

Inverse Hooke's law and complementary strain energy in coupled strain gradient elasticity

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The inverse Hooke's law and complementary strain energy density has been examined in the context of the theory of coupled gradient elasticity for second gradient materials. To this end, it was assumed that the potential energy density is a quadratic form of the strain and of the second gradient of displacement. Existence of the coupling term significantly complicates the problem. To avoid this complication the equation for the potential energy density was transformed in order to present it as an uncoupled quadratic form of a modified strain and the second gradient of displacement or of the strain and a modified second gradient of displacement. These transformations, which is in essence a block matrix diagonalization, lead to a decoupling of strains and strain gradient in the potential energy density and makes it possible to determine tensorial relations for the compliance tensors of fourth-, fifth-, and sixth-rank. Both modifications result in the same compliance tensors and are valid for an arbitrary material symmetry class. In the case of hemitropic materials, the compliance tensors have the same symmetry and the same form as the stiffness tensors and are characterized by eight independent constants, namely the two classical isotropic constants, five constants in the strain gradient part and one constant in the coupling term. Explicit expressions for these eight parameters are obtained from the tensorial relations for the compliance tensors and are compared with the direct solution of a linear system for the compliance's. All three solutions are identical, what we consider as a verification of the presented results.

KEYWORDS

Strain gradient elasticity, coupling fifth-rank tensor, complementary energy

1 | INTRODUCTION

A natural generalization of classical elasticity is the strain gradient elasticity which deals with those models of continuous media where a strain energy density depends on the first and second gradients of displacements. The motivation to introduce such an energy is the inability to describe size effects, surface and interface energies and to smooth out singularities in classical solutions, for example, around dislocations or boundary discontinuities.

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From the beginning of the 20th century, a variety of non-classical theories has been suggested to overcome the shortcomings of the classical theory of elasticity. The Cosserats [1] introduced first the rotation gradient and the associated coupled stresses in the motion equations and thus created the polar media.

The more general continua can be found after almost 50 years in the following papers [2–4]. These continua are called second gradient continua [4] or strain-gradient [3], where stored energy depends not only on strain, but also on higher derivatives of the displacement. More recently, the generalized continua was also applied to modeling materials at the micro- and nanometer scale [5, 6], to describing phenomena like dislocations [7], to composites with high contrast (at a lower scale) in material properties [8–12], to catching some phenomena in regions with a stress concentration [13], to including boundary and surface energies [14, 15] or to removing singularities, when discontinuities appear in the boundary conditions, for example, [16–19]. It has been shown in numerous papers [20–23] that the limitations of classical elasticity theory can be overcome with such gradient expansion.

Using the generalized continua involves higher gradients of the displacement leading to additional parameters that depend on the microstructure. The development of a general method for determining these additional parameters by using a computational approach was attempted [24,25].

In classical elasticity, eight symmetry classes can be distinguished [26], while in gradient elasticity 17 symmetry classes need to be distinguished [27,28]. For isotropic hexads C_6 we have index representations [29,30] and a spectral decomposition [31]. For anisotropic hexads matrix representations can be found in [32]. But this is only the strain gradient part of Hooke's law. At the same time, approximately 30% of all known crystals are not centro-symmetric [33], such that mirror operations are not in the materials symmetry groups, and the coupling tensor C_5 does not vanish. Even in the case of hemitropy, with $SO(3)$ as the symmetry group, an independent component remains in C_5 (see [34]). Nevertheless, the tensor C_5 is usually assumed to vanish, since this decoupling simplifies the analysis considerably.

Unfortunately, there is no mathematical or physical reasoning behind this, but the opposite is the case. Physically, with $C_5 = 0$, 30% of all known crystals are excluded. Also mathematically there are points in favor of C_5 : for analytic homogenization, often an asymptotic series expansion is used for Hooke's law. Then, C_5 is the first order correction term, and C_6 the second order correction term [35–37].

In the present paper we extend the results [38] for coupled strain gradient elasticity, where potential energy density was presented as uncoupled quadratic form of the strain and modified second gradient of displacement. This allows to evaluate explicit tensorial expression of the complementary energy in the coupled case. For the case of hemitropic materials, the compliance tensors S_4, S_5, S_6 are determined from the tensorial relations.

