lambda'(\psi)\partial_t\psi$. The regularity of ϑ from Proposition 2.4.1 carries over to the function u, by the uniform boundedness of $b'(\vartheta)$. This yields, that u is a *weak solution* of (2.26) in the sense of LIEBERMAN [26] or a *generalized solution* of (2.26) in the sense of LADYZHENSKAYA, SOLONNIKOV & URALTSEVA [24] and u is bounded, by (H5).

Furthermore, by (H5), there exists some constant $\tilde{\kappa} \in (0,1)$ such that

$$0 < \tilde{\kappa} \le a(t,x) \le \frac{1}{\tilde{\kappa}} < \infty,$$

for all $(t,x) \in J_{\max} \times \Omega$. Now we are in a position to use the arguments, which were successfully applied in [26, Theorem 6.44] to conclude that there exists a real number $\alpha \in (0,1/2)$ such that $u \in C^{\alpha,2\alpha}((0,T_{\max})\times\Omega)$, provided $f \in L_p(J_{\max}\times\Omega)$ and $p > (n+2)/2$. Here $C^{\alpha,2\alpha}((0,T_{\max})\times\Omega)$ is defined as

$$C^{\alpha,2\alpha}(\Omega_{T_{\max}}) := \{v \in C(\Omega_{T_{\max}}) : \sup_{(t,x),(s,y)\in\Omega_{T_{\max}}} \frac{|v(t,x)-v(s,y)|}{|t-s|^\alpha + |x-y|^{2\alpha}} < \infty\}.$$

and we have set $\Omega_{T_{\max}} = (0,T_{\max}) \times \Omega$ for the sake of readability. Actually, in [26, Theorem 6.44] the author assumes that f is bounded, but this assumption can be weaken to the condition, that $f \in L_p(J_{\max}\times\Omega)$ with the restriction $p > (n+2)/2$ (see [24, Chapter III]). By Proposition 2.4.1 it holds that $f = -\lambda'(\psi)\partial_t\psi \in L_r(J_{\max}\times\Omega)$, $r := \min\{p, 2(n+4)/n\}$. Consider the case $r = 2(n+4)/n$. Then it can be readily checked that

$$\frac{n+2}{2} < \frac{2(n+4)}{n} = r$$

provided $n \le 5$. The properties of the function b, namely (H5), then yield that $\vartheta = b^{-1}(u)$ is Hölder continuous, too. Therefore we may extend ϑ continuously to the closure $\overline{J_{\max}\times\Omega}$. In a next step we solve the initial-boundary value problem

$$\begin{aligned} \partial_t\vartheta - a(t,x)\Delta\vartheta &= g, \quad t \in J_{\max},\ x \in \Omega, \\ \partial_\nu\vartheta &= 0, \quad t \in J_{\max},\ x \in \partial\Omega, \\ \vartheta(0) &= \vartheta_0, \quad t = 0,\ x \in \Omega, \end{aligned} \tag{2.27}$$

with $g := -a(t,x)\lambda'(\psi)\partial_t\psi \in L_r(J_{\max}\times\Omega)$ and $r = 2(n+4)/n > (n+2)/2$. By Theorem 1.4.3 we obtain

$$\vartheta \in H^1_r(J_{\max}; L_r(\Omega)) \cap L_r(J_{\max}; H^2_r(\Omega)),$$

of (2.27), since

$$\vartheta_0 \in B^{2-2/p}_{pp}(\Omega) \hookrightarrow B^{2-2/r}_{rr}(\Omega), \quad p \ge r.$$

At this point we use equation (2.54) from the proof of Proposition 2.4.1 and (H5) to conclude $\partial_t\psi \in L_s(J_{\max}\times\Omega)$, with $s = \min\{p,q\}$ where q is restricted by

$$\frac{1}{q} \ge \frac{1}{r} - \frac{2}{n+4}.$$

For the case $r = 2(n+4)/n$, this yields

$$\frac{1}{q} \ge \frac{n-4}{2(n+4)},$$

i.e. q may be arbitrarily large in case $n \le 3$ and we may set $s = p$. Now we solve (2.27) again, this time with $g \in L_p(J_{\max}\times\Omega)$, to obtain

$$\vartheta \in H^1_p(J_{\max}; L_p(\Omega)) \cap L_p(J_{\max}; H^2_p(\Omega))$$

and therefore $\vartheta(T_{\max}) \in B^{2-2/p}_{pp}(\Omega)$ is well defined. Next, consider the equation

$$\partial_t\psi + \Delta^2\psi = \Delta\Phi'(\psi) - \Delta(\lambda'(\psi)\vartheta),$$

subject to the initial and boundary conditions $\psi(0) = \psi_0$ and $\partial_\nu \psi = \partial_\nu \Delta\psi = 0$. By Theorem 1.4.3 there exists a constant $M > 0$ such that

$$|\psi|_{Z^1(J_{\max})} \leq M(1 + |\Delta\Phi'(\psi)|_{X(J_{\max})} + |\Delta(\lambda'(\psi)\vartheta)|_{X(J_{\max})}). \tag{2.28}$$

We will first estimate the term $\Delta\Phi'(\psi) = \Delta\psi\Phi''(\psi) + |\nabla\psi|^2\Phi'''(\psi)$. Using the Gagliardo-Nirenberg inequality and (H3) we obtain

$$|\Phi''(\psi)\Delta\psi|_p \leq |\psi|_{2(\gamma+1)p}^{\gamma+1}|\Delta\psi|_{2p} \leq C|\psi|_{H_p^4}^{a+b(\gamma+1)}|\psi|_q^{1-a+(1-b)(\gamma+1)}, \tag{2.29}$$

where q will be chosen in such a way that $H_2^1(\Omega) \hookrightarrow L_q$, i.e. $\frac{n}{q} \geq \frac{n}{2} - 1$ and

$$(a + (\gamma+1)b)\left(4 - \frac{n}{p} + \frac{n}{q}\right) = 2 - \frac{n}{p} + \frac{n}{q}(\gamma+2).$$

The second term $\Phi'''(\psi)|\nabla\psi|^2$ will be treated in a similar way. The Gagliardo-Nirenberg inequality and (H3) again yield

$$|\Phi'''(\psi)|\nabla\psi|^2|_p \leq |\psi|_{2\gamma p}^{\gamma}|\nabla\psi|_{4p}^2 \leq C|\psi|_{H_p^4}^{2a+b\gamma}|\psi|_q^{2-2a+(1-b)\gamma}, \tag{2.30}$$

with $\frac{n}{q} \geq \frac{n}{2} - 1$ and

$$(2a + \gamma b)\left(4 - \frac{n}{p} + \frac{n}{q}\right) = 2 - \frac{n}{p} + (\gamma+2)\frac{n}{q}.$$

It turns out that the condition $\gamma < 3$ in case $n = 3$ ensures that either $a + (\gamma+1)b < 1$ and $2a + \gamma b < 1$ in (2.29) and (2.30), respectively. Integrating (2.29) and (2.30) with respect to t and using Hölders inequality as well as (2.25) we obtain the estimate

$$|\Delta\Phi'(\psi)|_X \leq C(1 + |\psi|_{Z^1}^\delta),$$

for some $\delta \in (0,1)$. It is easily seen that the term $\Delta(\lambda'(\psi)\vartheta)$ may be estimated in a similar way, by (H4) and since we have enough information of ψ, ϑ on $\bar{J}_{\max} \times \Omega$ (see also proof of Lemma 3.4.1). Therefore we may conclude from (2.28) that there exists a constant $M_1 > 0$ such that

$$|\psi|_{Z(J_{\max})} \leq M_1(1 + |\psi|_{Z^1(J_{\max})}^\delta), \ \delta \in (0,1).$$

This in turn implies that $|\psi|_{Z(J_{\max})}$ is bounded, i.e.

$$\psi \in H_p^1(J_{\max}; L_p(\Omega)) \cap L_p(J_{\max}; H_p^4(\Omega)) \hookrightarrow C(\bar{J}_{\max}; B_{pp}^{4-4/p}(\Omega)).$$

Hence, also $\psi(T_{\max}) \in B_{pp}^{4-4/p}(\Omega)$ is well defined. Therefore we may continue the solution (ψ, ϑ) beyond the point $T_{\max}$, which contradicts the assumption that $T_{\max} < \infty$. This in turn implies that the solution exists globally in time. We summarize these considerations in

Theorem 2.4.2. *Let $n \leq 3$, $p > (n+2)/2$, $p \geq 2$, $p \neq 3, 5$. Assume furthermore that (H1)-(H5) hold. Then for each $T_0 > 0$ there exists a unique solution (ψ, ϑ) of (2.2) on $J_0 = [0, T_0]$, with*

$$\psi \in H_p^1(J_0; L_p(\Omega)) \cap L_p(J_0; H_p^4(\Omega)) = Z^1(T_0)$$

and

$$\vartheta \in H_p^1(J_0; L_p(\Omega)) \cap L_p(J_0; H_p^2(\Omega)) = Z^2(T_0),$$

provided the data are subject to the following conditions.

(i) $\psi_0 \in B_{pp}^{4-4/p}(\Omega) = X_p^1$;

(ii) $\vartheta_0 \in B_{pp}^{2-2/p}(\Omega) = X_p^2$;

(iii) $\partial_\nu \Delta\psi_0 = 0$, *if* $p > 5$;

(iv) $\partial_\nu \psi_0 = 0$, *if* $p > 5/3$*;*

(v) $\partial_\nu \vartheta_0 = 0$, *if* $p > 3$.

The solution depends continuously on the given data and the map $(\psi_0, \vartheta_0) \mapsto (\psi, \vartheta)$ *defines a global semiflow on the natural phase manifold*

$$\mathcal{M}_p := \{(\psi_0, \vartheta_0) \in X_p^1 \times X_p^2 : \psi_0 \textit{ and } \vartheta_0 \textit{ satisfy } (iii) - (v)\}.$$

Remark: We want to point out, that the function $b(s) = -1/s$ for the classical Penrose-Fife model does not fit in this setting. Instead, we have to assume that there exists $\kappa \in (0,1)$ such that

$$0 < \kappa \le \vartheta(t,x) \le \frac{1}{\kappa} < \infty$$

for all $(t,x) \in J_{\max} \times \Omega$ and $b \in C^2(\mathbb{R}_+)$, $b'(s) > 0$, $s \in \mathbb{R}$. Then the statement of Theorem 2.4.2 remains true.

2.5 Asymptotic Behavior

Let $n \le 3$. In the following we will investigate the asymptotic behavior of global solutions of the homogeneous system

$$\begin{aligned} \partial_t \psi - \Delta\mu = 0, \quad \mu = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta, &\quad t > 0,\ x \in \Omega, \\ \partial_t \left(b(\vartheta) + \lambda(\psi)\right) - \Delta\vartheta = 0, &\quad t > 0,\ x \in \Omega, \\ \partial_\nu \mu = 0, &\quad t > 0,\ x \in \partial\Omega, \\ \partial_\nu \psi = 0, &\quad t > 0,\ x \in \partial\Omega, \\ \partial_\nu \vartheta = 0, &\quad t > 0,\ x \in \partial\Omega, \\ \psi(0) = \psi_0,\ \vartheta(0) = \vartheta_0, &\quad t = 0,\ x \in \Omega. \end{aligned} \tag{2.31}$$

To this end let $(\psi_0, \vartheta_0) \in \mathcal{M}_p$, $p \ge 2$ and denote by $(\psi(t), \vartheta(t))$ the unique global solution of (2.31). In the sequel we will make use of the following assumptions.

(H6) There is a constant $\sigma > 0$ such that

$$b'(\vartheta(t,x)) \ge \sigma > 0$$

for all $(t,x) \in \mathbb{R}_+ \times \Omega$ and $\vartheta \in L_\infty(\mathbb{R}_+ \times \Omega)$.

(H7) The functions Φ, λ and b are real analytic on $\mathbb{R}$.

Note that the boundary conditions $(2.31)_{3,5}$ yield

$$\int_\Omega \psi(t,x)\ dx \equiv \int_\Omega \psi_0(x)\ dx,$$

and

$$\int_\Omega (b(\vartheta(t,x)) + \lambda(\psi(t,x)))\ dx \equiv \int_\Omega (b(\vartheta_0(x)) + \lambda(\psi_0(x)))\ dx.$$

Replacing ψ by $\tilde\psi = \psi - c$, where $c := \frac{1}{|\Omega|}\int_\Omega \psi_0(x)\ dx$ we see that $\int_\Omega \tilde\psi\ dx \equiv 0$, if $\Phi(s)$ and $\lambda(s)$ are replaced by $\tilde\Phi(s) = \Phi(s+c)$ and $\tilde\lambda(s) = \lambda(s+c)$, respectively. Similarly we can achieve that

$$\int_\Omega (b(\vartheta(t,x)) + \lambda(\psi(t,x)))\ dx \equiv 0,$$

by another shift of λ, to be precise $\bar\lambda(s) := \lambda(s) - d$, where

$$d := \frac{1}{|\Omega|}\int_\Omega (b(\vartheta_0(x)) + \lambda(\psi_0(x)))\ dx.$$

With these modifications of the data we obtain the side conditions

$$\int_\Omega \psi(t,x)\;dx \equiv 0 \quad \text{and} \quad \int_\Omega (b(\vartheta(t,x)) + \lambda(\psi(t,x)))\;dx \equiv 0. \tag{2.32}$$

Recall from Section 2.4 the energy functional

$$E(u,v) = \frac{1}{2}|u|_2^2 + \int_\Omega \Phi(u)\;dx + \int_\Omega \beta(v)\;dx,$$

defined on the energy space $V = V_1 \times V_2$, where

$$V_1 = \left\{u \in H_2^1(\Omega) : \int_\Omega u\;dx = 0\right\}, \qquad V_2 = H_2^r(\Omega),\; r \in (0,1).$$

and V is equipped with the canonical norm $|(u,v)|_V := |u|_{H_2^1(\Omega)} + |v|_{H_2^r(\Omega)}$. It is convenient to embed V into a Hilbert space $H = H_1 \times H_2$ where

$$H_1 := \left\{u \in L_2(\Omega) : \int_\Omega u\;dx = 0\right\} \quad \text{and} \quad H_2 := L_2(\Omega).$$

Proposition 2.5.1. *Let $(\psi,\vartheta) \in \mathbb{E}_1$ be a global solution of (2.31) and assume (H1)-(H6). Then*

(i) $\psi \in L_\infty(\mathbb{R}_+; H_p^{2s}(\Omega)),\; s \in [0,1),\; p \in (1,\infty),\; \partial_t\psi \in L_2(\mathbb{R}_+ \times \Omega)$;

(ii) $\vartheta \in L_\infty(\mathbb{R}_+; H_2^1(\Omega)),\; \partial_t\vartheta \in L_2(\mathbb{R}_+ \times \Omega)$.

In particular the orbits $\psi(\mathbb{R}_+)$ and $\vartheta(\mathbb{R}_+)$ are relatively compact in $H_2^1(\Omega)$ and $H_2^r(\Omega)$, respectively, where $r \in [0,1)$.

Proof. Assertion (i) follows directly from (H6) and the proof of Proposition 2.4.1, which is given in the Appendix. Indeed, one may replace the interval $J_{\max}$ by $\mathbb{R}_+$, since the operator $-A^2 = -\Delta_N^2$ with domain

$$D(A^2) = \{u \in H_p^4(\Omega) : \partial_\nu u = \partial_\nu \Delta u = 0 \text{ on } \Gamma\},$$

generates an exponentially stable, analytic semigroup $e^{-A^2 t}$ in the space

$$\{u \in L_p(\Omega) : \int_\Omega u\;dx = 0\}.$$

Now we turn to (ii). We multiply $(2.31)_2$ by $\partial_t\vartheta$ and integrate by parts to the result

$$\int_\Omega b'(\vartheta(t,x))|\partial_t\vartheta(t,x)|^2\;dx + \frac{1}{2}\frac{d}{dt}|\nabla\vartheta(t)|_2^2 = -\int_\Omega \lambda'(\psi(t,x))\partial_t\psi(t,x)\partial_t\vartheta(t,x)\;dx.$$

Making use of (H6) and Young's inequality we obtain

$$C_1|\partial_t\vartheta|_{2,2}^2 + \frac{1}{2}|\nabla\vartheta(t)|_2^2 \le C_2(|\partial_t\psi|_{2,2}^2 + 1), \tag{2.33}$$

after integrating with respect to t. Then the claim follows from (i). □

By Assumption (H6), there exists some bounded interval $J_\vartheta \subset \mathbb{R}$ with $\vartheta(t,x) \in J_\vartheta$ for all $t \ge 0,\; x \in \Omega$. Therefore we may modify the nonlinearities b and β outside J_ϑ in such a way that $b,\beta \in C_b^3(\mathbb{R})$.

Unfortunately the energy functional E is not the right one, since we have to include the nonlinear side condition

$$\int_\Omega (\lambda(\psi) + b(\vartheta))\;dx = 0,$$

into our considerations. The linear constraint $\int_\Omega \psi \, dx = 0$ is part of the definition of the space H_1. For the nonlinear side condition we use a functional of Lagrangian type which is given by

$$L(u, v) = E(u, v) - \overline{v} F(u, v),$$

defined on V, where $F(u, v) := \int_\Omega (\lambda(u) + b(v)) \, dx$. Here we used the notation $\bar{w} = \frac{1}{|\Omega|} \int_\Omega w \, dx$ for a function $w \in L_1(\Omega)$. Concerning the differentiability of L we have the following result.

Proposition 2.5.2. *The functional L is twice continuously Fréchet differentiable on V and the derivatives are given by*

$$\langle L'(u, v), (h, k)\rangle_{V^*,V} = \langle E'(u, v), (h, k)\rangle_{V^*,V} - \overline{k} F(u, v) - \overline{v} \langle F'(u, v), (h, k)\rangle_{V^*,V} \tag{2.34}$$

and

$$\begin{aligned}\langle L''(u, v)(h_1, k_1), (h_2, k_2)\rangle_{V^*,V} = {} & \langle E''(u, v)(h_1, k_1), (h_2, k_2)\rangle_{V^*,V} \\ & - \overline{k_1} \langle F'(u, v), (h_2, k_2)\rangle_{V^*,V} - \overline{k_2} \langle F'(u, v), (h_1, k_1)\rangle_{V^*,V} \\ & - \overline{v} \langle F''(u, v)(h_1, k_1), (h_2, k_2)\rangle_{V^*,V},\end{aligned} \tag{2.35}$$

where $(h, k), (h_j, k_j) \in V$, $j = 1, 2$, and

$$\langle E'(u, v), (h, k)\rangle_{V^*,V} = \int_\Omega \nabla u \nabla h \, dx + \int_\Omega \Phi'(u) h \, dx + \int_\Omega \beta'(v) k \, dx,$$

$$\langle E''(u, v)(h_1, k_1), (h_2, k_2)\rangle_{V^*,V} = \int_\Omega \nabla h_1 \nabla h_2 \, dx + \int_\Omega \Phi''(u) h_1 h_2 \, dx + \int_\Omega \beta''(v) k_1 k_2 \, dx,$$

$$\langle F'(u, v), (h, k)\rangle_{V^*,V} = \int_\Omega \lambda'(u) h \, dx + \int_\Omega b'(v) k \, dx$$

and

$$\langle F''(u, v)(h_1, k_1), (h_2, k_2)\rangle_{V^*,V} = \int_\Omega \lambda''(u) h_1 h_2 \, dx + \int_\Omega b''(v) k_1 k_2 \, dx.$$

Proof. We only consider the first derivative, the second one is treated in a similar way. Since the bilinear form

$$a(u, v) := \int_\Omega \nabla u(x) \nabla v(x) \, dx \tag{2.36}$$

defined on $V_1 \times V_1$ is bounded and symmetric, the first term in E is twice continuously Fréchet differentiable. For the functional

$$G_1(u) := \int_\Omega \Phi(u) \, dx, \quad u \in V_1,$$

we argue as follows. With $u, h \in V_1$ it holds that

$$\begin{aligned}\Phi(u(x) + h(x)) - \Phi(u(x)) - \Phi'(u(x))h(x) &= \int_0^1 \frac{d}{dt} \Phi(u(x) + th(x)) \, dt - \int_0^1 \Phi'(u(x))h(x) \, dt \\ &= \int_0^1 \Big(\Phi'(u(x) + th(x)) - \Phi'(u(x))\Big) h(x) \, dt \\ &= \int_0^1 \int_0^t \frac{d}{ds} \Phi'(u(x) + sh(x))h(x) \, ds \, dt \\ &= \int_0^1 \int_0^t \Phi''(u(x) + sh(x))h(x)^2 \, ds \, dt \\ &= \int_0^1 \Phi''(u(x) + sh(x))h(x)^2 (1 - s) \, ds.\end{aligned}$$

From the growth condition (H3), Hölder's inequality and the Sobolev embedding theorem it follows that

$$\begin{aligned}\left|\int_\Omega \Big(\Phi(u(x)+h(x)) - \Phi(u(x)) - \Phi'(u(x))h(x)\Big)\, dx\right| &\le C\int_\Omega (1+|u(x)|^4+|h(x)|^4)|h(x)|^2\, dx\\ &\le C(1+|u|_6^4+|h|_6^4)|h|_6^2\\ &\le C(1+|u|_{V_1}^4+|h|_{V_1}^4)|h|_{V_1}^2.\end{aligned}$$

This proves that G_1 is Fréchet differentiable and also $G_1'(u) = \Phi'(u) \in L_{6/5}(\Omega) \hookrightarrow V_1^*$. The next step is the proof of the continuity of $G_1' : V_1 \to V_1^*$. We make again use of (H3), the Hölder inequality and the Sobolev embedding theorem to obtain

$$\begin{aligned}|G_1'(u)-G_1'(\bar u)|_{V_1^*} &\le C\left(\int_\Omega |\Phi'(u(x))-\Phi'(\bar u(x))|^{\frac{6}{5}}\, dx\right)^{\frac{5}{6}}\\ &\le C\left(\int_\Omega\int_0^1 |\Phi''(tu(x)+(1-t)\bar u(x))|^{\frac{6}{5}}|u(x)-\bar u(x)|^{\frac{6}{5}}\, dt\, dx\right)^{\frac{5}{6}}\\ &\le C\left(\int_\Omega (1+|u(x)|^{\frac{24}{5}}+|\bar u(x)|^{\frac{24}{5}})|u(x)-\bar u(x)|^{\frac{6}{5}}\, dx\right)^{\frac{5}{6}}\\ &\le C\left(\int_\Omega (1+|u(x)|^6+|\bar u(x)|^6)\, dx\right)^{\frac{2}{3}}\left(\int_\Omega |u(x)-\bar u(x)|^6\right)^{\frac{1}{6}}\\ &\le C(1+|u|_{V_1}^4+|\bar u|_{V_1}^4)|u-\bar u|_{V_1}.\end{aligned}$$

Actually this proves that G_1' is even locally Lipschitz continuous on V_1. The Fréchet differentiability of G_1' and the continuity of G_1'' can be proved in an analogue way. The fundamental theorem of differential calculus and the Sobolev embedding theorem yield the estimate

$$|\Phi'(u+h)-\Phi'(u)-\Phi''(u)h|_{V_1^*} \le C\left(\int_\Omega\int_0^1 |\Phi'''(u(x)+sh(x))|^{\frac{6}{5}}|h(x)|^{\frac{12}{5}}\, ds\, dx\right)^{\frac{5}{6}}.$$

We apply Assumption (H3) and Hölder's inequality to the result

$$\begin{aligned}|\Phi'(u+h)-\Phi'(u)-\Phi''(u)h|_{V_1^*} &\le C\left(\int_\Omega (1+|u(x)|^{\frac{18}{5}}+|h(x)|^{\frac{18}{5}})|h(x)|^{\frac{12}{5}}\, dx\right)^{\frac{5}{6}}\\ &\le C\left(\int_\Omega (1+|u(x)|^6+|h(x)|^6)\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |h(x)|^6\, dx\right)^{\frac{1}{3}}\\ &= C(1+|u|_{V_1}^3+|h|_{V_1}^3)|h|_{V_1}^2.\end{aligned}$$

Hence the Fréchet derivative is given by the multiplication operator $G_1''(u)$ defined by $G_1''(u)v = \Phi''(u)v$ for all $v \in V_1$ and $\Phi''(u) \in L_{3/2}(\Omega)$. We will omit the proof of continuity of G_1''. The way to show the C^2-property of the functional

$$G_2(u) := \int_\Omega \lambda(u(x))\, dx, \quad u \in V_1,$$

is identical to the one above, by Assumption (H4). Concerning the C^2-differentiability of the functionals

$$G_3(v) := \int_\Omega \beta(v(x))\, dx \quad\text{and}\quad G_4(v) := \int_\Omega b(v(x))\, dx, \quad v \in V_2,$$

one may adopt the proof for G_1 and G_2. In fact, this time it is easier, since β and b are assumed to be elements of the space $C_b^3(\mathbb{R})$ and so there is no need to apply Hölder's inequality. Nevertheless the embedding $H_2^r(\Omega) \hookrightarrow L_4(\Omega)$, valid for a sufficiently large $r \in (0,1)$, is crucial for the proof. We will skip the details.

Finally the product rule of differentiation yields that L is twice continuously Fréchet differentiable on $V_1 \times V_2$.

□

The corresponding stationary system to (2.31) will be of importance for the forthcoming calculations. Setting all time-derivatives in (2.31) equal to 0 yields

$$\Delta\mu = 0 \quad \text{and} \quad \Delta\vartheta = 0,$$

subject to the boundary conditions $\partial_\nu \mu = \partial_\nu \vartheta = 0$. Thus we have $\mu \equiv \mu_\infty = const$, $\vartheta \equiv \vartheta_\infty = const$ and there remains the nonlinear elliptic problem of second order

$$\begin{cases} -\Delta\psi_\infty + \Phi'(\psi_\infty) - \lambda'(\psi_\infty)\vartheta_\infty = \mu_\infty, & x \in \Omega, \\ \partial_\nu \psi_\infty = 0, \quad x \in \partial\Omega, \end{cases} \tag{2.37}$$

with the side conditions (2.32) for the unknowns ψ_∞ and ϑ_∞. The following proposition collects some properties of the functional L and the ω-limit set

$$\omega(\psi, \vartheta) := \{(\varphi, \theta) \in V_1 \times V_2 : \exists\ (t_n) \nearrow \infty\ s.t.\ (\psi(t_n), \vartheta(t_n)) \to (\varphi, \theta)\}.$$

Proposition 2.5.3. *Under Hypotheses (H1)-(H6) the following assertions are true.*

(i) *The ω-limit set is nonempty, connected and compact.*

(ii) *Each point $(\psi_\infty, \vartheta_\infty) \in \omega(\psi, \vartheta)$ is a strong solution of the stationary problem (2.37), where $\vartheta_\infty, \mu_\infty = const$ and $(\psi_\infty, \vartheta_\infty)$ satisfies the constraints (2.32) for the unknowns $\vartheta_\infty, \mu_\infty$.*

(iii) *The functional L is constant on $\omega(\psi, \vartheta)$ and each point $(\psi_\infty, \vartheta_\infty) \in \omega(\psi, \vartheta)$ is a critical point of L, i.e. $L'(\psi_\infty, \vartheta_\infty) = 0$ in V^*.*

Proof. The fact that $\omega(\psi, \vartheta)$ is nonempty, connected and compact follows from Proposition 2.5.1 and some well-known facts in the theory of dynamical systems.

Now we turn to (ii). Let $(\psi_\infty, \vartheta_\infty) \in \omega(\psi, \vartheta)$. Then there exists a sequence $(t_n) \nearrow +\infty$ such that $(\psi(t_n), \vartheta(t_n)) \to (\psi_\infty, \vartheta_\infty)$ in V as $n \to \infty$. Since $\partial_t \psi, \partial_t \vartheta \in L_2(\mathbb{R}_+ \times \Omega)$ it follows that $\psi(t_n + s) \to \psi_\infty$ and $\vartheta(t_n + s) \to \vartheta_\infty$ in $L_2(\Omega)$ for all $s \in [0, 1]$ and by relative compactness also in V. This can be seen as follows.

$$\begin{aligned} |\psi(t_n + s) - \psi_\infty|_2 &\le |\psi(t_n + s) - \psi(t_n)|_2 + |\psi(t_n) - \psi_\infty|_2 \\ &\le \int_{t_n}^{t_n+s} |\partial_t \psi(t)|_2\ dt + |\psi(t_n) - \psi_\infty|_2 \\ &\le s^{1/2} \left(\int_{t_n}^{t_n+s} |\partial_t \psi(t)|_2^2\ dt \right)^{1/2} + |\psi(t_n) - \psi_\infty|_2. \end{aligned}$$

Then, for $t_n \to \infty$ this yields $\psi(t_n + s) \to \psi_\infty$ for all $s \in [0, 1]$. The proof for ϑ is the same. Integrating (2.23) with $f_1 = f_2 = 0$ from t_n to $t_n + 1$ we obtain

$$E(\psi(t_n + 1), \vartheta(t_n + 1)) - E(\psi(t_n), \vartheta(t_n)) + \int_0^1 \int_\Omega \left(|\nabla\mu(t_n + s, x)|^2 + |\nabla\vartheta(t_n + s, x)|^2 \right)\ dx\ ds = 0.$$

Letting $t_n \to +\infty$ yields

$$|\nabla\mu(t_n + \cdot, \cdot)|, |\vartheta(t_n + \cdot, \cdot)| \to 0 \quad \text{in } L_2([0, 1] \times \Omega),$$

by Lebesgue's theorem of dominated convergence and (H6). This in turn yields a subsequence (t_{n_k}) such that $\nabla\mu(t_{n_k} + s), \nabla\vartheta(t_{n_k} + s) \to 0$ in $L_2(\Omega; \mathbb{R}^n)$ for a.e. $s \in [0, 1]$. Hence $\nabla\vartheta_\infty = 0$, since the gradient is a closed operator in $L_2(\Omega; \mathbb{R}^n)$. This in turn yields that ϑ_∞ is a constant. Furthermore the Poincaré-Wirtinger inequality implies that

$$\begin{aligned} &|\mu(t_{n_k} + s^*) - \mu(t_{n_l} + s^*)|_2 \\ &\quad \le C_p \Big(|\nabla\mu(t_{n_k} + s^*) - \nabla\mu(t_{n_l} + s^*)|_2 + \int_\Omega |\Phi'(\psi(t_{n_k} + s^*)) - \Phi'(\psi(t_{n_l} + s^*))|\ dx \\ &\qquad + \int_\Omega |\lambda'(\psi(t_{n_k} + s^*))\vartheta(t_{n_k} + s^*) - \lambda'(\psi(t_{n_l} + s^*))\vartheta(t_{n_l} + s^*)|\ dx, \end{aligned}$$

for some $s^* \in [0,1]$. Taking the limit $k,l \to \infty$ we see that $\mu(t_{n_k}+s^*)$ is a Cauchy sequence in $L_2(\Omega)$, hence it admits a limit, which we denote by μ_∞. In the same manner as for ϑ_∞ we therefore obtain $\nabla\mu_\infty = 0$, hence μ_∞ is a constant. Observe that the relation

$$\mu_\infty = \frac{1}{|\Omega|}\left(\int_\Omega (\Phi'(\psi_\infty) - \lambda'(\psi_\infty)\vartheta_\infty)\ dx\right)$$

is valid. Multiplying $(2.31)_1$ by a function $\varphi \in H_2^1(\Omega)$ and integrating by parts we obtain

$$(\mu(t_{n_k}+s^*),\varphi)_2 = (\nabla\psi(t_{n_k}+s^*),\nabla\varphi)_2+(\Phi'(\psi(t_{n_k}+s^*)),\varphi)_2-(\lambda'(\psi(t_{n_k}+s^*))\vartheta(t_{n_k}+s^*),\varphi)_2.$$

As $t_{n_k} \to \infty$ it follows that

$$(\mu_\infty,\varphi)_2 = (\nabla\psi_\infty,\nabla\varphi)_2 + (\Phi'(\psi_\infty),\varphi)_2 - \vartheta_\infty(\lambda'(\psi_\infty),\varphi)_2. \tag{2.38}$$

By the Lax-Milgram theorem the bounded, symmetric and elliptic form

$$a(u,v) := \int_\Omega \nabla u\nabla v\ dx,$$

defined on the space $V_1 \times V_1$ induces a bounded operator $A : V_1 \to V_1^*$ with nonempty resolvent, such that

$$a(u,v) = \langle Au, v\rangle_{V_1^*,V_1},$$

for all $(u,v) \in V_1 \times V_1$. It is well-known that the domain of the part A_p of the operator A in

$$\mathbb{X}_p := \{u \in L_p(\Omega) : \int_\Omega u\ dx = 0\}$$

is given by

$$D(A_p) = \{u \in \mathbb{X}_p : u \in H_p^2(\Omega),\ \partial_\nu u = 0\}.$$

Going back to (2.38) we obtain from (H3) and (H4) that $\psi_\infty \in D(A_q)$, where $q = 6/(\beta+2)$. Since $q > 6/5$ we may apply a bootstrap argument to conclude $\psi_\infty \in D(A_2)$. Integrating (2.38) by parts, assertion (iii) follows.

In order to prove (iii) we make use of (2.34) to obtain

$$\begin{aligned}\langle L'(\psi_\infty,\vartheta_\infty),(h,k)\rangle_{V^*,V} &= \langle E'(\psi_\infty,\vartheta_\infty),(h,k)\rangle_{V^*,V} - \vartheta_\infty\langle F'(\psi_\infty,\vartheta_\infty),(h,k)\rangle_{V^*,V}\\ &= \int_\Omega(-\Delta\psi_\infty + \Phi'(\psi_\infty))h\ dx + \int_\Omega \beta'(\vartheta_\infty)k\ dx\\ &\qquad\qquad - \vartheta_\infty\int_\Omega(\lambda'(\psi_\infty)h + b'(\vartheta_\infty)k)\ dx\\ &= \int_\Omega \mu_\infty h\ dx = 0,\end{aligned}$$

for all $(h,k) \in V$, since μ_∞ and ϑ_∞ are constant. A continuity argument finally yields the last statement of the proposition.

□

The following result is crucial for the proof of convergence.

Proposition 2.5.4 (Lojasiewicz-Simon inequality). *Let $(\psi_\infty,\vartheta_\infty) \in \omega(\psi,\vartheta)$ and assume (H1)-(H7). Then there exist constants $s \in (0,\frac{1}{2}], C,\delta > 0$ such that*

$$|L(u,v) - L(\psi_\infty,\vartheta_\infty)|^{1-s} \le C|L'(u,v)|_{V^*},$$

whenever $|(u,v) - (\psi_\infty,\vartheta_\infty)|_V \le \delta$.

Proof. We show first that $\dim N(L''(\psi_\infty, \vartheta_\infty)) < \infty$. By (2.35) we obtain

$$\begin{aligned}\langle L''(\psi_\infty,\vartheta_\infty)&(h_1,k_1),(h_2,k_2)\rangle_{V^*,V}\\ &= \int_\Omega \nabla h_1 \nabla h_2 \; dx + \int_\Omega \Phi''(\psi_\infty) h_1 h_2 \; dx + \int_\Omega \beta''(\vartheta_\infty) k_1 k_2 \; dx\\ &- \overline{k_1} \int_\Omega (\lambda'(\psi_\infty) h_2 + b'(\vartheta_\infty) k_2) \; dx\\ &- \overline{k_2} \int_\Omega (\lambda'(\psi_\infty) h_1 + b'(\vartheta_\infty) k_1) \; dx\\ &- \overline{\vartheta_\infty} \int_\Omega (\lambda''(\psi_\infty) h_1 h_2 + b''(\vartheta_\infty) k_1 k_2) \; dx.\end{aligned}$$

Since $\beta''(s) = b'(s) + s b''(s)$ and $\vartheta_\infty \equiv const$ we have

$$\begin{aligned}\langle L''(\psi_\infty,\vartheta_\infty)&(h_1,k_1),(h_2,k_2)\rangle_{V^*,V}\\ &= \int_\Omega \nabla h_1 \nabla h_2 \; dx + \int_\Omega \left(\Phi''(\psi_\infty) h_1 - \overline{k_1}\lambda'(\psi_\infty) - \vartheta_\infty \lambda''(\psi_\infty) h_1\right) h_2 \; dx\\ &+ \int_\Omega (b'(\vartheta_\infty)(k_1 - 2\overline{k_1}) - \overline{\lambda'(\psi_\infty) h_1}) k_2 \; dx\end{aligned}$$

for all $(h_j, k_j) \in V$. If $(h_1, k_1) \in N(L''(\psi_\infty, \vartheta_\infty))$, it follows that

$$b'(\vartheta_\infty)(k_1 - 2\overline{k_1}) - \overline{\lambda'(\psi_\infty) h_1} = 0.$$

It is obvious that a solution k_1 to this equation must be constant, hence it is given by

$$k_1 = -(b'(\vartheta_\infty))^{-1} \overline{\lambda'(\psi_\infty) h_1}, \tag{2.39}$$

where we also made use of (H6). Concerning h_1 we have

$$\langle A h_1, h_2 \rangle_{V_1^*, V_1} = \int_\Omega (k_1 \lambda'(\psi_\infty) + \vartheta_\infty \lambda''(\psi_\infty) h_1 - \Phi''(\psi_\infty) h_1) h_2 \; dx, \tag{2.40}$$

since k_1 is constant. By Proposition 2.5.3 it holds that $\psi_\infty \in D(A_2) \hookrightarrow L_\infty(\Omega)$, hence $A h_1 \in H_1$, which means that $h_1 \in D(A_2)$ and from (2.40) we obtain

$$A_2 h_1 + P(\Phi''(\psi_\infty) h_1 - \vartheta_\infty \lambda''(\psi_\infty) h_1 - k_1 \lambda'(\psi_\infty)) = 0,$$

where P denotes the projection $P : H_2 \to H_1$, defined by $Pu = u - \overline{u}$. It is an easy consequence of the embedding $D(A_2) \hookrightarrow L_\infty(\Omega)$ that the linear operator $B : H_1 \to H_1$ given by

$$B h_1 = P(\Phi''(\psi_\infty) h_1 - \vartheta_\infty \lambda''(\psi_\infty) h_1 - k_1 \lambda'(\psi_\infty))$$

is bounded. Here k_1 is given by (2.39). Furthermore the operator A_2 defined in the proof of Proposition 2.5.3 is invertible, hence $A_2^{-1} B : H_1 \to D(A_2)$ is a compact operator by compact embedding and this in turn yields that $(I + A_2^{-1} B)$ is a Fredholm operator. In particular it holds that $\dim N(I + A_2^{-1} B) < \infty$, whence $N(L''(\psi_\infty, \vartheta_\infty))$ is finite dimensional. Note that

$$N(L''(\psi_\infty, \vartheta_\infty)) \subset D(A_2) \times (H_2^r(\Omega) \cap L_\infty(\Omega)) \hookrightarrow L_\infty(\Omega) \times L_\infty(\Omega).$$

Furthermore, by Hypothesis (H7), the restriction of L' to the space $D(A_2) \times (H_2^r(\Omega) \cap L_\infty(\Omega))$ is analytic with values in $L_2(\Omega) \times L_2(\Omega)$. For the definition of analyticity in Banach spaces we refer to [9, Section 3]. Now the claim follows from [9, Corollary 3.11].

□

Remark: It is possible to consider the smaller space $\tilde{V} = V_1 \times (H_2^r(\Omega) \cap L_\infty(\Omega))$, $r < 1$, instead of $V = V_1 \times V_2$. In this case we replace the norm of $L'(u, v)$ on the right side of the Lojasiewicz-Simon inequality by the stronger norm $|L'(u, v)|_{V_1^* \times H_2}$ (cf. also [17]). Then all results of this section remain true, provided that $\vartheta(\mathbb{R}_+)$ is relatively compact in $L_\infty(\Omega)$.

Let us now state the main result of this section.

Theorem 2.5.5. *Let (ψ,ϑ) be a global solution of* (2.31) *and suppose that (H1)-(H7) hold. Then the limits*

$$\lim_{t\to\infty}\psi(t)=:\psi_\infty, \quad \text{and} \quad \lim_{t\to\infty}\vartheta(t)=:\vartheta_\infty=const$$

exist in $H_2^1(\Omega)$ and $H_2^r(\Omega)$, $r\in(0,1)$, respectively, and $(\psi_\infty,\vartheta_\infty)$ is a solution of the stationary problem (2.37).

Proof. Since by Proposition 2.5.3 the ω-limit set is compact, we may cover it by a union of *finitely* many balls with center $(\varphi_i,\theta_i)\in\omega(\psi,\vartheta)$ and radius $\delta_i>0$, $i=1,\dots,N$. Since $L(u,v)\equiv L_\infty$ on $\omega(\psi,\vartheta)$ and each (φ_i,θ_i) is a critical point of L, there are *uniform* constants $s\in(0,\frac{1}{2}]$, $C>0$ and an open set $U\supset\omega(\psi,\vartheta)$, such that

$$|L(u,v)-L_\infty|^{1-s}\le C|L'(u,v)|_{V^*}, \tag{2.41}$$

for all $(u,v)\in U$. After these preliminaries, we define $H:\mathbb{R}_+\to\mathbb{R}_+$ by

$$H(t):=(L(\psi(t),\vartheta(t))-L_\infty)^s.$$

The function H is nonincreasing and $\lim_{t\to\infty}H(t)=0$. A well known result in the theory of dynamical systems implies further that $\lim_{t\to\infty}\operatorname{dist}((\psi(t),\vartheta(t)),\omega(\psi,\vartheta))=0$, i.e. there exists $t^*\ge 0$, such that $(\psi(t),\vartheta(t))\in U$, whenever $t\ge t^*$. Next, we compute and estimate the time derivative of H. By (2.23) and Proposition 2.5.4 we obtain

$$\begin{aligned}-\frac{d}{dt}\,H(t)&=s\left(-\frac{d}{dt}\,L(\psi(t),\vartheta(t))\right)|L(\psi(t),\vartheta(t))-L_\infty|^{s-1}\\ &\ge C\frac{|\nabla\mu(t)|_2^2+|\nabla\vartheta(t)|_2^2}{|L'(\psi(t),\vartheta(t))|_{V^*}}\end{aligned} \tag{2.42}$$

Now we have to estimate the term $|L'(\psi(t),\vartheta(t))|_{V^*}$. For convenience we will write $\psi=\psi(t)$ and $\vartheta=\vartheta(t)$. From (2.34) we obtain

$$\begin{aligned}\langle L'(\psi,\vartheta),(h,k)\rangle_{V^*,V}&=\int_\Omega(-\Delta\psi+\Phi'(\psi))h\,dx+\int_\Omega\vartheta b'(\vartheta)k\,dx-\overline{\vartheta}\int_\Omega(\lambda'(\psi)h+b'(\vartheta)k)\,dx\\ &=\int_\Omega(\mu-\overline{\mu})h\,dx+\int_\Omega(\vartheta-\overline{\vartheta})\lambda'(\psi)h\,dx+\int_\Omega(\vartheta-\overline{\vartheta})b'(\vartheta)k\,dx\end{aligned} \tag{2.43}$$

An application of the Hölder and Poincaré inequality yields the estimates

$$|\int_\Omega(\vartheta-\overline{\vartheta})\lambda'(\psi)h\,dx|\le|\lambda'(\psi)|_\infty|\vartheta-\overline{\vartheta}|_2|h|_2\le c|\nabla\vartheta|_2|h|_2, \tag{2.44}$$

$$|\int_\Omega(\vartheta-\overline{\vartheta})b'(\vartheta)k\,dx|\le|b'(\vartheta)|_\infty|\vartheta-\overline{\vartheta}|_2|k|_2\le c|\nabla\vartheta|_2|k|_2 \tag{2.45}$$

and

$$|\int_\Omega(\mu-\overline{\mu})h\,dx|\le c|\nabla\mu|_2|h|_2, \tag{2.46}$$

whence we obtain the estimate

$$|L'(\psi(t),\vartheta(t))|_{V^*}\le C(|\nabla\mu(t)|_2+|\nabla\vartheta(t)|_2),$$

by taking the supremum over all functions $(h,k)\in V$ with norm less than 1 in (2.43)-(2.46). This in connection with (2.42) yields

$$-\frac{d}{dt}H(t)\ge C(|\nabla\mu(t)|_2+|\nabla\vartheta(t)|_2),$$

hence $\nabla\mu, \nabla\vartheta \in L_1([t^*,\infty), L_2(\Omega))$. Using the equation $\partial_t\psi = \Delta\mu$ we see that $\partial_t\psi \in L_1([t^*,\infty), H_2^1(\Omega)^*)$, hence the limit

$$\lim_{t\to\infty} \psi(t) =: \psi_\infty$$

exists in $H_2^1(\Omega)$ by Proposition 2.5.1. From equation $(2.31)_2$ it follows that $\partial_t e \in L_1([t^*,\infty); H_2^1(\Omega)^*)$, where $e := b(\vartheta) + \lambda(\psi)$, i.e. the limit $\lim_{t\to\infty} e(t)$ exists in $H_2^1(\Omega)^*$. This in turn yields that the limit

$$\lim_{t\to\infty} b(\vartheta(t)) =: b_\infty$$

exists in $H_2^1(\Omega)^*$ and by relative compactness (cf. Proposition 2.5.1) also in $L_2(\Omega)$. By the monotonicity assumption (H6) we obtain $\vartheta(t) = b^{-1}(b(\vartheta(t)))$ and thus the limit of $\vartheta(t)$ as t tends to infinity exists in $L_2(\Omega)$, again by (H6). From the relative compactness of the orbit $\vartheta(\mathbb{R}_+)$ it follows that the limit

$$\lim_{t\to\infty} \vartheta(t) =: \vartheta_\infty$$

also exists in $H_2^r(\Omega)$, $r \in [0,1)$. Finally Proposition 2.5.3 (ii) yields the last statement of the theorem.