Being able to invert the general, coupled Hookean law is important for extending the classical proofs of elasticity to the strain gradient case. For example, variational principles that involve the complementary energy and hence lower bound approximations (like the Reuss average) require the complementary strain energy. It is our hope that the formalization of the inversion helps to put the strain gradient theory on the same, firm mathematical foundations on which classical elasticity stands.

The presentation is organized as follows. In the next section we introduce notations used in the paper. The Section 3 contains a brief description of the block diagonalization of the composite stiffness in strain gradient elasticity, what leads to a decoupling of strains and a modified strain gradient (or the other way around) in the energy density equation and allows for the determination of the complementary strain energy in coupled strain gradient elasticity. In Section 4 the inverse constitutive tensors and the complementary strain energy in the coupled case are presented in the tensorial form. For the case of hemitropic materials, the compliance tensors S_4, S_5, S_6 are determined from the tensorial relations. The last section presents a concluding remark and a discussion.

2 | NOTATION

Scalars, vectors, second- and higher-rank tensors are denoted by italic letters (like a or A), bold minuscules (like \mathbf{a}), bold majuscules (like \mathbf{A}), and blackboard bold majuscules (like \mathbb{A}), respectively. The strain and strain gradient energy density is

$$w = \frac{1}{2} \mathbf{H}_2 \cdot \cdot \mathbf{C}_4 \cdot \cdot \mathbf{H}_2 + \mathbf{H}_2 \cdot \cdot \mathbf{C}_5 \cdot \cdot \cdot \mathbb{H}_3 + \frac{1}{2} \mathbb{H}_3 \cdot \cdot \cdot \mathbf{C}_6 \cdot \cdot \cdot \mathbb{H}_3, \quad (1)$$

where C_4, C_5, C_6 are the stiffness tensors and the strains and the second gradient of displacement are defined as:

$$\mathbf{H}_2 = \frac{1}{2} (\mathbf{u} \otimes \nabla + \nabla \otimes \mathbf{u}), \quad \mathbb{H}_3 = \mathbf{u} \otimes \nabla \otimes \nabla, \quad (2)$$

which are calculated from the displacement field $\mathbf{u}(\mathbf{x})$, where \mathbf{x} is the position vector of a material point. For convenience we drop the independent variable \mathbf{x} . “ \otimes ” denotes the dyadic product. ∇ is the nabla operator, with $\nabla_i = \frac{\partial}{\partial x_i} \mathbf{e}_i$, where \mathbf{e}_i denotes an orthonormal base vector. Repeated indices imply a summation. The nabla operator acts as follows on the displacement field \mathbf{u} :

$$\mathbf{u} \otimes \nabla = \frac{\partial u_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j = u_{i,j} \mathbf{e}_i \otimes \mathbf{e}_j. \tag{3}$$

The complementary elastic energy or stress energy density is given by

$$w^* = \frac{1}{2} \mathbf{T}_2 \cdot \cdot \mathbb{S}_4 \cdot \cdot \mathbf{T}_2 + \mathbf{T}_2 \cdot \cdot \mathbb{S}_5 \cdot \cdot \cdot \mathbb{T}_3 + \frac{1}{2} \mathbb{T}_3 \cdot \cdot \cdot \mathbb{S}_6 \cdot \cdot \cdot \mathbb{T}_3, \tag{4}$$

where $\mathbb{S}_4, \mathbb{S}_5, \mathbb{S}_6$ are the compliance tensors. The stresses and the double stresses are defined as

$$\mathbf{T}_2 = \frac{\partial w}{\partial \mathbf{H}_2} = \mathbb{C}_4 \cdot \cdot \mathbf{H}_2 + \mathbb{C}_5 \cdot \cdot \cdot \mathbb{H}_3, \tag{5}$$

$$\mathbb{T}_3 = \frac{\partial w}{\partial \mathbb{H}_3} = \mathbb{C}_5^T \cdot \cdot \mathbf{H}_2 + \mathbb{C}_6 \cdot \cdot \cdot \mathbb{H}_3. \tag{6}$$