□

2.6 Appendix

Proof of Proposition 2.3.1

Let $(u,v), (\bar u, \bar v) \in \mathbb{B}_R(u^*, v^*)$. By (2.6) it holds that $u, \bar u$ and $v, \bar v$ are uniformly bounded in $C^1(\overline{\Omega})$ and $C(\overline{\Omega})$, respectively. Furthermore, we will use the following inequality, which has been proven in [47, Lemma 6.2.3].

$$|f(w) - f(\bar w)|_{H_p^s(L_p)} \le \mu(T)(|w-\bar w|_{H_p^{s_0}(L_p)} + |w - \bar w|_{\infty,\infty}), \quad 0 < s < s_0 < 1, \tag{2.47}$$

valid for every $f \in C^{2-}(\mathbb{R})$ and all $w, \bar w \in \mathbb{B}_R^1(u^*) \cup \mathbb{B}_R^2(v^*)$. Here $\mu = \mu(T)$ denotes a function, with the property $\mu(T) \to 0$ as $T \to 0$.

(i) By Hölders inequality it holds that

$$\begin{aligned}
|\Delta\Phi'(u) - \Delta\Phi'(\bar u)|_{X(T)} &\le |\Delta u \Phi''(u) - \Delta\bar u\Phi''(\bar u)|_{X(T)} + ||\nabla u|^2\Phi'''(u) - |\nabla \bar u|^2\Phi'''(\bar u)|_{X(T)} \\
&\le |\Delta u|_{rp,rp}|\Phi''(u) - \Phi''(\bar u)|_{r'p,r'p} + |\Delta u - \Delta \bar u|_{rp,rp}|\Phi''(\bar u)|_{r'p,r'p} \\
&\quad + T^{1/p}\left(|\nabla u|^2_{\infty,\infty}|\Phi'''(u) - \Phi'''(\bar u)|_{\infty,\infty} + |\nabla u - \nabla\bar u|_{\infty,\infty}|\Phi'''(\bar u)|_{\infty,\infty}\right) \\
&\le T^{1/r'p}\left(|\Delta u|_{rp,rp}|\Phi''(u) - \Phi''(\bar u)|_{\infty,\infty} + |\Delta u - \Delta\bar u|_{rp,rp}|\Phi''(\bar u)|_{\infty,\infty}\right) \\
&\quad + T^{1/p}\left(|\nabla u|^2_{\infty,\infty}|\Phi'''(u) - \Phi'''(\bar u)|_{\infty,\infty} + |\nabla u - \nabla\bar u|_{\infty,\infty}|\Phi'''(\bar u)|_{\infty,\infty}\right),
\end{aligned}$$

since $u, \bar u \in C(J; C^1(\overline{\Omega}))$. We have

$$\Delta w \in H_p^{\theta_2/2}(J; H_p^{2(1-\theta_2)}(\Omega)) \hookrightarrow L_{rp}(J \times \Omega), \quad \theta_2 \in [0,1],$$

for every function $w \in Z^1(T)$, since $r > 1$ may be chosen close to 1. Therefore we obtain

$$|\Delta\Phi'(u) - \Delta\Phi'(\bar u)|_{X(T)} \le \mu_1(T)\,(R + |u^*|_1)\,|u - \bar u|_1,$$

due to the assumption $\Phi \in C^{4-}(\mathbb{R})$.

(ii) Here we will proceed in several steps. Firstly we consider the term $(\lambda'(\psi_0) - \lambda'(u))\Delta v - (\lambda'(\psi_0) - \lambda'(\bar u))\Delta\bar v$.

$$\begin{aligned}
&|(\lambda'(\psi_0) - \lambda'(u))\Delta v - (\lambda'(\psi_0) - \lambda'(\bar u))\Delta\bar v|_{X(T)} \\
&\qquad \le |(\lambda'(\psi_0) - \lambda'(u))\Delta(v - \bar v)|_{X(T)} + |(\lambda'(u) - \lambda'(\bar u))\Delta\bar v|_{X(T)} \\
&\qquad \le |\psi_0 - u|_{\infty,\infty}|v - \bar v|_{Z^2(T)} + |u - \bar u|_{\infty,\infty}|\bar v|_{Z^2(T)} \\
&\qquad \le (|\psi_0 - u^*|_{\infty,\infty} + |u^* - u|_{\infty,\infty})|v - \bar v|_{Z^2(T)} \\
&\qquad\qquad + |u - \bar u|_{Z^1(T)}(|\bar v - v^*|_{Z^2(T)} + |v^*|_{Z^2(T)}) \\
&\qquad \le C(\mu_2(T) + R)|(u,v) - (\bar u, \bar v)|_1,
\end{aligned}$$

since $\lambda \in C^{4-}(\mathbb{R})$. Next, we consider the term $\nabla(\lambda'(\psi_0) - \lambda'(u))\nabla v - \nabla(\lambda'(\psi_0) - \lambda'(\bar{u}))\nabla \bar{v}$. We obtain

$$\begin{aligned}&|\nabla(\lambda'(\psi_0) - \lambda'(u))\nabla v - \nabla(\lambda'(\psi_0) - \lambda'(\bar{u}))\nabla \bar{v}|_{X(T)} \\ &\qquad \le |\nabla(\lambda'(\psi_0) - \lambda'(u))|_\infty |\nabla(v-\bar{v})|_{X(T)} + |\nabla(\lambda'(u) - \lambda'(\bar{u}))|_\infty |\nabla \bar{v}|_{X(T)}.\end{aligned}$$

Since

$$\nabla(\lambda'(\psi_0) - \lambda'(u)) = \nabla\psi_0(\lambda''(\psi_0) - \lambda''(u)) + \lambda''(u)(\nabla\psi_0 - \nabla u),$$

and the same for $\nabla(\lambda'(u) - \lambda'(\bar{u}))$, we may argue as above, to conclude

$$\begin{aligned}&|\nabla(\lambda'(\psi_0) - \lambda'(u))|_{\infty,\infty} |\nabla(v-\bar{v})|_{X(T)} + |\nabla(\lambda'(u) - \lambda'(\bar{u}))|_{\infty,\infty} |\nabla \bar{v}|_{X(T)} \\ &\qquad \le (\mu_2(T) + R)|(u,v) - (\bar{u},\bar{v})|_1.\end{aligned}$$

Finally, we estimate the remaining part of (ii) with Hölder's inequality to the result

$$\begin{aligned}&|v\Delta(\lambda'(\psi_0) - \lambda'(u)) - \bar{v}\Delta(\lambda'(\psi_0) - \lambda'(\bar{u}))|_{X(T)} \\ &\qquad \le |v-\bar{v}|_{\infty,\infty}|\Delta(\lambda'(\psi_0) - \lambda'(u))|_{X(T)} + |\bar{v}|_{r'p,r'p}|\Delta(\lambda'(u) - \lambda'(\bar{u}))|_{rp,rp},\end{aligned} \tag{2.48}$$

where $1/r + 1/r' = 1$. For the first part, we obtain

$$\begin{aligned}|\Delta(\lambda'(\psi_0) - \lambda'(u))|_{X(T)} &\le |\Delta\psi_0|_p|\lambda''(\psi_0) - \lambda''(u)|_{\infty,\infty} + |\Delta\psi_0 - \Delta u|_p|\lambda''(u)|_{\infty,\infty} \\ &\quad + |\nabla\psi_0|^2_{\infty,\infty}|\lambda'''(\psi_0) - \lambda'''(u)|_{\infty,\infty} + |\lambda'''(u)|_{\infty,\infty}|\nabla\psi_0 - \nabla u|_{\infty,\infty} \\ &\le C(|\psi_0 - u|_{\infty,\infty} + |\nabla\psi_0 - \nabla u|_{\infty,\infty} + |\Delta\psi_0 - \Delta u|_{p,p}) \\ &\le C(\mu_2(T) + R),\end{aligned}$$

since $\psi_0 \in H^2_p(\Omega) \cap C^1(\overline{\Omega})$ and $\lambda \in C^{4-}(\mathbb{R})$. For the second term in (2.48) we obtain

$$\begin{aligned}|\Delta(\lambda'(u) - \lambda'(\bar{u}))|_{rp,rp} &\le |\Delta u|_{rp,rp}|\lambda''(u) - \lambda''(\bar{u})|_{\infty,\infty} + |\lambda''(\bar{u})|_{\infty,\infty}|\Delta u - \Delta\bar{u}|_{rp,rp} \\ &\quad + |\nabla u|^2_{\infty,\infty}|\lambda'''(u) - \lambda'''(\bar{u})|_{\infty,\infty} + |\lambda'''(\bar{u})|_{\infty,\infty}|\nabla u - \nabla\bar{u}|_{\infty,\infty} \\ &\le C|u-\bar{u}|_{Z^1(T)},\end{aligned}$$

since $u, \bar{u} \in C(J; C^1(\overline{\Omega}))$ and $r > 1$ can be chosen close enough to 1, due to the fact that $\bar{v} \in C(J; C(\overline{\Omega}))$. Finally, we observe

$$|\bar{v}|_{r'p,r'p} \le |\bar{v} - v^*|_{r'p,r'p} + |v^*|_{r'p,r'p} \le \mu_2(T) + R.$$

This yields (ii).

(iii) For simplicity we set $f(u,v) = a_0\lambda'(\psi_0) - a(v)\lambda'(u)$. Then we compute

$$\begin{aligned}&|f(u,v)\partial_t u - f(\bar{u},\bar{v})\partial_t\bar{u}|_{X(T)} \\ &\qquad \le |\partial_t u(f(u,v) - f(\bar{u},\bar{v}))|_{X(T)} + |f(\bar{u},\bar{v})(\partial_t u - \partial_t\bar{u})|_{X(T)} \\ &\qquad \le (|\partial_t u - \partial_t u^*|_{X(T)} + |\partial_t u^*|_{X(T)})|f(u,v) - f(\bar{u},\bar{v})|_{\infty,\infty} + |f(\bar{u},\bar{v})|_{\infty,\infty}|\partial_t u - \partial_t\bar{u}|_{X(T)} \\ &\qquad \le C(\mu_3(T) + R)|f(u,v) - f(\bar{u},\bar{v})|_{\infty,\infty} + |f(\bar{u},\bar{v})|_{\infty,\infty}|\partial_t u - \partial_t\bar{u}|_{X(T)}.\end{aligned} \tag{2.49}$$

Next we estimate

$$\begin{aligned}|f(u,v) - f(\bar{u},\bar{v})|_{\infty,\infty} &\le |a(v)(\lambda'(u) - \lambda'(\bar{u}))|_{\infty,\infty} + |\lambda'(\bar{u})(a(v) - a(\bar{v}))|_{\infty,\infty} \\ &\le |a(v)|_{\infty,\infty}|\lambda'(u) - \lambda'(\bar{u})|_{\infty,\infty} + |\lambda'(\bar{u})|_{\infty,\infty}|a(v) - a(\bar{v})|_{\infty,\infty} \\ &\le C(|u-\bar{u}|_{\infty,\infty} + |v-\bar{v}|_{\infty,\infty}) \le C|(u,v) - (\bar{u},\bar{v})|_1.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}|f(\bar{u},\bar{v})|_{\infty,\infty} &\le |a_0|_{\infty,\infty}|\lambda'(\psi_0) - \lambda'(\bar{u})|_{\infty,\infty} + |\lambda'(\bar{u})|_{\infty,\infty}|a_0 - a(\bar{v})|_{\infty,\infty} \\ &\le C(|\psi_0 - \bar{u}|_{\infty,\infty} + |\vartheta_0 - \bar{v}|_{\infty,\infty}) \\ &\le C(|\psi_0 - u^*|_{\infty,\infty} + |u^* - \bar{u}|_{\infty,\infty} + |\vartheta_0 - v^*|_{\infty,\infty} + |v^* - \bar{v}|_{\infty,\infty}) \\ &\le C(\mu_3(T) + R).\end{aligned}$$

The last two estimates together with (2.49) yield (iii).
(iv) The proof of this assertion follows the lines of (ii).
(v) We compute

$$\begin{aligned}|(a(v)-a(\bar v)f_2|_{X(T)} &\le |a(v)-a(\bar v)|_{\infty,\infty}|f_2|_{X(T)} \le |v-\bar v|_{\infty,\infty}|f_2|_{X(T)}\\ &\le \mu_5(T)|v-\bar v|_{Z^2(T)} \le \mu_5(T)|(u,v)-(\bar u,\bar v)|_1,\end{aligned}$$

since $f_2 \in X(T)$ is a fixed function, hence $|f_2|_{X(T)} \to 0$ as $T \to 0$.
(vi) By trace theory, we obtain

$$|\partial_\nu(\Phi'(u)-\Phi'(\bar u))|_{Y_1(T)} \le C|\Phi'(u)-\Phi'(\bar u)|_{H_p^{1/2}(J;L_p(\Omega))} + |\Phi'(u)-\Phi'(\bar u)|_{L_p(J;H_p^2(\Omega))}.$$

The second norm has already been estimated in (i), so it remains to estimate $\Phi'(u)-\Phi'(\bar u)$ in $H_p^{1/2}(J;L_p(\Omega))$. Here we will use (2.47), to obtain

$$\begin{aligned}|\Phi'(u)-\Phi'(\bar u)|_{H_p^{1/2}(L_p)} &\le \mu_6(T)(|u-\bar u|_{H_p^{s_0}(L_p)} + |u-\bar u|_{\infty,\infty})\\ &\le \mu_6(T)C|u-\bar u|_{Z^1(T)} \le \mu_6(T)C|(u,v)-(\bar u,\bar v)|_1,\end{aligned}$$

since $s_0 < 1$.
(vii) We may apply (ii) and trace theory, to conclude that it suffices to estimate

$$(\lambda'(\psi_0)-\lambda'(u))v - (\lambda'(\psi_0)-\lambda'(\bar u))\bar v = (\lambda'(\psi_0)-\lambda'(u))(v-\bar v) - (\lambda'(u)-\lambda'(\bar u))\bar v$$

in $H_p^{1/2}(J;L_p(\Omega))$. This yields

$$\begin{aligned}|(\lambda'(\psi_0)-\lambda'(u))(v-\bar v)|_{H_p^{1/2}(L_p)} &\le |\lambda'(\psi_0)-\lambda'(u)|_{H_p^{1/2}(L_p)}|v-\bar v|_{\infty,\infty} + |\lambda'(\psi_0)-\lambda'(u)|_{\infty,\infty}|v-\bar v|_{H_p^{1/2}(L_p)}\\ &\le (|\lambda'(\psi_0)-\lambda'(u^*)|_{H_p^{1/2}(L_p)} + |\lambda'(u^*)-\lambda'(u)|_{H_p^{1/2}(L_p)})|v-\bar v|_{Z^2(T)}\\ &\quad + (|\psi_0-u^*|_{\infty,\infty} + |u^*-u|_{\infty,\infty})|v-\bar v|_{Z^2(T)}\\ &\le \Big(|\lambda'(\psi_0)-\lambda'(u^*)|_{H_p^{1/2}(L_p)} + \mu(T)R + (\mu_7(T)+R)\Big)|v-\bar v|_{Z^2(T)},\end{aligned}$$

where $\mu = \mu(T)$ is from (2.47). Clearly $\lambda'(\psi_0)-\lambda'(u^*) \in {}_0H_p^{1/2}(J;L_p(\Omega))$, since ψ_0 does not depend on t and since $\lambda \in C^{4-}(\mathbb{R})$. Therefore it holds that

$$|\lambda'(\psi_0)-\lambda'(u^*)|_{H_p^{1/2}(L_p)} \to 0$$

as $T \to 0$. The second part $(\lambda'(u)-\lambda'(\bar u))\bar v$ can be treated as follows.

$$\begin{aligned}|(\lambda'(u)-\lambda'(\bar u))\bar v|_{H_p^{1/2}(L_p)} &\le |\lambda'(u)-\lambda'(\bar u)|_{H_p^{1/2}(L_p)}|\bar v|_{\infty,\infty} + |\lambda'(u)-\lambda'(\bar u)|_{\infty,\infty}|\bar v|_{H_p^{1/2}(L_p)}\\ &\le C(\mu(T)+R+\mu_7(T))|u-\bar u|_{Z^1(T)},\end{aligned}$$

where we applied again (2.47). This completes the proof of the proposition.

Proof of Proposition 2.4.1
Let $J_{\max}^\delta := [\delta, T_{\max}]$ for some small $\delta > 0$. Setting $A^2 = \Delta_N^2$ with domain

$$D(A^2) = \{u \in H_p^4(\Omega) : \partial_\nu u = \partial_\nu \Delta u = 0 \text{ on } \partial\Omega\},$$

the solution $\psi(t)$ of equation $(2.20)_1$ may be represented by the variation of parameters formula

$$\psi(t) = e^{-A^2t}\psi_0 + \int_0^t Ae^{-A^2(t-s)}\Big(\lambda'(\psi(s))\vartheta(s) - \Phi'(\psi(s))\Big)\,ds, \quad t \in J_{\max}, \tag{2.50}$$

where e^{-A^2t} denotes the analytic semigroup, generated by $-A^2 = -\Delta_N^2$ in $L_p(\Omega)$. By (H3), (H4) and (2.25) it holds that

$$\Phi'(\psi) \in L_\infty(J_{\max}; L_{q_0}(\Omega)) \quad \text{and} \quad \lambda'(\psi) \in L_\infty(J_{\max}; L_6(\Omega)),$$

with $q_0 = 6/(\gamma+2)$. We then apply $A^r,\ r \in (0,1)$, to (2.50) and make use of semigroup theory to obtain

$$\psi \in L_\infty(J^\delta_{\max}; H^{2r}_{q_0}(\Omega)), \tag{2.51}$$

valid for all $r \in (0,1)$, since $q_0 < 6$. It follows from (2.51) that $\psi \in L_\infty(J^\delta_{\max}; L_{p_1}(\Omega))$ if $2r - 3/q_0 \geq -3/p_1$, and

$$\Phi'(\psi) \in L_\infty(J^\delta_{\max}; L_{q_1}(\Omega)) \quad \text{as well as} \quad \lambda'(\psi) \in L_\infty(J^\delta_{\max}; L_{p_1}(\Omega)),$$

with $q_1 = p_1/(\gamma+2)$. Hence we have this time

$$\psi \in L_\infty(J^\delta_{\max}; H^{2r}_{q_1}(\Omega)), \quad r \in (0,1).$$

Iteratively we obtain a sequence $(p_n)_{n\in\mathbb{N}_0}$ such that

$$2r - \frac{3}{q_n} \geq -\frac{3}{p_{n+1}}, \quad n \in \mathbb{N}_0$$

with $q_n = p_n/(\gamma_1+2)$ and $p_0 = 6$. Thus the sequence $(p_n)_{n\in\mathbb{N}_0}$ may be recursively estimated by

$$\frac{1}{p_{n+1}} \geq \frac{\gamma+2}{p_n} - \frac{2r}{3},$$

for all $n \in \mathbb{N}_0$ and $r \in (0,1)$. From this definition it is not difficult to obtain the following estimate for $1/p_{n+1}$.

$$\begin{aligned} \frac{1}{p_{n+1}} &\geq \frac{(\gamma+2)^{n+1}}{p_0} - \frac{2r}{3}\sum_{k=0}^{n}(\gamma+2)^k \\ &= \frac{(\gamma+2)^{n+1}}{p_0} - \frac{2r}{3}\left(\frac{(\gamma+2)^{n+1}-1}{\gamma_1+1}\right) \\ &= (\gamma+2)^{n+1}\left(\frac{1}{p_0} - \frac{2r}{3\gamma+3}\right) + \frac{2r}{3\gamma+3}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{2.52}$$

By the assumption (H3) on γ we see that the term in brackets is negative if $r \in (0,1)$ is sufficiently close to 1 and therefore, after finitely many steps the entire right side of (2.52) is negative as well, whence we may choose p_n arbitrarily large or we may even set $p_n = \infty$ for $n \geq N$ and a certain $N \in \mathbb{N}_0$. In other words this means that for those $r \in (0,1)$ we have

$$\psi \in L_\infty(J^\delta_{\max}; H^{2r}_p(\Omega)), \tag{2.53}$$

for all $p \in [1,\infty]$. It is important, that we can achieve this result in *finitely* many steps!

Next we will derive an estimate for $\partial_t\psi$. For all forthcoming calculations we will use the abbreviation $\psi = \psi(t)$ and $\vartheta = \vartheta(t)$. Since we only have estimates on the interval $J^\delta_{\max}$, we will use the following solution formula.

$$\psi(t) = e^{-A^2(t-\delta)}\psi_\delta + \int_0^{t-\delta} Ae^{-A^2 s}\Big(\lambda'(\psi)\vartheta - \Phi'(\psi)\Big)(t-s)\, ds, \quad t \in J^\delta_{\max}$$

where $\psi_\delta := \psi(\delta)$. Differentiating with respect to t, we obtain

$$\partial_t\psi(t) = A\int_0^{t-\delta} e^{-A^2 s}(\lambda''(\psi)\vartheta\partial_t\psi + \lambda'(\psi)\partial_t\vartheta - \Phi''(\psi)\partial_t\psi)(t-s)\, ds + F(t,\psi_\delta,\vartheta_\delta), \tag{2.54}$$

for all $t \geq \delta$ and with

$$F(t,\psi_\delta,\vartheta_\delta) := Ae^{-A^2(t-\delta)}(\lambda'(\psi_\delta)\vartheta_\delta - \Phi'(\psi_\delta)) - A^2 e^{-A^2(t-\delta)}\psi_\delta.$$

Let us discuss the function F in detail. By trace theory we have $\psi_\delta \in B_{pp}^{4-4/p}(\Omega)$ and $\vartheta_\delta \in B_{pp}^{2-2/p}(\Omega)$. Since we assume $p > (n+2)/2$, it holds that $\psi_\delta, \vartheta_\delta \in L_\infty(\Omega)$. Furthermore, the semigroup $e^{-A^2 t}$ is analytic. Therefore there exist some constants $C > 0$ and $\omega \in \mathbb{R}$ such that

$$|F(t, \psi_\delta, \vartheta_\delta)|_{L_p(\Omega)} \leq C \left(\frac{1}{(t-\delta)^{1/2}} + \frac{1}{t-\delta} \right) e^{\omega t},$$

for all $t > \delta$. This in turn implies that

$$F(\cdot, \psi_\delta, \vartheta_\delta) \in L_p(J_{\max}^{\delta'} \times \Omega)$$

for all $p \in (1, \infty)$, where $0 < \delta < \delta' < T_{\max}$. We will now use equations $(2.31)_{1,2}$ to rewrite the integrand in (2.54) in the following way.

$$\begin{aligned}
(\lambda''(\psi)\vartheta - \Phi''(\psi))\partial_t \psi + \lambda'(\psi)\partial_t \vartheta &= (\lambda''(\psi)\vartheta - \Phi''(\psi))\Delta\mu + \frac{\lambda'(\psi)}{b'(\vartheta)}\Delta\vartheta - \frac{\lambda'(\psi)^2}{b'(\vartheta)}\Delta\mu \\
&= \operatorname{div}\left[\left(\lambda''(\psi)\vartheta - \frac{\lambda'(\psi)^2}{b'(\vartheta)} - \Phi''(\psi) \right) \nabla\mu \right] + \operatorname{div}\left[\frac{\lambda'(\psi)}{b'(\vartheta)} \nabla\vartheta \right] \\
&\quad - \nabla \left(\lambda''(\psi)\vartheta - \frac{\lambda'(\psi)^2}{b'(\vartheta)} - \Phi''(\psi) \right) \cdot \nabla\mu - \nabla \frac{\lambda'(\psi)}{b'(\vartheta)} \cdot \nabla\vartheta.
\end{aligned} \tag{2.55}$$

Thus we obtain a decomposition of the following form

$$(\lambda''(\psi)\vartheta - \Phi''(\psi))\partial_t \psi + \lambda'(\psi)\partial_t \vartheta = \operatorname{div}(f_\mu \nabla\mu + f_\vartheta \nabla\vartheta) + g_\mu \nabla\mu + g_\vartheta \nabla\vartheta + h_\mu \nabla\vartheta\nabla\mu + h_\vartheta |\nabla\vartheta|^2,$$

with

$$\begin{aligned}
f_\mu &:= \lambda''(\psi)\vartheta - \frac{\lambda'(\psi)^2}{b'(\vartheta)} - \Phi''(\psi), \quad f_\vartheta := \frac{\lambda'(\psi)}{b'(\vartheta)}, \\
g_\mu &:= -\left(\lambda'''(\psi)\vartheta - 2\frac{\lambda'(\psi)\lambda''(\psi)}{b'(\vartheta)} - \Phi''(\psi) \right) \nabla\psi, \quad g_\vartheta := -\frac{\lambda''(\psi)}{b'(\vartheta)} \nabla\psi, \\
h_\mu &:= \lambda''(\psi) - \frac{b''(\vartheta)\lambda'(\psi)^2}{b'(\vartheta)^2}, \quad h_\vartheta := \frac{b''(\vartheta)\lambda'(\psi)}{b'(\vartheta)^2}.
\end{aligned}$$

By Assumption (H5) and the first part of the proof it holds that $f_j, g_j, h_j \in L_\infty(J_{\max}^\delta \times \Omega)$ for each $j \in \{\mu, \vartheta\}$ and this in turn yields that

$$\begin{aligned}
\operatorname{div}(f_\mu \nabla\mu + f_\vartheta \nabla\vartheta) &\in L_2(J_{\max}^\delta; H_2^1(\Omega)^*), \\
g_\mu \cdot \nabla\mu + g_\vartheta \cdot \nabla\vartheta &\in L_2(J_{\max}^\delta \times \Omega), \\
h_\mu \nabla\vartheta \cdot \nabla\mu + h_\vartheta |\nabla\vartheta|^2 &\in L_1(J_{\max}^\delta \times \Omega),
\end{aligned}$$

where we also made use of (2.25). Setting

$$T_1 = Ae^{-A^2 t} * \operatorname{div}(f_\mu \nabla\mu + f_\vartheta \nabla\vartheta), \quad T_2 = Ae^{-A^2 t} * (g_\mu \cdot \nabla\mu + g_\vartheta \cdot \nabla\vartheta)$$

and

$$T_3 = Ae^{-A^2 t} * (h_\mu \nabla\vartheta \cdot \nabla\mu + h_\vartheta |\nabla\vartheta|^2),$$

we may rewrite (2.54) as

$$\partial_t \psi = T_1 + T_2 + T_3 + F(t, \psi_0, \vartheta_0).$$

Going back to (2.54) we obtain

$$\begin{aligned}
&T_1 \in H_2^{1/4}(J_{\max}^\delta; L_2(\Omega)) \cap L_2(J_{\max}^\delta; H_2^1(\Omega)) \hookrightarrow L_2(J_{\max}^\delta \times \Omega), \\
&T_2 \in H_2^{1/2}(J_{\max}^\delta; L_2(\Omega)) \cap L_2(J_{\max}^\delta; H_2^2(\Omega)) \hookrightarrow L_2(J_{\max}^\delta \times \Omega), \quad \text{and} \\
&F(\cdot, \psi_\delta, \vartheta_\delta) \in L_2(J_{\max}^{\delta'} \times \Omega).
\end{aligned}$$

Observe that we do not have full regularity for T_3 since A has no maximal regularity in $L_1(\Omega)$, but nevertheless we obtain

$$T_3 \in H_1^{1/2-}(J_{\max}^{\delta}; L_1(\Omega)) \cap L_1(J_{\max}^{\delta}; H_1^{2-}(\Omega)).$$

Here we used the notation $H_p^{s-} := H_p^{s-\varepsilon}$ and $\varepsilon > 0$ is sufficiently small. An application of the mixed derivative theorem then yields

$$H_1^{1/2-}(J_{\max}^{\delta}; L_1(\Omega)) \cap L_1(J_{\max}^{\delta}; H_1^{2-}(\Omega)) \hookrightarrow L_p(J_{\max}^{\delta}; L_2(\Omega)),$$

if $p \in (1, 8/7)$, whence

$$\partial_t \psi \in L_2(J_{\max}^{\delta'} \times \Omega) + L_p(J_{\max}^{\delta'}; L_2(\Omega))$$

for some $1 < p < 8/7$. Now we go back to (2.55) where we replace this time only $\partial_t \vartheta$ by the differential equation $(2.31)_2$ to obtain

$$\begin{aligned}(\lambda''(\psi)\vartheta - \Phi''(\psi))\partial_t\psi + \lambda'(\psi)\partial_t\vartheta &= \left(\lambda''(\psi)\vartheta - \Phi''(\psi) - \frac{\lambda'(\psi)^2}{b'(\vartheta)}\right)\partial_t\psi \\ &\quad + \operatorname{div}\left[\frac{\lambda'(\psi)}{b'(\vartheta)}\nabla\vartheta\right] - \frac{\lambda''(\psi)}{b'(\vartheta)}\nabla\psi\cdot\nabla\vartheta + \frac{\lambda'(\psi)b''(\vartheta)}{b'(\vartheta)^2}|\nabla\vartheta|^2 \\ &= f\partial_t\psi + \operatorname{div}[g\nabla\vartheta] + h\cdot\nabla\vartheta + k|\nabla\vartheta|^2.\end{aligned}$$

Rewrite (2.54) in the following way

$$\partial_t \psi = S_1 + S_2 + S_3 + S_4 + F(t, \psi_0, \vartheta_0), \tag{2.56}$$

where the functions S_j are defined in the same manner as T_j. Since $f, g, h \in L_\infty(J_{\max}^{\delta} \times \Omega)$ it follows again from regularity theory that

$$\begin{aligned}&S_1 \in H_2^{1/2}(J_{\max}^{\delta'}; L_2(\Omega)) \cap L_2(J_{\max}^{\delta'}; H_2^2(\Omega)) + H_p^{1/2}(J_{\max}^{\delta'}; L_2(\Omega)) \cap L_p(J_{\max}^{\delta'}; H_2^2(\Omega)),\\ &S_2 \in H_2^{1/4}(J_{\max}^{\delta'}; L_2(\Omega)) \cap L_2(J_{\max}^{\delta'}; H_2^1(\Omega)),\\ &S_3 \in H_2^{1/2}(J_{\max}^{\delta'}; L_2(\Omega)) \cap L_2(J_{\max}^{\delta'}; H_2^2(\Omega))\end{aligned}$$

and it can be readily checked that

$$H_p^{1/2}(J_{\max}^{\delta'}; L_2(\Omega)) \cap L_p(J_{\max}^{\delta'}; H_2^2(\Omega)) \hookrightarrow L_2(J_{\max}^{\delta'} \times \Omega),$$

whenever $p \in [1, 2]$. Now we turn our attention to the term $S_4 = Ae^{-A^2 t} * k|\nabla\vartheta|^2$. First we observe that by the mixed derivative theorem the embedding

$$Z_q := H_q^{1/2-}(J_{\max}^{\delta'}; L_1(\Omega)) \cap L_q(J_{\max}^{\delta'}; H_1^{2-}(\Omega)) \hookrightarrow L_2(J_{\max}^{\delta'} \times \Omega)$$

is valid, provided that $q \in (8/5, 2]$. Hence it holds that

$$|S_4|_{2,2} \le C|S_4|_{Z_q} \le C|k|\nabla\vartheta|^2|_{q,1} \le C|\nabla\vartheta|_{2q,2}^2,$$

with some constant $C > 0$. Taking the norm of $\partial_t\psi$ in $L_2(J_{\max}^{\delta'} \times \Omega)$ we obtain from (2.56)

$$|\partial_t\psi|_{2,2} \le C\left(\sum_{j=1}^{3}|S_j|_{2,2} + |\nabla\vartheta|_{2q,2}^2 + |F(\cdot, \psi_\delta, \vartheta_\delta)|_{2,2}\right).$$

The Gagliardo-Nirenberg inequality in connection with (2.25) yields the estimate

$$|\nabla\vartheta|_{2q,2}^2 \le c|\nabla\vartheta|_{2,2}^{2a}|\nabla\vartheta|_{\infty,2}^{2(1-a)} \le c|\nabla\vartheta|_{\infty,2}^{2(1-a)},$$

provided that $a = 1/q$. Multiply $(2.20)_2$ by $\partial_t\vartheta$ and integrate by parts to the result

$$\int_\Omega b'(\vartheta(t,x))|\partial_t\vartheta(t,x)|^2\,dx + \frac{1}{2}\frac{d}{dt}|\nabla\vartheta(t)|_2^2 = -\int_\Omega \lambda'(\psi(t,x))\partial_t\psi(t,x)\partial_t\vartheta(t,x)\,dx.$$

Making use of (H5) and Young's inequality we obtain

$$C_1|\partial_t\vartheta|_{2,2}^2 + \frac{1}{2}|\nabla\vartheta(t)|_2^2 \le C_2(|\partial_t\psi|_{2,2}^2 + |\nabla\vartheta_0|_2^2), \tag{2.57}$$

after integrating w.r.t. t. This in turn yields the estimate

$$|\nabla\vartheta|_{2q,2}^2 \le c|\nabla\vartheta|_{\infty,2}^{2(1-a)} \le c(1 + |\partial_t\psi|_{2,2}^{2(1-a)}).$$

In order to gain something from this inequality we require that $2(1-a) < 1$, i.e. q is restricted by $1 < q < 2$. Finally, if we choose $q \in (8/5, 2)$ and use the uniform boundedness of the L_2 norms of $S_j,\ j \in \{1,2,3\}$ we obtain

$$|\partial_t\psi|_{2,2} \le C(1 + |\partial_t\psi|_{2,2}^{2(1-a)}).$$

Since by construction $2(1-a) < 1$, it follows that the L_2-norm of $\partial_t\psi$ is bounded on $J_{\max}^{\delta'} \times \Omega$. In particular, this yields the statement for ϑ by equation (2.57).

Now we go back to (2.54) with δ replaced by δ'. By Assumption (H5), by the bounds $\partial_t\vartheta, \partial_t\psi \in L_2(J_{\max}^{\delta'}; L_2(\Omega))$ and by the first part of the proof we obtain

$$\lambda''(\psi)\vartheta\partial_t\psi + \lambda'(\psi)\partial_t\vartheta - \Phi''(\psi)\partial_t\psi \in L_2(J_{\max}^{\delta'}; L_2(\Omega)).$$

Since the operator $A^2 = \Delta^2$ with domain

$$D(A^2) = \{u \in H_p^4(\Omega) : \partial_\nu u = \partial_\nu\Delta u = 0\}$$

has the property of maximal L_p-regularity (cf. Theorem 1.4.3), we obtain from (2.54)

$$\partial_t\psi - F(\cdot, \psi_{\delta'}, \vartheta_{\delta'}) \in H_2^{1/2}(J_{\max}^{\delta'}; L_2(\Omega)) \cap L_2(J_{\max}^{\delta'}; H_2^2(\Omega)) \hookrightarrow L_r(J_{\max}^{\delta'}; L_r(\Omega)),$$

and the last embedding is valid for all $r \le 2(n+4)/n$. By the properties of the function F it follows

$$\partial_t\psi \in L_r(J_{\max}^{\delta''}; L_r(\Omega)),$$

for all $r \le 2(n+4)/n$ and some $0 < \delta'' < T_{\max}$. To obtain an estimate for the whole interval $J_{\max}$, we use the fact that we already have a local strong solution, i.e. $\partial_t\psi \in L_p(0, \delta''; L_p(\Omega))$, $p > (n+2)/2$. The proof is complete.

Chapter 3

The Non-Isothermal Cahn-Hilliard Equation with Dynamic Boundary Conditions

3.1 Derivation of the Model

The derivation of the classical non-isothermal Cahn-Hilliard equation

$$
\begin{aligned}
\partial_t\psi - \Delta\mu = 0, \quad \mu = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta, \quad t \in J,\ x \in \Omega,\\
\partial_t\vartheta + \lambda'(\psi)\partial_t\psi - \Delta\vartheta = 0, \quad t \in J,\ x \in \Omega,
\end{aligned}
\tag{3.1}
$$

follows the lines of Chapter 2. This time we start with a free energy functional of the form

$$F(\psi,\vartheta) = \int_\Omega \left(\frac{1}{2}|\nabla\psi|^2 + \Phi(\psi) - \lambda(\psi)\vartheta - \frac{1}{2}\vartheta^2\right)\,dx,$$

where we assume that the relative temperature ϑ varies in time and space. The chemical potential μ and the internal energy e are given by the variational derivatives

$$\mu = \frac{\delta F}{\delta\psi} = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta,$$

and

$$e = -\frac{\delta F}{\delta\vartheta} = \vartheta + \lambda(\psi).$$

To incorporate dynamics into these stationary equations, we assume that the order parameter ψ and the internal energy e are conserved quantities, subject to conservation laws. These read as follows

$$\partial_t\psi + \operatorname{div} j = 0, \quad \partial_t e + \operatorname{div} q = 0,$$

with the boundary conditions $(j|\nu) = (q|\nu) = 0$, where ν is the outer unit normal on $\Gamma = \partial\Omega$. Here q is the heat flux, which in this paper is assumed to be given by Fourier's law $q = -\nabla\vartheta$ and j denotes the phase flux of the order parameter ψ which is assumed to be of the form $j = -\nabla\mu$, a constitutive, but well accepted law. Since $(j|\nu) = 0$, we obtain $\partial_\nu\mu = 0$. Concerning ϑ we will use Robin boundary conditions, namely $\alpha\vartheta + \partial_\nu\vartheta = 0$, where $\alpha \geq 0$ is a constant. Since $(3.1)_1$ is an equation of fourth order, we need another boundary condition for ψ. Usually one uses further classical boundary conditions, e.g. $\partial_\nu\psi = 0$. Recently, to account for boundary effects, the authors in [22] proposed a dynamic boundary condition of the form

$$\partial_t\psi - \sigma_s\Delta_\Gamma\psi + \gamma\partial_\nu\psi + \kappa\psi = 0, \tag{3.2}$$

which we will use in our model, where $\sigma_s, \gamma > 0$, $\kappa \geq 0$. Hence, the system we investigate is

$$\begin{aligned} \partial_t\psi - \Delta\mu = f_1, \quad \mu = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta, &\quad t \in J,\ x \in \Omega,\\ \partial_t\vartheta + \lambda'(\psi)\partial_t\psi - \Delta\vartheta = f_2, &\quad t \in J,\ x \in \Omega,\\ \partial_\nu\mu = g_1, &\quad t \in J,\ x \in \Gamma,\\ \partial_t\psi - \sigma_s\Delta_\Gamma\psi + \gamma\partial_\nu\psi + \kappa\psi = g_2, &\quad t \in J,\ x \in \Gamma,\\ \alpha\vartheta + \partial_\nu\vartheta = g_3, &\quad t \in J,\ x \in \Gamma,\\ \psi(0) = \psi_0, &\quad t = 0,\ x \in \Omega,\\ \vartheta(0) = \vartheta_0, &\quad t = 0,\ x \in \Omega, \end{aligned} \tag{3.3}$$

where $\Omega \subset \mathbb{R}^n$ is open, bounded with compact boundary $\Gamma := \partial\Omega \in C^4$ and f_i,g_j,ψ_0,ϑ_0 are given functions in appropriate function spaces to be defined later. We are interested in solutions

$$\psi \in H^1_p(J; L_p(\Omega)) \cap L_p(J; H^4_p(\Omega)) =: Z^1,$$

$$\vartheta \in H^1_p(J; L_p(\Omega)) \cap L_p(J; H^2_p(\Omega)) =: Z^2,$$

with

$$\psi|_\Gamma \in H^1_p(J; W^{2-1/p}_p(\Gamma)) \cap L_p(J; W^{4-1/p}_p(\Gamma)) =: Z^1_\Gamma.$$

Let us explain, where the basic space $W^{2-1/p}_p(\Gamma)$ for $\psi|_\Gamma$ comes from. Taking the trace of ψ on Γ, this yields

$$\psi|_\Gamma \in W^{1-1/4p}_p(J; L_p(\Gamma)) \cap L_p(J; W^{4-1/p}_p(\Gamma)).$$

We apply the Laplace-Beltrami operator Δ_Γ to $\psi|_\Gamma$, to obtain

$$\Delta_\Gamma\psi|_\Gamma \in L_p(J; W^{2-1/p}_p(\Omega)).$$

If we treat the dynamical boundary condition as a heat equation on Γ, this will result in

$$\psi|_\Gamma \in H^1_p(J; W^{2-1/p}_p(\Gamma)) \cap L_p(J; W^{4-1/p}_p(\Gamma)),$$

since

$$\partial_\nu\psi \in W^{3/4-1/4p}_p(J; L_p(\Gamma)) \cap L_p(J; W^{3-1/p}_p(\Gamma)) \hookrightarrow L_p(J; W^{2-1/p}_p(\Omega)).$$

In the sequel we will use the notation

$$Z^1 \cap Z^1_\Gamma := \{u \in Z^1 : u|_\Gamma \in Z^1_\Gamma\}.$$

3.2 The Linear Problem

Before we deal with the linearized version of (3.3), we need some preliminaries. Firstly, we want to set $f_2, g_3, \vartheta_0 = 0$. For this we consider the system

$$\begin{aligned} \partial_t v - \Delta v = f_2, &\quad t \in J,\ x \in \Omega,\\ \alpha v + \partial_\nu v = g_3, &\quad t \in J,\ x \in \Gamma,\\ v(0) = \vartheta_0, &\quad t = 0,\ x \in \Omega. \end{aligned} \tag{3.4}$$

By Theorem 1.4.3 there is a unique solution $\vartheta_1 \in H^1_p(J; L_p(\Omega)) \cap L_p(J; H^2_p(\Omega))$ of (3.4), provided $f_2 \in L_p(J \times \Omega) =: X$, $\vartheta_0 \in B^{2-2/p}_{pp}(\Omega) =: X_p$,

$$g_3 \in W^{1/2-1/2p}_p(J; L_p(\Gamma)) \cap L_p(J; W^{1-1/p}_p(\Gamma)) =: Y_3,$$

and the compatibility condition $\alpha\vartheta_0 + \partial_\nu\vartheta_0 = g_3|_{t=0}$ is satisfied, whenever $p > 3$. Taking the latter for granted, we may set $f_2, g_3, \vartheta_0 = 0$ in (3.3).