This is the generalized Hookean law. Here the indices give the tensorial rank of $\mathbf{H}_2, \mathbb{H}_3, \mathbb{C}_4, \mathbb{C}_5, \mathbb{C}_6, \mathbb{S}_4, \mathbb{S}_5, \mathbb{S}_6$. The inverse is

$$\mathbf{H}_2 = \frac{\partial w^*}{\partial \mathbf{T}_2} = \mathbb{S}_4 \cdot \cdot \mathbf{T}_2 + \mathbb{S}_5 \cdot \cdot \cdot \mathbb{T}_3, \tag{7}$$

$$\mathbb{H}_3 = \frac{\partial w^*}{\partial \mathbb{H}_3} = \mathbb{S}_5^T \cdot \cdot \mathbf{T}_2 + \mathbb{S}_6 \cdot \cdot \cdot \mathbb{T}_3. \tag{8}$$

The dots are scalar contractions of the form

$$\mathbf{v}_1 \otimes \dots \otimes \mathbf{v}_k \underbrace{\cdot \dots \cdot}_{n \text{ dots}} \mathbf{w}_1 \otimes \dots \otimes \mathbf{w}_l = (\mathbf{v}_{k-n} \cdot \mathbf{w}_1) \dots (\mathbf{v}_k \cdot \mathbf{w}_n) \mathbf{v}_1 \otimes \mathbf{v}_{k-n-1} \otimes \mathbf{w}_{n+1} \otimes \dots \otimes \mathbf{w}_l. \tag{9}$$

For the double and triple scalar contractions in Equations (1) and (4) the associations are

$$\underbrace{\mathbf{e}_i \otimes \mathbf{e}_j \cdot \cdot \mathbf{e}_k \otimes \mathbf{e}_l}_{\text{bracket}} = \delta_{ik} \delta_{jl}, \tag{10}$$

$$\underbrace{\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \cdot \cdot \cdot \mathbf{e}_l \otimes \mathbf{e}_m \otimes \mathbf{e}_n}_{\text{bracket}} = \delta_{il} \delta_{jm} \delta_{kn}. \tag{11}$$

where δ_{ij} is the Kronecker symbol.

3 | BLOCK DIAGONALIZATIONS

In second-gradient elasticity, the potential energy density is a quadratic form of the strain \mathbf{H}_2 and the second gradient of displacement \mathbb{H}_3 , see Equation (1). The presence of the coupling tensor \mathbb{C}_5 significantly complicates the determination of conditions for positive definiteness of w as well as the calculation of the compliance tensors $\mathbb{S}_4, \mathbb{S}_5, \mathbb{S}_6$ needed for the definition of the complementary energy density. It has been shown [38] that it is possible to decouple \mathbf{H}_2 and \mathbb{H}_3 in w by a transformation of the elastic energy density, which leads to a block diagonalization of its matrix representation.

3.1 | Variant 1

To decouple the strain and strain gradient contributions, Equation (1) can be transformed by introducing a modified strain and a modified stiffness tensor of sixth-rank [38]:

$$w = \frac{1}{2} \mathbf{H}_2^m \cdot \mathbb{C}_4 \cdot \mathbf{H}_2^m + \frac{1}{2} \mathbb{H}_3 \cdots \mathbb{C}_6^m \cdots \mathbb{H}_3. \quad (12)$$

Here the superscript m denotes the modified strains and the modified stiffness tensor, which are

$$\mathbf{H}_2^m = \mathbf{H}_2 + \mathbb{H}_3 \cdots \mathbb{C}_5^T \cdots \mathbb{C}_4^{-1} \quad (13)$$

and

$$\mathbb{C}_6^m = \mathbb{C}_6 - \mathbb{C}_5^T \cdots \mathbb{C}_4^{-1} \cdots \mathbb{C}_5. \quad (14)$$

Given that the tensor \mathbb{C}_5 is symmetric with respect to the first two and to the last two indices (C_{ijklm}), the transposition of \mathbb{C}_5 is $C_{ijklm}^T = C_{klmij}$, that is, the first two and the last three entries are exchanged en bloc, such that

$$\mathbf{H}_2 : \mathbb{C}_5 \cdots \mathbb{H}_3 = \mathbb{H}_3 \cdots \mathbb{C}_5^T : \mathbf{H}_2 \quad (15)$$

holds. It has been shown [38] that such a transformation simplifies the analysis of definiteness of w and Equation (12) has been used to derive inequalities for the material parameters such that w is positive definite, including the coupling parameter in \mathbb{C}_5 . In the next Section it will be shown that the modified form of the potential energy density allows also to obtain tensorial expressions for the compliance tensors \mathbb{S}_4 , \mathbb{S}_5 , \mathbb{S}_6 and further to calculate the complementary energy density.

3.2 | Variant 2

The matrix representation of the potential energy density as block matrices can be obtain also by introducing a modified second gradient of displacement and modified stiffness tensor of fourth-rank:

$$w = \frac{1}{2} \mathbf{H}_2 \cdot \mathbb{C}_4^m \cdot \mathbf{H}_2 + \frac{1}{2} \mathbb{H}_3^m \cdots \mathbb{C}_6 \cdots \mathbb{H}_3^m. \quad (16)$$

As above the superscript m denotes the modified second gradient of displacement and the modified stiffness tensor, which are specified as

$$\mathbb{H}_3^m = \mathbb{H}_3 + \mathbf{H}_2 \cdots \mathbb{C}_5 \cdots \mathbb{C}_6^{-1} \quad (17)$$

and

$$\mathbb{C}_4^m = \mathbb{C}_4 - \mathbb{C}_5 \cdots \mathbb{C}_6^{-1} \cdots \mathbb{C}_5^T. \quad (18)$$

Remark 1. We should point out that all the equations for the potential energy density Equations (1), (12), (16) are identical and both block diagonalizations are applicable independently of the material symmetry class.

4 | INVERSE CONSTITUTIVE TENSORS AND THE COMPLEMENTARY STRAIN ENERGY

Both modified equations for the potential energy density Equations (12) and (16) can now be used to determine the compliance tensors \mathbb{S}_4 , \mathbb{S}_5 , \mathbb{S}_6 and hence for finding the stress energy density.

4.1 | Explicit expressions between stiffnesses and compliances

If the potential energy density is defined by Equation (12), using the inverse Hookean law for the modified strain and second gradient of displacement of form

$$\mathbf{T}_2 = \mathbb{C}_4^{-1} \cdot \mathbf{H}_2^m, \quad (19)$$

$$\mathbb{T}_3^m = (\mathbb{C}_6^m)^{-1} \cdot \mathbb{H}_3, \quad (20)$$

we can write down the complementary energy density like so:

$$w^* = \frac{1}{2} \mathbf{T}_2 \cdot \mathbb{C}_4^{-1} \cdot \mathbf{T}_2 + \frac{1}{2} \mathbb{T}_3^m \cdot \cdot (\mathbb{C}_6^m)^{-1} \cdot \cdot \mathbb{T}_3^m. \quad (21)$$

Here \mathbb{C}_6^m is defined in Equation (14) and \mathbb{T}_3^m is determined as

$$\mathbb{T}_3^m = \mathbb{T}_3 - \mathbb{C}_5^T \cdot \mathbb{C}_4^{-1} \cdot \mathbf{T}_2. \quad (22)$$

Comparing Equations (4), (21) the compliance tensors \mathbb{S}_4 , \mathbb{S}_5 , \mathbb{S}_6 can be defined as:

$$\mathbb{S}_4 = \mathbb{C}_4^{-1} + \mathbb{C}_4^{-1} \cdot \mathbb{C}_5 \cdot \cdot (\mathbb{C}_6^m)^{-1} \cdot \cdot \mathbb{C}_5^T \cdot \cdot \mathbb{C}_4^{-1}, \quad (23)$$

$$\mathbb{S}_5 = \mathbb{C}_4^{-1} \cdot \mathbb{C}_5 \cdot \cdot (\mathbb{C}_6^m)^{-1}, \quad (24)$$

$$\mathbb{S}_6 = (\mathbb{C}_6^m)^{-1}. \quad (25)$$

Using the same proceeding as above we can obtain the inverse Hookean law for the strain and modified second gradient of displacement for the second variant of the modified equation of the potential energy density Equation (16). In this case, the complementary energy density has form

$$w^* = \frac{1}{2} \mathbf{T}_2^m \cdot \cdot (\mathbb{C}_4^m)^{-1} \cdot \cdot \mathbf{T}_2^m + \frac{1}{2} \mathbb{T}_3 \cdot \cdot \mathbb{C}_6^{-1} \cdot \cdot \mathbb{T}_3, \quad (26)$$