Secondly we want to replace ϑ in $(3.3)_1$ by a term only depending on ψ and some given data. Therefore we will solve the inhomogeneous heat equation

$$\partial_t\vartheta - \Delta\vartheta = -\partial_t\lambda(\psi), \quad \vartheta(0) = 0 \tag{3.5}$$

with homogeneous Robin or Neumann boundary conditions. Suppose that we already know a solution $(\psi,\vartheta) \in (Z^1 \cap Z^1_\Gamma) \times Z^2$ of (3.3). Assuming that λ' is bounded we have $\partial_t\lambda(\psi) \in L_p(J\times\Omega)$. Let $A_K = -\Delta_K$, $K = R, N$, where R and N stand for Robin and Neumann boundary conditions, respectively. By $e^{-A_K t}$ we denote the bounded analytic semigroup, generated by $-A_K$ in $L_p(\Omega)$. The solution ϑ to (3.5) may then be represented by the variation of parameters formula

$$\vartheta(t) = -\int_0^t e^{-A_K(t-s)}\partial_t\lambda(\psi(s))\, ds.$$

Our aim is to split the derivative ∂_t into $\partial_t^{1/2}\partial_t^{1/2}$. But this is only possible if one applies ∂_t to a function with vanishing trace at $t = 0$. Since in general $\lambda(\psi_0) \neq 0$, we insert a function, say $w_0 \in Z^1$, with $w_0(0) = \lambda(\psi_0)$. For the existence of such a function w_0 we consider the initial boundary value problem

$$\begin{aligned} \partial_t w + \Delta^2 w &= 0, \quad t \in J,\ x \in \Omega,\\ \partial_\nu\Delta w &= g_1, \quad t \in J,\ x \in \Gamma,\\ \partial_\nu w &= g_2, \quad t \in J,\ x \in \Gamma,\\ w(0) &= \lambda(\psi_0), \quad t = 0,\ x \in \Omega. \end{aligned} \tag{3.6}$$

Let $e^{-\Delta^2_\Gamma t}$ denote the analytic semigroup, generated by $-\Delta^2_\Gamma$ in $L_p(\Gamma)$. We set $g_1 = 0$ if $p < 5$, $g_1 = e^{-\Delta^2_\Gamma t}\partial_\nu\Delta\lambda(\psi_0)$ if $p > 5$ and $g_2 = 0$ if $p < 5/3$, $g_2 = e^{-\Delta^2_\Gamma t}\partial_\nu\lambda(\psi_0)$ if $p > 5/3$.

The well-posedness of (3.6) is guaranteed by the following proposition.

Proposition 3.2.1. *Let $p > (n+4)/4$, $p \neq 5/3, 5$ and let $\psi_0 \in B^{4-4/p}_{pp}(\Omega)$. Then there exists a unique solution*

$$w_0 \in H^1_p(J; L_p(\Omega)) \cap L_p(J; H^4_p(\Omega))$$

of (3.6).

Proof. We will use Theorem 1.4.3 for the proof of the assertion. To this end we have to show

(i) $g_1 \in W_p^{1/4-1/4p}(J; L_p(\Gamma)) \cap L_p(J; W_p^{1-1/p}(\Gamma))$;

(ii) $g_2 \in W_p^{3/4-1/4p}(J; L_p(\Gamma)) \cap L_p(J; W_p^{3-1/p}(\Gamma))$;

(iii) $\lambda(\psi_0) \in B^{4-4/p}_{pp}(\Omega)$;

(iv) $\partial_\nu\Delta\lambda(\psi_0) = g_1|_{t=0}$, if $p > 5$;

(v) $\partial_\nu\lambda(\psi_0) = g_2|_{t=0}$, if $p > 5/3$.

By construction, the compatibility conditions (iv) and (v) are satisfied. We turn our attention to the initial value $\lambda(\psi_0)$. First observe that the embedding

$$B^{4-4/p}_{pp}(\Omega) \hookrightarrow C(\overline{\Omega}),$$

is true, provided that $p > (n+4)/4$. Furthermore we have

$$\begin{aligned} |\lambda(\psi_0(x)) - \lambda(\psi_0(y))| &= \Big|\int_0^1 \frac{d}{d\theta}\lambda(\theta\psi_0(x) + (1-\theta)\psi_0(y))\, d\theta\Big| \\ &\le |\psi_0(x) - \psi_0(y)| \int_0^1 |\lambda'(\theta\psi_0(x) + (1-\theta)\psi_0(y))|\, d\theta, \end{aligned}$$

for all $x, y \in \Omega$. Since $\psi_0 \in C(\overline{\Omega})$ and $\lambda' \in L_\infty(\mathbb{R})$ we obtain the estimate

$$|\lambda(\psi_0(x)) - \lambda(\psi_0(y))| \leq M|\psi_0(x) - \psi_0(y)|,$$

for some constant $M > 0$ which does not depend on x and y. Then (iii) follows from the definition of the spaces B^s_{pp} via differences at least if $4 - 4/p \in (0, 1)$. Consider the case $4 - 4/p \in (1, 2)$. Then it suffices to show that $\nabla\lambda(\psi_0) \in B^{3-4/p}_{pp}(\Omega) = W^{3-4/p}_p(\Omega)$. By Hölder's inequality for these spaces we obtain

$$|\nabla\lambda(\psi_0)|_{W^{3-4/p}_p} \leq |\nabla\psi_0|_{W^{3-4/p}_p} |\lambda'(\psi_0)|_{L_\infty} + |\lambda'(\psi_0)|_{W^{3-4/p}_{rp}} |\nabla\psi_0|_{L_{r'p}},$$

with $1/r + 1/r' = 1$. The first term is finite, so we may concentrate on the second one. By the arguments above there exists a constant $M > 0$ such that

$$|\lambda'(\psi_0)|_{W^{3-4/p}_{rp}} \leq M(1 + |\psi_0|_{W^{3-4/p}_{rp}}).$$

Sobolev embedding implies

$$\psi_0 \in B^{4-4/p}_{pp}(\Omega) \hookrightarrow W^{3-4/p}_{rp}(\Omega)$$

provided that $n/r' \leq p$. Furthermore we have $W^{4-4/p}_p(\Omega) \hookrightarrow L_{r'p}(\Omega)$ if $n/r' \geq 4 + n - 3p$, hence $4 + n - 3p \leq p$ or equivalently $p \geq (n + 4)/4$. This yields Assertion (iii) if $4 - 4/p \in (1, 2)$. The arguments for larger values of p are similar. We omit the details.

Finally, by trace theory, we obtain

$$\partial_\nu\lambda(\psi_0) \in B^{3-5/p}_{pp}(\Gamma) \quad \text{and} \quad \partial_\nu\Delta\lambda(\psi_0) \in B^{1-5/p}_{pp}(\Gamma),$$

if $p > 5/3$ and $p > 5$, respectively. It is well-known that the analytic semigroup $e^{-\Delta_\Gamma^2 t}$ has the property of maximal L_p-regularity and so (i) and (ii) are satisfied.

□

With the help of such a function w_0 we may write

$$\begin{aligned} \vartheta(t) &= -\int_0^t e^{-A_K(t-s)}\partial_t w_0(s)\, ds - \int_0^t e^{-A_K(t-s)}\partial_t(\lambda(\psi(s)) - w_0(s))\, ds + \vartheta_1(t) \\ &= \vartheta_2(t) - \partial_t^{1/2}(\partial_t + A_K)^{-1}\partial_t^{1/2}(\lambda(\psi(t)) - w_0(t)) + \vartheta_1(t), \end{aligned} \tag{3.7}$$

with $\vartheta_2(t) := -\int_0^t e^{-A_K(t-s)}\partial_t w_0(s)\, ds$. As we will see in Section 3, the splitting of the time derivative ∂_t yields a lower order term, compared to $(\partial_t + \Delta^2)\psi$. Thus it remains to solve the problem

$$\begin{aligned} \partial_t\psi + \Delta^2\psi = \Delta\Phi'(\psi) + \Delta(\lambda'(\psi)F(\psi)) - \Delta(\lambda'(\psi)\vartheta^*) + f, &\quad t \in J,\ x \in \Omega, \\ \partial_\nu\Delta\psi = \partial_\nu\Phi'(\psi) + \partial_\nu(\lambda'(\psi)F(\psi)) - \partial_\nu(\lambda'(\psi)\vartheta^*) + g, &\quad t \in J,\ x \in \Gamma, \\ \partial_t\psi - \sigma_s\Delta_\Gamma\psi + \gamma\partial_\nu\psi + \kappa\psi = h, &\quad t \in J,\ x \in \Gamma, \\ \psi(0) = \psi_0, &\quad t = 0,\ x \in \Omega, \end{aligned} \tag{3.8}$$

with $F(\psi) := \partial_t^{1/2}(\partial_t + A_K)^{-1}\partial_t^{1/2}(\lambda(\psi) - w_0)$ and $\vartheta^* = \vartheta_1 + \vartheta_2 \in Z^2$. The corresponding linear problem to (3.8) reads as follows

$$\begin{aligned} \partial_t u + \Delta^2 u = f, &\quad t \in J,\ x \in \Omega, \\ \partial_\nu\Delta u = g, &\quad t \in J,\ x \in \Gamma, \\ \partial_t u - \sigma_s\Delta_\Gamma u + \gamma\partial_\nu u + \kappa u = h, &\quad t \in J,\ x \in \Gamma, \\ u(0) = u_0, &\quad t = 0,\ x \in \Omega. \end{aligned} \tag{3.9}$$

Here is the main result on maximal L_p - regularity of (3.9).

Theorem 3.2.2. *Let $n \in \mathbb{N}$, $1 < p < \infty$, $p \neq 3, 5$ and let $\sigma_s, \gamma > 0$ and $\kappa \geq 0$ be constants. Suppose $\Omega \subset \mathbb{R}^n$ is bounded open with compact boundary $\Gamma = \partial\Omega \in C^4$ and let $J = [0, T]$. Then there is a unique solution u of* (3.9) *such that*

$$u \in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^4(\Omega)) = Z^1,$$

with

$$u|_\Gamma \in H_p^1(J; W_p^{2-1/p}(\Gamma)) \cap L_p(J; W_p^{4-1/p}(\Gamma)) = Z_\Gamma^1,$$

if and only if the data are subject to the following conditions.

(i) $f \in L_p(J \times \Omega) = X$,

(ii) $g \in W_p^{1/4-1/4p}(J; L_p(\Gamma)) \cap L_p(J; W_p^{1-1/p}(\Gamma)) =: Y_1$,

(iii) $h \in L_p(J; W_p^{2-1/p}(\Gamma)) =: Y_2$,

(iv) $u_0 \in \{w \in B_{pp}^{4-4/p}(\Omega) : w|_\Gamma \in B_{pp}^{4-3/p}(\Gamma)\} =: X_{p,\Gamma}$,

(v) $\partial_\nu \Delta u_0 = g|_{t=0}$, *if* $p > 5$.

Proof. This theorem is a special case of [34, Theorem 2.1]. □

3.3 Local Well-Posedness

In this section we will apply the contraction mapping principle to overcome the nonlinearities in

$$\begin{aligned} \partial_t u + \Delta^2 u &= \Delta G(u) + f, && t \in J,\ x \in \Omega, \\ \partial_\nu \Delta u &= \partial_\nu G(u) + g, && t \in J,\ x \in \Gamma, \\ \partial_t u - \sigma_s \Delta_\Gamma u + \gamma \partial_\nu u + \kappa u &= h, && t \in J,\ x \in \Gamma, \\ u(0) &= \psi_0, && t = 0,\ x \in \Omega, \end{aligned} \tag{3.10}$$

where

$$G(u) := \Phi'(u) + \lambda'(u)F(u) - \lambda'(u)\vartheta^*.$$

For that purpose let $\psi_0 \in X_{p,\Gamma}$, $f \in X$, $g \in Y_1$ and $h \in Y_2$, as well as $\vartheta_0 \in X_p$ be given, such that the compatibility condition

$$\partial_\nu \Delta \psi_0 = \partial_\nu(\Phi'(\psi_0) - \lambda'(\psi_0)\vartheta_0) + g|_{t=0}, \quad \text{if } p > 5,$$

is satisfied. We furthermore assume that $\lambda, \Phi \in C^{4-}(\mathbb{R})$ and we will use the embeddings

$$H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^4(\Omega)) \hookrightarrow C(J; B_{pp}^{4-4/p}(\Omega)) \hookrightarrow C(J \times \overline{\Omega}), \tag{3.11}$$

valid for $p > n/4 + 1$. For $[0, T] \subset [0, T_0]$ we define

$$\mathbb{E}_1 := \{w \in Z^1(T) : w|_\Gamma \in Z_\Gamma^1(T)\}, \quad \text{and} \quad {}_0\mathbb{E}_1 := \{w \in \mathbb{E}_1 : w|_{t=0} = 0\}$$

and

$$\mathbb{E}_0 := X(T) \times Y_1(T) \times Y_2(T), \quad \text{and} \quad {}_0\mathbb{E}_0 := \{(f, g, h) \in \mathbb{E}_0 : g|_{t=0} = 0\}.$$

The spaces $\mathbb{E}_1$ and $\mathbb{E}_0$ are endowed with canonical norms $|\cdot|_1$ and $|\cdot|_0$, respectively. Let furthermore $A := -\Delta_\Gamma$. By Theorem 3.2.2 there exists a unique solution $u^* \in \mathbb{E}_1$ of the linear system

$$\begin{aligned} \partial_t u + \Delta^2 u &= f, && t \in J,\ x \in \Omega, \\ \partial_\nu \Delta u &= g - e^{-A^2 t} g_0, && t \in J,\ x \in \Gamma, \\ \partial_t u - \sigma \Delta_\Gamma u + \gamma \partial_\nu u + \kappa u &= h, && t \in J,\ x \in \Gamma, \\ u(0) &= \psi_0, && t = 0,\ x \in \Omega, \end{aligned} \tag{3.12}$$

where $g_0 = 0$, if $p < 5$ and $g_0 = g|_{t=0} + \partial_\nu \Delta\psi_0$, if $p > 5$. We define a linear operator $\mathbb{L} : \mathbb{E}_1 \to \mathbb{E}_0$ by

$$\mathbb{L}w = \begin{bmatrix} \partial_t w + \Delta^2 w \\ \partial_\nu \Delta w \\ \partial_t w - \sigma_s \Delta_\Gamma w + \mu \partial_\nu w + \kappa w \end{bmatrix}.$$

Consider $\mathbb{L}$ as an operator from ${}_0\mathbb{E}_1$ to ${}_0\mathbb{E}_0$. Then by Theorem 3.2.2, $\mathbb{L}$ is bounded and bijective, i.e. an isomorphism. Hence, by the open-mapping theorem, $\mathbb{L}$ is invertible with bounded inverse.

Next define a nonlinear mapping $\tilde{G} : \mathbb{E}_1 \times {}_0\mathbb{E}_0 \to {}_0\mathbb{E}_0$ by means of

$$\tilde{G}(u^*, w) = \begin{bmatrix} \Delta G(u^* + w) \\ \partial_\nu G(u^* + w) - g_1 \\ 0 \end{bmatrix},$$

where $g_1 = 0$, if $p < 5$ and $g_1 = e^{-A^2 t} \partial_\nu G(u^*)|_{t=0}$ if $p > 5$. We will show in a subsequent proposition that the range of $\tilde{G}$ is indeed a subset of ${}_0\mathbb{E}_0$. For the moment, assume that this result is already at our disposal. It is then obvious that $u := u^* + w$ is a solution of (3.10) if and only if $\mathbb{L}w = \tilde{G}(u^*, w)$ or equivalently $w = \mathbb{L}^{-1}\tilde{G}(u^*, w)$. Define a ball $\mathbb{B}_R \subset {}_0\mathbb{E}_1$ by

$$\mathbb{B}_R := \mathbb{B}_R(0) := \{w \in {}_0\mathbb{E}_1 \ : \ |w|_1 \le R\}, \ R \in (0, 1],$$

and an operator $\mathcal{T} : \mathbb{B}_R \to {}_0\mathbb{E}_1$ by $\mathcal{T}w = \mathbb{L}^{-1}\tilde{G}(u^*, w)$. In order to apply the contraction mapping principle we have to ensure that $\mathcal{T}$ is a self-mapping, i.e. $\mathcal{T}\mathbb{B}_R \subset \mathbb{B}_R$ and that $\mathcal{T}$ defines a strict contraction on $\mathbb{B}_R$, i.e. there exists a number $\beta < 1$ with

$$|\mathcal{T}w - \mathcal{T}\bar{w}|_1 \le \beta |w - \bar{w}|_1, \quad w, \bar{w} \in \mathbb{B}_R.$$

We need some preliminaries to prove these properties. First we observe that all functions belonging to $\mathbb{B}_R$ are uniformly bounded on $J \times \Omega$. Indeed, by (3.11) it holds that

$$|w|_\infty \le M|w|_{Z^1} \le M|w|_1 \le MR \le M,$$

with a constant $M > 0$, independent of T, since $w|_{t=0} = 0$ for all $w \in \mathbb{B}_R$.

For all forthcoming considerations we define the shifted ball $\mathbb{B}_R(u^*) \subset \mathbb{E}_1$ by means of

$$\mathbb{B}_R(u^*) := \{w \in \mathbb{E}_1 : w = \tilde{w} + u^*, \ \tilde{w} \in \mathbb{B}_R\}.$$

Note that all functions $w \in \mathbb{B}_R(u^*)$ are uniformly bounded, too. In the sequel we will also make use of the well known estimate

$$|fg|_{H^s_p(L_p)} \le C(|f|_{L_{\sigma_1' p}(L_{r_1' p})} |g|_{H^s_{\sigma_1 p}(L_{r_1 p})} + |g|_{L_{\sigma_2' p}(L_{r_2' p})} |f|_{H^s_{\sigma_2 p}(L_{r_2 p})}), \tag{3.13}$$

where $1/\sigma_i + 1/\sigma_i' = 1/r_i + 1/r_i' = 1, \ i = 1, 2, \ s \in [0, 1]$. Indeed, this is a consequence of the definition of the spaces H^s_p via differences and Hölders inequality (cf. also [44]). Next, by [47, Lemma 6.2.3] there is a function $\mu(T) > 0$, with $\mu(T) \to 0$ as $T \to 0$, such that

$$|f(u) - f(v)|_{H^s_p(L_p)} \le \mu(T)(|u - v|_{H^{s_0}_p(L_p)} + |u - v|_\infty), \quad 0 < s < s_0 < 1, \tag{3.14}$$

for every $f \in C^{2-}(\mathbb{R})$ and all $u, v \in \mathbb{B}_R(u^*)$. Now we show that $F(w), \ w \in \mathbb{B}_R(u^*)$, represents a lower order term in (3.10), which is crucial to establish the desired properties of the operator $\mathcal{T}$.

By the mixed-derivative theorem we obtain

$$w_0 \in Z^1 \hookrightarrow H^{3/4}_p(J; H^1_p(\Omega)) \hookrightarrow H^s_p(J; H^1_p(\Omega)),$$

for every $s \in (0, 3/4)$. By (3.13) we see that $\lambda(w) \in H^s_p(J; H^1_p(\Omega)), \ s \in (0, 3/4)$, too. Thus $(\lambda(w) - w_0) \in {}_0H^s_p(J; H^1_p(\Omega))$, and for $1/2 < s < 3/4$ and every $\eta \in {}_0H^s_p(J; H^1_p(\Omega))$ we obtain

$$(\partial_t + A_K)^{-1}\partial_t^{1/2}\eta \in {}_0H^{s+1/2}_p(J; H^1_p(\Omega)) \cap {}_0H^{s-1/2}_p(J; H^3_p(\Omega)) \hookrightarrow {}_0H^{s+\theta-1/2}_p(J; H^{3-2\theta}_p(\Omega)),$$

for each $\theta \in (0,1)$ and $s \in (1/2, 3/4)$, where the latter embedding is due to the mixed-derivative theorem. Finally it holds

$$\partial_t^{1/2}(\partial_t + A_K)^{-1}\partial_t^{1/2} : \ {}_0H_p^s(J; H_p^1(\Omega)) \to \ {}_0H_p^{s+\theta-1}(J; H_p^{3-2\theta}(\Omega)), \quad \theta \in (1-s, 1). \tag{3.15}$$

The following proposition shows that the Lipschitz property of λ carries over to F.

Proposition 3.3.1. *Let $p > n/4 + 1$, $\lambda \in C^{2-}(\mathbb{R})$ and $J = [0,T] \subset [0,T_0]$. Then there exists a function $\mu(T) > 0$, with $\mu(T) \to 0$ as $T \to 0$, such that for every $s \in [\frac{1}{2}, \frac{3}{4})$, and all $u, v \in \mathbb{B}_R(u^*)$ the estimate*

$$\begin{aligned}|F(u) - F(v)|_{H_p^{s-1/2}(H_p^2)} + |\nabla F(u) - \nabla F(v)|_{H_p^{s-1/2}(H_p^1)}& \\ + |\Delta F(u) - \Delta F(v)|_{H_p^{s-1/2}(L_p)} &\le \mu(T)|u-v|_1\end{aligned}$$

is valid.

Proof. By (3.15) it suffices to show that

$$|\lambda(u) - \lambda(v)|_{H_p^s(H_p^1)} \le \mu(T)|u - v|_1.$$

Obviously $|\lambda(u) - \lambda(v)|_{H_p^s(H_p^1)} \le C(|\lambda(u) - \lambda(v)|_{H_p^s(L_p)} + |\nabla u\lambda'(u) - \nabla v\lambda'(v)|_{H_p^s(L_p)})$ and

$$|\nabla u\lambda'(u) - \nabla v\lambda'(v)|_{H_p^s(L_p)} \le C(|\nabla u(\lambda'(u) - \lambda'(v))|_{H_p^s(L_p)} + |\lambda'(v)(\nabla u - \nabla v)|_{H_p^s(L_p)}).$$

Now (3.13) yields

$$\begin{aligned}|\nabla u(\lambda'(u) - \lambda'(v))|_{H_p^s(L_p)} \le C(T_0)\Big(&|\nabla u|_{L_{r'p}(L_{r'p})}|\lambda'(u) - \lambda'(v)|_{H_{rp}^s(L_{rp})} \\ &+ T^{1/\sigma' p}|\lambda'(u) - \lambda'(v)|_\infty |\nabla u|_{H_{\sigma p}^s(L_{\sigma p})}\Big),\end{aligned}$$

as well as

$$\begin{aligned}&|\lambda'(v)(\nabla u - \nabla v)|_{H_p^s(L_p)} \\ &\qquad \le C(T_0)\Big(|\nabla u - \nabla v|_{L_{r'p}(L_{r'p})}|\lambda'(v)|_{H_{rp}^s(L_{rp})} + T^{1/\sigma' p}|\nabla u - \nabla v|_{H_{\sigma p}^s(L_{\sigma p})}\Big).\end{aligned}$$

Again by (3.13) we see that $|\lambda'(v)|_{H_{rp}^s(L_{rp})} \le T^{1/\rho' p}|\lambda'(v)|_{H_{r\rho p}^s(L_{r\rho p})}$. Observe that the embedding $Z^1 \hookrightarrow H_{rp}^s(L_{rp})$ holds, whenever $s \le 1/r$. To meet this requirement we set $r = 4/3$, i.e. $r' = 4$. Hence for sufficiently small $\rho > 1$ and by (3.14) we obtain the desired estimate. □

The next proposition collects all the facts we need to show the desired properties of the operator $\mathcal{T}$ defined above.

Proposition 3.3.2. *Let $p > (n+4)/4$, $\lambda, \Phi \in C^{4-}(\mathbb{R})$ and $J = [0,T] \subset [0,T_0]$. Then there exists a constant $C > 0$, independent of T, and functions $\mu_j = \mu_j(T)$ with $\mu_j(T) \to 0$ as $T \to 0$, $j = 1, \ldots, 4$, such that for all $u, v \in \mathbb{B}_R(u^*)$ the following statements hold.*

(i) $|\Delta\Phi'(u) - \Delta\Phi'(v)|_{X(T)} \le \mu_1(T)|u - v|_1$,

(ii) $|(\Delta(\lambda'(u)F(u)) - \Delta(\lambda'(v)F(v))|_{X(T)} \le \mu_2(T)|u - v|_1$,

(iii) $|\partial_\nu\Phi'(u) - \partial_\nu\Phi'(v)|_{Y_1(T)} \le \mu_3(T)|u - v|_1$,

(iv) $|\partial_\nu(\lambda'(u)F(u)) - \partial_\nu(\lambda'(v)F(v))|_{Y_1(T)} \le \mu_4(T)|u - v|_1$.

For each fixed $\eta \in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega))$ we have *(v)* $|(\Delta(\lambda'(u)\eta) - \Delta(\lambda'(v)\eta)|_{X(T)} \leq C|\eta|_{Z^2}|u-v|_1$,

(vi) $|(\partial_\nu(\lambda'(u)\eta) - \partial_\nu(\lambda'(v)\eta)|_{Y_1(T)} \leq C|\eta|_{Z^2}|u-v|_1$.

The proof is given in the Appendix.

Note that we have also shown that $\Delta G(w) \in X(T)$ and $\partial_\nu G(w) \in Y_1(T)$ for each $w \in \mathbb{B}_R(u^*)$, where G was defined in (3.10). Thus the operator $\mathcal{T} : \mathbb{B}_R \to {}_0\mathbb{E}_1$ is well defined. With our previous considerations, it is now easy to verify the self-mapping property as well as the strict contraction property of $\mathcal{T}$. Let $w \in \mathbb{B}_R$. Then we obtain

$$\begin{aligned}|\mathcal{T}w|_1 &= |\mathbb{L}^{-1}\tilde{G}(u^*,w)|_1 \leq |\mathbb{L}^{-1}||\tilde{G}(u^*,w)|_0\\ &\leq C(|\tilde{G}(u^*,w)-\tilde{G}(u^*,0)|_0 + |\tilde{G}(u^*,0)|_0)\\ &\leq C(|\Delta G(u^*+w) - \Delta G(u^*)|_{X(T)} + |\partial_\nu(G(u^*+w)-G(u^*))|_{Y_1(T)}\\ &\qquad + |\Delta G(u^*)|_{X(T)} + |\partial_\nu G(u^*)|_{Y_1(T)} + |g_1|_{Y_1(T)}.\end{aligned}$$

By Proposition 3.3.2 there exists a function $\mu(T)$, with $\mu(T) \to 0$ as $T \to 0$, such that

$$|\Delta G(u^*+w) - \Delta G(u^*)|_{X(T)} + |\partial_\nu(G(u^*+w)-G(u^*))|_{Y_1(T)} \leq \mu(T)|w|_1 \leq \mu(T)R,$$

since $w + u^* \in \mathbb{B}_R(u^*)$. Thus we see that $|\mathcal{T}w|_1 \leq R$, if $T > 0$ is sufficiently small. We remark that g_1 and $G(u^*)$ are fixed functions, hence $|g_1|_{Y_1(T)}, |\Delta G(u^*)|_{X(T)}, |\partial_\nu G(u^*)|_{Y_1(T)} \to 0$ as $T \to 0$. This shows that $\mathcal{T}\mathbb{B}_R \subset \mathbb{B}_R$. Furthermore for all $w, \bar{w} \in \mathbb{B}_R$ we have

$$\begin{aligned}|\mathcal{T}w - \mathcal{T}\bar{w}|_1 &= |\mathbb{L}^{-1}(\tilde{G}(u^*,w) - \tilde{G}(u^*,\bar{w}))|_1 \leq |\mathbb{L}^{-1}||\tilde{G}(u^*,w) - \tilde{G}(u^*,\bar{w})|_0\\ &\leq C(|\Delta G(u^*+w) - \Delta G(u^*+\bar{w})|_{X(T)} + |\partial_\nu G(u^*+w) - \partial_\nu G(u^*+\bar{w})|_{Y_1(T)}).\end{aligned}$$

It is a consequence of Proposition 3.3.2 that

$$|\Delta G(u^*+w) - \Delta G(u^*+\bar{w})|_{X(T)} + |\partial_\nu G(u^*+w) - \partial_\nu G(u^*+\bar{w})|_{Y_1(T)} \leq \mu(T)|w-\bar{w}|_1,$$

hence $\mathcal{T} : \mathbb{B}_R \to \mathbb{B}_R$ is a strict contraction, if $T > 0$ is sufficiently small. Thus the contraction-mapping principle yields a unique fixed-point w^* of $\mathcal{T}$, i.e. a solution $\psi \in \mathbb{E}_1$ of (3.8), which depends continuously on the given data $f \in X$, $g \in Y_1$, $h \in Y_2$ and $\psi_0 \in X_{p,\Gamma}$. Since $\partial_t\lambda(\psi) = \partial_t\psi\lambda'(\psi) \in L_p(J \times \Omega)$, there is a unique solution $\vartheta \in Z^2(T)$ of

$$\begin{aligned}\partial_t v - \Delta v &= -\partial_t\lambda(\psi) + f_2, & t \in J,\ x \in \Omega,\\ \alpha v + \partial_\nu v &= g_3, & t \in J,\ x \in \Gamma,\\ v(0) &= \vartheta_0, & t = 0,\ x \in \Omega.\end{aligned}$$

Finally we see that $(\psi, \vartheta) \in \mathbb{E}_1 \times Z^2(T)$ is the unique solution of (3.3) on the interval $[0,T]$. We summarize these considerations in

Theorem 3.3.3. *Let $p > (n+4)/4$, $p \neq 3, 5$ and let $\sigma_s, \gamma > 0$, $\alpha, \kappa \geq 0$ be constants. Assume furthermore that $\lambda, \Phi \in C^{4-}(\mathbb{R})$. Then there exists an interval $J = [0,T] \subset [0,T_0]$ and a unique solution (ψ, ϑ) of (3.3) on J, with*

$$\psi \in H^1_p(J; L_p(\Omega)) \cap L_p(J; H^4_p(\Omega)) = Z^1(T), \quad \vartheta \in H^1_p(J; L_p(\Omega)) \cap L_p(J; H^2_p(\Omega)) = Z^2(T),$$

and

$$\psi|_\Gamma \in H^1_p(J; W^{2-1/p}_p(\Gamma)) \cap L_p(J; W^{4-1/p}_p(\Gamma)) = Z^1_\Gamma(T),$$

provided the data are subject to the following conditions.

(i) $f_1, f_2 \in L_p(J \times \Omega) = X$,

(ii) $g_1 \in W^{1/4-1/4p}_p(J; L_p(\Gamma)) \cap L_p(J; W^{1-1/p}_p(\Gamma)) = Y_1$,

(iii) $g_2 \in L_p(J; W_p^{2-1/p}(\Gamma)) = Y_2$,

(iv) $g_3 \in W_p^{1/2-1/2p}(J; L_p(\Gamma)) \cap L_p(J; W_p^{1-1/p}(\Gamma)) = Y_3$,

(v) $\psi_0 \in \{u \in B_{pp}^{4-4/p}(\Omega) : u|_\Gamma \in B_{pp}^{4-3/p}(\Gamma)\} = X_{p,\Gamma}$,

(vi) $\vartheta_0 \in B_{pp}^{2-2/p}(\Omega) = X_p$,

(vii) $\partial_\nu \Delta\psi_0 = \partial_\nu(\Phi'(\psi_0) - \lambda'(\psi_0)\vartheta_0) - g_1|_{t=0}$, *if* $p > 5$,

(viii) $\alpha\vartheta_0 + \partial_\nu\vartheta_0 = g_3|_{t=0}$, *if* $p > 3$.

The solution depends continuously on the given data and if the data are independent of t*, the map* $(\psi_0, \vartheta_0) \mapsto (\psi, \vartheta)$ *defines a local semiflow on the natural phase manifold* $\mathcal{M}_p \subset X_{p,\Gamma} \times X_p$*, defined by (vii) and (viii).*

3.4 Global Well-Posedness

Throughout this section we assume that $n \leq 3$ and that the potential Φ satisfies the growth conditions

$$\Phi(s) \geq -\frac{\eta}{2}s^2 - c_0, \quad c_0 > 0, \; s \in \mathbb{R}, \tag{3.16}$$

where $\eta < \lambda_1$, with λ_1 being the smallest nontrivial eigenvalue of the negative Laplacian on Ω with Neumann boundary conditions,

$$|\Phi'(s)| \leq (c_1\Phi(s) + c_2 s^2 + c_3)^\theta, \quad \text{for all} \;\; s \in \mathbb{R}, \tag{3.17}$$

with some constants $c_i > 0$, $\theta \in (0,1)$ and

$$|\Phi'''(s)| \leq C(1 + |s|^\beta), \quad s \in \mathbb{R}, \tag{3.18}$$

with $\beta < 3$ in case $n = 3$. Furthermore let $\lambda', \lambda'', \lambda''' \in L_\infty(\mathbb{R})$.

Remark: (i) The conditions (3.16)-(3.18) are certainly fulfilled, if Φ is a polynomial of degree $2m$, $m = 1, 2$. Then we may set $\theta = 1 - 1/2m$ in (3.17).
(ii) As we will see we may omit (3.17) if $f_1 = g_1 = 0$.

A successive application of Theorem 3.3.3 yields a maximal time interval of existence $J_{max} = [0, T_{max}) \subset [0, T_0]$ for the solution $\psi \in \mathbb{E}_1$ of (3.10). If $T_{max} < T_0$, this interval is characterized by the following two equivalent conditions

$$\lim_{t \to T_{max}} \psi(t) \quad \text{does not exist in } X_{p,\Gamma}$$

and

$$|\psi|_{Z^1(T_{max})} + |\psi_{|\Gamma}|_{Z^1_\Gamma(T_{max})} = \infty.$$

First of all, we need some a priori estimates for ψ. We multiply $(3.3)_1$ by μ and $(3.3)_2$ by ϑ. Integration by parts and the boundary conditions $(3.3)_{3,4,5}$ lead to the energy-equation

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(|\nabla\psi|_2^2 + |\vartheta|_2^2 + \frac{\sigma_s}{\gamma}|\nabla_\Gamma\psi|_{2,\Gamma}^2 + \frac{\kappa}{\gamma}|\psi|_{2,\Gamma}^2 + 2\int_\Omega \Phi(\psi)\, dx\Big)& \\ + |\nabla\mu|_2^2 + \frac{1}{\gamma}|\partial_t\psi|_{2,\Gamma}^2 + |\nabla\vartheta|_2^2 + \alpha|\vartheta|_{2,\Gamma}^2& \\ = \int_\Omega (f_1\mu + f_2\vartheta)\, dx + \int_\Gamma (g_1\mu + \frac{1}{\gamma}g_2\partial_t\psi + g_3\vartheta)\, d\Gamma.&\end{aligned} \tag{3.19}$$

The Poincaré-Wirtinger inequality as well as the embedding $H_2^1(\Omega) \hookrightarrow L_2(\Gamma)$ imply

$$\int_\Omega f_1\mu\, dx \leq C|f_1|_2(|\nabla\mu|_2 + |\int_\Omega \mu\, dx|) \quad \text{and} \quad \int_\Gamma g_1\mu\, dx \leq C|g_1|_{2,\Gamma}(|\nabla\mu|_2 + |\int_\Omega \mu\, dx|). \tag{3.20}$$

By the definition of the chemical potential μ, by the divergence theorem and by the dynamic boundary condition $(3.3)_4$ we have

$$\int_\Omega \mu\ dx = \int_\Omega (\Phi'(\psi) - \lambda'(\psi)\vartheta) + \frac{1}{\gamma}\int_\Gamma (\partial_t\psi + \kappa\psi - g_2)\ dx,$$

hence using (3.17) and the Cauchy-Schwarz inequality

$$|\int_\Omega \mu\ dx| \le C(|\vartheta|_2 + |\partial_t\psi|_{2,\Gamma} + |\psi|_{2,\Gamma} + |g_2|_{2,\Gamma}) + \int_\Omega (c_1\Phi(\psi) + c_2|\psi|^2 + c_3)^\theta\ dx.$$

For simplicity, we set

$$E(\psi,\vartheta) := \frac{1}{2}|\nabla\psi|_2^2 + \frac{1}{2}|\vartheta|_2^2 + \frac{1}{2}\frac{\sigma_s}{\gamma}|\nabla_\Gamma\psi|_{2,\Gamma}^2 + \frac{1}{2}\frac{\kappa}{\gamma}|\psi|_{2,\Gamma}^2 + \int_\Omega \Phi(\psi)\ dx.$$

From (3.19), (3.20) and by Hölder's inequality, Young's inequality as well as by the Poincaré-Wirtinger inequality we obtain the estimate

$$\begin{aligned}\frac{d}{dt}\ E(\psi,\vartheta) &+ C(|\nabla\mu|_2^2 + |\partial_t\psi|_{2,\Gamma}^2 + |\nabla\vartheta|_2^2 + \alpha|\vartheta|_{2,\Gamma}^2)\\ &\le C_1E(\psi,\vartheta) + C_2(|f_1|_2^q + |f_2|_2^2 + |g_1|_{2,\Gamma}^q + |g_2|_{2,\Gamma}^2 + |g_3|_{2,\Gamma}^2 + 1),\end{aligned}$$

where $q := \max\{2, \frac{1}{1-\theta}\}$. Observe that the functional E is bounded from below. Indeed, by (3.16) we obtain

$$E(\psi(t),\vartheta(t)) \ge \frac{1}{2}(|\nabla\psi|_2^2 - \eta|\psi|_2^2) - c_0|\Omega| \ge \frac{\lambda_1 - \eta}{2\lambda_1}|\nabla\psi|_2^2 - c \ge -c, \quad c > 0, \tag{3.21}$$

where we used again the Poincaré inequality, since

$$|\int_\Omega \psi(t)\ dx| \le \int_\Omega |\psi_0|\ dx + |f_1|_{L_1(J\times\Omega)} + |g_1|_{L_1(J\times\Gamma)}. \tag{3.22}$$

Then Gronwall's lemma yields the estimate

$$E(\psi,\vartheta) \le C\left(E(\psi_0,\vartheta_0) + \int_0^{T_{max}} (|f_1|_2^q + |f_2|_2^2 + |g_1|_{2,\Gamma}^q + |g_2|_{2,\Gamma}^2 + |g_3|_{2,\Gamma}^2 + 1)\ dt\right),$$

and by (3.22) we obtain among other things the a priori estimate $\psi \in L_\infty(J_{\max}; H_2^1(\Omega))$ again with the help of the Poincaré-Wirtinger inequality. The following lemma is the key to obtain global existence.

Lemma 3.4.1. *Suppose $p \ge 2$, $n \le 3$ and let $\psi \in \mathbb{E}_1$ be the solution of (3.10). Then there exist constants $m, C > 0$ and $\delta \in (0,1)$, independent of $T > 0$, such that*

$$|\Delta G(\psi)|_{X(T)} + |\partial_\nu G(\psi)|_{Y_1(T)} \le C(1 + |\psi|_{Z^1(T)}^\delta |\psi|_{L_\infty(J;H_2^1(\Omega))}^m).$$

The proof is given in the Appendix. Observe that by maximal L_p-regularity the estimate

$$|\psi|_{Z^1(T)} + |\psi|_{Z_\Gamma^1(T)} \le M(|\Delta G(\psi)|_{X(T)} + |\partial_\nu G(\psi)|_{Y_1(T)} + |f|_X + |g|_{Y_1} + |h|_{Y_2} + |\psi_0|_{X_{p,\Gamma}}),$$

for the solution ψ of (3.10) is valid, with a constant $M = M(T_0) > 0$. Then it follows from Lemma 3.4.1 that

$$|\psi|_{Z^1(T)} + |\psi_{|\Gamma}|_{Z_\Gamma^1(T)} \le M(1 + |\psi|_{Z^1(T)}^\delta),$$

hence $|\psi|_{Z^1(T)}$ is bounded, since $\delta < 1$. This in turn yields the boundedness of $|\psi_{|\Gamma}|_{Z_\Gamma^1(T)}$ and therefore global existence of the solution (ψ,ϑ) of (3.3), since ϑ solves the heat-equation $\partial_t\vartheta - \Delta\vartheta = -\partial_t\lambda(\psi)$, subject to Robin boundary and initial conditions. We summarize these considerations in

Theorem 3.4.2. *Let $n \leq 3$, $p \geq 2$, $p \neq 3,5$, $\theta \in (0,1)$ from (3.17), $q = \max\{2, \frac{1}{1-\theta}\}$ and $J_0 = [0, T_0]$. Assume furthermore that $\lambda', \lambda'', \lambda''' \in L_\infty(\mathbb{R})$ and let Φ satisfy (3.16)-(3.18). Then there exists a unique global solution (ψ, ϑ) of (3.3) on J_0, with*

$$\psi \in H^1_p(J_0; L_p(\Omega)) \cap L_p(J_0; H^4_p(\Omega)), \quad \vartheta \in H^1_p(J_0; L_p(\Omega)) \cap L_p(J_0; H^2_p(\Omega)),$$

and

$$\psi|_\Gamma \in H^1_p(J_0; W^{2-1/p}_p(\Gamma)) \cap L_p(J_0; W^{4-1/p}_p(\Gamma)),$$

if the data are subject to the following conditions.

(i) $f_1, f_2 \in L_p(J_0 \times \Omega)$, $f_1 \in L_q(J_0; L_2(\Omega))$,

(ii) $g_1 \in W^{1/4-1/4p}_p(J_0; L_p(\Gamma)) \cap L_p(J_0; W^{1-1/p}_p(\Gamma)) \cap L_q(J_0; L_2(\Gamma))$,

(iii) $g_2 \in L_p(J_0; W^{2-1/p}_p(\Gamma))$,

(iv) $g_3 \in W^{1/2-1/2p}_p(J_0; L_p(\Gamma)) \cap L_p(J_0; W^{1-1/p}_p(\Gamma))$,

(v) $\psi_0 \in \{u \in B^{4-4/p}_{pp}(\Omega) : u|_\Gamma \in B^{4-3/p}_{pp}(\Gamma)\} = X_{p,\Gamma}$,

(vi) $\vartheta_0 \in B^{2-2/p}_{pp}(\Omega) = X_p$,

(vii) $\partial_\nu \Delta \psi_0 = \partial_\nu(\Phi'(\psi_0) - \lambda'(\psi_0)\vartheta_0) - g_1|_{t=0}$, *if* $p > 5$,

(viii) $\alpha\vartheta_0 + \partial_\nu \vartheta_0 = g_3|_{t=0}$, *if* $p > 3$.

The solution depends continuously on the given data and if the data are independent of t, the map $(\psi_0, \vartheta_0) \mapsto (\psi, \vartheta)$ defines a global semiflow on the natural phase manifold $\mathcal{M}_p \subset X_{p,\Gamma} \times X_p$, defined by (vii) and (viii).