with

$$\mathbf{T}_2^m = (\mathbb{C}_4^m)^{-1} \cdot \mathbf{H}_2, \quad (27)$$

$$\mathbb{T}_3 = \mathbb{C}_6^{-1} \cdot \cdot \mathbb{H}_3^m. \quad (28)$$

where \mathbb{C}_4^m is defined in Equation (18) and \mathbf{T}_2^m is determined as

$$\mathbf{T}_2^m = \mathbf{T}_2 - \mathbb{C}_5 \cdot \cdot \mathbb{C}_6^{-1} \cdot \cdot \mathbb{T}_3. \quad (29)$$

Comparing Equations (4) and (26), the compliance tensors \mathbb{S}_4 , \mathbb{S}_5 , \mathbb{S}_6 are

$$\mathbb{S}_4 = (\mathbb{C}_4^m)^{-1}, \quad (30)$$

$$\mathbb{S}_5 = (\mathbb{C}_4^m)^{-1} \cdot \mathbb{C}_5 \cdot \cdot \mathbb{C}_6^{-1}, \quad (31)$$

$$\mathbb{S}_6 = \mathbb{C}_6^{-1} + \mathbb{C}_6^{-1} \cdot \cdot \mathbb{C}_5^T \cdot \cdot (\mathbb{C}_4^m)^{-1} \cdot \cdot \mathbb{C}_5 \cdot \cdot \mathbb{C}_6^{-1}. \quad (32)$$

Remark 2. It should be noted that all three equations for the complementary energy density (4), (21), and (26) are identical.

Remark 3. Expressions for the compliance tensors \mathbb{S}_4 , \mathbb{S}_5 , \mathbb{S}_6 obtained by different variants (Equations (23)–(25) and (30)–(32)) are identical, which will be demonstrated now.

Indeed, Equation (31) can be rewritten in the form

$$\mathbb{S}_5 = (\mathbb{C}_4^m)^{-1} \cdot \mathbb{C}_5 \cdot \mathbb{C}_6^{-1} \cdot \mathbb{C}_5 \cdot \mathbb{C}_6^m \cdot (\mathbb{C}_6^m)^{-1}, \quad (33)$$

which after triple contraction $\mathbb{C}_6^{-1} \cdot \mathbb{C}_6^m$, where $\mathbb{C}_6^m = \mathbb{C}_6 - \mathbb{C}_5^T \cdot \mathbb{C}_4^{-1} \cdot \mathbb{C}_5$, leads to

$$\mathbb{S}_5 = (\mathbb{C}_4^m)^{-1} \cdot \mathbb{C}_5 \cdot \left(\mathbb{I}_6 - \mathbb{C}_6^{-1} \cdot \mathbb{C}_5^T \cdot \mathbb{C}_4^{-1} \cdot \mathbb{C}_5 \right) \cdot (\mathbb{C}_6^m)^{-1}. \quad (34)$$

After triple contraction in $\mathbb{C}_5 \cdot \left(\mathbb{I}_6 - \mathbb{C}_6^{-1} \cdot \mathbb{C}_5^T \cdot \mathbb{C}_4^{-1} \cdot \mathbb{C}_5 \right)$ we obtain

$$\mathbb{S}_5 = (\mathbb{C}_4^m)^{-1} \cdot \left(\mathbb{C}_5 - \mathbb{C}_5 \cdot \mathbb{C}_6^{-1} \cdot \mathbb{C}_5^T \cdot \mathbb{C}_4^{-1} \cdot \mathbb{C}_5 \right) \cdot (\mathbb{C}_6^m)^{-1}, \quad (35)$$

which can be transformed as

$$\mathbb{S}_5 = (\mathbb{C}_4^m)^{-1} \cdot \left(\mathbb{I}_4 - \mathbb{C}_5 \cdot \mathbb{C}_6^{-1} \cdot \mathbb{C}_5^T \cdot \mathbb{C}_4^{-1} \right) \cdot \mathbb{C}_5 \cdot (\mathbb{C}_6^m)^{-1}. \quad (36)$$

Given that $\mathbb{C}_4^{-1} = (\mathbb{C}_4^m)^{-1} \cdot \left(\mathbb{I}_4 - \mathbb{C}_5 \cdot \mathbb{C}_6^{-1} \cdot \mathbb{C}_5^T \cdot \mathbb{C}_4^{-1} \right)$, latter equation is identical to Equation (24).