3.5 Asymptotic Behavior

In this section we study the asymptotic behavior of a global solution (ψ, ϑ) of the system

$$\begin{aligned}
\partial_t\psi - \Delta\mu = 0, \quad \mu = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta, &\quad t > 0,\ x \in \Omega,\\
\partial_t\vartheta + \lambda'(\psi)\partial_t\psi - \Delta\vartheta = 0, &\quad t > 0,\ x \in \Omega,\\
\partial_\nu \mu = 0, &\quad t > 0,\ x \in \Gamma,\\
\alpha\vartheta + \partial_\nu\vartheta = 0, &\quad t > 0,\ x \in \Gamma,\\
\partial_t\psi - \sigma_s\Delta_\Gamma\psi + \gamma\partial_\nu\psi + \kappa(\psi - g) = 0, &\quad t > 0,\ x \in \Gamma,\\
\psi(0) = \psi_0, &\quad t = 0,\ x \in \Omega,\\
\vartheta(0) = \vartheta_0, &\quad t = 0,\ x \in \Omega,
\end{aligned} \tag{3.23}$$

where $\alpha \geq 0$, $\sigma_s, \gamma > 0$, $\kappa \geq 0$, $g \in \mathbb{R}$, $(\psi_0, \vartheta_0) \in \mathcal{M}_2$. For the forthcoming considerations, we need the following assumptions. Let $\lambda', \lambda'', \lambda''' \in L_\infty(\mathbb{R})$ and let Φ satisfy (3.16) as well as (3.18).

The main tool will be the *Lojasiewicz-Simon inequality* (see Proposition 3.5.4), which leads to the convergence result. We define two functionals $E_K(u, v)$ by means of

$$E_R(u,v) = \frac{1}{2}\left(\int_\Omega |\nabla u|^2 + \int_\Omega |v|^2 + \frac{\sigma_s}{\gamma}\int_\Gamma |\nabla_\Gamma u|^2 + \frac{\kappa}{\gamma}\int_\Gamma |u|^2\right) - \frac{\kappa g}{\gamma}\int_\Gamma u \, d\Gamma + \int_\Omega \Phi(u)$$

and

$$E_N(u,v) = E_R(u,v) - \bar{v}\int_\Omega (\lambda(u) + v)\, dx =: E_R(u,v) - \bar{v}F(u,v),$$

in case of Robin or Neumann boundary conditions, i.e. $\alpha > 0$ or $\alpha = 0$. Here we use the abbreviation

$$\overline{w} = \frac{1}{|\Omega|}\int_\Omega w(x)dx$$

for the mean value of a function $w \in L_1(\Omega)$. The reason for the modification of the energy functional in case of Neumann boundary conditions is that we have the additional side condition

$$\int_\Omega (\lambda(\psi(t,x)) + \vartheta(t,x))\ dx \equiv c^* = const,$$

for all $t \geq 0$. We will see below that w.l.o.g. we may assume $c^* = 0$. A suitable energy space both for E_N and E_R will be $V = V_1 \times V_2$, where

$$V_1 := \left\{ u \in H_2^1(\Omega) : u|_\Gamma \in H_2^1(\Gamma),\ \int_\Omega u = 0 \right\} \quad \text{and} \quad V_2 := L_2(\Omega).$$

Note that the condition $\int_\Omega u = 0$ is compatible with our system. This might be seen by integrating $(3.23)_1$ and invoking the boundary condition $(3.23)_3$. We obtain $\int_\Omega \psi = \int_\Omega \psi_0$. If we replace the solution ψ by $\tilde{\psi} = \psi - c$, with $c = \frac{1}{|\Omega|}\int_\Omega \psi_0$, we see that $\tilde{\psi}$ satisfies again (3.23) provided $\Phi(s)$ and $\lambda(s)$ are replaced by $\Phi_1(s) := \Phi(s+c)$ and $\lambda_1(s) := \lambda(s+c)$, respectively and the constant $g \in \mathbb{R}$ has to be replaced by $g - \kappa c = const$. In the sequel we will still denote this shifted constant by g. In a similar way we can achieve that in case $\alpha = 0$ we have $\int_\Omega(\vartheta + \lambda(\psi)) = 0$. Indeed this follows by a shift of λ, i.e. $\tilde{\lambda}(s) := \lambda(s) - d$, where $d = \frac{1}{|\Omega|}\int_\Omega(\vartheta_0 + \lambda(\psi_0))$. It is suitable to embed V into a Hilbert space $H = H_1 \times V_2$, with

$$H_1 := \left\{ u \in L_2(\Omega) : \int_\Omega u = 0 \right\}.$$

Proposition 3.5.1. *The functionals E_R and E_N are twice continuously Fréchet differentiable on V and the derivatives are given by*

$$\begin{aligned} \langle E_R'(u,v),(h,k)\rangle_{V^*,V} &= \int_\Omega (\nabla u \nabla h + vk)\ dx \\ &\quad + \frac{\sigma_s}{\gamma}\int_\Gamma \nabla_\Gamma u \nabla_\Gamma h\ d\Gamma + \frac{\kappa}{\gamma}\int_\Gamma (u-g)h\ d\Gamma + \int_\Omega \Phi'(u)h\ dx, \end{aligned} \tag{3.24}$$

$$\begin{aligned} \langle E_R''(u,v)(h_1,k_1),(h_2,k_2)\rangle_{V^*,V} &= \int_\Omega (\nabla h_1 \nabla h_2 + k_1 k_2)\ dx \\ &\quad + \frac{\sigma_s}{\gamma}\int_\Gamma \nabla_\Gamma h_1 \nabla_\Gamma h_2\ d\Gamma + \frac{\kappa}{\gamma}\int_\Gamma h_1 h_2\ d\Gamma + \int_\Omega \Phi''(u)h_1 h_2\ dx, \end{aligned} \tag{3.25}$$

where $(h,k),(h_j,k_j) \in V$, $j = 1,2$, and

$$\langle E_N'(u,v),(h,k)\rangle_{V^*,V} = \langle E_R'(u,v),(h,k)\rangle_{V^*,V} - \overline{k}F(u,v) - \overline{v}\langle F'(u,v),(h,k)\rangle_{V^*,V}, \tag{3.26}$$

$$\begin{aligned} \langle E_N''(u,v)(h_1,k_1),(h_2,k_2)\rangle_{V^*,V} &= \langle E_R''(u,v)(h_1,k_1),(h_2,k_2)\rangle_{V^*,V} \\ &\quad - \overline{k_1}\langle F'(u,v),(h_2,k_2)\rangle_{V^*,V} - \overline{k_2}\langle F'(u,v),(h_1,k_1)\rangle_{V^*,V} \\ &\quad - \overline{v}\langle F''(u,v)(h_1,k_1),(h_2,k_2)\rangle_{V^*,V}, \end{aligned} \tag{3.27}$$

where

$$\langle F'(u,v),(h,k)\rangle_{V^*,V} = \int_\Omega \lambda'(u)h\ dx + \int_\Omega k\ dx$$

and

$$\langle F''(u,v)(h_1,k_1),(h_2,k_2)\rangle_{V^*,V} = \int_\Omega \lambda''(u)h_1 h_2\ dx.$$

Proof. Here we may follow the lines of the proof of Proposition 2.5.2. Indeed we set $b(s) = s$, hence $\beta(s) = \frac{1}{2}s^2$ and we use the bilinear form

$$a((u_1, v_1), (u_2, v_2)) = \int_\Omega \nabla u_1 \nabla u_2 \, dx + \int_\Omega v_1 v_2 + \frac{\sigma_s}{\gamma} \int_\Gamma \nabla_\Gamma u_1 \nabla_\Gamma u_2 \, d\Gamma + \frac{\kappa}{\gamma} \int_\Gamma u_1 u_2 \, d\Gamma, \quad (3.28)$$

defined on the space $V \times V$, which is bounded, symmetric and elliptic, by the Poincaré inequality. $\square$

Next we compute the derivative of $E_K(\psi(\cdot), \vartheta(\cdot))$, $K \in \{N, R\}$, with respect to time to obtain

$$\frac{d}{dt} \, E_K(\psi(t), \vartheta(t)) = -|\nabla \mu|_2^2 - |\nabla \vartheta|_2^2 - \frac{1}{\gamma} |\partial_t \psi|_{2,\Gamma}^2 - \alpha |\vartheta|_{2,\Gamma}^2. \quad (3.29)$$

This is a consequence of (3.19) with $g_2 = \kappa g = const$. By (3.21) the functionals $E_K(\psi(\cdot), \vartheta(\cdot))$ are bounded from below. This can be seen as follows.

$$\begin{aligned} E_R(\psi(t), \vartheta(t)) &\geq \frac{\varepsilon}{2} |\nabla \psi(t)|_2^2 + \frac{(1-\varepsilon)\lambda_1 - \eta}{2} |\psi(t)|_2^2 + |\vartheta(t)|_2^2 + \frac{\sigma_s}{\gamma} |\nabla_\Gamma \psi(t)|_{2,\Gamma}^2 \\ &\quad + \frac{\kappa}{\gamma} |\psi(t)|_{2,\Gamma}^2 - \frac{\kappa g}{\gamma} \int_\Gamma \psi(t) \, d\Gamma \\ &\geq (\frac{\varepsilon}{2} - \delta) |\nabla \psi(t)|_2^2 + \frac{(1-\varepsilon)\lambda_1 - \eta}{2} |\psi(t)|_2^2 + |\vartheta(t)|_2^2 + \frac{\sigma_s}{\gamma} |\nabla_\Gamma \psi(t)|_{2,\Gamma}^2 \\ &\quad + \frac{\kappa}{\gamma} |\psi(t)|_{2,\Gamma}^2 - C(\delta), \end{aligned} \quad (3.30)$$

where $\delta, \varepsilon > 0$ are sufficiently small, such that $\frac{\varepsilon}{2} - \delta > 0$ and $(1-\varepsilon)\lambda_1 - \eta > 0$. Here we used (3.21), the Poincaré inequality, Hölder's inequality, Young's inequality and the estimate

$$-\frac{\kappa g}{\gamma} \int_\Gamma \psi(t) \, d\Gamma \geq -\frac{\kappa g}{\gamma} |\psi(t)|_{1,\Gamma} \geq -C |\psi(t)|_{2,\Gamma} \geq -C |\nabla \psi(t)|_2 \geq -\delta |\nabla \psi(t)|_2^2 - C(\delta),$$

with the trace map $H_2^1(\Omega) \hookrightarrow L_2(\Gamma)$. The proof for E_N is the same, since by construction $F(\psi(t), \vartheta(t)) \equiv 0$ for all $t \geq 0$. Then (3.29) and (3.30) yield

$$\psi \in C_b(\mathbb{R}_+; V_1) \quad \text{and} \quad \vartheta \in C_b(\mathbb{R}_+; V_2). \quad (3.31)$$

Proposition 3.5.2. *Let* $(\psi, \vartheta) \in \mathbb{E}_1 \times Z^2$ *be a global solution of* (3.23). *Then* $(\psi(t), \vartheta(t))$ *has relatively compact range in* V.

Proof. We already know that a global solution is bounded in V. To prove the relative compactness of the orbit $\psi(\mathbb{R}_+)$ we will proceed in two steps. First we consider the operator $A_p := \Delta^2$ in $L_p(\Omega)$, with domain

$$D(A_p) = \{w \in H_p^4(\Omega) : \Delta w = 0 \text{ and } \partial_\nu w = 0 \text{ on } \Gamma\}.$$

By [10, Proof of Proposition 5.2 (b)] we have $L_p(\Omega) = N(A_p) \oplus R(A_p)$ and the semigroup, generated by A_p is exponentially stable on $R(A_p)$. Let P be the corresponding projection onto $N(A_p)$ and set $Q = I - P$. Consider the evolution equation

$$\partial_t \psi_1 + A_p \psi_1 = Q(\Phi'(\psi) - \lambda'(\psi)\vartheta), \; t > 0, \quad \psi_1(0) = \psi_{10}, \quad (3.32)$$

where ψ_{10} denotes the solution of the elliptic problem

$$\begin{cases} \Delta \psi_{10} = \psi_0, \; x \in \Omega, \\ \partial_\nu \psi_{10} = 0, \; x \in \Gamma. \end{cases} , \qquad \int_\Omega \psi_{10} = 0.$$

Since ψ_0 has mean 0, the compatibility condition is fulfilled. The solvability of (3.32) has already been studied in [10] by applying the results from [11].

Choosing $p = 2$ if $\beta \in (0,1]$ and $p = 6/(\beta+2)$ if $\beta \geq 1$ and using the fact that $\lambda' \in L_\infty(\mathbb{R})$, we obtain $Q(\Phi'(\psi) + \lambda'(\psi)\vartheta) \in C_b(\mathbb{R}_+; R(A_p))$. Thus by semigroup-theory this yields $\psi_1 \in C_b(\mathbb{R}_+; H_p^r(\Omega))$, for each $r < 4$ and by compact embedding the orbit $\Delta\psi_1(\mathbb{R}_+)$ is relatively compact in $H_2^1(\Omega)$. Next we split ψ by means of $\psi = \Delta\psi_1 + \psi_2$. From [10, Proof of Proposition 5.2] it follows immediately that the orbit $\psi_2(\mathbb{R}_+)$ is relatively compact in V_1. Therefore the orbit of ψ is also relatively compact in V_1, since the trace of $\Delta\psi_1$ on Γ vanishes.

Now let $e = \vartheta + \lambda(\psi)$. Then e solves the following system

$$\begin{aligned} \partial_t e - \Delta e &= -\Delta\lambda(\psi), \quad t > 0,\ x \in \Omega,\\ \alpha e + \partial_\nu e &= \alpha\lambda(\psi) + \partial_\nu\lambda(\psi), \quad t > 0,\ x \in \Gamma,\\ e(0) &= e_0, \quad t = 0,\ x \in \Omega, \end{aligned}$$

where $e_0 := \vartheta_0 + \lambda(\psi_0)$. By (3.31) we see that $-\Delta\lambda(\psi) \in C_b(\mathbb{R}_+; H_2^1(\Omega)^*)$. The Laplacian generates an exponentially stable analytic semigroup in $H_2^1(\Omega)^*$, if $\alpha > 0$. In case $\alpha = 0$ the semigroup is exponentially stable on $\hat{H}_2^1(\Omega)^*$, where

$$\hat{H}_2^1(\Omega) := \{w \in H_2^1(\Omega) : \int_\Omega w = 0\}.$$

Therefore semigroup-theory implies that $e \in C_b(\mathbb{R}_+; H_2^r(\Omega))$, for every $r \in (0,1)$. One more time we use (3.31) to obtain

$$\vartheta = e - \lambda(\psi) \in C_b(\mathbb{R}_+; H_2^r(\Omega)),$$

hence by compact embedding the orbit $\vartheta(\mathbb{R}_+)$ is relatively compact in V_2.

□

The next proposition collects some properties of the functionals $E_K : V \to \mathbb{R}$.

Proposition 3.5.3. *Let (ψ, ϑ) be a global solution of (3.23) with $h = \frac{1}{|\Omega|}\int \psi_0$ and suppose that Φ satisfies (3.16) as well as (3.18). Let further $K \in \{N, R\}$. Then the following statements hold.*

(i) The functions $E_K(\psi(\cdot), \vartheta(\cdot))$ are nonincreasing and the limits

$$\lim_{t\to\infty} E_K(\psi(t), \vartheta(t)) =: E_K^\infty$$

exist.

(ii) The ω-limit set

$$\omega(\psi, \vartheta) := \{(\varphi, \theta) \in V : \exists\ (t_n)_{n\in\mathbb{N}} \nearrow \infty,\ s.t.\ (\psi(t_n), \vartheta(t_n)) \to (\varphi, \theta)\ in\ V\}$$

is nonempty, compact, connected and E_K is constant on $\omega(\psi, \vartheta)$.

(iii) For every $(\psi_\infty, \vartheta_\infty) \in \omega(\psi, \vartheta)$ it holds that $\vartheta_\infty = const$ and $(\psi_\infty, \vartheta_\infty)$ is a strong solution of the stationary problem

$$\begin{cases} -\Delta\psi_\infty + \Phi'(\psi_\infty) - \lambda'(\psi_\infty)\vartheta_\infty = const, & x \in \Omega,\\ -\sigma_s\Delta_\Gamma\psi_\infty + \gamma\partial_\nu\psi_\infty + \kappa(\psi_\infty - g) = 0, & x \in \Gamma, \end{cases} \tag{3.33}$$

where $\vartheta_\infty = const$, if $\alpha = 0$ and $\vartheta_\infty = 0$, if $\alpha > 0$.

(iv) Every $(\psi_\infty, \vartheta_\infty) \in \omega(\psi, \vartheta)$ is a critical point of E_K, i.e. $E_K'(\psi_\infty, \vartheta_\infty) = 0$.

Proof. By (3.29) the functions $E_N(\psi(\cdot), \vartheta(\cdot))$ and $E_R(\psi(\cdot), \vartheta(\cdot))$ are nonincreasing, hence the limits $\lim_{t\to\infty} E_K(\psi(t), \vartheta(t))$ exist, since $E_K(\psi(\cdot), \vartheta(\cdot))$ are bounded from below. This yields (i). By Proposition 3.5.2, the solution (ψ, ϑ) has relatively compact range in V. Therefore, by well-known results, the ω-limit set is nonempty, compact and connected. The fact that E_K is constant on the ω-limit set, follows easily from continuity of E_K on V and (i).

Now let $(\psi_\infty, \vartheta_\infty) \in \omega(\psi, \vartheta)$ and let $(t_n)_{n\in\mathbb{N}} \nearrow \infty$, such that $(\psi(t_n), \vartheta(t_n)) \to (\psi_\infty, \vartheta_\infty)$ in V, as $n \to \infty$. This yields

$$\vartheta(t_n) + \lambda(\psi(t_n)) =: e(t_n) \to e_\infty := \vartheta_\infty + \lambda(\psi_\infty) \quad \text{in } L_2(\Omega).$$

Since $\nabla\mu, \nabla\vartheta \in L_2(\mathbb{R}_+ \times \Omega; \mathbb{R}^n)$ and $\partial_t\psi|_\Gamma \in L_2(\mathbb{R}_+ \times \Gamma)$ it holds that $\psi(t_n + s) \to \psi_\infty$ in $H_2^1(\Omega)^*$, $\psi(t_n + s)|_\Gamma \to \psi_\infty|_\Gamma$ in $L_2(\Gamma)$ and $e(t_n + s) \to e_\infty$ in $H_2^1(\Omega)^*$ for all $s \in [0, 1]$. We will prove this exemplarily for ψ in $H_2^1(\Omega)^*$. First of all, note that $\partial_t\psi \in L_2(\mathbb{R}_+; H_2^1(\Omega)^*)$, by equation $(3.23)_1$. Therefore we obtain

$$\begin{aligned}
|\psi(t_n + s) - \psi_\infty|_{H_2^1(\Omega)^*} &\le |\psi(t_n + s) - \psi(t_n)|_{H_2^1(\Omega)^*} + |\psi(t_n) - \psi_\infty|_{H_2^1(\Omega)^*} \\
&\le \int_{t_n}^{t_n+s} |\partial_t\psi|_{H_2^1(\Omega)^*}\, dt + |\psi(t_n) - \psi_\infty|_{H_2^1(\Omega)^*} \\
&\le s^{1/2} \left(\int_{t_n}^{t_n+s} |\partial_t\psi|^2_{H_2^1(\Omega)^*}\, dt \right)^{1/2} + |\psi(t_n) - \psi_\infty|_{H_2^1(\Omega)^*}.
\end{aligned}$$

Taking the limit as $t_n \to \infty$, this yields the claim. By the relative compactness of $\psi(\mathbb{R}_+)$ in V_1 it follows that $\psi(t_n+s) \to \psi_\infty$ in V_1 for all $s \in [0, 1]$. Hence, for all $s \in [0, 1]$, we have $\vartheta(t_n+s) \to \vartheta_\infty$ first in $H_2^1(\Omega)^*$ and then by relative compactness also in $L_2(\Omega)$. Integrating (3.29) with respect to t from t_n to $t_n + 1$ we obtain

$$\begin{aligned}
&E_K(\psi(t_n + 1), \vartheta(t_n + 1)) - E_K(\psi(t_n), \vartheta(t_n)) \\
&\quad + \int_0^1 \left(|\nabla\mu(t_n + s)|_2^2 + |\nabla\vartheta(t_n + s)|_2^2 + \frac{1}{\gamma}|\partial_t\psi(t_n + s)|^2_{2,\Gamma} + \alpha|\vartheta(t_n + s)|^2_{2,\Gamma} \right) ds = 0.
\end{aligned}$$

Letting $t_n \to \infty$ yields

$$|\nabla\mu(t_n + \cdot)|_2^2 + |\nabla\vartheta(t_n + \cdot)|_2^2 + \frac{1}{\gamma}|\partial_t\psi(t_n + \cdot)|^2_{2,\Gamma} + \alpha|\vartheta(t_n + \cdot)|^2_{2,\Gamma} \to 0$$

in $L_2(0, 1)$. This in turn yields a subsequence (t_{n_k}) such that

$$|\nabla\mu(t_{n_k} + s)|_2^2 + |\nabla\vartheta(t_{n_k} + s)|_2^2 + \frac{1}{\gamma}|\partial_t\psi(t_{n_k} + s)|^2_{2,\Gamma} + \alpha|\vartheta(t_{n_k} + s)|^2_{2,\Gamma} \to 0$$

as $k \to \infty$ for a.e. $s \in [0, 1]$. It follows that $\vartheta_\infty \in H_2^1(\Omega)$ and $\nabla\vartheta_\infty = 0$, since the gradient is a closed operator in $L_2(\Omega; \mathbb{R}^n)$, hence ϑ_∞ is constant. In particular, if $\alpha > 0$ then $\vartheta_\infty = 0$. The Poincaré-Wirtinger inequality yields a constant $C_p > 0$ such that

$$\begin{aligned}
&|\mu(t_{n_k} + s^*) - \mu(t_{n_l} + s^*)|_2 \\
&\quad \le C_p\Big(|\nabla\mu(t_{n_k} + s^*) - \nabla\mu(t_{n_l} + s^*)|_2 + \int_\Omega |\Phi'(\psi(t_{n_k} + s^*)) - \Phi'(\psi(t_{n_l} + s^*))|\, dx \\
&\qquad + \int_\Omega |\lambda'(\psi(t_{n_k} + s^*))\vartheta(t_{n_k} + s^*) - \lambda'(\psi(t_{n_l} + s^*))\vartheta(t_{n_l} + s^*)|\, dx \\
&\qquad + \int_\Gamma |\partial_t\psi(t_{n_k} + s^*) - \partial_t\psi(t_{n_l} + s^*)|\, d\Gamma + \int_\Gamma |\psi(t_{n_k} + s^*) - \psi(t_{n_l} + s^*)|\, d\Gamma \Big),
\end{aligned}$$

for some $s^* \in [0, 1]$. Taking the limit $k, l \to \infty$ we see that $\mu(t_{n_k} + s^*)$ is a Cauchy sequence in $L_2(\Omega)$, hence it admits a limit, which we denote by μ_∞. In the same manner as for ϑ_∞ we therefore obtain $\nabla\mu_\infty = 0$, hence μ_∞ is a constant. Observe that the relation

$$\mu_\infty = \frac{1}{|\Omega|} \left(\int_\Omega (\Phi'(\psi_\infty) - \lambda'(\psi_\infty)\vartheta_\infty)\, dx + \frac{\kappa}{\gamma} \int_\Gamma (\psi_\infty - g)\, d\Gamma \right)$$

is valid. Multiplying $(3.23)_1$ by a function $\varphi \in H_2^1(\Omega) \cap H_2^1(\Gamma)$ and integrating by parts we obtain

$$(\mu(t_{n_k} + s^*), \varphi)_2 = (\nabla\psi(t_{n_k} + s^*), \nabla\varphi)_2 + \frac{\sigma}{\gamma}(\nabla_\Gamma\psi(t_{n_k} + s^*), \nabla_\Gamma\varphi)_{2,\Gamma} + \frac{\kappa}{\gamma}(\psi(t_{n_k} + s^*), \varphi)_{2,\Gamma}$$
$$+ (\Phi'(\psi(t_{n_k} + s^*)), \varphi)_2 - (\lambda'(\psi(t_{n_k} + s^*))\vartheta(t_{n_k} + s^*), \varphi)_2 + \frac{1}{\gamma}(\partial_t\psi(t_{n_k} + s^*), \varphi)_{2,\Gamma} - \frac{\kappa}{\gamma}(g, \varphi)_{2,\Gamma},$$

where $(\cdot,\cdot)_2$ and $(\cdot,\cdot)_{2,\Gamma}$ are the inner products in $L_2(\Omega)$ and $L_2(\Gamma)$, respectively. As $t_{n_k} \to \infty$ it follows that

$$\begin{aligned}(\mu_\infty, \varphi)_2 = (\nabla\psi_\infty, \nabla\varphi)_2 &+ \frac{\sigma_s}{\gamma}(\nabla_\Gamma\psi_\infty, \nabla_\Gamma\varphi)_{2,\Gamma} \\ &+ \frac{\kappa}{\gamma}(\psi_\infty - g, \varphi)_{2,\Gamma} + (\Phi'(\psi_\infty), \varphi)_2 - \vartheta_\infty(\lambda'(\psi_\infty), \varphi)_2. \end{aligned} \tag{3.34}$$

By the Lax-Milgram theorem the bounded, symmetric and elliptic form

$$a(u, v) := \int_\Omega \nabla u \nabla v \; dx + \frac{\sigma}{\gamma}\int_\Gamma \nabla_\Gamma u \nabla_\Gamma v \; d\Gamma + \frac{\kappa}{\gamma}\int_\Gamma uv \; d\Gamma,$$

defined on the space $V_1 \times V_1$ induces a bounded operator $A : V_1 \to V_1^*$ with nonempty resolvent set, such that

$$a(u, v) = \langle Au, v\rangle_{V_1^*, V_1},$$

for all $(u, v) \in V_1 \times V_1$. For $\kappa = 0$, consider the part A_p of the operator A in

$$\mathbb{X}_p^0 := \{u \in L_p(\Omega) : \int_\Omega u \; dx = 0\}.$$

In case $\kappa > 0$ we consider the part A_p of the operator A in $L_p(\Omega)$. It has been shown in [10] that the domain $D(A_p)$ of A_p in $\mathbb{X}_p^0$ is given by

$$D(A_p) = \{u \in \mathbb{X}_p^0 : u \in H_p^2(\Omega),\; u|_\Gamma \in W_p^{3-1/p}(\Gamma),\; -\sigma_s\Delta_\Gamma u + \gamma\partial_\nu u = 0\}.$$

With the same methods it can be verified that the domain of A_p in L_p in case $\kappa > 0$ is

$$D(A_p) = \{u \in L_p(\Omega) : u \in H_p^2(\Omega),\; u|_\Gamma \in W_p^{3-1/p}(\Gamma),\; -\sigma_s\Delta_\Gamma u + \gamma\partial_\nu u + \kappa u = 0\}.$$

At this point we want to remark that the condition $\int_\Omega u \; dx = 0$ is not needed to compute the domain of the operator A_p, since in case $\kappa > 0$ the kernel of the solution operator is trivial.

We go back to (3.34). If $\kappa = 0$, we obtain from the growth condition (3.18) and the bound on λ' that $\psi_\infty \in D(A_q)$, where $q = 6/(\beta + 2)$, since μ_∞ and ϑ_∞ are constant. Since $q > 6/5$ we may apply a bootstrap argument to conclude $\psi_\infty \in D(A_2)$. Integrating (3.34) by parts, assertion (iii) follows. In case $\kappa > 0$ we define the new function $\psi_\infty^1 := \psi_\infty - g$. It follows that

$$\begin{aligned}(\mu_\infty, \varphi)_2 = (\nabla\psi_\infty^1, \nabla\varphi)_2 &+ \frac{\sigma_s}{\gamma}(\nabla_\Gamma\psi_\infty^1, \nabla_\Gamma\varphi)_{2,\Gamma} \\ &+ \frac{\kappa}{\gamma}(\psi_\infty^1, \varphi)_{2,\Gamma} + (\tilde{\Phi}'(\psi_\infty^1), \varphi)_2 - \vartheta_\infty(\tilde{\lambda}'(\psi_\infty^1), \varphi)_2, \end{aligned} \tag{3.35}$$

since g is constant, where $\tilde{\Phi}$ and $\tilde{\lambda}$ are defined by

$$\tilde{\Phi}(s) = \Phi(s + g) \quad \text{and} \quad \tilde{\lambda}(s) := \lambda(s + g)$$

for all $s \in \mathbb{R}$. The same arguments as in case $\kappa = 0$ yield that $\psi_\infty^1 \in D(A_2)$, hence (iii) follows after integrating by parts. Finally, assertion (iv) follows from (iii) and integration by parts.

□

Assuming in addition that Φ is real analytic and that in case of Neumann boundary conditions λ is real analytic too, we obtain the following result.

Proposition 3.5.4 (Lojasiewicz-Simon inequality)**.** *Let $K \in \{N, R\}$ and let $(\psi_\infty, \vartheta_\infty) \in \omega(\psi, \vartheta)$. Assume that Φ is real analytic and in case $K = N$, λ is real analytic too, and let* (3.18) *as well as $\lambda', \lambda'', \lambda''' \in L_\infty(\mathbb{R})$ hold. Then there exist constants $s \in (0, \frac{1}{2}], C, \delta > 0$ such that*

$$|E_K(u, v) - E_K(\psi_\infty, \vartheta_\infty)|^{1-s} \le C|E_K'(u, v)|_{V^*},$$

whenever $|(u, v) - (\psi_\infty, \vartheta_\infty)|_V \le \delta$.

Proof. The proof follows the lines of the proof of Proposition 2.5.4. The only difference is that one has to use the bilinear form, which is defined in (3.28). We skip the details.

□

Now we are in a position to state our main result concerning the asymptotic behavior of solutions of the Cahn-Hilliard equation.

Theorem 3.5.5. *Let (ψ, ϑ) be a global solution of the Cahn-Hilliard equation* (3.23) *with $h = \frac{1}{|\Omega|}\int_\Omega \psi_0$ and suppose that Φ satisfies conditions* (3.16) *as well as* (3.18) *and let $\lambda', \lambda'', \lambda''' \in L_\infty(\mathbb{R})$. Assume that Φ is real analytic, and that λ is real analytic, if $\alpha = 0$. Then the limits*

$$\lim_{t\to\infty} \psi(t) =: \psi_\infty, \quad \textit{and} \quad \lim_{t\to\infty} \vartheta(t) = \vartheta_\infty = const$$

exist in V_1 and V_2, respectively, and $(\psi_\infty, \vartheta_\infty)$ is a solution of the stationary problem (3.33).

Proof. Proposition 3.5.4 yields, that for every $(\varphi, \theta) \in \omega(\psi, \vartheta)$ there exist constants $s \in (0, \frac{1}{2}]$, $C > 0$ and $\delta > 0$ such that

$$|E_K(u, v) - E_K(\varphi, \theta)|^{1-s} \le C|E_K'(u, v)|_{V^*},$$

whenever $|(u, v) - (\varphi, \theta)|_V \le \delta$. By Proposition 3.5.3 (iii) the ω-limit set $\omega(\psi, \vartheta)$ is compact, hence we may cover it by a union of *finitely* many balls with center (φ_i, θ_i) and radius δ_i, $i = 1, \ldots, N$. Since $E_K(u, v) \equiv E_K^\infty$ on $\omega(\psi, \vartheta)$, there are *uniform* constants $s \in (0, \frac{1}{2}]$, $C > 0$ and an open set $U \supset \omega(\psi, \vartheta)$, with

$$|E_K(u, v) - E_K^\infty|^{1-s} \le C|E_K'(u, v)|_{V^*}, \tag{3.36}$$

for all $(u, v) \in U$. After these preliminaries, we define the function $H : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$H(t) := (E_K(\psi(t), \vartheta(t)) - E_K^\infty)^s.$$

By Proposition 3.5.3 the function H is nonincreasing and $\lim_{t\to\infty} H(t) = 0$. A well known result in the theory of dynamical systems implies further that $\lim_{t\to\infty} \operatorname{dist}((\psi(t), \vartheta(t)), \omega(\psi, \vartheta)) = 0$, i.e. there exists $t^* \ge 0$, such that $(\psi(t), \vartheta(t)) \in U$, whenever $t \ge t^*$. Next, we compute and estimate the time derivative of H. By (3.29) and (3.36) we obtain

$$\begin{aligned}
-\frac{d}{dt}\, H(t) &= s\left(-\frac{d}{dt}\, E_K(\psi(t), \vartheta(t))\right)|E_K(\psi(t), \vartheta(t)) - E_K^\infty|^{s-1} \\
&\ge C\frac{|\nabla\mu|_2^2 + |\nabla\vartheta|_2^2 + |\partial_t\psi|_{2,\Gamma}^2 + \alpha|\vartheta|_{2,\Gamma}^2}{|E_K'(\psi(t), \vartheta(t))|_{V^*}}.
\end{aligned} \tag{3.37}$$

By Proposition 3.5.1, the Poincaré inequality and integration by parts we obtain

$$\begin{aligned}
&|\langle E_N'(\psi, \vartheta), (h, k)\rangle_{V^*,V}| \\
&\quad = |\int_\Omega (-\Delta\psi + \Phi'(\psi))h\, dx + \int_\Omega \vartheta k\, dx - \overline{\vartheta}\int_\Omega (\lambda'(\psi)h + k)\, dx - \frac{1}{\gamma}\int_\Gamma \partial_t\psi h\, d\Gamma| \\
&\quad = |\int_\Omega (\mu - \overline{\mu})h\, dx + \int_\Omega (\vartheta - \overline{\vartheta})\lambda'(\psi)h\, dx + \int_\Omega (\vartheta - \overline{\vartheta})k\, dx - \frac{1}{\gamma}\int_\Gamma \partial_t\psi h\, d\Gamma| \\
&\quad \le |\nabla\mu|_2|h|_2 + |\nabla\vartheta|_2(|k|_2 + |h|_2) + |\partial_t\psi|_{2,\Gamma}|h|_{2,\Gamma},
\end{aligned} \tag{3.38}$$

since $\lambda' \in L_\infty(\mathbb{R})$. For E_R we have

$$
\begin{aligned}
|\langle E_R'(\psi,\vartheta),(h,k)\rangle_{V^*,V}| &= |\int_\Omega (\mu-\overline{\mu})h + \int_\Omega \vartheta(k+\lambda'(\psi)h) - \frac{1}{\gamma}\int_\Gamma \partial_t\psi h| \\
&\le C(|\nabla\mu|_2|h|_2 + (|h|_2+|k|_2)(|\nabla\vartheta|_2 + \alpha^{1/2}|\vartheta|_{2,\Gamma}) + |\partial_t\psi|_{2,\Gamma}|h|_{2,\Gamma}), \qquad (3.39)
\end{aligned}
$$

respectively, where $\bar\mu = \frac{1}{|\Omega|}\int_\Omega \mu$. Here we made also use of a version of the Poincaré inequality, namely

$$|w|_2 \le C(|\nabla w|_2 + |w|_{2,\Gamma}), \quad w \in H_2^1(\Omega).$$

If we take the supremum in (3.38) and (3.39) over all $(h,k) \in V$ with norm less than 1 this results in

$$|E_K'(\psi(t),\vartheta(t))|_{V^*} \le C(|\nabla\mu(t)|_2 + |\nabla\vartheta(t)|_2 + \alpha^{1/2}|\vartheta(t)|_{2,\Gamma} + |\partial_t\psi(t)|_{2,\Gamma}).$$

Hence from (3.37) it follows that

$$-\frac{d}{dt}\,H(t) \ge C(|\nabla\mu(t)|_2 + |\nabla\vartheta(t)|_2 + \alpha^{1/2}|\vartheta(t)|_{2,\Gamma} + |\partial_t\psi(t)|_{2,\Gamma})$$

and this in turn implies that $|\nabla\mu|, |\nabla\vartheta| \in L_1([t^*,\infty), L_2(\Omega))$, $\partial_t\psi \in L_1([t^*,\infty); L_2(\Gamma))$ and $\vartheta \in L_1([t^*,\infty); L_2(\Gamma))$, the latter in case of Robin boundary conditions. It follows from the equations that $\partial_t\psi, \partial_t e \in L_1([t^*,\infty), H_2^1(\Omega)^*)$, where as before we have set $e := \vartheta + \lambda(\psi)$. Hence the limits

$$\lim_{t\to\infty}\psi(t) =: \psi_\infty \quad \text{and} \quad \lim_{t\to\infty} e(t) =: e_\infty,$$

exist in $H_2^1(\Omega)^*$. By relative compactness of the orbit $\psi(\mathbb{R}_+)$ the first limit exists in $H_2^1(\Omega)$. Then, by the conditions on λ it holds that

$$\vartheta(t) = e(t) - \lambda(\psi(t)) \to e_\infty - \lambda(\psi_\infty) =: \vartheta_\infty,$$

in $H_2^1(\Omega)^*$ and then in $L_2(\Omega)$, by relative compactness. The remaining part of the proof follows from Proposition 3.5.3 (iii).

□

We close this section with a remark. In Theorems 3.4.2 and 3.5.5 we assumed that $\lambda' \in L_\infty(\mathbb{R})$. This is not the case if one considers for example the function $\lambda(s) = s^2 + c_1$, which is sometimes used in the literature, instead of the linear function $\lambda(s) = s + c_2$. However, at least for the homogeneous system (3.23) it is possible to derive higher order a priori estimates for the local solution ψ of Theorem 3.3.3 on the maximal interval of existence $J_{\max}$ under the assumption

$$|\lambda'(s)| \le c(1+|s|), \quad s \in \mathbb{R}.$$

In particular one may adopt the technique used in Proposition 3.5.2 for the proof of relative compactness of the orbits, in combination with the bootstrap argument of Proposition 2.4.1, given in the Appendix of Chapter 2, to obtain $\psi \in L_\infty(J_{\max}\times\Omega)$, hence $\lambda'(\psi) \in L_\infty(J_{\max}\times\Omega)$, which is enough to ensure, that Theorems 3.4.2 and 3.5.5 are still valid. Actually this has already been proven in the thesis of Vergara [45] for classical boundary conditions. But this result remains true for dynamic boundary conditions.

3.6 Appendix

(a) Proof of Proposition 3.3.2

(i) By Hölders inequality it holds that

$$
\begin{aligned}
|\Delta\Phi'(u) - \Delta\Phi'(v)|_{p,p} &\le |\Delta u\Phi''(u) - \Delta v\Phi''(v)|_{p,p} + ||\nabla u|^2\Phi'''(u) - |\nabla v|^2\Phi'''(v)|_{p,p} \\
&\le |\Delta u|_{rp,rp}|\Phi''(u) - \Phi''(v)|_{r'p,r'p} + |\Delta u - \Delta v|_{rp,rp}|\Phi''(v)|_{r'p,r'p} \\
&\quad + |\nabla u|^2_{2\sigma p,2\sigma p}|\Phi'''(u) - \Phi'''(v)|_{\sigma'p,\sigma'p} + ||\nabla u|^2 - |\nabla v|^2|_{\sigma p,\sigma p}|\Phi'''(v)|_{\sigma'p,\sigma'p} \\
&\le T^{1/r'p}\,(|\Delta u|_{rp,rp}|\Phi''(u) - \Phi''(v)|_{\infty,\infty} + |\Delta u - \Delta v|_{rp,rp}|\Phi''(v)|_{\infty,\infty}) \\
&\quad + T^{1/\sigma'p}\left(|\nabla u|^2_{2\sigma p,2\sigma p}|\Phi'''(u) - \Phi'''(v)|_{\infty,\infty} + ||\nabla u|^2 - |\nabla v|^2|_{\sigma p,\sigma p}|\Phi'''(v)|_{\infty,\infty}\right),
\end{aligned}
$$

where we also used the fact that all functions belonging to $\mathbb{B}_R(u^*)$ are uniformly bounded. We have

$$\nabla w \in H_p^{3\theta_1/4}(J; H_p^{3(1-\theta_1)}(\Omega)) \hookrightarrow L_{2\sigma p}(J \times \Omega), \quad \theta_1 \in [0,1],$$

and

$$\Delta w \in H_p^{\theta_2/2}(J; H_p^{2(1-\theta_2)}(\Omega)) \hookrightarrow L_{rp}(J \times \Omega), \quad \theta_2 \in [0,1],$$

for every function $w \in \mathbb{B}_R(u^*)$, since $r, \sigma > 1$ may be chosen close to 1. Therefore we have

$$|\Delta\Phi'(u) - \Delta\Phi'(v)|_{p,p} \leq \mu_1(T)\,(R + |u^*|_1)\,|u - v|_1,$$

due to the assumption $\Phi \in C^{4-}(\mathbb{R})$. The function μ_1 is given by $\mu_1(T) = \max\{T^{1/r'p}, T^{1/\sigma'p}\}$. This yields (i).

(ii) Firstly we observe that

$$\Delta(\lambda'(w)F(w)) = (\Delta w\lambda''(w) + |\nabla w|^2\lambda'''(w))F(w) + 2\lambda''(w)\nabla w \cdot \nabla F(w) + \lambda'(w)\Delta F(w),$$

for all $w \in \mathbb{B}_R(u^*)$. Secondly by (3.15), the embeddings

$$F(w) \in {}_0H_p^{s+\theta-1}(H_p^{3-2\theta}) \hookrightarrow L_{2p}(J \times \Omega) \quad \text{and} \quad \nabla F(w) \in {}_0H_p^{s+\theta-1}(H_p^{2-2\theta}) \hookrightarrow L_{4p/3}(J \times \Omega),$$

with $s \in \left[\frac{1}{2}, \frac{3}{4}\right)$, $\theta \in [0,1]$, are valid, whenever

$$p > \frac{2}{2s+1}(\frac{n}{4} + \frac{1}{2}) \quad \text{and} \quad p > \frac{1}{s}(\frac{n}{8} + \frac{1}{4}),$$

respectively. It is obvious, that these conditions are fulfilled for every $s \in \left[\frac{1}{2}, \frac{3}{4}\right)$, whenever $p > n/4 + 1$. An easy computation shows that $\nabla w \in L_{4p}(J \times \Omega)$ and $\Delta w \in L_{2p}(J \times \Omega)$, if $p > n/4 + 1$ (here we use strict embeddings). If $1/\sigma + 1/\sigma' = 1$ and $\sigma > 1$ is sufficiently small, then Hölder's inequality and Proposition 3.3.1 lead to the estimate

$$\begin{aligned}
|\lambda''(u)\Delta uF(u) - \lambda''(v)\Delta vF(v)|_{p,p} &\leq C|\Delta uF(u) - \Delta vF(v)|_{p,p} + |\lambda''(u) - \lambda''(v)|_{\infty,\infty}|\Delta vF(v)|_{p,p} \\
&\leq T^{1/2\sigma'p}(|\Delta u|_{2\sigma p,2\sigma p}|F(u) - F(v)|_{2p,2p} + |\Delta u - \Delta v|_{2\sigma p,2\sigma p}|F(v)|_{2p,2p} \\
&\qquad + |\lambda''(u) - \lambda''(v)|_{\infty,\infty}|\Delta v|_{2\sigma p,2\sigma p}|F(v)|_{2p,2p}) \\
&\leq \mu_2(T)(1 + |u^*|_1)|u - v|_1.
\end{aligned}$$

In a similar way we obtain

$$|\lambda'''(u)|\nabla u|^2F(u) - \lambda'''(v)|\nabla v|^2F(v)|_{p,p} \leq \mu_2(T)(1 + |u^*|_1)|u - v|_1,$$

$$|\lambda''(u)\nabla u\nabla F(u) - \lambda''(v)\nabla v\nabla F(v)|_{p,p} \leq \mu_2(T)(1 + |u^*|_1)|u - v|_1,$$

as well as

$$|\lambda'(u)\Delta F(u) - \lambda'(v)\Delta F(v)|_{p,p} \leq \mu_2(T)(1 + |u^*|_1)|u - v|_1,$$

for all $u, v \in \mathbb{B}_R(u^*)$. This proves (ii).