Accounting for the identity $\mathbb{C}_4^{-1} \cdot \mathbb{C}_5 \cdot (\mathbb{C}_6^m)^{-1} = (\mathbb{C}_4^m)^{-1} \cdot \mathbb{C}_5 \cdot \mathbb{C}_6^{-1}$ (Equations (24) and (31)), Equation (23) can be rewritten as

$$\mathbb{S}_4 = \mathbb{C}_4^{-1} + (\mathbb{C}_4^m)^{-1} \cdot \mathbb{C}_5 \cdot \mathbb{C}_6^{-1} \cdot \mathbb{C}_5^T \cdot \mathbb{C}_4^{-1}, \quad (37)$$

and factoring out $(\mathbb{C}_4^m)^{-1}$ leads to

$$\mathbb{S}_4 = (\mathbb{C}_4^m)^{-1} \cdot \left(\mathbb{C}_4^m \cdot \mathbb{C}_4^{-1} + \mathbb{C}_5 \cdot \mathbb{C}_6^{-1} \cdot \mathbb{C}_5^T \cdot \mathbb{C}_4^{-1} \right). \quad (38)$$

Replacing \mathbb{C}_4^m in brackets by $(\mathbb{C}_4 - \mathbb{C}_5 \cdot \mathbb{C}_6^{-1} \cdot \mathbb{C}_5^T)$ in accordance with Equation (18), gives the tensor \mathbb{S}_4 as

$$\mathbb{S}_4 = (\mathbb{C}_4^m)^{-1} \cdot \left(\mathbb{I}_4 - \mathbb{C}_4^{-1} \cdot \mathbb{C}_5 \cdot \mathbb{C}_6^{-1} \cdot \mathbb{C}_5^T + \mathbb{C}_5 \cdot \mathbb{C}_6^{-1} \cdot \mathbb{C}_5^T \cdot \mathbb{C}_4^{-1} \right). \quad (39)$$

We can see that the above equation is identical to Equation (30).

It is possible to apply to Equation (32) the analogical transformation. Accounting for the identity of Equations (24) and (31), Equation (32) can be represented as

$$\mathbb{S}_6 = \mathbb{C}_6^{-1} + \mathbb{C}_6^{-1} \cdot \mathbb{C}_5^T \cdot \mathbb{C}_4^{-1} \cdot \mathbb{C}_5 \cdot (\mathbb{C}_6^m)^{-1} \quad (40)$$

and factoring out $(\mathbb{C}_6^m)^{-1}$ leads to \mathbb{S}_6 in the following form,

$$\mathbb{S}_6 = \left(\mathbb{C}_6^{-1} \cdot (\mathbb{C}_6^m) + \mathbb{C}_6^{-1} \cdot \mathbb{C}_5^T \cdot \mathbb{C}_4^{-1} \cdot \mathbb{C}_5 \right) \cdot (\mathbb{C}_6^m)^{-1}. \quad (41)$$

Substituting expression for $(\mathbb{C}_6^m)^{-1}$ (Equation (14)) in bracket we have

$$\mathbb{S}_6 = \left(\mathbb{I}_6 - \mathbb{C}_6^{-1} \cdot \mathbb{C}_5^T \cdot \mathbb{C}_4^{-1} \cdot \mathbb{C}_5 + \mathbb{C}_6^{-1} \cdot \mathbb{C}_5^T \cdot \mathbb{C}_4^{-1} \cdot \mathbb{C}_5 \right) \cdot (\mathbb{C}_6^m)^{-1}, \quad (42)$$

which is identical Equation (25).

Remark 4. It is seen from Equations (23)–(25) and (30)–(32) that for the special case of uncoupled strain gradient elasticity if tensor $C_5 = 0$ the compliance tensors of the fourth- and sixth-rank are equal to the inverse stiffness tensors of the same rank $S_4 = C_4^{-1}$ and $S_6 = C_6^{-1}$.