(iii) This is an easy consequence of (i) and (3.14), since by trace-theory (cf. [11]) we obtain

$$|\partial_\nu\Phi'(u) - \partial_\nu\Phi'(v)|_{Y_1(T)} \leq C\left(|\Phi'(u) - \Phi'(v)|_{H_p^{1/2}(L_p)} + |\Phi'(u) - \Phi'(v)|_{L_p(H_p^2)}\right).$$

(iv) In a similar way as in (iii) we obtain

$$\begin{aligned}
&|\partial_\nu(\lambda'(u)F(u)) - \partial_\nu(\lambda'(v)F(v))|_{Y_1(T)} \\
&\qquad \leq C\left(|\lambda'(u)F(u) - \lambda'(v)F(v)|_{H_p^{1/2}(L_p)} + |\lambda'(u)F(u) - \lambda'(v)F(v)|_{L_p(H_p^2)}\right).
\end{aligned}$$

The desired Lipschitz-estimate for the second term follows from (ii). The first term will be rewritten in the usual way, i.e.

$$\begin{aligned}|\lambda'(u)F(u)-\lambda'(v)F(v)|_{H_p^{1/2}(L_p)}\\ \le \left(|(\lambda'(u)-\lambda'(v))F(u)|_{H_p^{1/2}(L_p)}+|\lambda'(v)(F(u)-F(v))|_{H_p^{1/2}(L_p)}\right).\end{aligned}$$

Applying (3.13) we obtain

$$\begin{aligned}&|(\lambda'(u)-\lambda'(v))F(u)|_{H_p^{1/2}(L_p)}\\ &\le C\left(|\lambda'(u)-\lambda'(v)|_{H_{rp}^{1/2}(L_{rp})}|F(u)|_{L_{r'p}(L_{r'p})}+T^{1/\sigma'p}|\lambda'(u)-\lambda'(v)|_\infty|F(u)|_{H_{\sigma p}^{1/2}(L_{\sigma p})}\right),\end{aligned}$$

as well as

$$\begin{aligned}&|\lambda'(v)(F(u)-F(v))|_{H_p^{1/2}(L_p)}\\ &\le C\left(|\lambda'(v)|_{H_{rp}^{1/2}(L_{rp})}|F(u)-F(v)|_{L_{r'p}(L_{r'p})}+T^{1/\sigma'p}|\lambda'(v)|_\infty|F(u)-F(v)|_{H_{\sigma p}^{1/2}(L_{\sigma p})}\right).\end{aligned}$$

Let $w\in\mathbb{B}_R(u^*)$. It is obvious that $F(w)\in H_{\sigma p}^{1/2}(J;L_{\sigma p}(\Omega))$, since $\sigma>1$ may be chosen arbitrarily close to 1. So it remains to check if $\lambda'(w)\in H_{rp}^{1/2}(J;L_{rp}(\Omega))$ and $F(w)\in L_{r'p}(J;L_{r'p}(\Omega))$. It holds

$$H_p^\theta(J;H_p^{4(1-\theta)}(\Omega))\hookrightarrow H_{rp}^{1/2}(J;L_{rp}(\Omega))\quad\text{and}\quad H_p^{s+\theta-1}(J;H_p^{3-2\theta}(\Omega))\hookrightarrow L_{r'p}(J;L_{r'p}(\Omega))$$

if $p>\frac{2}{r'}(\frac{n}{4}+1)$ and $p>\frac{2}{r(2s+1)}(\frac{n}{2}+1)$, respectively. Thus we set $r'=r=2$. Now the claim follows from (3.14) and Proposition 3.3.1.

(v) With the help of Hölder's inequality we compute

$$\begin{aligned}&|(\Delta\lambda'(u)-\Delta\lambda'(v))\eta|_{p,p}\\ &\qquad\le|\eta|_{Z^2}(|\Delta u\lambda''(u)-\Delta v\lambda''(v)|_{2p,2p}+||\nabla u|^2\lambda'''(u)-|\nabla v|^2\lambda'''(v)|_{2p,2p}),\end{aligned}\tag{3.40}$$

for each $\eta\in Z^2$. Since $\lambda\in C^{4-}(\mathbb{R})$, it follows from the uniform boundedness of $u,v\in\mathbb{B}_R(u^*)$ that

$$\begin{aligned}|\Delta u\lambda''(u)-\Delta v\lambda''(v)|_{2p,2p}&\le|\lambda''(u)-\lambda''(v)|_{\infty,\infty}|\Delta u|_{2p,2p}+|\lambda''(v)|_{\infty,\infty}|\Delta u-\Delta v|_{2p,2p}\\ &\le C(1+|u^*|_1)|u-v|_1.\end{aligned}$$

In a similar way the second term in (3.40) can be treated, obtaining

$$||\nabla u|^2\lambda'''(u)-|\nabla v|^2\lambda'''(v)|_{2p,2p}\le C(1+|u^*|_1)|u-v|_1.$$

Furthermore we have

$$\begin{aligned}|(\nabla u\lambda''(u)-\nabla v\lambda''(v))\nabla\eta|_{p,p}&\le|\nabla\eta|_{4p/3,4p/3}|\nabla u\lambda''(u)-\nabla v\lambda''(v)|_{4p,4p}\\ &\le|\eta|_{Z^2}(|\lambda''(u)-\lambda''(v)|_\infty|\nabla u|_{4p,4p}+|\lambda''(v)|_\infty|\nabla u-\nabla v|_{4p,4p})\\ &\le C|\eta|_{Z^2}(1+|u^*|_1)|u-v|_1\end{aligned}$$

and

$$|(\lambda'(u)-\lambda'(v))\Delta\eta|_{p,p}\le|\lambda'(u)-\lambda'(v)|_{\infty,\infty}|\Delta\eta|_{p,p}\le C|\eta|_{Z^2}|u-v|_1,$$

by Hölder's inequality and the Lipschitz-property of λ',λ''.

(vi) Finally we apply trace-theory to obtain

$$|\partial_\nu(\lambda'(u)\eta)-\partial_\nu(\lambda'(v)\eta)|_{Y_1(T)}\le C\left(|(\lambda'(u)-\lambda'(v))\eta|_{H_p^{1/2}(L_p)}+|(\lambda'(u)-\lambda'(v))\eta|_{L_p(H_p^2)}\right).$$

The estimate for the second term is clear by (v). Again we will use (3.13) to estimate the first term. This yields

$$\begin{aligned}&|(\lambda'(u)-\lambda'(v))\eta|_{H_p^{1/2}(L_p)}\\&\qquad\le C(T_0)\left(|\lambda'(u)-\lambda'(v)|_{H_{rp}^{1/2}(L_{rp})}|\eta|_{L_{r'p}(L_{r'p})}+|\lambda'(u)-\lambda'(v)|_{L_{\sigma'p}(L_{\sigma'p})}|\eta|_{H_{\sigma p}^{1/2}(L_{\sigma p})}\right).\end{aligned}$$

As in (iv), it follows that $r=2$. Furthermore we have

$$H_p^\theta(J;H_p^{2(1-\theta)}(\Omega))\hookrightarrow L_{2p}(J;L_{2p}(\Omega)),$$

if $p>\frac{n}{4}+1$. Last but not least we apply (3.14). The proof is complete.

(b) Proof of Lemma 3.4.1

Step 1. We start with $\Delta\Phi'(\psi)=\Delta\psi\Phi''(\psi)+|\nabla\psi|^2\Phi'''(\psi)$. Using the Gagliardo-Nirenberg inequality and (3.18) we obtain

$$|\Phi''(\psi)\Delta\psi|_p\le|\psi|_{2(\beta+1)p}^{\beta+1}|\Delta\psi|_{2p}\le C|\psi|_{H_p^4}^{a+b(\beta+1)}|\psi|_q^{1-a+(1-b)(\beta+1)},\tag{3.41}$$

where q will be chosen in such a way that $H_2^1(\Omega)\hookrightarrow L_q$, i.e. $\frac{n}{q}\ge\frac{n}{2}-1$ and

$$(a+(\beta+1)b)\left(4-\frac{n}{p}+\frac{n}{q}\right)=2-\frac{n}{p}+\frac{n}{q}(\beta+2).$$

The second term $\Phi'''(\psi)|\nabla\psi|^2$ will be treated in a similar way. The Gagliardo-Nirenberg inequality and (3.18) again yield

$$|\Phi'''(\psi)|\nabla\psi|^2|_p\le|\psi|_{2\beta p}^{\beta}|\nabla\psi|_{4p}^2\le C|\psi|_{H_p^4}^{2a+b\beta}|\psi|_q^{2-2a+(1-b)\beta},\tag{3.42}$$

with $\frac{n}{q}\ge\frac{n}{2}-1$ and

$$(2a+\beta b)\left(4-\frac{n}{p}+\frac{n}{q}\right)=2-\frac{n}{p}+(\beta+2)\frac{n}{q}.$$

It turns out that the condition $\beta<3$ in case $n=3$ ensures that either $a+(\beta+1)b<1$ and $2a+\beta b<1$ in (3.41) and (3.42), respectively. Integrating (3.41) and (3.42) with respect to t and using Hölders inequality we obtain the desired estimate. Now we estimate $\partial_\nu\Phi'(\psi)$ in Y_1. By trace-theory we obtain

$$|\partial_\nu\Phi'(\psi)|_{Y_1}\le C(|\Phi'(\psi)|_{H_p^{1/2}(L_p)}+|\Phi'(\psi)|_{L_p(H_p^2)}).$$

The estimate in $L_p(H_p^2)$ follows from the considerations above. By the mean-value theorem and (3.18) we obtain

$$|\Phi'(\psi)|_{H_p^{1/2}(L_p)}\le C(|\psi|_{L_{(\beta+2)p}(L_{(\beta+2)p})}^{\beta+2}+|\psi|_{H_p^{1/2}(L_p)}+|\psi|_{L_{\sigma'p}(L_{r'p})}^{\beta+1}|\psi|_{H_{\sigma p}^{1/2}(L_{rp})}),\tag{3.43}$$

where $1/\sigma+(\beta+1)/\sigma'=1/r+(\beta+1)/r'=1$. This follows similarly to (3.13) from the characterization of H_p^s via differences and Hölders inequality. The Gagliardo-Nirenberg inequality implies

$$|\psi|_{L_{\sigma'p}(L_{r'p})}\le c|\psi|_{Z^1}^a|\psi|_{L_\infty(H_2^1)}^{1-a},$$

if $a\in[0,1/\sigma']$ and $a(3-n/p+n/2)\ge n/2-1-n/r'p$. Therefore we set

$$a=1/\sigma'=[n/2-1-n/r'p]_+/(3-n/p+n/2)$$

and choose $r'=(\beta+1)n/p$ if $p\le n$, $r'=2(\beta+1)n/p$ if $n<p\le 2n$ and $r'=\infty$ if $p>2n$. Observe that

$$Z^1\hookrightarrow H_p^{1-\theta}(H_p^{4\theta})\hookrightarrow H_p^s(L_{rp}),$$

if $s = 1 - n/4r'p$. Hence complex interpolation yields

$$|\psi|_{H^{1/2}_{\sigma p}(L_{rp})} \le c|\psi|^b_{Z^1}|\psi|^{1-b}_{L_{\tau p}(L_{rp})},$$

provided $b = 1/2s$ and

$$1/\sigma \ge b + (1-b)/\tau. \tag{3.44}$$

Finally we apply the Gagliardo-Nirenberg inequality one more time to obtain

$$|\psi|_{L_{\tau p}(L_{rp})} \le c|\psi|^d_{Z^1}|\psi|^{1-d}_{L_\infty(H^1_2)},$$

with $d(3 - n/p + n/2) \ge n/2 - 1 - n/rp$ and $d \in [0, 1/\tau]$. We set

$$d = 1/\tau = [n/2 - 1 - n/rp]_+/(3 - n/p + n/2).$$

Suppose now that (3.44) holds. Then we have

$$(\beta+1)a + b + (1-b)d = \frac{\beta+1}{\sigma'} + b + \frac{1-b}{\tau} \le \frac{\beta+1}{\sigma'} + \frac{1}{\sigma} = 1.$$

Hence the desired estimate follows if the inequality (3.44) is strict. We have to distinguish three cases, namely $p \le n$, $n < p \le 2n$ and $p > 2n$. In the first case we have $r' = (\beta+1)n/p$, $s = (4\beta+3)/(4\beta+4)$ and $b = (2\beta+2)/(4\beta+3)$. Since $p \ge 2$, (3.44) is equivalent to

$$\frac{6\beta+3}{4\beta+3} - (\beta+1)[n/2 - 1/(\beta+1) - 1]_+ \ge 0.$$

We see that $[n/2 - 1/(\beta+1) - 1]_+ = 0$, if either $n = 1, 2$ or $n = 3$ and $\beta \le 1$; then we are done. So let $n = 3$ and $\beta > 1$. An easy calculation shows that

$$\frac{6\beta+3}{4\beta+3} > \frac{\beta-1}{2},$$

for all $1 < \beta < 3$. In the second case we have $r' = 2(\beta+1)n/p$, $s = (8\beta+7)/(8\beta+8)$ and $b = (4\beta+4)/(8\beta+7)$, thus (3.44) is equivalent to

$$\frac{4\beta+3}{8\beta+7}(3 - n/p + n/2) - (\beta+1)[n/2 - 1/2(\beta+1) - 1]_+ \ge \frac{4\beta+3}{8\beta+7}[n/2 - 1/2 - n/p]_+.$$

Since $p \le 2n$ we see that this inequality is strict for $n = 1, 2$. If $n = 3$ and due to $p > n$, the inequality reduces to

$$\frac{4\beta+3}{8\beta+7} \ge \beta/7$$

and we have again strict inequality, if $\beta < 3$. In the last case, we have $r' = \infty$, $s = 1$ and $b = 1/2$. Therefore (3.44) is equivalent to

$$(3 - n/p + n/2) - 2(\beta+1)[n/2 - 1]_+ \ge [n/2 - n/p - 1]_+.$$

Again for $n = 1, 2$ we have $[n/2 - 1]_+ = [n/2 - n/p - 1]_+ = 0$, thus we set $n = 3$. Since $p > 2n$ this yields

$$4 - (\beta+1) \ge 0,$$

hence strict inequality if $\beta < 3$. Note that the second term on the right hand side of (3.43) is dominated by the third term. Furthermore the desired estimate for the first term is a simple consequence of the Gagliardo-Nirenberg inequality as long as $q > (\beta+1)n/4$.

Step 2. Next we estimate the term $\Delta(\lambda'(\psi)F(\psi))$ in X. Observe that

$$\Delta(\lambda'(\psi)F(\psi)) = F(\psi)(\Delta\psi\lambda''(\psi) + |\nabla\psi|^2\lambda'''(\psi)) + 2\lambda''(\psi)\nabla\psi\nabla F(\psi) + \lambda'(\psi)\Delta F(\psi).$$

As in section 2 we will solve the homogeneous heat equation (3.5). Obviously $|F(\psi)|_{L_p(L_p)} \leq C(1+|\psi|_{L_p(L_p)})$, for every $1 < p < \infty$, since $|\lambda(s)| \leq C(1+|s|)$. Applying the Gagliardo-Nirenberg one more time we obtain

$$\begin{aligned}|\lambda''(\psi)F(\psi)\Delta\psi|_{L_p(L_p)} &\leq C(|\Delta\psi|_{L_{3p/2}(L_{3p/2})} + |\Delta\psi|_{L_{3p/2}(L_{3p/2})}|\psi|_{L_{3p}(L_{3p})}) \\ &\leq C(|\psi|^a_{Z^1}|\psi|^{1-a}_{L_\infty(L_q)} + |\psi|^{a+b}_{Z^1}|\psi|^{2-(a+b)}_{L_\infty(L_q)}),\end{aligned}$$

if $a(4-n/p+n/q) = 2-2n/3p+n/q$ and $b(4-n/p+n/q) = n/q-n/3p$ and $a \in [1/2,1]$, $b \in [0,1]$. The two latter conditions are fulfilled if $q \leq 3p$. We require furthermore $a < 2/3$ and $b < 1/3$. This leads to the condition $q > n/2$, which is true. Then we also have $a+b < 1$. In a similar way we estimate $\lambda'''(\psi)F(\psi)|\nabla\psi|^2$, to obtain

$$\begin{aligned}|\lambda'''(\psi)F(\psi)|\nabla\psi|^2|_{L_p(L_p)} &\leq C(1+|\psi|_{L_{3p}(L_{3p})})|\nabla\psi|^2_{L_{3p}(L_{3p})} \\ &\leq C(|\psi|^{2a}_{Z^1}|\psi|^{2(1-a)}_{L_\infty(L_q)} + |\psi|^{2a+b}_{Z^1}|\psi|^{3-(2a+b)}_{L_\infty(L_q)}),\end{aligned} \quad (3.45)$$

whenever $a(4-n/p+n/q) = 1+n/q-n/3p$ and $b(4-n/p+n/q) = n/q-n/3p$ and $a \in [1/4,1]$, $b \in [0,1]$. The two latter conditions are satisfied if $q \leq 3p$. It is easy to verify that $a \leq 1/3$ and $b < 1/3$, whenever $q \geq 2n$, i.e. $q = 6$. Finally it holds $2a+b < 1$.

Note that the representation of $F(\psi)$ implies

$$|\nabla F(\psi)|_{L_{2p}(L_{2p})} \leq C(1+|\nabla\lambda(\psi)|_{L_{2p}(L_{2p})}) \leq C(1+|\nabla\psi|_{L_{2p}(L_{2p})}),$$

hence by the inequalities of Hölder and Gagliardo-Nirenberg we obtain

$$\begin{aligned}|\lambda''(\psi)\nabla F(\psi)\nabla\psi|_{L_p(L_p)} &\leq |\nabla F(\psi)|_{L_{2p}(L_{2p})}|\nabla\psi|_{L_{2p}(L_{2p})} \\ &\leq C(1+|\nabla\psi|^2_{L_{2p}(L_{2p})}) \leq C(1+|\psi|^{2a}_{Z^1}|\psi|^{2(1-a)}_{L_\infty(L_q)}),\end{aligned}$$

with

$$2a\left(4-\frac{n}{p}+\frac{n}{q}\right) = 2+\frac{2n}{q}-\frac{n}{p}, \quad a \in [1/4,1].$$

Since $q \leq 3p$ we see that $a \geq 1/4$. Furthermore we have $2a < 1$, if $n < 6$. The estimate of $\lambda'(\psi)\Delta F(\psi)$ in L_p is more involved. With the help of (3.15) with $s = \theta = 1/2$, we obtain

$$|\lambda'(\psi)\Delta F(\psi)|_{L_p(L_p)} \leq C(1+|\lambda(\psi)|_{H^{1/2}_p(H^1_p)}) \leq C(1+|\psi|_{H^{1/2}_p(L_p)} + |\nabla\psi\lambda'(\psi)|_{H^{1/2}_p(L_p)}).$$

A similar estimate for ψ in $H^{1/2}_p(L_p)$ has already been done in (3.43). For the term $\nabla\psi\lambda'(\psi)$ we will use (3.13). This leads to

$$|\nabla\psi\lambda'(\psi)|_{H^{1/2}_p(L_p)} \leq c(|\nabla\psi|_{H^{1/2}_p(L_p)} + |\nabla\psi|_{L_{\sigma' p}(L_{r'p})}|\psi|_{H^{1/2}_{\sigma p}(L_{rp})}),$$

since $\lambda' \in L_\infty(\mathbb{R})$. We will use the same strategy as in (3.43). First we observe that complex-interpolation and the Gagliardo-Nirenberg inequality lead to the desired estimate for $|\nabla\psi|_{H^{1/2}_p(L_p)}$. Secondly we make again use of the Gagliardo-Nirenberg inequality to obtain

$$|\nabla\psi|_{L_{\sigma' p}(L_{r'p})} \leq c|\psi|^a_{Z^1}|\psi|^{1-a}_{L_\infty(H^1_2)},$$

where $a = 1/\sigma' = (n/2-n/r'p)/(3+n/2-n/p)$. Complex interpolation yields

$$|\psi|_{H^{1/2}_{\sigma p}(L_{rp})} \leq c|\psi|^b_{H^s_p(L_{rp})}|\psi|^{1-b}_{L_{\tau p}(L_{rp})}, \quad (3.46)$$

where $b = 1/2s$, $s = 1-n/4r'p$ and

$$1/\sigma \geq b+(1-b)/\tau. \quad (3.47)$$

One more time the Gagliardo-Nirenberg inequality leads to the estimate

$$|\psi|_{L_{\tau p}(L_{rp})} \leq c|\psi|^{d}_{Z^1}|\psi|^{1-d}_{L_\infty(H^1_2)}, \tag{3.48}$$

with $d = 1/\tau = [n/2 - 1 - n/rp]_+/(3 - n/p + n/2)$. Finally we have to check if (3.47) is valid and if in this case the inequality is strict. We distinguish two cases. If $p \leq n$ we set $r' = 2n/p$ and if $p > n$ we set $r' = 2$ (then $r' \in [2, 3]$). In the first case we have $s = 7/8$, $b = 4/7$ and $[n/2 - 1/2 - n/p]_+ = 0,\ n = 1, 2, 3$, since $p \geq 2$. Thus (3.47) is equivalent to the condition

$$p \geq \frac{6n}{25 - 4n}\ ,$$

which is always fulfilled and strict inequality holds if $n \leq 3$. In the second case we have $r' = 2$, thus $s = 1 - n/8p > 7/8$. We set $s = 7/8$ and therefore $b = 4/7$. Then (3.47) is equivalent to

$$18 - 4n + n/p \geq 6[n/2 - 1 - n/2p]_+.$$

This inequality is obviously fulfilled and additionally strict, if $n = 1, 2$. So let $n = 3$. Then $n/2 - 1 - n/2p > 0$ and we obtain $1 + 4/p > 0$, which is certainly true.

The next estimate will be done for the term $\partial_\nu(\lambda'(\psi)F(\psi))$ in Y_1. Again we use trace-theory to obtain

$$|\partial_\nu(\lambda'(\psi)F(\psi))|_{Y_1} \leq C(|\lambda'(\psi)F(\psi)|_{H^{1/2}_p(L_p)} + |\lambda'(\psi)F(\psi)|_{L_p(H^2_p)}).$$

The estimate of $\lambda'(\psi)F(\psi)$ in $L_p(H^2_p)$ has already been done. Making use of (3.13) and (3.15), with $s = 1/2$ and L_p instead of H^1_p, we obtain

$$|\lambda'(\psi)F(\psi)|_{H^{1/2}_p(L_p)} \leq C\left(1 + |\psi|_{H^{1/2}_p(L_p)} + (1 + |\psi|_{L_{\sigma' p}(L_{r' p})})(1 + |\psi|_{H^{1/2}_{\sigma p}(L_{rp})})\right),$$

since $\lambda', \lambda'' \in L_\infty(\mathbb{R})$. Therefore the estimate follows immediately from Step 1 .

Step 3. Last but not least we have to consider $\Delta(\lambda'(\psi)\eta)$ in X and $\partial_\nu(\lambda'(\psi)\eta)$ in Y_1, where $\eta \in Z^2$ is a fixed function. We compute

$$\Delta(\lambda'(\psi)\eta) = \eta(\Delta\psi\lambda''(\psi) + |\nabla\psi|^2\lambda'''(\psi)) + 2\lambda''(\psi)\nabla\psi\nabla\eta + \lambda'(\psi)\Delta\eta.$$

Since $p \geq 2$ we have $Z^2 \hookrightarrow L_{3p}(J \times \Omega)$, hence the estimate for the first term follows from Step 2. Moreover by (3.45) we obtain

$$|\nabla\psi\nabla\eta|_{L_p(L_p)} \leq |\nabla\psi|_{L_{3p}(L_{3p})}|\nabla\eta|_{L_{3p/2}(L_{3p/2})} \leq c|\psi|^{\delta}_{Z^1(T)}|\psi|^{1-\delta}_{L_\infty(J;H^1_2(\Omega))},\ \delta < 1,$$

since $\nabla\eta$ is a fixed function and

$$H^{1/2}_p(J; L_p(\Omega)) \cap L_p(J; H^1_p(\Omega)) \hookrightarrow L_{3p/2}(J \times \Omega).$$

Finally the last term $\lambda'(\psi)\Delta\eta$ is dominated by the fixed function $\Delta\eta \in L_p(J\times\Omega)$, since $\lambda' \in L_\infty(\mathbb{R})$. A last time we apply trace-theory to obtain

$$|\partial_\nu(\lambda'(\psi)\eta)|_{Y_1} \leq C(|\lambda'(\psi)\eta|_{H^{1/2}_p(L_p)} + |\lambda'(\psi)\eta|_{L_p(H^2_p)})$$

and then (3.13) leads to

$$|\lambda'(\psi)\eta|_{H^{1/2}_p(L_p)} \leq C(|\eta|_{H^{1/2}_p(L_p)} + |\eta|_{L_{\sigma' p}(L_{r' p})}|\psi|_{H^{1/2}_{\sigma p}(L_{rp})}).$$

Since $\eta \in L_\infty(J; H^1_p(\Omega))$ for all $p \geq 2$ we may choose $r' \in [2, 3]$, i.e. $r \in [3/2, 2]$ and σ' may be arbitrarily large. Then the claim follows from (3.46) and (3.48). The proof is complete.

Chapter 4

A Generalized Cahn-Hilliard Equation based on a Microforce Balance

4.1 Derivation of the Model

We start again with the derivation of the classical Cahn-Hilliard equation. Consider the free energy functional of the form

$$\mathcal{F}(\psi) = \int_\Omega \left(\frac{1}{2}|\nabla\psi|^2 + \Phi(\psi) \right) \, dx, \tag{4.1}$$

where Ω is a bounded, open and connected subset of $\mathbb{R}^n$ with boundary $\Gamma := \partial\Omega \in C^3$. We assume that the order parameter ψ is a conserved quantity. The according conservation law reads

$$\partial_t \psi + \operatorname{div} j = 0, \tag{4.2}$$

where j is a vector field representing the phase flux of the order parameter. The next step is to combine the two quantities j and μ. Similar to Fourier's law in the derivation of the heat equation one typically assumes that j is given by

$$j = -\nabla\mu, \tag{4.3}$$

a *postulated* relation. Finally we have to derive an equation for μ. The chemical potential μ is given by the *variational derivative* of $\mathcal{F}$, i.e.

$$\mu = \frac{\delta\mathcal{F}}{\delta\psi} = -\Delta\psi + \Phi'(\psi).$$

If $\mathcal{F}$ is of the form (4.1) this yields the classical Cahn-Hilliard equation.

In the early nineties Gurtin [16] proposed a *generalized* Cahn-Hilliard equation, which is based on the following objections:

- Fundamental physical laws should account for the work associated with each operative kinematical process;
- There is no clear separation of the balance law (4.2) and the constitutive equation (4.3);
- Forces that are associated with microscopic configurations of atoms are not considered in the derivation of the classical Cahn-Hilliard equation.

According to Gurtin there should exist so called 'microforces' whose work accompanies changes in the order parameter ψ. The microforce system is characterized by the microstress $\xi \in \mathbb{R}^n$ and scalar quantities π and γ which represent internal and external microforces, respectively. The main assumption in [16] is that ξ, π and γ satisfy the (local) *microforce balance*

$$\operatorname{div}\xi + \pi + \gamma = 0, \tag{4.4}$$

which can be motivated from a static point of view, see [16] for more details. In a next step we want to derive constitutive equations, which relate the quantities j, the flux of the order parameter, ξ and π to the fields ψ and μ. The technique used in [16] for this derivation is based on the balance equation (4.4) and a (local) dissipation inequality, which is a direct consequence of the first and the second law of thermodynamics, that is, the energy balance

$$\frac{d}{dt}\int_\Omega e\ dx = -\int_{\partial\Omega} q\cdot\nu\ d\sigma + \int_\Omega r\ dx + \mathcal{W}(\Omega) + \mathcal{M}(\Omega),$$

and

$$\frac{d}{dt}\int_\Omega S\ dx \geq -\int_{\partial\Omega} \frac{q}{\theta}\cdot\nu\ d\sigma + \int_\Omega \frac{r}{\theta}\ dx,$$

cf. [16, Appendix A]. The second law of thermodynamics is also known as the *Clausius-Duhem inequality.* Here e is the internal energy, S is the entropy, θ is the absolute temperature, q is the heat flux, r is the heat supply, $\mathcal{W}(\Omega)$ is the rate of working on Ω of all forces exterior to Ω and $\mathcal{M}(\Omega)$ is the rate at which energy is added to Ω by mass transport. Let F be the free energy density, depending on the vector $z = (\psi, \nabla\psi, \mu, \nabla\mu, \partial_t\psi)$. Then the second law of thermodynamics (in its mechanical version as considered by Gurtin [16]) reads

$$\frac{d}{dt}\int_\Omega F(z)\ dx \leq -\int_{\partial\Omega} \mu j(z)\cdot\nu\ d\sigma + \int_{\partial\Omega} \xi\cdot\nu\partial_t\psi\ d\sigma + \int_\Omega \mu m\ dx + \int_\Omega \gamma\partial_t\psi\ dx,$$

with m being the external mass supply. Making use of Green's formula, we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega F(z)\ dx \leq &-\int_\Omega (\nabla\mu\cdot j(z) + \mu\operatorname{div} j)\ dx\\ &+\int_\Omega (\operatorname{div}\xi\partial_t\psi + \xi\cdot\nabla\partial_t\psi)\ dx + \int_\Omega \mu m\ dx + \int_\Omega \gamma\partial_t\psi\ dx.\end{aligned}$$

in presence of external mass supply m, (4.2) will be modified to

$$\partial_t\psi + \operatorname{div} j = m. \tag{4.5}$$

In view of (4.4) and (4.5) we obtain the dissipation inequality

$$\frac{d}{dt}\int_\Omega F(z)\ dx \leq \int_\Omega (\mu\partial_t\psi - j\cdot\nabla\mu - \pi\partial_t\psi + \xi\cdot\nabla\partial_t\psi)\ dx.$$

This in turn yields the following local dissipation inequality

$$\partial_t F(z) \leq \mu\partial_t\psi - j\cdot\nabla\mu - \pi\partial_t\psi + \xi\cdot\nabla\partial_t\psi,$$

for all fields ψ and μ, this means, we have

$$(\partial_\psi F + \pi - \mu)\dot\psi + (\partial_{\nabla\psi}F - \xi)\cdot\nabla\dot\psi + \partial_\mu F\dot\mu + \partial_{\nabla\mu}F\nabla\dot\mu + \partial_{\dot\psi}F\ddot\psi + \nabla\mu\cdot j \leq 0, \tag{4.6}$$

where $\dot u = \partial_t u$ and $\ddot u = \partial_t^2 u$ for a smooth function u. This local inequality needs to be satisfied for all smooth fields ψ and μ. Hence we have necessarily

$$F(z) = F(\psi, \nabla\psi) \quad \text{and} \quad \xi(\psi, \nabla\psi) = \partial_{\nabla\psi}F(\psi, \nabla\psi)$$

and there remains the inequality

$$(\partial_\psi F + \pi - \mu)\dot\psi + \nabla\mu\cdot j \leq 0$$

whose general solution is given by (cf. [16, Appendix B])

$$\partial_\psi F + \pi - \mu = -\beta\dot{\psi} - c \cdot \nabla\mu \quad \text{and} \quad j = -a\dot{\psi} - B\nabla\mu,$$

with *constitutive moduli* $\beta(z)$ (scalar), $a(z), c(z)$ (vectors), $B(z)$ (matrix) and the constraint that the tensor

$$\begin{bmatrix} \beta & c^T \\ a & B \end{bmatrix} \tag{4.7}$$

is positive semidefinite. We assume that β is constant and a, c and B do only depend on x instead of z, whence we deal with an approximation of the constitutive moduli $\beta(z), a(z), B(z)$. In particular, if the free energy density F is given by $F(\psi, \nabla\psi) = \frac{1}{2}|\nabla\psi|^2 + \Phi(\psi)$ we obtain the following *Cahn-Hilliard-Gurtin equations*.

$$\begin{aligned} \partial_t\psi - \operatorname{div}(B\nabla\mu) - \operatorname{div}(a\partial_t\psi) &= f, \quad t \in J, \ x \in \Omega, \\ \mu - c \cdot \nabla\mu + \Delta\psi - \beta\partial_t\psi - \Phi'(\psi) &= g, \quad t \in J, \ x \in \Omega, \end{aligned} \tag{4.8}$$

where $\Omega \subset \mathbb{R}^n$ is open, bounded with compact boundary $\Gamma = \partial\Omega \in C^3$. In this chapter, we are interested in solutions of (4.8) subject to the Neumann boundary conditions $\partial_\nu\psi = 0$ and $B\nabla\mu \cdot \nu = 0$, having optimal regularity in the sense

$$\psi \in H_p^1(J; H_p^1(\Omega)) \cap L_p(J; H_p^3(\Omega)),$$

and

$$\mu \in L_p(J; H_p^2(\Omega)).$$

We impose the following assumptions on the data $a, c \in C_{ub}^1(\overline{\Omega}; \mathbb{R}^n)$ and $B \in C_{ub}^1(\overline{\Omega}; \mathbb{R}^{n\times n})$.

$$\operatorname{div} a(x) = \operatorname{div} c(x) = 0, \quad \text{for all } x \in \Omega, \text{ and } a(x) \cdot \nu(x) = c(x) \cdot \nu(x) = 0, \quad \text{for all } x \in \Gamma, \tag{4.9}$$

$$B(x)\tau(x) \cdot \nu(x) = 0, \quad \text{for all } x \in \Gamma \text{ and all } \tau(x) \in T_x\Gamma, \tag{4.10}$$

where $T_x\Gamma$ denotes the tangential space in a point $x \in \Gamma$ on Γ.

Finally we want to emphasize that for the special case $B = I$, $a = c = 0$ and $\beta = 0$, we obtain the classical Cahn-Hilliard equation.

4.2 The Linear Cahn-Hilliard-Gurtin Problem in $\mathbb{R}^n$

In this section we will solve the full space problem

$$\begin{aligned} \partial_t u - \operatorname{div}(a\partial_t u) &= \operatorname{div}(B\nabla\mu) + f, \quad t > 0, \ x \in \mathbb{R}^n, \\ \mu - c \cdot \nabla\mu &= \beta\partial_t u - \Delta u + g, \quad t > 0, \ x \in \mathbb{R}^n, \\ u(0) &= u_0, \quad t = 0, \ x \in \mathbb{R}^n, \end{aligned} \tag{4.11}$$

where $\beta \in \mathbb{R}_+$, $a, c \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n\times n}$ is symmetric and positive definite. Set $A = \beta B - \frac{1}{2}(a \otimes c + c \otimes a)$, where $a \otimes c = (a_i c_j)_{i,j=1}^n$. In the sequel we assume the following condition on the matrix A.

(A) There is a constant $\varepsilon > 0$, such that $(A\xi|\xi) \geq \varepsilon|\xi|^2$ for all $\xi \in \mathbb{R}^n$.

Here is the main result on optimal L_p-regularity of (4.11).

Theorem 4.2.1. *Let $1 < p < \infty$ and assume that (A) holds true. Then (4.11) admits a unique solution*

$$u \in H_p^1(J; H_p^1(\mathbb{R}^n)) \cap L_p(J; H_p^3(\mathbb{R}^n)) =: Z^1,$$

$$\mu \in L_p(J; H_p^2(\mathbb{R}^n)) =: Z^2,$$

if the data is subject to the following conditions.

(i) $f \in L_p(J; L_p(\mathbb{R}^n)) =: X^1$,

(ii) $g \in L_p(J; H_p^1(\mathbb{R}^n)) =: X^2$,

(iii) $u_0 \in B_{pp}^{3-2/p}(\mathbb{R}^n) =: X_p$.

Proof. We apply the operator $(I-\Delta)^{-1/2}$ to both equations in (4.11) and define the new functions $w = (I-\Delta)^{-1/2}u$, $\eta = (I-\Delta)^{-1/2}\mu$, $\tilde{f} = (I-\Delta)^{-1/2}f$, $\tilde{g} = (I-\Delta)^{-1/2}g$ and $w_0 = (I-\Delta)^{-1/2}u_0$. Then it holds that

$$\tilde{f} \in L_p(J; H_p^1(\mathbb{R}^n)), \quad \tilde{g} \in L_p(J; H_p^2(\mathbb{R}^n)),$$
$$w_0 \in B_{pp}^{4-2/p}(\mathbb{R}^n)$$

and we are looking for a solution (w, η) of the system

$$\begin{aligned} w_t - \operatorname{div}(aw_t) &= \operatorname{div}(B\nabla\eta) + \tilde{f}, \quad t > 0,\ x \in \mathbb{R}^n, \\ \eta - c\cdot\nabla\eta &= \beta w_t - \Delta w + \tilde{g}, \quad t > 0,\ x \in \mathbb{R}^n, \\ w(0) &= w_0, \quad t = 0,\ x \in \mathbb{R}^n, \end{aligned} \tag{4.12}$$

in the regularity class

$$w \in H_p^1(J; H_p^2(\mathbb{R}^n)) \cap L_p(J; H_p^4(\mathbb{R}^n)),$$
$$\eta \in L_p(J; H_p^3(\mathbb{R}^n)).$$

In a next step we want to eliminate the functions $\tilde{g}$ and w_0. To achieve this, let w^* be the unique solution of the problem

$$\begin{aligned} \beta w_t^* - \Delta w^* &= -\tilde{g}, \quad t > 0,\ x \in \mathbb{R}^n, \\ w^*(0) &= w_0, \quad t = 0,\ x \in \mathbb{R}^n, \end{aligned}$$

with regularity

$$w^* \in H_p^1(J; L_p(\mathbb{R}^n)) \cap L_p(J; H_p^2(\mathbb{R}^n)),$$

if and only if $\tilde{g} \in L_p(J \times \mathbb{R}^n)$ and $w_0 \in B_{pp}^{2-2/p}(\mathbb{R}^n)$. Here J denotes the interval $[0, T]$. If we even have $\tilde{g} \in L_p(J; H_p^2(\mathbb{R}^n))$ and $w_0 \in B_{pp}^{4-2/p}(\mathbb{R}^n)$ then by regularity theory we obtain

$$w^* \in H_p^1(J; H_p^2(\mathbb{R}^n)) \cap L_p(J; H_p^4(\mathbb{R}^n)).$$

The pair of functions $(v, \eta) = (w - w^*, \eta)$ should now solve the problem

$$\begin{aligned} \partial_t v - \operatorname{div}(a\partial_t v) &= \operatorname{div}(B\nabla\eta) + F, \quad t > 0,\ x \in \mathbb{R}^n, \\ \eta - c\cdot\nabla\eta &= \beta\partial_t v - \Delta v, \quad t > 0,\ x \in \mathbb{R}^n, \\ v(0) &= 0, \quad t = 0,\ x \in \mathbb{R}^n, \end{aligned} \tag{4.13}$$

where F is defined by

$$F = \tilde{f} + w_t^* - \operatorname{div}(aw_t^*) \in L_p(J; H_p^1(\mathbb{R}^n)).$$

In order to solve (4.13) we take the Laplace transform in the time variable and the Fourier transform in the spatial variable to obtain

$$\begin{aligned} \lambda(1 - i(a|\xi))\hat{v} &= -(B\xi|\xi)\hat{\eta} + \hat{F}, \\ (1 - i(c|\xi))\hat{\eta} &= (\beta\lambda + |\xi|^2)\hat{v}, \end{aligned}$$

and $(\cdot|\cdot)$ denotes the inner product in $\mathbb{C}^n$. This system of algebraic equations can be written in matrix form

$$\underbrace{\begin{bmatrix} \lambda(1 - i(a|\xi)) & (B\xi|\xi) \\ -(\beta\lambda + |\xi|^2) & (1 - i(c|\xi)) \end{bmatrix}}_{M(\lambda,\xi)} \begin{bmatrix} \hat{v} \\ \hat{\eta} \end{bmatrix} = \begin{bmatrix} \hat{F} \\ 0 \end{bmatrix},$$

where $\lambda \in \Sigma_\phi,\ \phi > \pi/2$ and $\xi \in \mathbb{R}^n$ such that $|\lambda| + |\xi| \neq 0$. Hence the unique solution to this equation is given by

$$\begin{bmatrix} \hat{v} \\ \hat{\eta} \end{bmatrix} = \frac{1}{m(\lambda,\xi)} \begin{bmatrix} (1 - i(c|\xi)) & -(B\xi|\xi) \\ (\beta\lambda + |\xi|^2) & \lambda(1 - i(a|\xi)) \end{bmatrix} \begin{bmatrix} \hat{F} \\ 0 \end{bmatrix},$$

provided that

$$m(\lambda,\xi) := \det M(\lambda,\xi) \neq 0.$$

Let $v_0, v_1 \in {}_0H_p^1(J; H_p^2(\mathbb{R}^n)) \cap L_p(J; H_p^4(\mathbb{R}^n))$ be the unique solutions of

$$\begin{aligned} \partial_t(I-\Delta)v_0 + \Delta^2 v_0 &= F - c\cdot\nabla F, \quad t>0,\ x\in\mathbb{R}^n, \\ v_0(0) &= 0, \end{aligned}$$

and

$$\begin{aligned} \partial_t(I-\Delta)v_1 + \Delta^2 v_1 &= (I-\Delta)^{1/2}F, \quad t>0,\ x\in\mathbb{R}^n, \\ v_1(0) &= 0. \end{aligned}$$

Therefore it holds that

$$\partial_t(I-\Delta)v + \Delta^2 v = T(\partial_t(I-\Delta) + \Delta^2)v_0,$$

and

$$(I-\Delta)^{3/2}\eta = T(I-\Delta)(\beta\partial_t - \Delta)v_1,$$

where T is defined by its Fourier-Laplace symbol

$$\hat{T}(\lambda,\xi) = \frac{\lambda(1+|\xi|^2) + |\xi|^4}{m(\lambda,\xi)}.$$

The assertion of the theorem follows if we can show that T is a bounded operator from $L_p(J; L_p(\mathbb{R}^n))$ to $L_p(J; L_p(\mathbb{R}^n))$. This will be a consequence of the classical Mikhlin multiplier theorem and the Kalton-Weis Theorem 1.3.1. We recall the classical Mikhlin condition

(M) $$\max_{|\alpha|\le[n/2]+1} \sup_{\xi\in\mathbb{R}^n} |\xi|^{|\alpha|}|\partial_\xi^\alpha \hat{T}(\lambda,\xi)| < \infty,$$

where $\alpha \in \mathbb{N}_0^n$ is a multiindex and $[s]$ denotes the largest integer not exceeding $s \in \mathbb{R}$. Firstly we show that the symbol $\hat{T}(\lambda,\xi)$ is uniformly bounded for all $\lambda \in \Sigma_\phi$ and $\xi \in \mathbb{R}^n$, with $|\lambda| + |\xi| \neq 0$.

Consider the function $\tilde{m}(\lambda,\xi) := m(\lambda,\xi)/\lambda$ given by

$$\tilde{m}(\lambda,\xi) = 1 - (a|\xi)(c|\xi) + \beta(B\xi|\xi) - i(a+c|\xi) + \beta(B\xi|\xi)|\xi|^2/\lambda = z_1(\xi) + z_2(\lambda,\xi),$$

where $z_2 := \beta(B\xi|\xi)|\xi|^2/\lambda$. Let $\phi_j = \arg z_j$; then a short computation shows that

$$|z_1 + z_2| \ge C(\phi_1,\phi_2)(|z_1| + |z_2|),$$

provided that $|\phi_1 - \phi_2| < \pi$. Here

$$C(\phi_1,\phi_2) := \frac{1}{\sqrt{2}}\min\{1, (1+\cos(\phi_1-\phi_2))^{1/2}\}.$$

From (A) and the Cauchy-Schwarz inequality we obtain

$$\left|\frac{(a+c|\xi)}{1-(a|\xi)(c|\xi)+\beta(B\xi|\xi)}\right| \le C|a+c|\frac{|\xi|}{1+|\xi|^2} \le C|a+c| < \infty,$$

hence $|\phi_1| \le \sigma < \pi/2$ for all $\xi \in \mathbb{R}^n$. Since $|\phi_2| = |\arg\lambda| \le \phi$ we have

$$|\phi_1 - \phi_2| \le \sigma + \phi < \pi,$$

provided $\phi > \pi/2$ is sufficiently close to $\pi/2$ and this in turn yields together with (A)

$$|\tilde{m}(\lambda,\xi)| = |z_1 + z_2| \geq C(|z_1| + |z_2|) \geq C(1 + |\xi|^2 + |\xi|^4/|\lambda|)$$

or equivalently

$$|m(\lambda,\xi)| \geq C(|\lambda|(1 + |\xi|^2) + |\xi|^4), \tag{4.14}$$

hence $|\hat{T}(\lambda,\xi)| \leq C$ for all such λ and ξ from above.

In the next step we will verify (M) for $|\alpha| = 1$, uniformly in $\lambda \in \Sigma_\phi$. Observe that

$$\partial_{\xi_j}\hat{T}(\lambda,\xi) = \frac{m(\lambda,\xi)(2\lambda\xi_j + 4\xi_j|\xi|^2) - (\lambda(1 + |\xi|^2) + |\xi|^4)\partial_{\xi_j}m(\lambda,\xi)}{m(\lambda,\xi)^2}. \tag{4.15}$$

The derivative of $m(\lambda,\xi)$ is given by

$$\partial_{\xi_j}m(\lambda,\xi) = \lambda\Big(2\beta(B\xi|e_j) - i(a_j + c_j) - a_j(c|\xi) - c_j(a|\xi)\Big) + 2\beta\Big(\xi_j(B\xi|\xi) + (B\xi|e_j)|\xi|^2\Big),$$

and this yields

$$|\partial_{\xi_j}m(\lambda,\xi)| \leq C(|\lambda|(1 + |\xi|) + |\xi|^3).$$

Young's inequality implies furthermore that

$$|m(\lambda,\xi)| \leq C(|\lambda|(1 + |\xi|^2) + |\xi|^4)$$

and thus we obtain from (4.14) and (4.15) the estimate

$$|\partial_{\xi_j}\hat{T}(\lambda,\xi)| \leq C\frac{|\lambda|(1 + |\xi|) + |\xi|^3}{|\lambda|(1 + |\xi|^2) + |\xi|^4},$$

whence we see that

$$|\xi||\partial_{\xi_j}\hat{T}(\lambda,\xi)| \leq C < \infty,$$

for all $\lambda \in \Sigma_\phi$ and $\xi \in \mathbb{R}^n$, with $|\lambda| + |\xi| \neq 0$. Inductively it follows that (M) is fulfilled for each multiindex $\alpha \in \mathbb{N}_0^n$, uniformly in $\lambda \in \Sigma_\phi$. The classical Mikhlin multiplier theorem then implies that $\hat{T}$ is a Fourier multiplier in $L_p(\mathbb{R}^n)$ w.r.t the variable ξ and this yields a holomorphic uniformly bounded family $\{T(\lambda)\}_{\lambda\in\Sigma_\phi} \subset \mathcal{B}(L_p(\mathbb{R}^n))$, $\phi > \pi/2$. By [15, Theorem 3.2] this family is also $\mathcal{R}$-bounded in $L_p(J \times \mathbb{R}^n)$. Finally, since the operator ∂_t admits a bounded $\mathcal{H}^\infty$-calculus with angle $\pi/2$ we obtain from Theorem 1.3.1 that T is bounded in $L_p(J; L_p(\mathbb{R}^n))$. For the functions $u = (I - \Delta)^{1/2}w$ and $\mu = (I - \Delta)^{1/2}\eta$, this yields

$$u \in H_p^1(J; H_p^1(\mathbb{R}^n)) \cap L_p(J; H_p^3(\mathbb{R}^n)),$$

and

$$\mu \in L_p(J; H_p^2(\mathbb{R}^n)).$$

The proof is complete.

□

It is possible to extend Theorem 4.2.1 to the case of variable coefficients with a small deviation from constant ones. To prove this result we write the coefficients in the form

$$a(x) = a^0 + a^1(x), \quad c(x) = c^0 + c^1(x) \quad \text{and} \quad B(x) = B^0 + B^1(x),$$

where $|a^1|_{L_\infty(\mathbb{R}^n;\mathbb{R}^n)} + |c^1|_{L_\infty(\mathbb{R}^n;\mathbb{R}^n)} + |B^1|_{L_\infty(\mathbb{R}^n,\mathbb{R}^{n\times n})} \leq \omega$, with some constant $\omega > 0$ and $a^1, c^1 \in W_\infty^1(\mathbb{R}^n;\mathbb{R}^n)$, $B^1 \in W_\infty^1(\mathbb{R}^n;\mathbb{R}^{n\times n})$. Furthermore we require $\operatorname{div} a^1(x) = \operatorname{div} c^1(x) = 0$ for all $x \in \mathbb{R}^n$ and we assume that (β, a^0, c^0, B^0) satisfy condition (A). Then also $(\beta, a(x), c(x), B(x))$ satisfy (A) with a possibly smaller constant $\varepsilon > 0$, provided $\omega > 0$ is sufficiently small. Note that due to the uniform boundedness of the data, the norms of the solution operators are uniform as well. Therefore we may cut the interval $J = [0, T]$ into pieces $J_i = [i\delta, i\delta + \delta]$ for some small $\delta > 0$.

We then solve the problem successively on J_i. W.l.o.g. we may treat the first interval J_0. For this purpose we define the spaces Z^j_δ, X^j_δ, $j = 1, 2$ as the restriction of the spaces Z^j, X^j to the interval J_0. Note that w.l.o.g. we may assume $u_0 = 0$

Let S denote the solution operator for the constant coefficient case from Theorem 4.2.1 and denote by T that of the perturbed problem. Assume that we already know a solution to the perturbed problem. Then it is easy to verify the identity

$$T = S + SBT, \quad \text{where} \quad B\begin{bmatrix} u \\ \mu \end{bmatrix} = \begin{bmatrix} \operatorname{div}(a^1(x)\partial_t u) + \operatorname{div}(B^1(x)\nabla\mu) \\ c^1(x)\cdot\nabla\mu \end{bmatrix}, \tag{4.16}$$

and $(u, \mu) \in {}_0Z^1_\delta \times Z^2_\delta$ is the solution of the perturbed problem. From the assumption on the coefficients we obtain the following estimate

$$\begin{aligned} \left| B\begin{bmatrix} u \\ \mu \end{bmatrix} \right|_{X^1_\delta \times X^2_\delta} \leq C\Big(&|a^1|_\infty |u|_{Z^1_\delta} + (|B^1|_\infty + |c^1|_\infty)|\mu|_{Z^2_\delta} \\ &+ |\partial_t u|_{L_p(J_0\times\Omega)} + |\mu|_{L_p(J_0;H^1_p(\Omega))}\Big). \end{aligned} \tag{4.17}$$

The task is to estimate the terms $|\partial_t u|_{L_p(J_0\times\Omega)}$ and $|\mu|_{L_p(J_0;H^1_p(\Omega))}$, since they are not of lower order with respect to the variable t. To this end we consider the elliptic problem

$$\mu - (a + c)\cdot\nabla\mu + \operatorname{div}(a(c\cdot\nabla\mu)) - \operatorname{div}(\beta B\nabla\mu) = \operatorname{div}(a\Delta u) - \Delta u + \tilde f$$

which results, if we replace $\partial_t u$ in $(4.11)_1$ by the second equation in $(4.11)_2$, where

$$\tilde f := \beta f + a\cdot\nabla g - g \in L_p(J\times\mathbb{R}^n)$$

is a fixed function. For this elliptic problem we obtain the following a priori estimate.

Proposition 4.2.2. *There exists a constant $M > 0$ such that*

$$|\mu|_{L_p(J_0;H^1_p(\mathbb{R}^n))} + |\partial_t u|_{L_p(J_0;L_p(\mathbb{R}^n))} \leq M(|u|_{L_p(J_0;H^2_p(\mathbb{R}^n))} + |\tilde f|_{L_p(J_0;L_p(\mathbb{R}^n))} + |g|_{L_p(J_0;L_p(\mathbb{R}^n))}).$$

Proof. First we show that the L_p-realization A_0 of the differential operator

$$\mathcal{A}_0(D)w = c\cdot\nabla w + a\cdot\nabla w - \operatorname{div}(a(c\cdot\nabla w)) + \operatorname{div}(\beta B\nabla w)$$

with domain $D(A_0) = L_p(J_0; H^2_p(\mathbb{R}^n))$ is dissipative. To this end, we compute

$$\begin{aligned} &\operatorname{Re}\int_{\mathbb{R}^n} A_0 w\ \bar w|w|^{p-2}\,dx \\ &= \operatorname{Re}\left(\int_{\mathbb{R}^n}(c+a)\cdot\nabla w\ \bar w|w|^{p-2}\,dx + \int_{R^n}(\operatorname{div}(\beta B\nabla w) - \operatorname{div}(a(c\cdot\nabla w)))\bar w|w|^{p-2}\,dx\right) \\ &= \frac{1}{p}\int_{\mathbb{R}^n}(c+a)\nabla|w|^p\,dx - \operatorname{Re}\int_{\mathbb{R}^n}(\beta B\nabla w - a(c\cdot\nabla w))\cdot\nabla(\bar w|w|^{p-2})\,dx \\ &= -\int_{\mathbb{R}^n}|w|^{p-4}\operatorname{Re}\left(\frac{p}{2}(\tilde B\nabla w\cdot\nabla\bar w)|w|^2 + \left(\frac{p}{2}-1\right)(\tilde B\nabla w\cdot\nabla w)\bar w^2\right)\,dx \end{aligned}$$

for each $w \in H^2_p(\mathbb{R}^n)$, where $\tilde B := \beta B - \frac{1}{2}(a\otimes c + c\otimes a)$. Here we used integration by parts, and the fact that $\operatorname{div} a(x) = \operatorname{div} c(x) = 0$. To estimate the integral, we set $\nabla w = u + iv$ and $w = b_1 + ib_2$, with $u, v \in \mathbb{R}^n$ and $b_j \in \mathbb{R}$. This yields

$$\begin{aligned} &\operatorname{Re}\left(\frac{p}{2}(\tilde B\nabla w\cdot\nabla\bar w)|w|^2 + \left(\frac{p}{2}-1\right)(\tilde B\nabla w\cdot\nabla w)\bar w^2\right) \\ &= \frac{p}{2}(\tilde Bu|u)(b_1^2+b_2^2) + \frac{p}{2}(\tilde Bv|v)(b_1^2+b_2^2) + (\frac{p}{2}-1)(\tilde Bu|u)(b_1^2-b_2^2) \\ &\qquad - (\frac{p}{2}-1)(\tilde Bv|v)(b_1^2-b_2^2) + 4(\frac{p}{2}-1)(\tilde Bu|v)b_1b_2 \\ &= (p-1)(\tilde Bu|u)b_1^2 + (\tilde Bu|u)b_2^2 + (\tilde Bv|v)b_1^2 + (p-1)(\tilde Bv|v)b_2^2 + 2(p-2)(\tilde Bu|v)b_1b_2 \\ &= (p-1)\Big((\tilde Bu|u)b_1^2 + (\tilde Bv|v)b_2^2 + 2(\tilde Bu|v)b_1b_2\Big) + (\tilde Bu|u)b_2^2 + (\tilde Bv|v)b_1^2 - 2(\tilde Bu|v)b_1b_2. \end{aligned}$$

Thanks to the Cauchy-Schwarz inequality, Young's inequality and Assumption (A), it follows that $(\tilde{B}u|u)b_1^2 + (\tilde{B}v|v)b_2^2 + 2(\tilde{B}u|v)b_1b_2$ is nonnegative. To see this, we estimate as follows

$$|2(\tilde{B}u|v)b_1b_2| \le 2\sqrt{(\tilde{B}u|u)}\sqrt{(\tilde{B}v|v)}|b_1||b_2| \le b_1^2(\tilde{B}u|u) + b_2^2(\tilde{B}v|v).$$

By the same arguments, the term $(\tilde{B}u|u)b_2^2 + (\tilde{B}v|v)b_1^2 - 2(\tilde{B}u|v)b_1b_2$ is nonnegative too. This yields

$$\begin{aligned}\operatorname{Re}\Big(\frac{p}{2}(\tilde{B}\nabla w\cdot\nabla\bar{w})|w|^2 + \Big(\frac{p}{2}-1\Big)(\tilde{B}\nabla w\cdot\nabla w)\bar{w}^2\Big)&\\ \ge \min\{1,(p-1)\}\Big[((\tilde{B}u|u)+(\tilde{B}v|v))(b_1^2+b_2^2)\Big]&\\ \ge \varepsilon\min\{1,(p-1)\}(|u|^2+|v|^2)(b_1^2+b_2^2) = \varepsilon\min\{1,(p-1)\}|\nabla w|^2|w|^2,&\end{aligned}$$

by condition (A). This shows that A_0 is dissipative. Next we split the operator $\mathcal{A}_0(D) = \mathcal{A}_0^{\#}(D) + \mathcal{A}_0^{low}(D)$, with

$$\mathcal{A}_0^{\#}(D)w = \beta B : \nabla^2 w - (\nabla^2 w)c\cdot a,$$

and

$$\mathcal{A}_0^{low}(D)w = \beta\,\operatorname{Div} B\cdot\nabla w - \nabla c\nabla w\cdot a + (a+c)\cdot\nabla w,$$

where we used again the property $\operatorname{div} a(x) = 0$. Here $\operatorname{Div} A$ denotes the divergence of a matrix A, defined by

$$\operatorname{Div} A = \left(\sum_{j=1}^{n}\partial_j(a_{ij})\right)_{i=1,\dots,n} \in \mathbb{R}^n.$$

Furthermore we use the notation $B:\nabla^2 w = \sum_{i,j} b_{ij}\partial_i\partial_j w$. By condition (A) it is easily seen that the principal part $\mathcal{A}_0^{\#}(D)$ of $\mathcal{A}_0(D)$ is parameter elliptic in the sense of [12, Definition 5.1]. Note that the coefficients in the lower order terms are smooth. By [12, Theorem 5.7] there exists some $\lambda > 0$ such that $\lambda - A_0$ is $\mathcal{R}$-sectorial, hence also sectorial. This in turn yields that A_0 is the generator of a contraction semigroup in $L_p(J_0; L_p(\mathbb{R}^n))$, by the Lumer-Phillips Theorem and the dissipativity of A_0. In particular the operator $(I - A_0)$ is invertible. Consider the equation

$$\mu_1 - \mathcal{A}_0(D)\mu_1 = -\Delta u + \tilde{f}. \tag{4.18}$$

By the above considerations the solution $\mu_1 \in L_p(J_0; H_p^2(\mathbb{R}^n))$ of (4.18) is unique and satisfies the estimate $|\mu_1|_{L_p(J_0;H_p^2(\mathbb{R}^n))} \le C(|u|_{L_p(J_0;H_p^2(\mathbb{R}^n))} + |\tilde{f}|_{L_p(J_0;L_p(\mathbb{R}^n))})$ for some constant $C > 0$. Then the function $\mu_2 = \mu - \mu_1$ solves the equation

$$\mu_2 - \mathcal{A}_0(D)\mu_2 = \operatorname{div}(a\Delta u). \tag{4.19}$$

In a next step, we want to write $\mu_2 = \operatorname{div}\mu_3 + \mu_4$ for some suitable functions μ_3, μ_4. To this end, we consider firstly the following equations

$$\mu_3^j - \mathcal{A}_0(D)\mu_3^j = a_j\Delta u, \quad j\in\{1,...,n\}, \tag{4.20}$$

where a_j is the j^{th} component of the vector a. Each equation admits a unique solution $\mu_3^j \in L_p(J_0; H_p^2(\mathbb{R}^n))$ and we have the estimate

$$|\mu_3^j|_{L_p(J_0;H_p^2(\mathbb{R}^n))} \le C|u|_{L_p(J_0;H_p^2(\mathbb{R}^n))}$$

for each $j \in \{1,...,n\}$ and some constant $C > 0$ at our disposal. Setting $\mu_3 = [\mu_3^1,\dots,\mu_3^n]^{\mathsf{T}}$ and applying the divergence operator to the system of equations (4.20), we obtain

$$\operatorname{div}\mu_3 - \mathcal{A}_0(D)(\operatorname{div}\mu_3) = \operatorname{div}(a\Delta u) - [\mathcal{A}_0(D),\operatorname{div}]\mu_3,$$

where $[\mathcal{A}_0(D), \operatorname{div}]\mu_3$ denotes the commutator of $\mathcal{A}_0(D)$ and div, i.e.

$$[\mathcal{A}_0(D), \operatorname{div}]\mu_3 := \mathcal{A}_0(D)(\operatorname{div}\mu_3) - \operatorname{div}(\mathcal{A}_0(D)\mu_3),$$

which is in fact an operator of second order. Let μ_4 denote the unique solution of

$$\mu_4 - \mathcal{A}_0(D)\mu_4 = -[\mathcal{A}_0(D), \operatorname{div}]\mu_3,$$

with the estimate

$$|\mu_4|_{L_p(J_0;H_p^2(\mathbb{R}^n))} \leq C|[\mathcal{A}_0(D), \operatorname{div}]\mu_3|_{L_p(J_0;L_p(\mathbb{R}^n))} \leq C|u|_{L_p(J;H_p^2(\mathbb{R}^n))},$$

by the estimate for μ_3. Finally, by the uniqueness of the solution μ_2 of (4.19) we may conclude that $\mu_2 = \operatorname{div}\mu_3 + \mu_4$. This in turn yields the desired estimate for μ, since $\mu = \mu_1 + \operatorname{div}\mu_3 + \mu_4$. To estimate $\partial_t u$ in $L_p(J; L_p(\mathbb{R}^n))$, we make use of equation $(4.11)_2$. This completes the proof. $\square$

Now we go back to (4.17) to obtain the estimate

$$\left|B\begin{bmatrix}u\\ \mu\end{bmatrix}\right|_{X_\delta^1\times X_\delta^2} \leq C\Big(|a^1|_\infty|u|_{Z_\delta^1} + (|B^1|_\infty + |c^1|_\infty)|\mu|_{Z_\delta^2} + |u|_{L_p(J_0;H_p^2(\mathbb{R}^n))} + |f|_{L_p(J_0;L_p(\mathbb{R}^n))} + |g|_{L_p(J_0;H_p^1(\mathbb{R}^n))}\Big).$$

We use the mixed derivative theorem to obtain

$$Z_\delta^1 = H_p^1(J_0; H_p^1(\mathbb{R}^n)) \cap L_p(J_0; H_p^3(\mathbb{R}^n)) \hookrightarrow H_p^{1/2}(J_0; H_p^2(\mathbb{R}^n)) \hookrightarrow L_{2p}(J_0; H_p^2(\mathbb{R}^n)).$$

This in turn yields

$$|u|_{L_p(J_0;H_p^2(\mathbb{R}^n))} \leq \delta^{1/2p}|u|_{L_{2p}(J_0;H_p^2(\mathbb{R}^n))} \leq C\delta^{1/2p}|u|_{Z_\delta^1}.$$

If we choose $\delta > 0$ and $\omega > 0$ small enough, we obtain from (4.16) the estimate

$$|(u,\mu)|_{Z_\delta^1\times Z_\delta^2} \leq M(|f|_{X_\delta^1} + |g|_{X_\delta^2} + |u_0|_{X_p}),$$

for the solution of the perturbed problem. Therefore the operator $L \in \mathcal{B}(Z_\delta^1 \times Z_\delta^1; X_\delta^1 \times X_\delta^2 \times X_p)$ which is defined by the first lines of the left hand side of (4.11) is injective and has closed range, i.e. it is a semi Fredholm operator. To show surjectivity of L we apply a continuation argument for semi Fredholm operators, which is due to Kato [21]. Let L_τ be the corresponding operator to (4.11) with data

$$(\beta_\tau, a_\tau, c_\tau, B_\tau) := (1-\tau)(\beta, a^0, c^0, B^0) + \tau(\beta, a, c, B), \quad \tau \in [0,1].$$

By Theorem 4.2.1 the operator L_0 is bijective, since the data (β, a^0, c^0, B^0) satisfy Assumption (A). Furthermore, the data $(\beta_\tau, a_\tau, c_\tau, B_\tau)$ satisfy (A) too, by the smallness of $\omega > 0$. It is also clear that a_τ and c_τ are divergence free vector fields and (a_τ, c_τ, B_τ) enjoy the same regularity as (a, c, B). Hence each operator L_τ is injective and has closed range, by the above calculation. Finally, the continuity property of the Fredholm index yields that the index of L_τ is zero for each $\tau \in [0,1]$. This proves that $L_1 = L$ is also surjective. Therefore we have the following result.

Corollary 4.2.3. *Let $a_1, c_1 \in W_\infty^1(\mathbb{R}^n; \mathbb{R}^n)$ and $B_1 \in W_\infty^1(\mathbb{R}^n; \mathbb{R}^{n\times n})$ with $\operatorname{div} a_1(x) = \operatorname{div} c_1(x) = 0$ for all $x \in \mathbb{R}^n$. Then Theorem 4.2.1 remains valid in case of variable coefficients*

$$a(x) = a^0 + a^1(x), \quad c(x) = c^0 + c^1(x) \quad \text{and} \quad B(x) = B^0 + B^1(x),$$

provided that (β, a^0, c^0, B^0) satisfy (A) and

$$|a^1|_{L_\infty(\mathbb{R}^n;\mathbb{R}^n)} + |c^1|_{L_\infty(\mathbb{R}^n;\mathbb{R}^n)} + |B^1|_{L_\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})} \leq \omega,$$

with $\omega > 0$ being sufficiently small.

4.3 The Linear Cahn-Hilliard-Gurtin Problem in $\mathbb{R}^n_+$

Set $x = (x', y) \in \mathbb{R}^{n-1} \times \mathbb{R}_+$ and consider the half space problem

$$\begin{aligned}
\partial_t u - \operatorname{div}(a\partial_t u) &= \operatorname{div}(B\nabla\mu) + f, \quad t \in J,\ x' \in \mathbb{R}^{n-1},\ y > 0,\\
\mu - c\cdot\nabla\mu &= \beta\partial_t u - \Delta u + g, \quad t \in J,\ x' \in \mathbb{R}^{n-1},\ y > 0,\\
B\nabla\mu\cdot\nu &= h_1, \quad t \in J,\ x' \in \mathbb{R}^{n-1},\ y = 0,\\
\partial_y u &= h_2, \quad t \in J,\ x' \in \mathbb{R}^{n-1},\ y = 0,\\
u(0) &= u_0, \quad t = 0,\ x' \in \mathbb{R}^{n-1},\ y > 0,
\end{aligned} \tag{4.21}$$

where $J = [0, T]$, ν is the outer unit normal at $x \in \partial\mathbb{R}^n_+$, i.e. $\nu = [0, \dots, 0, -1]^{\mathsf{T}}$, and the data (β, a, c, B) are subject to Assumption (A). Due to the conditions (4.9) and (4.10) it holds that $a = (a_0, 0) \in \mathbb{R}^{n-1} \times \mathbb{R}$, $c = (c_0, 0) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and

$$B = \begin{bmatrix} B_0 & 0 \\ 0 & B_2 \end{bmatrix},$$

where $B_0 \in \mathbb{R}^{(n-1)\times(n-1)}$ is symmetric and $B_2 \in \mathbb{R}$. We assume that B is positive definite, hence $B_2 > 0$.

The main result on optimal regularity of (4.21) reads as follows.

Theorem 4.3.1. *Let $1 < p < \infty$ and assume that (A) and (4.9) and (4.10) hold true. Then (4.21) admits a unique solution*

$$u \in H^1_p(J; H^1_p(\mathbb{R}^n_+)) \cap L_p(J; H^3_p(\mathbb{R}^n_+)) =: Z^1,$$

$$\mu \in L_p(J; H^2_p(\mathbb{R}^n_+)) =: Z^2,$$

if and only if the data is subject to the following conditions.

(i) $f \in L_p(J; L_p(\mathbb{R}^n_+)) =: X^1$,

(ii) $g \in L_p(J; H^1_p(\mathbb{R}^n_+)) =: X^2$,

(iii) $h_1 \in L_p(J; W^{1-1/p}_p(\mathbb{R}^{n-1})) =: Y^1$,

(iv) $h_2 \in W^{1-1/2p}_p(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; W^{2-1/p}_p(\mathbb{R}^{n-1})) =: Y^2$,

(v) $u_0 \in B^{3-2/p}_{pp}(\mathbb{R}^n) =: X_p$.

(vi) $\partial_y u_0 = h_2|_{t=0}$ *if* $p > 3/2$.

Proof. The necessity part follows from the equations and trace theory, cf. Theorem 1.4.3, so we can turn to the sufficiency part. We want to remark that due to the structure of the matrix B, the boundary condition $B\nabla\mu\cdot\nu|_{y=0}$ becomes

$$B\nabla\mu\cdot\nu|_{y=0} = B_2\partial_y\mu|_{y=0},$$

with $B_2 > 0$ since B is assumed to be positive definite. Now we want to set $h_1 = h_2 = u_0 = 0$. For this purpose we first solve the elliptic problem

$$\begin{aligned}
(I - \Delta_{x'})\eta - \partial_y^2\eta &= 0, \quad x' \in \mathbb{R}^{n-1},\ y > 0,\\
\partial_y\eta &= h_1/B_2, \quad x' \in \mathbb{R}^{n-1},\ y = 0.
\end{aligned} \tag{4.22}$$

Define $\tilde{L} := (I - \Delta_{x'})^{1/2}$ in $L_p(\mathbb{R}^{n-1})$, with $D(\tilde{L}) = H^1_p(\mathbb{R}^{n-1})$ and let L denote the natural extension of $\tilde{L}$ to $L_p(J; L_p(\mathbb{R}^{n-1}))$, that is $D(L) = L_p(J; H^1_p(\mathbb{R}^{n-1}))$ and $Lu = \tilde{L}u$ for each $u \in D(L)$. Then the unique solution η of (4.22) is given by

$$\eta(y) = -L^{-1}e^{-Ly}(h_1/B_2).$$

Since B_2 is constant, it holds that $h_1/B_2 \in L_p(J; W_p^{1-1/p}(\mathbb{R}^{n-1})) = D_L(1-1/p,p)$, hence $e^{-Ly}(h_1/B_2) \in D(L)$ and therefore $\eta \in L_p(J; H_p^2(\mathbb{R}^n_+))$, with $B_2\partial_y\eta|_{y=0} = h_1$. In order to remove h_2 and u_0, we solve the initial boundary value problem

$$\begin{aligned} \beta\partial_t v - \Delta_{x'} v - \partial_y^2 v = 0, &\quad t \in J,\ x' \in \mathbb{R}^{n-1},\ y > 0, \\ \partial_y v = h_2, &\quad t \in J,\ x' \in \mathbb{R}^{n-1},\ y = 0, \\ v(0) = u_0, &\quad t = 0,\ x' \in \mathbb{R}^{n-1},\ y > 0. \end{aligned} \tag{4.23}$$

To this end we extend $u_0 \in B_{pp}^{3-2/p}(\mathbb{R}^n_+)$ to a function $\tilde{u}_0 \in B_{pp}^{3-2/p}(\mathbb{R}^n)$ and solve the heat equation

$$\beta\partial_t\tilde{v} - \Delta\tilde{v} = 0,\ t \in J,\ x \in \mathbb{R}^n, \quad \tilde{v}(0) = \tilde{u}_0,\ t = 0,\ x \in \mathbb{R}^n,$$

in $L_p(J; H_p^1(\mathbb{R}^n))$. This yields a solution

$$\tilde{v} \in H_p^1(J; H_p^1(\mathbb{R}^n)) \cap L_p(J; H_p^3(\mathbb{R}^n)).$$

If $v_1 := P\tilde{v}$ denotes the restriction of $\tilde{v}$ to the half space $\mathbb{R}^n_+$, the function $v_2 := v - v_1$ should solve the initial boundary value problem

$$\begin{aligned} \beta\partial_t v_2 - \Delta_{x'} v_2 - \partial_y^2 v_2 = 0, &\quad t \in J,\ x' \in \mathbb{R}^{n-1},\ y > 0, \\ \partial_y v_2 = \bar{h}_2, &\quad t \in J,\ x' \in \mathbb{R}^{n-1},\ y = 0, \\ v_2(0) = 0, &\quad t = 0,\ x' \in \mathbb{R}^{n-1},\ y > 0, \end{aligned} \tag{4.24}$$

where $\bar{h}_2 := h_2 - \partial_y v_1|_{y=0}$. Set $v_3 = (I - \Delta_{x'})^{1/2} v_2$. Then v_3 is a solution of

$$\begin{aligned} \beta\partial_t v_3 - \Delta_{x'} v_3 - \partial_y^2 v_3 = 0, &\quad t \in J,\ x' \in \mathbb{R}^{n-1},\ y > 0, \\ \partial_y v_3 = h_3, &\quad t \in J,\ x' \in \mathbb{R}^{n-1},\ y = 0, \\ v_3(0) = 0, &\quad t = 0,\ x' \in \mathbb{R}^{n-1},\ y > 0. \end{aligned} \tag{4.25}$$

with $h_3 = (I - \Delta_{x'})^{1/2}\bar{h}_2 \in {}_0W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^{n-1}))$. We define $L = (\beta\partial_t - \Delta_{x'})^{1/2}$ with natural domain

$$D(L) = {}_0H_p^{1/2}(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; H_p^1(\mathbb{R}^{n-1}_+)).$$

Then, the unique solution v_3 of (4.25) is given by

$$v_3(y) = -L^{-1}e^{-Ly}h_3,$$

and $h_3 \in D_L(1-1/p,p)$. This yields

$$v_3 \in {}_0H_p^1(J; L_p(\mathbb{R}^n_+)) \cap L_p(J; H_p^2(\mathbb{R}^n_+)).$$

On the other hand, if we consider the function $v_4 := \partial_y v_2$ as the solution of

$$\begin{aligned} \beta\partial_t v_4 - \Delta_{x'} v_4 - \partial_y^2 v_4 = 0, &\quad t \in J,\ x' \in \mathbb{R}^{n-1},\ y > 0, \\ v_4 = \bar{h}_2, &\quad t \in J,\ x' \in \mathbb{R}^{n-1},\ y = 0, \\ v_4(0) = 0, &\quad t = 0,\ x' \in \mathbb{R}^{n-1},\ y > 0, \end{aligned} \tag{4.26}$$

we obtain $v_4(y) = e^{-Ly}\bar{h}_2$ and $\bar{h}_2 \in D_L(2-1/p,p)$. This yields

$$v_4 \in {}_0H_p^1(J; L_p(\mathbb{R}^n_+)) \cap L_p(J; H_p^2(\mathbb{R}^n_+)).$$

From the regularity of v_3 and v_4 we may conclude that

$$v_2 \in {}_0H_p^1(J; H_p^1(\mathbb{R}^n_+)) \cap L_p(J; H_p^3(\mathbb{R}^n_+)).$$

Now the functions $u_1 := u - v$ and $\mu_1 := \mu - \eta$, with $v = v_1 + v_2$, should solve the system

$$\begin{aligned} \partial_t u_1 - \operatorname{div}(a\partial_t u_1) = \operatorname{div}(B\nabla\mu_1) + f_1, \quad t \in J,\ x' \in \mathbb{R}^{n-1},\ y > 0, \\ \mu_1 - c\cdot\nabla\mu_1 = \beta\partial_t u_1 - \Delta u_1 + g_1, \quad t \in J,\ x' \in \mathbb{R}^{n-1},\ y > 0, \\ B_2\partial_y\mu_1 = 0, \quad t \in J,\ x' \in \mathbb{R}^{n-1},\ y = 0, \\ \partial_y u_1 = 0, \quad t \in J,\ x' \in \mathbb{R}^{n-1},\ y = 0, \\ u_1(0) = 0, \quad t = 0,\ x' \in \mathbb{R}^{n-1},\ y > 0, \end{aligned} \tag{4.27}$$

with some modified data $f_1 \in X^1$ and $g_1 \in X^2$. In a next step we extend the functions f_1 and g_1 to $J \times \mathbb{R}^n$ by even reflection, i.e. we set

$$f_2(t,x',y) = \begin{cases} f_1(t,x',y), & \text{if } y \geq 0 \\ f_1(t,x',-y), & \text{if } y \leq 0 \end{cases} \quad \text{and} \quad g_2(t,x',y) = \begin{cases} g_1(t,x',y), & \text{if } y \geq 0 \\ g_1(t,x',-y), & \text{if } y \leq 0 \end{cases}.$$

Thanks to Theorem 4.2.1 we can solve the full space problem

$$\begin{aligned} \partial_t u_2 - \operatorname{div}(a\partial_t u_2) = \operatorname{div}(B\nabla\mu_2) + f_2, \quad t \in J,\ x \in \mathbb{R}^n, \\ \mu_2 - c\cdot\nabla\mu_2 = \beta\partial_t u_2 - \Delta u_2 + g_2, \quad t \in J,\ x \in \mathbb{R}^n, \\ u_2(0) = 0, \quad t = 0,\ x \in \mathbb{R}^n, \end{aligned} \tag{4.28}$$

since $f_2 \in L_p(J \times \mathbb{R}^n)$ and $g_2 \in L_p(J; H^1_p(\mathbb{R}^n))$. This yields a unique solution

$$u_2 \in H^1_p(J; H^1_p(\mathbb{R}^n)) \cap L_p(J; H^3_p(\mathbb{R}^n)) \quad \text{and} \quad \mu_2 \in L_p(J; H^2_p(\mathbb{R}^n)),$$

by Theorem 4.2.1. At this point we emphasize that the equations $(4.27)_{1,2}$ are invariant w.r.t. even reflection in the variable y, since $a_1 = c_1 = 0$ and $B_1 = 0$. This in turn implies that the solution (u_2, μ_2) is symmetric, w.r.t the variable y and this yields necessarily, $\partial_y u_2|_{y=0} = \partial_y \mu_2|_{y=0} = 0$. Denoting by P the restriction of the solution (u_2, μ_2) to the half space $\mathbb{R}^n_+$, it follows that $(u_1, \mu_1) = P(u_2, \mu_2)$ is the unique solution of (4.27) and therefore $u = v + u_1$ and $\mu = \eta + \mu_1$ is the unique solution of (4.21). The proof is complete.

□

As in Section 4.2, we may extend Theorem 4.3.1 to the case of variable coefficients with a small deviation from constant ones. The arguments are similar to those in the proof of Corollary 4.2.3. Indeed it suffices to show that there is a version of Proposition 4.2.2 for the half space case. Assume that we have given coefficients

$$a(x) = a^0 + a^1(x), \quad c(x) = c^0 + c^1(x) \quad \text{and} \quad B(x) = B^0 + B^1(x),$$

where $|a^1|_{L_\infty(\mathbb{R}^n_+;\mathbb{R}^n)} + |c^1|_{L_\infty(\mathbb{R}^n_+;\mathbb{R}^n)} + |B^1|_{L_\infty(\mathbb{R}^n_+,\mathbb{R}^{n\times n})} \leq \omega$, with some constant $\omega > 0$ and $a^1, c^1 \in W^1_\infty(\mathbb{R}^n_+;\mathbb{R}^n)$, $B^1 \in W^1_\infty(\mathbb{R}^n_+;\mathbb{R}^{n\times n})$. Furthermore we require $\operatorname{div} a^1(x) = \operatorname{div} c^1(x) = 0$ for all $x \in \mathbb{R}^n_+$,

$$(a^0|e_n) = (a^1(x)|e_n) = (c^0|e_n) = (c^1(x)|e_n) = 0$$

for all $x \in \partial\mathbb{R}^n_+$, and we assume that (β, a^0, c^0, B^0) satisfy condition (A). Here $e_n = [0,\ldots,0,-1]^\mathsf{T}$. Finally, let B^0 satisfy (4.10). Extending the data to the whole of $\mathbb{R}^n$ we may w.l.o.g. assume that $f = g = u_0 = 0$. Then we have the following result.

Proposition 4.3.2. *There exists a constant $M > 0$ such that*

$$|\mu|_{L_p(J_0;H^1_p(\mathbb{R}^n_+))} + |\partial_t u|_{L_p(J_0;L_p(\mathbb{R}^n_+))} \leq M(|u|_{L_p(J_0;H^2_p(\mathbb{R}^n_+))} + |h_1|_{L_p(J_0;W^{1-1/p}_p(\mathbb{R}^{n-1}))}).$$

Proof. Extending the data f, g and u_0 to the whole of $\mathbb{R}^n$ and solving the full space problem with Corollary 4.2.3 we may assume that $f = g = u_0 = 0$. The corresponding elliptic boundary value problem for μ reads

$$\mu - c\cdot\nabla\mu - a\cdot\nabla\mu + \operatorname{div}(a(c\cdot\nabla\mu)) - \operatorname{div}(\beta B\nabla\mu) = \operatorname{div}(a\Delta u) - \Delta u, \quad B\nabla\mu\cdot\nabla\nu = h_1. \tag{4.29}$$

It is not difficult to show that the L_p-realization A_0 of the differential operator

$$\mathcal{A}_0(D)w = c \cdot \nabla w + a \cdot \nabla w - \operatorname{div}(a(c \cdot \nabla w)) + \operatorname{div}(\beta B \nabla w)$$

with domain

$$D(A_0) = \{v \in L_p(J_0; H^2_p(\mathbb{R}^n_+)) : B\nabla v \cdot \nu = 0\},$$

is dissipative in $L_p(J_0; L_p(\mathbb{R}^n_+))$. In fact, we may exactly follow the lines of the proof of Proposition 4.2.2 since there appear no boundary terms. This is due to the boundary conditions $B\nabla v \cdot \nu = 0,\ v \in D(A_0)$, and (4.9). To prove invertibility of $I - A_0$ we use following identity.

$$\begin{aligned}\operatorname{div}(a(c \cdot \nabla\mu)) &= a \cdot \nabla(c \cdot \nabla\mu) = a \cdot (\nabla c \nabla\mu) + a \cdot (\nabla^2\mu c)\\ &= a \cdot (\nabla c\nabla\mu) + \frac{1}{2}(a \otimes c + c \otimes a)\nabla^2\mu\\ &= a \cdot (\nabla c\nabla\mu) - \frac{1}{2}[\operatorname{Div}(a \otimes c + c \otimes a)] \cdot \nabla\mu + \frac{1}{2}\operatorname{div}[(a \otimes c + c \otimes a)\nabla\mu].\end{aligned}$$

Owing to this identity, we may write $\mathcal{A}_0(D) = \mathcal{A}_1(D) + \mathcal{A}_1^{low}(D)$, with

$$\mathcal{A}_1(D)\mu = \operatorname{div}(\tilde{B}\nabla\mu),$$

and

$$\mathcal{A}_1^{low}(D)\mu = \frac{1}{2}[\operatorname{Div}(a \otimes c + c \otimes a)] \cdot \nabla\mu - a \cdot (\nabla c\nabla\mu) + (a + c) \cdot \nabla\mu.$$

Here the matrix $\tilde{B}$ is given by $\beta B - \frac{1}{2}(a \otimes c + c \otimes a)$. Observe that

$$\tilde{B}\nabla\mu \cdot \nu = \nabla\mu \cdot \tilde{B}\nu = \nabla\mu \cdot B\nu = B\nabla\mu \cdot \nu = h_1,$$

by the assumption (4.9) on the vector fields a, c and since the matrices B and $\tilde{B}$ are symmetric. Consider the linear elliptic problem with a *conormal* boundary condition

$$\begin{aligned} w - \operatorname{div}(\tilde{B}\nabla w) = f, &\quad (x', y) \in \mathbb{R}^{n-1} \times \mathbb{R}_+,\\ \tilde{B}\nabla w \cdot \nu = g, &\quad x' \in \mathbb{R}^{n-1},\ y = 0.\end{aligned} \tag{4.30}$$

Elliptic problems of this type have been extensively studied in the literature and it is well-known that (4.30) admits a unique solution $w \in H^2_p(\mathbb{R}^n_+)$ if and only if $(f, g) \in L_p(\mathbb{R}^n_+) \times W^{1-1/p}_p(\mathbb{R}^{n-1})$. In addition, there exists a constant $M > 0$ such that the estimate

$$|w|_{L_p(J_0;H^2_p(\mathbb{R}^n_+))} \le M(|f|_{L_p(J_0;L_p(\mathbb{R}^n_+))} + |g|_{L_p(J_0;W^{1-1/p}_p(\mathbb{R}^{n-1}))})$$

holds, i.e. we have maximal regularity of type L_p for (4.30). Then, by perturbation theory, there exists $\lambda_0 \ge 0$ such that

$$\begin{aligned} (1 + \lambda_0)w - \mathcal{A}_1^{low}(D)w - \operatorname{div}(\tilde{B}\nabla w) = f, &\quad (x', y) \in \mathbb{R}^{n-1} \times \mathbb{R}_+,\\ \tilde{B}\nabla w \cdot \nu = g, &\quad x' \in \mathbb{R}^{n-1},\ y = 0,\end{aligned} \tag{4.31}$$

has a unique solution $w \in H^2_p(\mathbb{R}^n_+)$ if and only if $(f, g) \in L_p(\mathbb{R}^n_+) \times W^{1-1/p}_p(\mathbb{R}^{n-1})$. Setting $g = 0$, this yields the invertibility of the operator $I - A_0$, since A_0 is dissipative. Here we used the canonical extension of the differential operators from the basic space $L_p(\mathbb{R}^n_+)$ to $L_p(J_0; L_p(\mathbb{R}^n_+))$.

After these considerations we go back to (4.29). For $h_1 \in L_p(J_0; W^{1-1/p}_p(\mathbb{R}^{n-1}))$, let $\mu_1 \in L_p(J_0; H^2_p(\mathbb{R}^n_+))$ be the unique solution of the boundary value problem

$$\begin{aligned} \mu_1 - \operatorname{div}(B\nabla\mu_1) = 0, &\quad x' \in \mathbb{R}^{n-1},\ y > 0,\\ B\nabla\mu_1 \cdot \nu = h_1, &\quad x' \in \mathbb{R}^{n-1},\ y = 0,\end{aligned}$$

with the estimate

$$|\mu_1|_{L_p(J_0;H^2_p(\mathbb{R}^n_+))} \le C|h_1|_{L_p(J_0;W^{1-1/p}_p(\mathbb{R}^{n-1}))},$$

for some constant $C > 0$. Then the function $\mu_2 := \mu - \mu_1$ is the unique solution of

$$\mu_2 - \mathcal{A}_0(D)\mu_2 = \operatorname{div}(a\Delta u) - \Delta u - (I - \mathcal{A}_0(D))\mu_1, \quad B\nabla\mu_2 \cdot \nu = 0.$$

Define the function $\mu_3 \in L_p(J; H^2_p(\mathbb{R}^n_+))$ to be the unique solution of

$$\mu_3 - \mathcal{A}_0(D)\mu_3 = -\Delta u - (I - \mathcal{A}_0(D))\mu_1, \quad B\nabla\mu_3 \cdot \nu = 0,$$

subject to the estimate

$$|\mu_3|_{L_p(J_0;H^2_p(\mathbb{R}^n_+))} \le C(|u|_{L_p(J_0;H^2_p(\mathbb{R}^n_+))} + |h_1|_{L_p(J_0;W^{1-1/p}_p(\mathbb{R}^{n-1}))}),$$

for some constant $C > 0$. Then the new function $\mu_4 := \mu_2 - \mu_3$ solves

$$\mu_4 - \mathcal{A}_0(D)\mu_4 = \operatorname{div}(a\Delta u), \quad B\nabla\mu_4 \cdot \nu = 0.$$

Define the boundary operator $\mathcal{B}(D)$ by

$$\mathcal{B}(D)v = B\nabla v \cdot \nu.$$

Then we solve the system of equations

$$\mu^j_5 - \mathcal{A}_0(D)\mu^j_5 = a_j\Delta u, \quad \mathcal{B}(D)\mu^j_5 = 0, \tag{4.32}$$

for each $j = 1, \dots, n$ to obtain solutions $\mu^j_5 \in L_p(J_0; H^2_p(\mathbb{R}^n_+))$ with the estimate

$$|\mu^j_5|_{L_p(J_0;H^2_p(\mathbb{R}^n_+))} \le C_j|u|_{L_p(J_0;H^2_p(\mathbb{R}^n_+))},$$

with some constants $C_j > 0$. Applying the divergence operator to the system (4.32) yields

$$\operatorname{div}\mu_5 - \mathcal{A}_0(D)(\operatorname{div}\mu_5) = \operatorname{div}(a\Delta u) + [\mathcal{A}_0(D), \operatorname{div}]\mu_5, \quad \mathcal{B}(D)(\operatorname{div}\mu_5) = [\mathcal{B}(D), \operatorname{div}]\mu_5, \tag{4.33}$$

where $[\mathcal{A}_0(D), \operatorname{div}]$ and $[\mathcal{B}(D), \operatorname{div}]$ denote the commutators of div and $\mathcal{A}(D)$ or $\mathcal{B}(D)$, respectively. Observe that the estimates

$$|[\mathcal{A}_0(D), \operatorname{div}]\mu_5|_{L_p(J_0;L_p(\mathbb{R}^n_+))} \le C_1|\mu_5|_{L_p(J_0;H^1_p(\mathbb{R}^n_+))}$$

and

$$|[\mathcal{B}(D), \operatorname{div}]\mu_5|_{L_p(J_0;W^{1-1/p}_p(\mathbb{R}^{n-1}))} \le C_2|\mu_5|_{L_p(J_0;H^1_p(\mathbb{R}^n_+))},$$

for some constants $C_1, C_2 > 0$ hold. Hence we may conclude that there exists a function $\mu_6 \in L_p(J_0; H^2_p(\mathbb{R}^n_+))$ such that $\mu_4 = \operatorname{div}\mu_5 + \mu_6$ and μ_6 satisfies the estimate

$$|\mu_6|_{L_p(J_0;H^2_p(\mathbb{R}^n_+))} \le C|u|_{L_p(J_0;H^2_p(\mathbb{R}^n_+))}.$$

This implies

$$|\mu_4|_{L_p(J_0;H^1_p(\mathbb{R}^n_+))} \le C|u|_{L_p(J_0;H^2_p(\mathbb{R}^n_+))},$$

with some constant $C > 0$. Since $\mu = \mu_1 + \mu_3 + \mu_4$ this yields a constant $M > 0$ such that

$$|\mu|_{L_p(J_0;H^1_p(\mathbb{R}^n_+))} \le M(|u|_{L_p(J_0;H^2_p(\mathbb{R}^n_+))} + |h_1|_{L_p(J_0;W^{1-1/p}_p(\mathbb{R}^{n-1}))}).$$

The desired estimate for $\partial_t u$ follows from $(4.34)_2$. The proof is complete.

□

The continuation argument in the proof of Corollary 4.2.3 yields the following result.

Corollary 4.3.3. *Let $a_1, c_1 \in W^1_\infty(\mathbb{R}^n_+; \mathbb{R}^n)$ and $B_1 \in W^1_\infty(\mathbb{R}^n_+; \mathbb{R}^{n\times n})$ with $\operatorname{div} a_1(x) = \operatorname{div} c_1(x) = 0$ for all $x \in \mathbb{R}^n_+$. Then Theorem 4.3.1 remains valid in case of variable coefficients*

$$a(x) = a^0 + a^1(x), \quad c(x) = c^0 + c^1(x) \quad \textit{and} \quad B(x) = B^0 + B^1(x),$$

provided that (β, a^0, c^0, B^0) satisfy (A), (4.10),

$$(a^0|e_n) = (a^1(x)|e_n) = (c^0|e_n) = (c^1(x)|e_n) = 0, \quad x \in \partial\mathbb{R}^n_+,$$

and

$$|a^1|_{L_\infty(\mathbb{R}^n_+;\mathbb{R}^n)} + |c^1|_{L_\infty(\mathbb{R}^n_+;\mathbb{R}^n)} + |B^1|_{L_\infty(\mathbb{R}^n_+;\mathbb{R}^{n\times n})} \le \omega,$$

with $\omega > 0$ being sufficiently small.

4.4 Localization

In this section we prove the well-posedness of the system

$$\begin{aligned} \partial_t u - \operatorname{div}(a\partial_t u) &= \operatorname{div}(B\nabla\mu) + f, \quad t > 0,\ x \in \Omega,\\ \mu - c\cdot\nabla\mu &= \beta\partial_t u - \Delta u + g, \quad t > 0,\ x \in \Omega,\\ B\nabla\mu\cdot\nu &= h_1, \quad t > 0,\ x \in \Gamma,\\ \partial_\nu u &= h_2, \quad t > 0,\ x \in \Gamma,\\ u(0) &= u_0, \quad t = 0,\ x \in \Omega, \end{aligned} \tag{4.34}$$

where $\Omega \subset \mathbb{R}^n$ is a domain, with compact boundary $\Gamma := \partial\Omega \in C^3$ and $\nu = \nu(x)$ is the outer unit normal in a point $x \in \Gamma$. We assume that the data a, c and B enjoy the regularity $a, c \in C^1_{ub}(\overline{\Omega}; \mathbb{R}^n)$ and $B \in C^1_{ub}(\overline{\Omega}; \mathbb{R}^{n\times n})$. Suppose furthermore that the data $(\beta, a(x), c(x), B(x))$ are subject to Assumption (A) for every $x \in \overline{\Omega}$ and satisfy the conditions (4.9) and (4.10).

Let us recall some general properties of variable transformations. Suppose $\Omega \subset \mathbb{R}^n$ is a domain with compact C^m-boundary Γ, $m \in \mathbb{N}$ and let $x_0 \in \Gamma$. Without loss of generality, we may assume that $x_0 = 0$ and that $\nu(x_0) = [0, \ldots, 0, -1] \in \mathbb{R}^n$. This can always been achieved by a composition of a translation and a rotation in $\mathbb{R}^n$. We remark that such affine mappings of $\mathbb{R}^n$ onto itself leave invariant all function spaces under consideration. They also preserve ellipticity, i.e. (A) and the conditions (4.9)-(4.10). By definition of a C^m-boundary, there exists an open neighborhood $U = U_1 \times U_2 \subset \mathbb{R}^n$ of x_0 with $U_1 \subset \mathbb{R}^{n-1}$ and $U_2 \subset \mathbb{R}$ as well as a function $\rho \in C^m(\overline{U_1})$ such that

$$\begin{aligned} \Gamma \cap U &= \{x = (x', x_n) \in U : x_n = \rho(x')\},\\ \Omega \cap U &= \{x = (x', x_n) \in U : x_n > \rho(x')\}. \end{aligned}$$

Define $g : \overline{U} \to \mathbb{R}^n$ by

$$g_k(x) = x'_k, \text{ if } k = 1, \ldots, n-1 \quad \text{and} \quad g_n(x) = x_n - \rho(x'). \tag{4.35}$$

Clearly, $g \in C^m(\overline{U}; \mathbb{R}^n)$ is one-to-one and satisfies $\Omega \cap U = \{x \in U : g_n(x) > 0\}$ as well as $\Gamma \cap U = \{x \in U : g_n(x) = 0\}$. By extending ρ to a function $\tilde{\rho} \in C^m(\mathbb{R}^{n-1})$ with compact support and defining $\tilde{g}$ by (4.35), with ρ replaced by $\tilde{\rho}$, we get a C^m-diffeomorphism $\tilde{g}$ of $\mathbb{R}^n$ onto itself, extending g and satisfying $\tilde{g}(x) = x$ for sufficiently large $|x|$. Also $\tilde{g}$ is a C^m-diffeomorphic mapping from $\Omega_0 := \{x \in \mathbb{R}^n : x_n > \tilde{\rho}(x')\}$ onto $\mathbb{R}^n_+$. For the Jacobian $D\tilde{g}(x)$, one obtains

$$D\tilde{g}(x) = \begin{bmatrix} E_{n-1} & 0 \\ -\nabla_{x'}\tilde{\rho}(x') & 1 \end{bmatrix},\ x \in \mathbb{R}^n,$$

which entails $\det D\tilde{g}(x) = 1$ for all $x \in \mathbb{R}^n$ and $D\tilde{g}(0) = E_n$. Given a function $v \in H^m_p(\mathbb{R}^n_+)$, we define the pull back Θv on Ω_0 by $\Theta v(x) = v(\tilde{g}(x))$. Since $\det D\tilde{g} \equiv 1$ and the derivatives of $\tilde{g}$ and $\tilde{g}^{-1}$ up to order m are bounded, the transformation formula for the Lebesgue integral shows that Θ induces isomorphisms $\Theta^{(p)} : H^k_p(\mathbb{R}^n_+) \to H^k_p(\Omega_0)$ for each $p \in (1, \infty)$ and $k \in \{0, \ldots, m\}$.
We are going to prove the following

Theorem 4.4.1. *Let $1 < p < \infty$, $J = [0, T]$ and assume that (A),* (4.9) *and* (4.10) *hold. Suppose furthermore that $a, c \in C^1_{ub}(\Omega; \mathbb{R}^n)$ and $B \in C^1_{ub}(\Omega; \mathbb{R}^{n\times n})$. Then* (4.34) *admits a unique solution*

$$u \in H^1_p(J; H^1_p(\Omega)) \cap L_p(J; H^3_p(\Omega)) = Z^1, \quad \mu \in L_p(J; H^2_p(\Omega)) = Z^2,$$

if and only if the data are subject to the following conditions.

(i) $f \in L_p(J; L_p(\Omega)) = X^1$,

(ii) $g \in L_p(J; H^1_p(\Omega)) = X^2$,

(iii) $h_1 \in L_p(J; W^{1-1/p}_p(\Gamma)) = Y^1$,

(iv) $h_2 \in W_p^{1-1/2p}(J; L_p(\Gamma)) \cap L_p(J; W_p^{2-1/p}(\Gamma)) = Y^2$,

(v) $u_0 \in B_{pp}^{3-2/p}(\Omega) = X_p$,

(vi) $\partial_\nu u_0 = h_2|_{t=0}$ *if* $p > 3/2$.

Proof. Note that due to the uniform continuity of the data, the norms of the solution operators for the full space or half space case are uniform as well. Therefore we may cut the interval $J = [0, T]$ into pieces $J_i = [i\delta, i\delta + \delta]$ for some small $\delta > 0$. We then solve the problem successively on J_i. W.l.o.g. we may treat the first interval J_0. For this purpose we define the spaces Z_δ^j, X_δ^j, Y_δ^j, $j = 1, 2$ as the restriction of the spaces Z^j, X^j and Y^j to the interval J_0. Furthermore we may assume that $g = h_2 = u_0 = 0$, by solving the linear heat equation

$$\begin{aligned}
\beta\partial_t u - \Delta u &= -g, \quad t \in J_0,\ x \in \Omega,\\
\partial_\nu u &= h_2, \quad t \in J_0,\ x \in \partial\Omega,\\
u(0) &= u_0, \quad t = 0,\ x \in \Omega.
\end{aligned}$$

We cover $\overline{\Omega}$ by finitely many open sets U_k, $k = 1, ..., N$, which are subject to the following conditions.

(i) $U_k \cap \Gamma = \emptyset$ and $U_k = B_{r_k}(x_k)$ for all $k = 1, ..., N_1$;

(ii) $U_k \cap \Gamma \neq \emptyset$ for $k = N_1 + 1, ..., N$.

We choose next a partition of unity $\{\varphi_k\}_{k=1}^N$ such that $\sum_{k=1}^N \varphi_k = 1$ on $\overline{\Omega}$, $0 \le \varphi_k(x) \le 1$ and $\operatorname{supp} \varphi_k \subset U_j$. Note that (u, μ) is a solution of (4.34) if and only if

$$\begin{aligned}
\partial_t u_k - \operatorname{div}(a\partial_t u_k) &= \operatorname{div}(B\nabla\mu_k) + f_k + F_k(u,\mu), \quad t \in [0,\delta],\ x \in \Omega \cap U_k,\ 1 \le k \le N,\\
\mu_k - c\cdot\nabla\mu_k &= \beta\partial_t u_k - \Delta u_k + G_k(u,\mu), \quad t \in [0,\delta],\ x \in \Omega \cap U_k,\ 1 \le k \le N\\
B\nabla\mu_k \cdot \nu &= h_{1k} + (B\nabla\varphi_k \cdot \nu)\mu, \quad t \in [0,\delta],\ x \in \Gamma \cap U_k,\ N_1 + 1 \le k \le N\\
\partial_\nu u_k &= u\partial_\nu\varphi_k, \quad t \in [0,\delta],\ x \in \Gamma \cap U_k,\ N_1 + 1 \le k \le N\\
u_k(0) &= 0, \quad t = 0,\ x \in \Omega \cap U_k.
\end{aligned} \tag{4.36}$$

Here we have set $u_k = u\varphi_k$, $\mu_k = \mu\varphi_k$, $f_k = f\varphi_k$ and $h_{1k} = h_1\varphi_k$. The terms $F_k(u, \mu)$ and $G_k(u, \mu)$ are defined by

$$F_k(u,\mu) = -(a\cdot\nabla\varphi_k)\partial_t u - (\operatorname{div} B\cdot\nabla\varphi_k)\mu - 2B\nabla\varphi_k\cdot\nabla\mu - (B : \nabla^2\varphi_k)\mu,$$

and

$$G_k(u,\mu) = -(c\cdot\nabla\varphi_k)\mu + 2\nabla u\nabla\varphi_k + u\Delta\varphi_k.$$

In case $k = 1, ..., N_1$ we have no boundary conditions, i.e. we only have to consider the first two equations in (4.36). The aim is to derive an extension of the coefficients $(a(x), c(x), B(x))$ from each ball $U_k = B_{r_k}(x_k)$ to the whole of $\mathbb{R}^n$ in order to treat these local problems with the help of Corollary 4.2.3 for all $k = 1, \dots, N_1$. To achieve this we have to find an extension such that $\operatorname{div}\tilde{a}(x) = \operatorname{div}\tilde{c}(x) = 0$, $x \in \mathbb{R}^n$, for the extended coefficients $\tilde{a}$ and $\tilde{c}$.

We will now show how to construct such an extension. First of all note that w.l.o.g. we may assume $x_k = 0$, $k = 1, \dots, N_1$, after a translation in $\mathbb{R}^n$. We use the following ansatz for the extension $\tilde{a}$ of a.

$$\tilde{a}^k(x) = \begin{cases} a(x), & x \in \overline{B_{r_k}(0)},\\ a\left(\frac{r_k^2 x}{r^2}\right) - 2\left(\sum_{j=1}^n \xi_j a_j\right)\xi + R(r,\xi)\xi, & x \in \mathbb{R}^n \setminus \overline{B_{r_k}(0)}, \end{cases} \tag{4.37}$$

where $r = |x|$, $\xi = x/|x|$ and ξ_j, a_j denote the components of ξ and a, respectively. The scalar valued function $R = R(r, \xi)$ will be defined later. We require $\operatorname{div}\tilde{a}^k(x) = 0$ for all $x \in \mathbb{R}^n$. Clearly this condition is fulfilled for all $x \in \overline{B_{r_k}(0)}$. So we have to compute $\operatorname{div} \tilde{a}^k(x)$ for $x \in \mathbb{R}^n \setminus \overline{B_{r_k}(0)}$. First of all we have by the chain rule

$$\begin{aligned}
\operatorname{div}\left[a\left(\frac{r_k^2 x}{r^2}\right)\right] &= \partial_i\left[a_i\left(\frac{r_k^2 x}{r^2}\right)\right] = (\partial_j a_i)\left(\frac{r_k^2 x}{r^2}\right)\cdot\left(\frac{r_k^2\delta_{ji}}{r^2} - 2\frac{r_k^2\xi_j\xi_i}{r^2}\right)\\
&= \frac{r_k^2}{r^2}\left((\operatorname{div} a)\left(\frac{r_k^2 x}{r^2}\right) - 2\xi_j\xi_i(\partial_j a_i)\left(\frac{r_k^2 x}{r^2}\right)\right)\\
&= -2\frac{r_k^2}{r^2}\sum_{i,j}\xi_i\xi_j\partial_j a_i\left(\frac{r_k^2 x}{r^2}\right),
\end{aligned} \tag{4.38}$$

since $\operatorname{div} a(x) = 0$. Here we made use of sum convention for the sake of readability and δ_{ji} denotes the Kronecker symbol. In a next step we compute $\operatorname{div}((\xi_j a_j)\xi)$. For convenience we suppress the argument $\left(\frac{r_k^2 x}{r^2}\right)$ of a. This yields

$$\begin{aligned}
\operatorname{div}((\xi_j a_j)\xi) &= \partial_i\left(\frac{x_i x_j a_j}{r^2}\right)\\
&= \frac{(n+1)x_j a_j}{r^2} - \frac{2x_i^2 x_j a_j}{r^4} + \xi_i\xi_j\partial_i a_j\\
&= \frac{(n-1)x_j a_j}{r^2} + \xi_i\xi_j\partial_m a_j\cdot\left(\frac{\delta_{im} r_k^2}{r^2} - 2\frac{x_i x_m r_k^2}{r^4}\right)\\
&= \frac{(n-1)x_j a_j}{r^2} + \frac{r_k^2}{r^2}\left(\xi_i\xi_j\partial_i a_j - 2\xi_i^2\xi_m\xi_j\partial_m a_j\right)\\
&= \frac{(n-1)x_j a_j}{r^2} + \frac{r_k^2}{r^2}\left(\xi_i\xi_j\partial_i a_j - 2\xi_m\xi_j\partial_m a_j\right)\\
&= \frac{(n-1)\xi_j a_j}{r} - \sum_{i=1}^n \frac{r_k^2}{r^2}(\xi_i\xi_j\partial_i a_j).
\end{aligned} \tag{4.39}$$

These calculations imply the identity

$$\operatorname{div} \tilde{a}^k(x) = -2\frac{(n-1)}{r}\sum_{j=1}^n \xi_j a_j + \operatorname{div}(R\xi).$$

Finally we have to compute the divergence of $R\xi$, where $R = R(r,\xi)$. We obtain

$$\begin{aligned}
\operatorname{div}(R\xi) &= \partial_i(R\xi_i) = \xi_i\partial_i R + R\cdot\left(\frac{n}{r} - \frac{\xi_i^2}{r}\right)\\
&= \xi_i\partial_i R + \frac{n-1}{r}R\\
&= \xi_i\left(\xi_i\partial_r R + \partial_{\xi_m} R\cdot\left(\frac{\delta_{im}}{r} - \frac{\xi_m\xi_i}{r}\right)\right) + \frac{n-1}{r}R\\
&= \partial_r R + \frac{1}{r}\xi_i\partial_{\xi_i} R - \frac{1}{r}\xi_i^2\xi_m\partial_{\xi_m} R + \frac{n-1}{r}R\\
&= \partial_r R + \frac{n-1}{r}R
\end{aligned} \tag{4.40}$$

Since we require $\operatorname{div} \tilde{a}(x) = 0$ for all $x \in \mathbb{R}^n$, it follows that $R = R(r,\xi),\ r \geq r_k$ must be a solution of the ordinary differential equation

$$\partial_r R + \frac{(n-1)}{r}R = 2\frac{(n-1)}{r}(\xi\cdot a), \quad r \geq r_k.$$

The compatibility condition $\tilde{a}^k(x) = a(x)$ for all $x \in \mathbb{R}^n$ with $|x| = r_k$ and (4.37) yield the initial condition

$$R_k(\xi) := R(r_k,\xi) = 2(\xi\cdot a(r_k\xi)), \quad \xi = x/|x|.$$

The function $R = R(r, \xi)$ can be explicitly computed to the result

$$R(r,\xi) = \frac{r_k^{n-1}}{r^{n-1}} R_k(\xi) + \frac{2(n-1)}{r^{n-1}} \int_{r_k}^{r} s^{n-2}(\xi \cdot a)\, ds, \quad r \geq r_k.$$

With the help of (4.37) we may extend the coefficients a and c in each ball $U_k = B_{r_k}(x_k),\ k = 1, \ldots, N_1$ to the whole of $\mathbb{R}^n$, such that $\operatorname{div} \tilde{a}^k(x) = \operatorname{div} \tilde{c}^k(x) = 0$ for all $x \in \mathbb{R}^n$ and all $k = 1, \ldots, N_1$. At this point we want to emphasize that for an arbitrarily small number $\omega > 0$ we have

$$|\tilde{a}^k(x) - a(x_k)| + |\tilde{c}^k(x) - c(x_k)| \leq \omega$$

provided $r_k > 0$ is sufficiently small. For the coefficient matrix $B(x)$ we use the extension method from [12], i.e. we set

$$\tilde{B}^k(x) = \begin{cases} B(x), & x \in \overline{B_{r_k}(x_k)}, \\ B\left(x_k + r_k \frac{x - x_k}{|x - x_k|^2}\right), & x \in \mathbb{R}^n \setminus \overline{B_{r_k}(x_k)}. \end{cases} \tag{4.41}$$

This yields again $|B^k(x) - B(x)| \leq \omega$ for an arbitrarily small $\omega > 0$ and each $x \in \mathbb{R}^n$, provided $r_k > 0$ is sufficiently small. Hence for each chart $U_k,\ k = 1, \ldots, N_1$ we have coefficients, which fit into the setting of Corollary 4.2.3. Therefore we obtain solution operators $S_k^F \in \mathcal{B}(X^1 \times X^2;\, {}_0Z^1 \times Z^2)$ of (4.36) such that

$$\begin{bmatrix} u_k \\ \mu_k \end{bmatrix} = S_k^F \begin{bmatrix} f_k + F_k(u,\mu) \\ G_k(u,\mu) \end{bmatrix}, \tag{4.42}$$

for each $k = 1, \ldots, N_1$.

For the remaining charts U_k, $k = N_1 + 1, \ldots, N$ we obtain problems in crooked half spaces with inhomogeneous Neumann boundary conditions. For the further analysis we have to understand how to treat (4.34) in such a setting. To this end we fix a point $x_0 \in \partial\Omega$ and a ball $B_{r_0}(x_0)$ with radius $r_0 > 0$ around x_0. After a composition of a translation and a rotation in $\mathbb{R}^n$, we may assume that $x_0 = 0$ and $\nu(x_0) = [0, \ldots, 0, -1] = e_n$. Consider a graph $\rho \in C^3(\mathbb{R}^{n-1})$, having compact support, such that

$$\{(x', x_n) \in \overline{B_{r_0}(x_0)} \subset \mathbb{R}^n : x_n = \rho(x')\} = \partial\Omega \cap \overline{B_{r_0}(x_0)}.$$

Note that by decreasing the size of the charts we may assume that $|\nabla_{x'}\rho|_\infty$ is as small as we like, since $\nabla_{x'}\rho(0) = 0$. We set furthermore

$$G = \{(x', x_n) \in \mathbb{R}^n : x_n > \rho(x')\}.$$

We want to achieve that $\operatorname{div} a(x) = \operatorname{div} c(x) = 0$ for all $x \in G$ in order to apply Corollary 4.3.3, after a transformation of the crooked half space to $\mathbb{R}^n_+$. For the time being, we only know that $\operatorname{div} a(x) = \operatorname{div} c(x) = 0$ for all $x \in B_{r_0}(x_0) \cap G$. So we have to extend the coefficients a and c in a suitable way. To this end we transform the crooked boundary $\partial\Omega \cap B_{r_0}(x_0)$ to a straight line in $\mathbb{R}^{n-1}$. This will be done with the help of a transformation, which was introduced at the beginning of this section. Let $u(x', x_n) = v(g(x)) = v(x', x_n - \rho(x'))$ and $\mu(x) = \eta(g(x)) = \eta(x', x_n - \rho(x'))$, $x' \in B_{r_0}(x_0) \cap \mathbb{R}^{n-1}$. Then the differential operators $a \cdot \nabla u$ and $c \cdot \nabla \mu$ transform as follows.

$$a(x) \cdot \nabla u(x) = a(x) \cdot (Dg(x)^{\mathsf{T}} \nabla v(g(x))) = (Dg(x)a(x)) \cdot \nabla v(g(x)) = \bar{a}(g(x)) \cdot \nabla v(g(x)),$$

and

$$c(x) \cdot \nabla \mu(x) = c(x) \cdot (Dg^{\mathsf{T}}(x) \nabla \eta(g(x))) = (Dg(x)c(x)) \cdot \nabla \eta(g(x)) = \bar{c}(g(x)) \cdot \nabla \eta(g(x)),$$

with $\bar{a}(x) := Dg(x)a(g^{-1}(x))$ and $\bar{c}(x) = Dg(x)c(g^{-1}(x))$. Similarly we obtain

$$\operatorname{div}(B\nabla\mu) = \operatorname{div}(\bar{B}\nabla\eta),$$

where $\bar{B}(x) := Dg(x)B(g^{-1}(x))Dg^{\mathsf{T}}(x)$ and the matrix Dg is given by

$$Dg(x) = \begin{bmatrix} E_{n-1} & 0 \\ -\nabla_{x'}\rho(x') & 1 \end{bmatrix}, \; x' \in B_{r_0}(x_0) \cap \mathbb{R}^{n-1},$$

where E_{n-1} is the identity matrix in $\mathbb{R}^{(n-1)\times(n-1)}$. The Laplace operator is transformed as follows

$$\Delta u = \Delta v + |\nabla_{x'}\rho|^2 \partial_y^2 v - 2\nabla_{x'}\rho\nabla\partial_y v - \Delta_{x'}\rho\partial_y v,$$

and the normal ν at ∂G is given by

$$\nu(x', \rho(x')) = \frac{1}{\sqrt{1+|\nabla_{x'}\rho|^2}} \begin{bmatrix} \nabla_{x'}\rho \\ -1 \end{bmatrix}.$$

Therefore $\sqrt{1+|\nabla_{x'}\rho(x')|^2}(Dg^{\mathsf{T}})^{-1}\nu = [0, \dots, 0, -1]^{\mathsf{T}} = e_n$, hence the transformed boundary conditions are $\bar{B}\nabla\eta \cdot e_n = \sqrt{1+|\nabla\rho(x')|^2}\Theta^{-1}h_1$ and

$$\nabla v \cdot e_n = \frac{\Theta^{-1}h_2}{\sqrt{1+|\nabla_{x'}\rho|^2}} - \frac{\nabla_{x'}\rho \cdot \nabla_{x'}v}{1+|\nabla_{x'}\rho|^2}.$$

Here Θ^{-1} denotes the push forward operator, the inverse of the pull back operator.

Note that the set $\Theta^{-1}(B_{r_0}(x_0) \cap \Omega) \cap \mathbb{R}^n_+$ is not a hemisphere. Nevertheless we may choose a radius $0 < r_1 < r_0$ such that

$$B_{r_1}(x_0) \cap \mathbb{R}^n_+ \subset \Theta^{-1}(B_{r_0}(x_0) \cap \Omega).$$

By construction, the transformed coefficients satisfy $\operatorname{div} \bar{a}(x) = \operatorname{div} \bar{c}(x) = 0$ for all $x \in B_{r_1}(x_0) \cap \mathbb{R}^n_+$ and $\bar{a} \cdot e_n = \bar{c} \cdot e_n = 0$ for all $x \in B_{r_1}(x_0) \cap \mathbb{R}^{n-1}$. Firstly we extend the coefficients $\bar{a}$ and $\bar{c}$ to the whole ball $B_{r_1}(x_0)$ by even reflection in the tangential coordinates and by odd reflection w.r.t. the variable y, i.e. we set

$$\hat{a}_{x'}(x', y) = \begin{cases} \bar{a}_{x'}(x', y), & y \geq 0, \\ \bar{a}_{x'}(x', -y), & y \leq 0, \end{cases}$$

and $\hat{a}_y(x', y) = \bar{a}_y(x', y)$ if $(x', y) \in B_{r_1}(x_0) \cap \mathbb{R}^n_+$, $\hat{a}_y(x', y) = -\bar{a}_y(x', -y)$ if $(x', y) \in B_{r_1}(x_0) \cap \mathbb{R}^n_-$ and in the same way for $\bar{c}$. By the property $\bar{a} \cdot e_n = \bar{c} \cdot e_n = 0$ for all $x \in B_{r_1}(x_0) \cap \mathbb{R}^{n-1}$ it holds that $\hat{a}, \hat{c} \in W^1_\infty(B_{r_1}(x_0))$ and $\operatorname{div} \hat{a}(x) = \operatorname{div} \hat{c}(x) = 0$ for all $x \in B_{r_1}(x_0)$. Now we are in a position to use the extension (4.37) in order to extend $\hat{a}$ and $\hat{c}$ to the whole of $\mathbb{R}^n$, such that the divergence condition $\operatorname{div} \tilde{a}(x) = \operatorname{div} \tilde{c}(x) = 0$ is preserved. It is furthermore clear by the structure of (4.37) that $\tilde{a} \cdot e_n = \tilde{c} \cdot e_n = 0$ for all $x \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$. The coefficient matrix $\bar{B}$ can be extended to a matrix $\tilde{B}$ by the technique in [12, Proof of Theorem 8.2]. Then the condition $\tilde{B}(x_0)\tau(x_0) \cdot e_n$ holds for all $\tau(x_0) \in T_{x_0}\mathbb{R}^{n-1}$. We reverse the transformation to the crooked half space. This yields the following problem

$$\begin{aligned} \partial_t u - \operatorname{div}(a\partial_t u) &= \operatorname{div}(B\nabla\mu) + f, \quad t > 0, \; x \in G, \\ \mu - c \cdot \nabla\mu &= \beta\partial_t u - \Delta u + g, \quad t > 0, \; x \in G, \\ B\nabla\mu \cdot \nu &= h_1, \quad t > 0, \; x \in \partial G, \\ \partial_\nu u &= h_2, \quad t > 0, \; x \in \partial G, \\ u(0) &= 0, \quad t = 0, \; x \in G. \end{aligned} \tag{4.43}$$

The coefficients (β, a, c, B) satisfy (A) and have the properties $\operatorname{div} a(x) = \operatorname{div} c(x) = 0$ for all $x \in G$ and $a(x) \cdot \nu(x) = c(x) \cdot \nu(x) = 0$ for all $x \in \partial G$. Furthermore the matrix B satisfies $B(x_0)\tau(x_0) \cdot \nu(x_0) = 0$ for $x_0 \in \partial G$. In order to solve (4.43) we transform it again to the half space $\mathbb{R}^n_+$ by the procedure described above. Suppose that we already know a solution $(u, \mu) \in {}_0Z^1 \times Z^2$

of (4.43). The transformation from G to $\mathbb{R}^n_+$ then yields

$$\begin{aligned} \partial_t v - \operatorname{div}(\tilde{a}\partial_t v) &= \operatorname{div}(\tilde{B}\nabla\eta) + \Theta^{-1} f, \quad t>0,\ x\in\mathbb{R}^n_+, \\ \eta - \tilde{c}\cdot\nabla\eta &= \beta\partial_t v - \Delta v + \mathcal{C}_1(x,D)v + \Theta^{-1}g, \quad t>0,\ x\in\mathbb{R}^n_+, \\ \tilde{B}\nabla\eta\cdot e_n &= \sqrt{1+|\nabla_{x'}\rho(x')|^2}\Theta^{-1}h_1, \quad t>0,\ x'\in\mathbb{R}^{n-1},\ y=0, \\ -\partial_y v &= \frac{\Theta^{-1}h_2}{\sqrt{1+|\nabla_{x'}\rho|^2}} + \mathcal{C}_2(x',D)v, \quad t>0,\ x'\in\mathbb{R}^{n-1},\ y=0, \\ v(0) &= 0, \quad t=0,\ x\in\mathbb{R}^n_+, \end{aligned} \tag{4.44}$$

where the differential operators $\mathcal{C}_1(x,D)$ and $\mathcal{C}_2(x',D)$ are defined by

$$\mathcal{C}_1(x,D)v = -|\nabla_{x'}\rho|^2\partial_y^2 v + 2\nabla_{x'}\rho\nabla\partial_y v + \Delta_{x'}\rho\partial_y v,$$

and

$$\mathcal{C}_2(x',D)v = -\frac{\nabla_{x'}\rho\cdot\nabla_{x'}v}{1+|\nabla_{x'}\rho|^2}.$$

From the extension method above it follows that

$$|\tilde{a}(x)-a(x_0)|_{L_\infty(\mathbb{R}^n_+;\mathbb{R}^n)} + |\tilde{c}(x)-c(x_0)|_{L_\infty(\mathbb{R}^n_+;\mathbb{R}^n)} + |\tilde{B}(x)-B(x_0)|_{L_\infty(\mathbb{R}^n_+;\mathbb{R}^{n\times n})} \le \omega,$$

where we can choose $\omega>0$ arbitrarily small, provided $r_0>0$ is sufficiently small. By Corollary 4.3.3 there exists a solution operator $S^H \in \mathcal{B}(X^1_\delta \times X^2_\delta \times Y^1_\delta \times {}_0Y^2_\delta; {}_0Z^1_\delta \times Z^2_\delta)$ of (4.44), i.e.

$$\begin{bmatrix} u \\ \mu \end{bmatrix} = \Theta S^H \begin{bmatrix} \Theta^{-1}f \\ \mathcal{C}_1(x,D)\Theta^{-1}u \\ \sqrt{1+|\nabla_{x'}\rho(x')|^2}\Theta^{-1}h_1 \\ \frac{\Theta^{-1}h_2}{\sqrt{1+|\nabla_{x'}\rho|^2}} + \mathcal{C}_2(x',D)\Theta^{-1}u \end{bmatrix}. \tag{4.45}$$

Since the solution operator is bounded and Θ is a C^3-diffeomorphism, we obtain the estimate

$$|(u,\mu)|_{Z^1_\delta\times Z^2_\delta} \le M(|f|_{X^1_\delta} + |g|_{X^2_\delta} + |h_1|_{Y^1_\delta} + |h_2|_{Y^2_\delta} + |u|_{L_p(J_0;H^2_p(G))} + |\nabla_{x'}\rho|_\infty |u|_{Z^1_\delta}).$$

We remind that the norm of the solution operator S^H does not depend on the length $\delta>0$ of the interval J_0, since we have time trace 0. This means we may again use the embeddings

$${}_0Z^1_\delta = {}_0H^1_p(J_0;H^1_p(G)) \cap L_p(J_0;H^3_p(G)) \hookrightarrow {}_0H^{1/2}_p(J_0;H^2_p(G)) \hookrightarrow L_{2p}(J_0;H^2_p(G)),$$

to obtain $|u|_{L_p(J_0;H^2_p(G))} \le \delta^{1/2p}|u|_{Z^1_\delta}$. Since $|\nabla_{x'}\rho|_\infty$ may be arbitrarily small, we obtain

$$|(u,\mu)|_{Z^1_\delta\times Z^2_\delta} \le M(|f|_{X^1_\delta} + |g|_{X^2_\delta} + |h_1|_{Y^1_\delta} + |h_2|_{Y^2_\delta}).$$

This means the operator $L: {}_0Z^1_\delta \times Z^2_\delta \to X^1_\delta \times X^2_\delta \times Y^1_\delta \times Y^2_\delta$ defined by

$$L(u,\mu) = \begin{bmatrix} \partial_t u - \operatorname{div}(a\partial_t u) - \operatorname{div}(B\nabla\mu) \\ \mu - (c\cdot\nabla\mu) - \beta\partial_t u + \Delta u \\ (B\nabla\mu\cdot\nu) \\ \partial_\nu u \end{bmatrix},$$

is injective and has closed range, i.e. it is a semi-Fredholm operator. To show surjectivity, we apply again the homotopy argument to the set of data

$$(\beta, a_\tau, c_\tau, B_\tau) = (1-\tau)(\beta,0,0,E_n) + \tau(\beta,a,c,B), \quad \tau\in[0,1],$$

where E_n is the identity matrix in $\mathbb{R}^{n\times n}$. We claim that the corresponding operator L_0 is bijective. Then L_1 is surjective, since each operator L_τ is injective and has closed range, by the above calculations. Therefore we have to consider the system

$$\begin{aligned}
\partial_t u &= \Delta\mu + f, \quad t\in J_0,\ x\in G,\\
\mu &= \beta\partial_t u - \Delta u + g, \quad t\in J_0,\ x\in G,\\
\partial_\nu \mu &= h_1, \quad t\in J_0,\ x\in \partial G,\\
\partial_\nu u &= h_2, \quad t\in J_0,\ x\in \partial G,\\
u(0) &= 0, \quad t=0,\ x\in G.
\end{aligned} \tag{4.46}$$

Multiply the first equation by β and substitute $\beta\partial_t u$ by the second equation. This yields the elliptic problem

$$\mu - \beta\Delta\mu = \beta f + g - \Delta u, \quad \partial_\nu\mu = h_1.$$

This problem admits a unique solution $\mu \in L_p(J_0; H_p^2(G))$ provided $\beta f + g - \Delta u \in L_p(J_0; L_p(G))$ and $h_1 \in L_p(J_0; W_p^{1-1/p}(\partial G))$. Denoting by $\mathcal{S}$ the corresponding solution operator, we may write

$$\mu = -\mathcal{S}\Delta u + \mathcal{S}(\beta f + g, h_1) =: -\mathcal{S}\Delta u + \mu_0,$$

with $\mu_0 \in L_p(J_0; H_p^2(G))$. Now we go back to $(4.46)_2$ to obtain the initial boundary value problem

$$\begin{aligned}
\beta\partial_t u - \Delta u &= \mu_0 - \mathcal{S}\Delta u - g, \quad t\in J_0,\ x\in G,\\
\partial_\nu u &= h_2, \quad t\in J_0,\ x\in\partial G,\\
u(0) &= 0, \quad t=0,\ x\in G,
\end{aligned} \tag{4.47}$$

for the function u. If $u \in {}_0Z^1_\delta$ is given, then

$$\mathcal{S}\Delta u \in {}_0H_p^{1/2}(J_0; H_p^2(G)) \cap L_p(J_0; H_p^3(G)).$$

This means that the term $\mathcal{S}\Delta u$ is of lower order and a Neumann series argument yields a unique solution $u^* \in {}_0Z^1$ of (4.47). By Kato's continuation argument it follows that the crooked half space problem (4.43) admits a unique solution $(u,\mu) \in {}_0Z^1 \times Z^2$ first on a small interval $J_0 = [0,\delta]$ and then also on the whole interval $J = [0,T]$ by a successive application of the above procedure.

We may use this result for the charts U_k, $k = N_1+1,\ldots,N$ which intersect the boundary $\partial\Omega$, to obtain solution operators $S_k^H \in \mathcal{B}(X^1_\delta \times X^2_\delta \times Y^1_\delta \times {}_0Y^2_\delta; {}_0Z^1_\delta \times Z^2_\delta)$ such that

$$\begin{bmatrix} u_k \\ \mu_k \end{bmatrix} = S_k^H \begin{bmatrix} f_k + F_k(u,\mu) \\ G_k(u,\mu) \\ h_{1k} + (B\nabla\varphi_k\cdot\nu)\mu \\ u\partial_\nu\varphi_k \end{bmatrix}, \tag{4.48}$$

for each $k \in \{N_1+1,\ldots,N\}$. Summing (4.42) and (4.48) over all charts U_k, $k=1,\ldots,N$ yields

$$\begin{bmatrix} u \\ \mu \end{bmatrix} = \sum_{k=1}^N S_k^F \begin{bmatrix} f_k + F_k(u,\mu) \\ G_k(u,\mu) \end{bmatrix} + \sum_{k=1}^N S_k^H \begin{bmatrix} f_k + F_k(u,\mu) \\ G_k(u,\mu) \\ h_{1k} + (B\nabla\varphi_k\cdot\nu)\mu \\ u\partial_\nu\varphi_k \end{bmatrix}, \tag{4.49}$$

since $\{\varphi_k\}_{k=1}^N$ is a partition of unity. By the boundedness of the solution operators we obtain the estimate

$$|(u,\mu)|_{Z^1_\delta\times Z^2_\delta} \le M(|f|_{X^1_\delta} + |h_1|_{Y^1_\delta} + |u|_{L_p(J_0;H_p^2(\Omega))} + |\partial_t u|_{L_p(J_0;L_p(\Omega))} + |\mu|_{L_p(J_0;H_p^1(\Omega))}), \tag{4.50}$$

for some constant $M>0$. The term $|u|_{L_p(J_0;H_p^2(\Omega))}$ may be estimated by $\delta^{1/2p}C|u|_{Z^1_\delta}$ with some constant $C>0$, while for the last two terms in (4.50) we need an estimate like that of Proposition 4.3.2 but here for the domain Ω. The arguments for a general domain $\Omega\subset\mathbb{R}^n$ are similar to those in the proof of Proposition 4.3.2. Indeed, it suffices to show that the L_p-realization A_0 of the differential operator

$$\mathcal{A}_0(D)w = c\cdot\nabla w + a\cdot\nabla w - \operatorname{div}(a(c\cdot\nabla w)) + \operatorname{div}(\beta B\nabla w)$$

with domain

$$D(A_0) = \{v \in L_p(J_0; H_p^2(\Omega)) : B\nabla v\cdot\nu = 0 \text{ on } \partial\Omega\},$$

is dissipative in $L_p(J_0; L_p(\Omega))$. But due to the assumptions $\operatorname{div} a(x) = \operatorname{div} c(x) = 0,\ x\in\Omega$, and $a(x)\cdot\nu(x) = c(x)\cdot\nu(x) = 0,\ x\in\partial\Omega$, this follows immediately from the proof of Proposition 4.2.2. Hence, choosing $\delta > 0$ small enough we therefore obtain

$$|(u,\mu)|_{Z_\delta^1\times Z_\delta^2} \le M(|f|_{X_\delta^1} + |g|_{X_\delta^2} + |h_1|_{Y_\delta^1} + |h_2|_{Y_\delta^2} + |u_0|_{X_p}), \tag{4.51}$$

for the solution $(u,\mu)\in Z_\delta^1\times Z_\delta^2$ of (4.34). Now we may again employ the continuation argument of Kato to see that the solution operator to (4.34) is bijective. This can be done as in the case of a crooked half space. The proof is complete

□

4.5 Local Well-Posedness

We are going to solve the semilinear problem

$$\begin{aligned}
\partial_t\psi - \operatorname{div}(a\partial_t\psi) &= \operatorname{div}(B\nabla\mu) + f, \quad t>0,\ x\in\Omega,\\
\mu - c\cdot\nabla\mu &= \beta\partial_t\psi - \Delta\psi + \Phi'(\psi) + g, \quad t>0,\ x\in\Omega,\\
B\nabla\mu\cdot\nu &= h_1, \quad t>0,\ x\in\Gamma,\\
\partial_\nu\psi &= h_2, \quad t>0,\ x\in\Gamma,\\
\psi(0) &= \psi_0, \quad t=0,\ x\in\Omega,
\end{aligned} \tag{4.52}$$

where the data (β, a, c, B) are subject to Assumption (A), (4.9) and (4.10) and let $a, c\in C_{ub}^1(\overline{\Omega};\mathbb{R}^n)$, $B\in C_{ub}^1(\overline{\Omega};\mathbb{R}^{n\times n})$. To this end let $f\in X^1$, $g\in X^2$, $h_j\in Y_j,\ j=1,2$ and $\psi_0\in X_p$ be given such that the compatibility condition $\partial_\nu\psi_0 = h_2|_{t=0}$ if $p>3/2$ is satisfied. Applying Theorem 4.4.1 we may define a pair of functions $(u^*, v^*)\in Z^1\times Z^2$ as the unique solution of

$$\begin{aligned}
u_t^* - \operatorname{div}(au_t^*) &= \operatorname{div}(B\nabla v^*) + f, \quad t>0,\ x\in\Omega,\\
v^* - c\cdot\nabla v^* &= \beta u_t^* - \Delta u^* + g, \quad t>0,\ x\in\Omega,\\
B\nabla v^*\cdot\nu &= h_1, \quad t>0,\ x\in\Gamma,\\
\partial_\nu u^* &= h_2, \quad t>0,\ x\in\Gamma,\\
u^*(0) &= \psi_0, \quad t=0,\ x\in\Omega.
\end{aligned} \tag{4.53}$$

We set

$$\begin{aligned}
\mathbb{E}_1 &= Z^1(T)\times Z^2(T), \quad {}_0\mathbb{E}_1 = \{(u,v)\in\mathbb{E}_1 : u|_{t=0} = 0\},\\
\mathbb{E}_0 &= X^1(T)\times X^2(T)\times Y_1(T)\times Y_2(T), \quad {}_0\mathbb{E}_0 = \{(f,g,h_1,h_2)\in\mathbb{E}_0 : h_2|_{t=0} = 0\}
\end{aligned}$$

and denote by $|\cdot|_1$ and $|\cdot|_0$ the canonical norms in $\mathbb{E}_1$ and $\mathbb{E}_0$, respectively. Following the lines of Chapters 2 & 3 we define a linear operator $\mathbb{L}:\mathbb{E}_1\to\mathbb{E}_0$ by

$$\mathbb{L}(u,v) = \begin{bmatrix} \partial_t u - \operatorname{div}(a\partial_t u) - \operatorname{div}(B\nabla v)\\ v - c\cdot\nabla v - \beta\partial_t u + \Delta u\\ B\nabla v\cdot\nu\\ \partial_\nu u \end{bmatrix}$$

and a nonlinear function $G: {}_0\mathbb{E}_1\times\mathbb{E}_1\to {}_0\mathbb{E}_0$ by

$$G((u,v),(u^*,v^*)) = \begin{bmatrix} 0\\ \Phi'(u+u^*)\\ 0\\ 0 \end{bmatrix}$$

Again we consider $\mathbb{L}$ as an operator from ${}_0\mathbb{E}_1$ to ${}_0\mathbb{E}_0$. Hence Theorem 4.4.1 yields that $\mathbb{L}$ is a bounded isomorphism and by the open mapping theorem $\mathbb{L}$ is invertible with bounded inverse $\mathbb{L}^{-1}$. It is easily seen that $(\psi, \mu) := (u + u^*, v + v^*)$ is a solution of (4.52) if and only if

$$\mathbb{L}(u, v) = G((u, v), (u^*, v^*)) \text{ or equivalently } (u, v) = \mathbb{L}^{-1}G((u, v), (u^*, v^*)).$$

Consider a ball $\mathbb{B}_R \subset \mathbb{E}_1$ where $R \in (0, 1]$ will be fixed later. To apply the contraction mapping principle we furthermore define a nonlinear operator by $\mathcal{T}(u, v) := \mathbb{L}^{-1}G((u, v), (u^*, v^*))$. As in Chapters 2 & 3 we have to show that $\mathcal{T}\mathbb{B}_R \subset \mathbb{B}_R$ and that there exists a constant $\kappa < 1$ such that the contractive inequality

$$|\mathcal{T}(u, v) - \mathcal{T}(\bar{u}, \bar{v})|_1 \le \kappa|(u, v) - (\bar{u}, \bar{v})|_1 \tag{4.54}$$

holds for all $(u, v), (\bar{u}, \bar{v}) \in \mathbb{B}_R$. We first care about the contraction mapping property. By Hölder's inequality and with the assumption $\Phi \in C^{3-}(\mathbb{R})$ we obtain

$$\begin{aligned}
|\mathcal{T}(u, v) - \mathcal{T}(\bar{u}, \bar{v})|_1 &\le |\mathbb{L}^{-1}||G((u, v), (u^*, v^*)) - G((\bar{u}, \bar{v}), (u^*, v^*))|_0 \\
&\le M|\Phi'(u + u^*) - \Phi'(\bar{u} + u^*)|_{X_2(T)} \\
&\le M\Big(|\Phi'(u + u^*) - \Phi'(\bar{u} + u^*)|_{p,p} \\
&\qquad\qquad + |\nabla(\Phi'(u + u^*) - \Phi'(\bar{u} + u^*))|_{p,p}\Big) \\
&\le M\Big(|u - \bar{u}|_{p,p} + |\nabla(u + u^*)|_{rp,rp}|\Phi''(u + u^*) - \Phi''(\bar{u} + u^*)|_{r'p,r'p} \\
&\qquad\qquad + |\Phi''(\bar{u} + u^*)|_{r'p,r'p}|\nabla u - \nabla\bar{u}|_{rp,rp}\Big) \\
&\le MT^{1/r'p}\Big(|u - \bar{u}|_{rp,rp} + |\nabla u - \nabla\bar{u}|_{rp,rp}\Big) \\
&\le \kappa(T)|(u, v) - (\bar{u}, \bar{v})|_1,
\end{aligned} \tag{4.55}$$

where $\kappa = \kappa(T)$ is a function with the property that $\kappa(T) \to 0$ as $T \to 0$. and a constant $M > 0$ which does not depend on T, since time traces are equal to 0 at $t = 0$, whenever they exist. Here we made use of the embedding

$$H_p^1(J; H_p^1(\Omega)) \cap L_p(J; H_p^3(\Omega)) \hookrightarrow C(J \times \overline{\Omega}),$$

provided $p > (n+2)/3$. Furthermore, since in the above calculation we may chose $r > 1$ arbitrarily close to 1, it holds that

$$H_p^1(J; H_p^1(\Omega)) \cap L_p(J; H_p^3(\Omega)) \hookrightarrow L_{rp}(J; H_{rp}^1(\Omega)).$$

Thus, if T is sufficiently small we obtain (4.54). The self mapping property can be shown in a similar way. The above computation yields

$$\begin{aligned}
|\mathcal{T}(u, v)|_1 &\le |\mathcal{T}(u, v) - \mathcal{T}(0, 0)|_1 + |\mathcal{T}(0, 0)|_1 \\
&\le \kappa(T)|(u, v)|_1 + M|G((0, 0), (u^*, v^*))|_0 \\
&\le \kappa(T)|(u, v)|_1 + M|\Phi'(u^*)|_{X_2(T)} \\
&\le \kappa(T)R + M|\Phi'(u^*)|_{X_2(T)}.
\end{aligned} \tag{4.56}$$

Since $\Phi'(u^*)$ is a fixed function in $X_2(T)$ it follows that $|\Phi'(u^*)|_{X_2(T)} \to 0$ as $T \to 0$, whence $\mathcal{T}\mathbb{B}_R \subset \mathbb{B}_R$, provided that $T > 0$ is small enough. The contraction mapping principle yields a unique fixed point $(\tilde{u}, \tilde{v}) \in {}_0\mathbb{E}_1$ or equivalently $(\psi, \mu) := (\tilde{u} + u^*, \tilde{v} + v^*) \in \mathbb{E}_1$ is the unique local solution of (4.52). Therefore we have the following result.

Theorem 4.5.1. *Let $1 < p < \infty$ and $p > (n + 2)/3$. Assume furthermore that $\Phi \in C^{3-}(\mathbb{R})$ and let (A) as well as* (4.9),(4.10) *be satisfied. Suppose that $a, c \in C_{ub}^1(\overline{\Omega}; \mathbb{R}^n)$ and $B \in C_{ub}^1(\overline{\Omega}; \mathbb{R}^{n\times n})$. Then there exists an interval $J = [0, T] \subset [0, T_0]$ and a unique solution (ψ, μ) of* (4.52) *on J, with*

$$\psi \in H_p^1(J; H_p^1(\Omega)) \cap L_p(J; H_p^3(\Omega)) = Z^1(T)$$

and

$$\mu \in L_p(J; H_p^2(\Omega)) = Z^2(T),$$

provided that the data are subject to the following conditions.

(i) $f \in L_p(J_0; L_p(\Omega)) = X^1$,

(ii) $g \in L_p(J_0; H_p^1(\Omega)) = X^2$,

(iii) $h_1 \in L_p(J_0; W_p^{1-1/p}(\Gamma)) = Y^1$,

(iv) $h_2 \in W_p^{1-1/2p}(J_0; L_p(\Gamma)) \cap L_p(J_0; W_p^{2-1/p}(\Gamma)) = Y^2$,

(v) $\psi_0 \in B_{pp}^{3-2/p}(\Omega) = X_p$,

(vi) $\partial_\nu \psi_0 = h_2|_{t=0}$ *if* $p > 3/2$.

The solution depends continuously on the given data and if the data are independent of t*, the map* $\psi_0 \mapsto \psi(t),\ t \in \mathbb{R}_+$*, defines a local semiflow on the natural phase manifold*

$$\mathcal{M}_p := \{\psi_0 \in X_p : \psi_0 \textit{ satisfies } (vi)\}.$$

4.6 Global Well-Posedness

Throughout this section, we assume that $p \geq 2$ and $n \leq 3$. Furthermore, we will need the following assumption.

(H) There exists a constant $\varepsilon > 0$ such that

$$\beta z_0^2 + (a + c|z_1)z_0 + (Bz_1|z_1) \geq \varepsilon(z_0^2 + |z_1|^2),$$

for all $(z_0, z_1) \in \mathbb{R} \times \mathbb{R}^n$,

This condition is crucial in order to obtain some energy estimates, which will be used in the proof of global well-posedness. We will show in the Appendix, that (H) already implies (A). Assume furthermore that the data (β, a, c, B) satisfy (4.9), (4.10) and let $a, c \in C_{ub}^1(\overline{\Omega}; \mathbb{R}^n)$ and $B \in C_{ub}^1(\overline{\Omega}; \mathbb{R}^{n\times n})$.

A successive application of Theorem 4.5.1 yields a maximal interval of existence $J_{\max} = [0, T_{\max})$ for the solution $(\psi, \vartheta) \in \mathbb{E}_1$ of (4.52). In order to prove the global existence of ψ on $\mathbb{R}_+$, we have to verify that $|\psi|_{Z^1(T)}$ is uniformly bounded for all $T \in I$ and all compact intervals $I \subset \mathbb{R}_+$. The embedding

$$Z^1(T) \hookrightarrow C([0,T]; B_{pp}^{3-2/p}(\Omega))$$

then yields that the limit $\lim_{t\to T_{\max}} |\psi(t)|_{X_p}$ exists, which means that we can continue the solution ψ beyond $T_{\max}$. Then it follows from the equations that μ exists globally, too. In other words this means that $T_{\max} = +\infty$. Let $J_0 = [0, T_0]$ and let $T \in J_0$. The open mapping theorem yields the estimate

$$\begin{aligned} |\psi|_{Z^1(T)} + |\mu|_{Z^2(T)} &\leq M(T_0)\Big(|\Phi'(\psi)|_{X^2(T)} + |f|_{X^1(T_0)} + |g|_{X^2(T_0)} \\ &\qquad + |h_1|_{Y^1(T_0)} + |h_2|_{Y^2(T_0)} + |\psi_0|_{X_p}\Big) \\ &\leq M(T_0)\Big(1 + |\Phi'(\psi)|_{X^2(T)}\Big) \end{aligned} \tag{4.57}$$

for the local solution $(\psi, \mu) \in \mathbb{E}_1$ of (4.52). First of all we will derive an a priori estimate for ψ. To do so we multiply $(4.52)_1$ by μ, $(4.52)_2$ by $-\partial_t\psi$ and integrate by parts to obtain

$$\int_\Omega \Big(\partial_t\psi\mu + (B\nabla\mu|\nabla\mu) + (a|\nabla\mu)\partial_t\psi\Big)\, dx = \int_\Omega \mu f\, dx + \int_\Gamma \mu h_1\, d\Gamma \tag{4.58}$$

and

$$\int_\Omega \Big(-\partial_t\psi\mu+(c|\nabla\mu)\partial_t\psi+\beta|\partial_t\psi|^2+\frac{1}{2}\frac{\partial}{\partial t}|\nabla\psi|^2+\frac{\partial}{\partial t}\Phi(\psi)\Big)\,dx=\int_\Gamma \partial_t\psi h_2\,d\Gamma-\int_\Omega \partial_t\psi g\,dx. \quad (4.59)$$

Adding (4.58) and (4.59) yields the equation

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}|\nabla\psi|_2^2+\int_\Omega \Phi(\psi)\,dx\right)&+\beta|\partial_t\psi|_2^2+(a+c|\partial_t\psi\nabla\mu)_2+(B\nabla\mu|\nabla\mu)_2\\ &=\int_\Omega \mu f\,dx+\int_\Gamma \mu h_1\,d\Gamma+\int_\Gamma \partial_t\psi h_2\,d\Gamma-\int_\Omega \partial_t\psi g\,dx. \quad (4.60)\end{aligned}$$

From Assumption (H) with $z_0=\partial_t\psi$ and $z_1=\nabla\mu$ it follows that

$$\beta|\partial_t\psi|_2^2+(a+c|\partial_t\psi\nabla\mu)_2+(B\nabla\mu|\nabla\mu)_2\geq\varepsilon(|\partial_t\psi|_2^2+|\nabla\mu|_2^2).$$

For the first and the second integral in (4.60) we apply Hölder's inequality as well as the Poincaré-Wirtinger inequality to obtain

$$\int_\Omega \mu f\,dx\leq C|f|_2\left(|\nabla\mu|_2+|\int_\Omega \mu\,dx|\right)\quad\text{and}\quad\int_\Gamma \mu h_1\,d\Gamma\leq C|h_1|_{2,\Gamma}\left(|\nabla\mu|_2+|\int_\Omega \mu\,dx|\right).$$

The integral $\int_\Omega \mu\,dx$ can be computed in the following way. Assuming that $\operatorname{div} c=0$ in Ω and $(c|\nu)=0$ as well as $(a|\nu)=0$ on Γ we have

$$\int_\Omega (c|\nabla\mu)\,dx=\int_\Gamma (c|\nu)\mu\,d\Gamma-\int_\Omega \mu\operatorname{div} c\,dx=0,$$

hence it follows from $(4.52)_1$, $(4.52)_2$ and the boundary conditions that

$$\begin{aligned}\int_\Omega \mu\,dx&=\beta\int_\Omega \partial_t\psi\,dx+\int_\Omega \Phi'(\psi)\,dx+\int_\Omega g\,dx\\ &=\int_\Omega \Phi'(\psi)\,dx+\int_\Omega g\,dx+\beta\left(\int_\Omega f\,dx+\int_\Gamma h_1\,d\Gamma\right).\end{aligned}$$

With the additional assumption

$$|\Phi'(s)|\leq(c_1\Phi(s)+c_2s^2+c_3)^\theta,\quad\text{for all } s\in\mathbb{R}, \quad (4.61)$$

with some constants $c_i>0$, $\theta\in(0,1)$, we obtain

$$|\int_\Omega \mu\,dx|\leq\int_\Omega (c_1\Phi(\psi)+c_2|\psi|^2+c_3)^\theta\,dx+c(|g|_1+|h_1|_{1,\Gamma}+|f|_1).$$

By the last estimate, Young's inequality and the Poincaré inequality it holds that

$$\int_\Omega \mu f\,dx+\int_\Gamma \mu h_1\,d\Gamma\leq C(\delta)\left(|\nabla\psi|_2^2+\int_\Omega \Phi(\psi)\,dx+|f|_2^q+|h_1|_{2,\Gamma}^q+|g|_2^2+1\right)+\delta|\nabla\mu|_2^2, \quad (4.62)$$

where $q:=\max\{2,\frac{1}{1-\theta}\}$ and $\delta>0$ may be arbitrarily small. For the term $\int_\Omega \partial_t\psi g\,dx$ in (4.60) we apply Young's inequality one more time to obtain

$$\int_\Omega \partial_t\psi g\,dx\leq\delta|\partial_t\psi|_2^2+C(\delta)|g|_2^2. \quad (4.63)$$

Integrating (4.60) with respect to t and choosing $\delta>0$ small enough, we obtain together with (4.62) and (4.63) the estimate

$$\begin{aligned}&\frac{1}{2}|\nabla\psi(t)|_2^2+\int_\Omega \Phi(\psi(t))\,dx+C_1(|\partial_t\psi|_{2,2}^2+|\nabla\mu|_{2,2}^2)\\ &\qquad\leq C_2\left(\int_0^t\left(\frac{1}{2}|\nabla\psi(\tau)|_2^2+\Phi(\psi(\tau))\right)\,d\tau+|f|_{q,2}^q+|h_1|_{q,2,\Gamma}^q+|g|_{2,2}^2+1\right)\\ &\qquad\qquad+\int_0^t\int_\Gamma \partial_t\psi h_2\,d\Gamma\,d\tau. \quad (4.64)\end{aligned}$$

In order to treat the last double integral, we have to assume more regularity for the function h_2. To be precise, we assume that

$$h_2 \in H_p^1(J; L_p(\Gamma)) \cap L_p(J; W_p^{2-1/p}(\Gamma)) \hookrightarrow C(J; L_p(\Gamma)).$$

Due to this fact, we may integrate the last term in (4.64) by parts to the result

$$\int_0^t \int_\Gamma \partial_t \psi h_2 \, d\Gamma \, d\tau = \int_\Gamma \psi(t) h_2(t) \, d\Gamma - \int_\Gamma \psi_0 h_2|_{t=0} \, d\Gamma - \int_0^t \int_\Gamma \psi \partial_t h_2 \, d\Gamma \, d\tau, \tag{4.65}$$

where we also made use of Fubini's theorem. For the first term we use Young's inequality, the embedding $H_2^1(\Omega) \hookrightarrow L_2(\Gamma)$ and the fact that

$$\int_\Omega \psi(t) \, dx = \int_\Omega \psi_0 \, dx + \int_0^t \int_\Omega f \, dx \, d\tau + \int_0^t \int_\Gamma h_1 \, d\Gamma \, d\tau. \tag{4.66}$$

This yields

$$\begin{aligned} \int_\Gamma \psi(t) h_2(t) \, d\Gamma &\le \delta |\psi(t)|^2_{H_2^1(\Omega)} + C(\delta) |h_2(t)|^2_{2,\Gamma} \\ &\le \eta C |\nabla \psi(t)|_2^2 + C(\eta) \left(|h_2|^2_{\infty,2,\Gamma} + |f|_{1,1} + |h_1|_{1,1,\Gamma} + |\psi_0|_1 \right). \end{aligned}$$

Next, by Theorem 4.5.1 (vi) it holds that $h_2|_{t=0} = \partial_\nu \psi_0 \in B_{pp}^{2-3/p}(\Gamma) \hookrightarrow L_2(\Gamma)$, if $p > 3/2$ and by trace theory, we obtain

$$B_{pp}^{3-2/p}(\Omega) \hookrightarrow B_{pp}^{3-3/p}(\Gamma) \hookrightarrow L_2(\Gamma).$$

These embeddings ensure that the integral $\int_\Gamma \psi_0 h_2|_{t=0} \, d\Gamma$ converges. Finally, concerning the last term in (4.65) we use Young's inequality one more time to the result

$$\begin{aligned} \int_0^t \int_\Gamma \psi \partial_t h_2 \, d\Gamma \, d\tau &\le \frac{1}{2} \int_0^t |\psi(\tau)|^2_{H_2^1(\Omega)} \, d\tau + \frac{1}{2} |\partial_t h_2|^2_{2,2,\Gamma} \\ &\le C \int_0^t |\nabla \psi(\tau)|_2^2 \, d\tau + C(T_0, f, h_1, \partial_t h_2, \psi_0), \end{aligned}$$

where we used again (4.66). Set

$$E(u) = \frac{1}{2} |\nabla u|_2^2 + \int_\Omega \Phi(u) \, dx, \quad u \in H_2^1(\Omega).$$

Then by the above estimates there exist some constants $C_j > 0$ such that

$$E(\psi(t)) + C_1 (|\partial_t \psi|^2_{2,2} + |\nabla \mu|^2_{2,2}) \le C_2 \int_0^t E(\psi(\tau)) \, d\tau + C_3(T_0, f, g, h_1, h_2, \partial_t h_2, \psi_0),$$

provided that $\delta > 0$ is sufficiently small. Assume that Φ satisfies the additional condition

$$\Phi(s) \ge -\frac{\eta}{2} s^2 - c_0, \quad s \in \mathbb{R}, \tag{4.67}$$

where $c_0 > 0$ and $\eta < \lambda_1$, with λ_1 being the first nontrivial eigenvalue of the negative Neumann Laplacian. With the help of (4.67) it follows that $E(u)$ is bounded from below for all $u \in H_2^1(\Omega)$, hence we may apply Gronwall's lemma to the result that $E(\psi(\cdot))$ is bounded on $J_{\max} = [0, T_{\max})$. Applying (4.67) one more time and using the fact that $|\int_\Omega \psi(t,x) \, dx| \le C$ it holds that

$$\psi \in L_\infty(J_{\max}; H_2^1(\Omega)).$$

Applying the same arguments as in the proof of Lemma 3.4.1, we obtain an inequality of the form

$$|\Phi'(\psi)|_{X^2(T)} \le C(1 + |\psi|^\delta_{Z^1(T)} |\psi|^m_{L_\infty(J; H_2^1(\Omega))}),$$

for some constants $C > 0$, $m > 0$ and $\delta \in (0,1)$, provided that the potential $\Phi \in C^3(\mathbb{R})$ satisfies the growth condition

$$|\Phi'''(s)| \le c_0(1+|s|^\gamma), \quad s \in \mathbb{R}, \tag{4.68}$$

with $\gamma < 3$ in case $n = 3$ and some constant $c_0 > 0$. Hence it follows from (4.57) that $|\psi|_{Z^1(T)}$ is bounded with respect to $T \in [0, T_0]$ and this means that we may continue the solution beyond $T_{\max}$. Thus we obtain the following result on global well-posedness.

Theorem 4.6.1. *Let $p \ge 2$, $n \le 3$, $q = \max\{2, \frac{1}{1-\theta}\}$, with θ from (4.61), and let Hypotheses (H) as well as (4.9),(4.10) hold. Suppose that $a, c \in C^1_{ub}(\overline{\Omega}; \mathbb{R}^n)$ and $B \in C^1_{ub}(\overline{\Omega}; \mathbb{R}^{n\times n})$. Assume furthermore that Φ satisfies (4.61), (4.67) and (4.68). Then there exists a unique global solution (ψ, μ) of (4.52) on $J_0 = [0, T_0]$, with*

$$\psi \in H^1_p(J_0; H^1_p(\Omega)) \cap L_p(J_0; H^3_p(\Omega))$$

and

$$\mu \in L_p(J_0; H^2_p(\Omega)),$$

provided that the data are subject to the following conditions.

(i) $f \in L_p(J_0; L_p(\Omega)) \cap L_q(J_0; L_2(\Omega))$,

(ii) $g \in L_p(J_0; H^1_p(\Omega))$,

(iii) $h_1 \in L_p(J_0; W^{1-1/p}_p(\Gamma)) \cap L_q(J_0; L_2(\Gamma))$,

(iv) $h_2 \in H^1_p(J_0; L_p(\Gamma)) \cap L_p(J_0; W^{2-1/p}_p(\Gamma))$,

(v) $\psi_0 \in B^{3-2/p}_{pp}(\Omega)$,

(vi) $\partial_\nu \psi_0 = h_2|_{t=0}$, *if* $p > 3/2$.

The solution depends continuously on the given data and if the data are independent of t, the map $\psi_0 \mapsto \psi(t)$, $t \in \mathbb{R}_+$, defines a global semiflow on the natural phase manifold $\mathcal{M}_p$.

4.7 Asymptotic Behavior

In this last section we will give a qualitative analysis of global solutions of the Cahn-Hilliard-Gurtin system

$$\begin{aligned} \partial_t\psi - \operatorname{div}(a\partial_t\psi) &= \operatorname{div}(B\nabla\mu), \quad t>0,\ x\in\Omega,\\ \mu - c\cdot\nabla\mu &= \beta\partial_t\psi - \Delta\psi + \Phi'(\psi), \quad t>0,\ x\in\Omega,\\ B\nabla\mu\cdot\nu &= 0, \quad t>0,\ x\in\Gamma,\\ \partial_\nu\psi &= 0, \quad t>0,\ x\in\Gamma,\\ \psi(0) &= \psi_0, \quad t=0,\ x\in\Omega. \end{aligned} \tag{4.69}$$

To be more precise we will show that each trajectory converges to a stationary point, i.e. to a solution of the corresponding stationary system. The so called *Lojasiewicz-Simon inequality* will play an important role in the proof of this assertion. Assume that the data (β, a, c, B) satisfy (4.9), (4.10) and (H). Suppose furthermore that $a, c \in C^1_{ub}(\overline{\Omega}; \mathbb{R}^n)$ and $B \in C^1_{ub}(\overline{\Omega}; \mathbb{R}^{n\times n})$. Let $\psi_0 \in \mathcal{M}_2$ and let (ψ, μ) be the unique global solution of (4.69). We recall from Section 4.6 the energy functional

$$E(u) = \frac{1}{2}|\nabla u|_2^2 + \int_\Omega \Phi(u)\, dx,$$

defined on the energy space

$$V := \{u \in H^1_2(\Omega) : \int_\Omega u\, dx = 0\}.$$

Note that due to $(4.69)_1$ and the boundary condition $(4.69)_3$ we obtain $\int_\Omega \psi\, dx \equiv \int_\Omega \psi_0\, dx$, since $(a|\nu) = 0$ on Γ. If we perform a shift of ψ by means of $\tilde{\psi} = \psi - c$, where $c := \int_\Omega \psi_0\, dx$, it follows that $\tilde{\psi}$ is again a solution of (4.69), if we replace the physical potential Φ by $\tilde{\Phi}(s) = \Phi(s + c)$ and additionally it holds that $\int_\Omega \tilde{\psi}\, dx = 0$. It follows from (4.60) that in the homogeneous case $E(\psi(\cdot))$ satisfies the equation

$$\frac{d}{dt}E(\psi(t)) + \beta|\partial_t\psi(t)|_2^2 + (a + c|\partial_t\psi(t)\nabla\mu(t))_2 + (B\nabla\mu(t)|\nabla\mu(t))_2 = 0,$$

for all $t \in \mathbb{R}_+$. Making again use of Hypothesis (H) we obtain the inequality

$$\frac{d}{dt}E(\psi(t)) + \varepsilon\left(|\partial_t\psi(t)|_2^2 + |\nabla\mu(t)|_2^2\right) \le 0, \tag{4.70}$$

which holds for all $t \in \mathbb{R}_+$. Integrating with respect to t and making use of (4.67) as well as of the Poincaré inequality we obtain the a priori estimates

$$\psi \in L_\infty(\mathbb{R}_+; H_2^1(\Omega)) \quad \text{and} \quad \partial_t\psi, |\nabla\mu| \in L_2(\mathbb{R}_+ \times \Omega).$$

Proposition 4.7.1. *The orbit $\{\psi(t)\}_{t\in\mathbb{R}_+}$ is relatively compact in V.*

Proof. We rewrite equation $(4.69)_2$ as follows

$$\beta\partial_t\psi - \Delta\psi + \psi = \mu - \overline{\mu} - (c(x)|\nabla\mu) + \overline{\mu} + \psi - \Phi'(\psi),$$

where $\overline{\mu} = \frac{1}{|\Omega|}\int_\Omega \Phi'(\psi)\, dx$. By the energy estimates above and the Poincaré-Wirtinger inequality it holds that

$$f := \mu - \overline{\mu} + (c|\nabla\mu) \in L_2(\mathbb{R}_+; L_2(\Omega)).$$

Furthermore we have

$$g := \overline{\mu} + \psi - \Phi'(\psi) \in L_\infty(\mathbb{R}_+; L_q(\Omega)),$$

where $q = 6/(\gamma+2)$ is determined by the growth condition (4.68) on Φ. The operator $A := -\Delta + I$ with domain

$$D(A) = \{u \in H_p^2(\Omega) : \partial_\nu u = 0 \text{ on } \Gamma\}$$

generates an exponentially stable, analytic C_0-semigroup $\{T(t)\}_{t\in\mathbb{R}_+}$ in $L_p(\Omega)$. Therefore

$$T(\cdot) * f \in H_2^1(\mathbb{R}_+; L_2(\Omega)) \cap L_2(\mathbb{R}_+; H_2^2(\Omega)) \hookrightarrow C_0(\mathbb{R}_+; H_2^1(\Omega)).$$

For the function g we apply elementary semigroup theory to obtain

$$T(\cdot) * g \in C_b(\mathbb{R}_+; H_q^s(\Omega)),$$

for each $s \in (0, 2)$. The space $H_q^s(\Omega)$ embeds compactly into $H_2^1(\Omega)$, if s is chosen close enough to 2. This completes the proof of relative compactness, since $\psi_0 \in H_2^2(\Omega)$. □

The following proposition provides some crucial properties of the ω-limit set

$$\omega(\psi) = \{\varphi \in V :\ \exists\ (t_n) \nearrow \infty,\ s.t.\ \psi(t_n) \to \varphi \text{ in } V\}.$$

Proposition 4.7.2. *Suppose that (ψ, μ) is a global solution of* (4.69) *and let Φ satisfy Hypotheses* (4.67) *and* (4.68). *Then the following statements hold.*

(i) The mapping $t \mapsto E(\psi(t))$ is nonincreasing and the limit $\lim_{t\to\infty} E(\psi(t)) =: E_\infty \in \mathbb{R}$ exists.

(ii) The ω-limit set $\omega(\psi) \subset V$ is nonempty, connected, compact and E is constant on $\omega(\psi)$.

(iii) *Every $\psi_\infty \in \omega(\psi)$ is a strong solution (in the sense of L_2) of the stationary problem*

$$\begin{aligned} -\Delta\psi_\infty + \Phi'(\psi_\infty) &= \mu_\infty, \quad x \in \Omega, \\ \partial_\nu \psi_\infty &= 0, \quad x \in \Gamma, \end{aligned} \tag{4.71}$$

where $\mu_\infty = \frac{1}{|\Omega|}\int_\Omega \Phi'(\psi_\infty)\, dx = const.$

(iv) *Each $\psi_\infty \in \omega(\psi)$ is a critical point of E, i.e. $E'(\psi_\infty) = 0$ in V^*, where V^* is the topological dual space of V.*

Proof. From the inequality (4.70) it follows that $E(\psi(\cdot))$ is nonincreasing with respect to t. Furthermore by (4.67) it follows that $E(u)$ is bounded from below for all $u \in V$. This proves (i). Assertion (ii) follows easily from well-known facts in the theory of dynamical systems.

Let $\psi_\infty \in \omega(\psi)$. Then there exists a sequence $(t_n) \nearrow +\infty$ such that $\psi(t_n) \to \psi_\infty$ in V as $n \to \infty$. Since $\partial_t\psi \in L_2(\mathbb{R}_+ \times \Omega)$ it follows that $\psi(t_n + s) \to \psi_\infty$ in $L_2(\Omega)$ for all $s \in [0, 1]$ and by relative compactness also in V. This can be seen as in Chapter 2. Integrating (4.70) from t_n to $t_n + 1$ we obtain

$$E(\psi(t_n + 1)) - E(\psi(t_n)) + \varepsilon \int_0^1 \int_\Omega \left(|\nabla\mu(t_n + s, x)|^2 + |\partial_t\psi(t_n + s, x)|^2\right)\, dx\, ds \le 0.$$

Letting $t_n \to +\infty$ yields

$$|\nabla\mu(t_n + \cdot, \cdot)|, \partial_t\psi(t_n + \cdot, \cdot) \to 0 \quad \text{in } L_2([0, 1] \times \Omega).$$

This in turn yields a subsequence (t_{n_k}) such that $|\nabla\mu(t_{n_k} + s)|, \partial_t\psi(t_{n_k} + s) \to 0$ in $L_2(\Omega)$ for a.e. $s \in [0, 1]$. We fix such an s, say $s^* \in [0, 1]$. The Poincaré-Wirtinger inequality implies that

$$\begin{aligned} &|\mu(t_{n_k} + s^*) - \mu(t_{n_l} + s^*)|_2 \\ &\quad \le C_p \left(|\nabla\mu(t_{n_k} + s^*) - \nabla\mu(t_{n_l} + s^*)|_2 + \int_\Omega |\Phi'(\psi(t_{n_k} + s^*)) - \Phi'(\psi(t_{n_l} + s^*))|\, dx\right), \end{aligned}$$

since $\int_\Omega \mu\, dx = \int_\Omega \Phi'(\psi)\, dx$. Letting $k, l \to \infty$ and making use of (4.68) it follows that $\mu(t_{n_k} + s^*)$ is a Cauchy sequence in $L_2(\Omega)$, hence it admits a limit, which we denote by μ_∞. Since the gradient is a closed operator in $L_2(\Omega; \mathbb{R}^n)$ it holds that $\mu_\infty \in H_2^1(\Omega)$ and $\nabla\mu_\infty = 0$. Thus $\mu_\infty = const.$ and we have the identity $\mu_\infty = \frac{1}{|\Omega|}\int_\Omega \Phi'(\psi_\infty)\, dx$. Finally we multiply $(4.69)_2$ by a function $\varphi \in V$ in $L_2(\Omega)$ to the result

$$\begin{aligned} &(\mu(t_{n_k} + s^*), \varphi)_2 + (c \cdot \nabla\mu(t_{n_k} + s^*), \varphi)_2 \\ &\qquad = \beta(\partial_t\psi(t_{n_k} + s^*), \varphi)_2 - (\Delta\psi(t_{n_k} + s^*), \varphi)_2 + (\Phi'(\psi(t_{n_k} + s^*)), \varphi)_2. \end{aligned} \tag{4.72}$$

Taking the limit $t_{n_k} \to \infty$ we obtain

$$a(\psi(t_{n_k} + s^*), \varphi) \to (\mu_\infty - \Phi'(\psi_\infty), \varphi)_2,$$

where $a : V \times V \to \mathbb{R}$ is the form defined in Section 2.5 and $(\cdot, \cdot)_2$ denotes the scalar product in $L_2(\Omega)$. Since $\Phi'(\psi_\infty) \in L_q(\Omega)$ with $q = 6/(\beta + 2)$ it follows that $\psi_\infty \in D(A_q) = \{u \in H_q^2(\Omega) : \partial_\nu u = 0\}$, where A_q is the part of the operator A in $L_q(\Omega)$ which is induced by the form $a(u, v)$. Observe that $q > 6/5$ by assumption, whence we may apply a bootstrap argument to conclude $\psi_\infty \in H_2^2(\Omega)$ and $\partial_\nu\psi_\infty = 0$ on Γ. Going back to (4.72) we obtain for $(t_{n_k}) \nearrow \infty$ the identity

$$(\nabla\psi_\infty, \nabla\varphi)_2 + (\Phi'(\psi_\infty), \varphi)_2 = (\mu_\infty, \varphi)_2,$$

for all functions $\varphi \in V$. This yields (iii) after integration by parts. Assertion (iv) follows from (iii) and again via integration by parts, since by Proposition 2.5.2 the first Fréchet derivative of E is given by

$$\langle E'(u), h\rangle_{V^*, V} = \int_\Omega \nabla u \nabla h\, dx + \int_\Omega \Phi'(u) h\, dx.$$

□

The next proposition is the key for the proof of the convergence of the orbit $\psi(t)$ towards a stationary state as $t \to \infty$.

Proposition 4.7.3 (Lojasiewicz-Simon inequality)**.** *Let $\varphi \in V$ be a critical point of the functional E. Assume in addition that Φ is real analytic. Then there exist constants $s \in (0, \frac{1}{2}]$, $C, \delta > 0$ such that*

$$|E(u) - E(\varphi)|^{1-s} \leq C|E'(u)|_{V^*},$$

whenever $|u - \varphi|_V \leq \delta$.

Proof. The proof follows the lines of the proof of Proposition 2.5.4. We skip the details. □

Now we are in a position to state the main result of this section.

Theorem 4.7.4. *Let Φ satisfy the conditions* (4.67) *and* (4.68)*. Assume in addition that Φ is real analytic. Then the limit*

$$\lim_{t\to\infty} \psi(t) =: \psi_\infty$$

exists in V and ψ_∞ is a strong solution of the stationary problem (4.71)*.*

Proof. Since each element $\varphi \in \omega(\psi)$ is a critical point of E, Proposition 4.7.3 implies that the Lojasiewicz-Simon inequality is valid in some neighborhood of $\varphi \in \omega(\psi)$. By Proposition 4.7.2 (ii) the ω-limit set is compact, hence there exists $N \in \mathbb{N}$ such that

$$\bigcup_{j=1}^{N} B_{\delta_j}(\varphi_j) \supset \omega(\psi),$$

where $B_{\delta_j}(\varphi_j) \subset V$ are open balls with center $\varphi_i \in \omega(\psi)$ and radius δ_i. Additionally in each ball the Lojasiewicz-Simon inequality is valid. It follows from Proposition 4.7.2 (i) and (ii) that the energy functional E is constant on $\omega(\psi)$, i.e. $E(\varphi) = E_\infty$, for all $\varphi \in \omega(\psi)$. Thus there exists an open set $U \supset \omega(\psi)$ and *uniform* constants $s \in (0, \frac{1}{2}]$ $C, \delta > 0$ with

$$|E(u) - E_\infty|^{1-s} \leq C|E'(u)|_{V^*},$$

for all $u \in U$. A well-known result in the theory of dynamical systems sates that the ω-limit set is an attractor for the orbit $\{\psi(t)\}_{t\in\mathbb{R}_+}$. To be precise this means

$$\lim_{t\to\infty} \operatorname{dist}(\psi(t), \omega(\psi)) = 0 \quad \text{in } V.$$

This implies that there exists some time $t^* \geq 0$ such that $\psi(t) \in U$ for all $t \geq t^*$ and thus the Lojasiewicz-Simon inequality holds for the solution $\psi(t)$, i.e.

$$|E(\psi(t)) - E_\infty|^{1-s} \leq C|E'(\psi(t))|_{V^*}, \quad t \geq t^*. \tag{4.73}$$

Define a function $H : \mathbb{R}_+ \to \mathbb{R}_+$ by $H(t) = (E(\psi(t) - E_\infty)^s$. Then with (4.70) and (4.73) it holds that

$$\begin{aligned} -\frac{d}{dt}H(t) &= (E(\psi(t)) - E_\infty)^{s-1}\left(-\frac{d}{dt}E(\psi(t))\right) \\ &\geq \varepsilon\frac{|\partial_t\psi(t)|_2^2 + |\nabla\mu(t)|_2^2}{(E(\psi(t)) - E_\infty)^{1-s}} \\ &\geq C_\varepsilon\frac{|\partial_t\psi(t)|_2^2 + |\nabla\mu(t)|_2^2}{|E'(\psi(t))|_{V^*}} \end{aligned} \tag{4.74}$$

Following the lines of Section 2.5 the first Frechét derivative of E in V reads

$$\langle E'(u), h\rangle_{V^*,V} = \int_\Omega \nabla u \nabla h \; dx + \int_\Omega \Phi'(u)h \; dx,$$

for all $(u,h) \in V \times V$. Setting $u = \psi(t)$ and making use of $(4.69)_2$ we obtain with the help of Hölder's inequality, Poincaré's inequality and integration by parts

$$\begin{aligned}\langle E'(\psi(t)), h\rangle_{V^*,V} &= \int_\Omega (\mu(t) - \bar{\mu}(t))h \, dx - \int_\Omega c \cdot \nabla\mu(t)h \, dx - \beta \int_\Omega \partial_t\psi(t)h \, dx \\ &\le C(|\nabla\mu(t)|_2 + |\partial_t\psi(t)|_2)|h|_2,\end{aligned} \tag{4.75}$$

since $\operatorname{div} c(x) = 0,\ x \in \Omega$ and $(c(x)|\nu(x)) = 0,\ x \in \partial\Omega$. Taking the supremum in (4.75) over all functions $h \in V$ with norm less than 1 it follows that

$$|E'(\psi(t))|_{V^*} \le C(|\nabla\mu(t)|_2 + |\partial_t\psi(t)|_2).$$

We insert this estimate into (4.74) to obtain

$$-\frac{d}{dt}H(t) \ge C_\varepsilon(|\nabla\mu(t)|_2 + |\partial_t\psi(t)|_2).$$

Integrating this inequality from t^* to ∞ it follows that $|\partial_t\psi(\cdot)|_2, |\nabla\mu(\cdot)|_2 \in L_1(\mathbb{R}_+)$, since $H(t) > 0$. This implies that the limit $\lim_{t\to\infty}\psi(t) =: \psi_\infty$ exists firstly in $L_2(\Omega)$ but by relative compactness also in V. Finally, by Proposition 4.7.2 (iii) the limit ψ_∞ is a solution of the stationary problem (4.71). The proof is complete.

□

4.8 Appendix

For $(z_0, z_1) \in \mathbb{R} \times \mathbb{R}^n$ Hypothesis (H) reads

$$\beta z_0^2 + (d|z_1)z_0 + (Bz_1|z_1) \ge \varepsilon(z_0^2 + |z_1|^2),$$

where $d := a + c$. Observe that the left side of this inequality can be rewritten as

$$\left(\sqrt{\beta}z_0 + \frac{1}{2\sqrt{\beta}}(d|z_1)\right)^2 + \left(\left(B - \frac{1}{4\beta}(d \otimes d)\right) z_1 \middle| z_1\right).$$

For a fixed $z_1 \in \mathbb{R}^n$ we choose $z_0 \in \mathbb{R}$ in such a way that the squared bracket is equal to 0. Thus we obtain the estimate

$$(\beta Bz_1|z_1) - \frac{1}{4}((d \otimes d)z_1|z_1) \ge \varepsilon\beta|z_1|^2,$$

valid for all $z_1 \in \mathbb{R}^n$. By the definition of d it holds that

$$d \otimes d = a \otimes c + c \otimes a + a \otimes a + c \otimes c,$$

hence we obtain the identity

$$\begin{aligned}\beta B - \frac{1}{2}(a \otimes c + c \otimes a) &= \beta B - \frac{1}{4}(d \otimes d) + \frac{1}{4}(a \otimes a + c \otimes c - a \otimes c - c \otimes a) \\ &= \beta B - \frac{1}{4}(d \otimes d) + \frac{1}{4}(a - c) \otimes (a - c).\end{aligned}$$

Since the matrix $(a - c) \otimes (a - c)$ is positive semi-definite we finally obtain the assertion.

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Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbstständig und ohne fremde Hilfe verfasst, keine anderen als die von mir angegebenen Quellen und Hilfsmittel benutzt und die den benutzten Werken wörtlich und inhaltlich entnommenen Stellen als solche kenntlich gemacht habe.

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Curriculum Vitae

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Publications

1. Prüss, J. ; Wilke, M., *Maximal L_p - Regularity and Long - Time Behavior of the Non-Isothermal Cahn-Hilliard Equation with Dynamic Boundary Conditions*, Oper. Theory Adv. Appl. **168** (2006), 209-236.
2. Abels, H. ; Wilke, M., *Convergence to Equilibrium for the Cahn-Hilliard Equation with a Logarithmic Free Energy*, Nonlinear Anal. **67** (2007), 3176-3193.
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Halle, 24 August 2007