4.2 | Relations for compliance tensors of hemitropic material

We apply the results presented in the previous subsection to determine the relations for the compliance tensors S_4, S_5, S_6 in the case of hemitropic materials. The stiffnesses C_4, C_5, C_6 are in case of hemitropy characterized by eight independent parameters and have the following form [3,31] [39]

$$C_4 = [\lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})] \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \quad (43)$$

$$C_5 = [\kappa (\varepsilon_{imk} \delta_{jl} + \varepsilon_{ilk} \delta_{jm} + \varepsilon_{jmk} \delta_{il} + \varepsilon_{jlk} \delta_{im})] \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \otimes \mathbf{e}_m \quad (44)$$

$$\begin{aligned} C_6 = & [c_1 (\delta_{jk} \delta_{im} \delta_{nl} + \delta_{jk} \delta_{in} \delta_{ml} + \delta_{ji} \delta_{kl} \delta_{mn} + \delta_{jl} \delta_{ik} \delta_{mn}) \\ & + c_2 (\delta_{ji} \delta_{km} \delta_{nl} \delta_{jm} \delta_{ki} \delta_{nl} + \delta_{ji} \delta_{kn} \delta_{ml} + \delta_{jn} \delta_{ik} \delta_{ml}) \\ & + c_3 (\delta_{jm} \delta_{kl} \delta_{in} \delta_{jl} \delta_{in} \delta_{km} + \delta_{jn} \delta_{im} \delta_{kl} + \delta_{jl} \delta_{im} \delta_{nk}) \\ & + c_4 (\delta_{jn} \delta_{il} \delta_{km} \delta_{jm} \delta_{kn} \delta_{il}) \\ & + c_5 \delta_{il} \delta_{jk} \delta_{mn}] \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \otimes \mathbf{e}_m \otimes \mathbf{e}_n, \end{aligned} \quad (45)$$

where δ_{ij} is the Kronecker symbol and ε_{ijk} the Levi-Civita permutation symbol, λ and μ are Lamé's coefficients, and κ and $c_{1,2,3,4,5}$ are the higher order material parameters. One can check that the following index symmetries hold [40]

$$H_{ij} = H_{ji}, \quad (46)$$

$$H_{ijk} = H_{ikj}, \quad (47)$$

$$C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}, \quad (48)$$

$$C_{ijklm} = C_{jiklm} = C_{ijkml}, \quad (49)$$

$$C_{ijklmn} = C_{lmnijk} = C_{ikjlmn} = C_{ijklnm}. \quad (50)$$

Substituting Equations (43)–(45) in relations for compliance tensors Equations (23)–(25) and accounting for Equation (14) we obtain that tensors S_4, S_5, S_6 have the same symmetry and the same structure as tensors C_4, C_5, C_6 and are characterized by eight independent constants $s_{1,2,3,4,5,6,7,8}$

$$S_4 = [s_1 \delta_{ij} \delta_{kl} + s_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})] \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \quad (51)$$

$$S_5 = [s_8 (\varepsilon_{imk} \delta_{jl} + \varepsilon_{ilk} \delta_{jm} + \varepsilon_{jmk} \delta_{il} + \varepsilon_{jlk} \delta_{im})] \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \otimes \mathbf{e}_m \quad (52)$$

$$\begin{aligned} S_6 = & [s_3 (\delta_{jk} \delta_{im} \delta_{nl} + \delta_{jk} \delta_{in} \delta_{ml} + \delta_{ji} \delta_{kl} \delta_{mn} + \delta_{jl} \delta_{ik} \delta_{mn}) \\ & + s_4 (\delta_{ji} \delta_{km} \delta_{nl} \delta_{jm} \delta_{ki} \delta_{nl} + \delta_{ji} \delta_{kn} \delta_{ml} + \delta_{jn} \delta_{ik} \delta_{ml}) \\ & + s_5 (\delta_{jm} \delta_{kl} \delta_{in} \delta_{jl} \delta_{in} \delta_{km} + \delta_{jn} \delta_{im} \delta_{kl} + \delta_{jl} \delta_{im} \delta_{nk}) \\ & + s_6 (\delta_{jn} \delta_{il} \delta_{km} \delta_{jm} \delta_{kn} \delta_{il}) \\ & + s_7 \delta_{il} \delta_{jk} \delta_{mn}] \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \otimes \mathbf{e}_m \otimes \mathbf{e}_n, \end{aligned} \quad (53)$$

where $s_{1,2,5,6,8}$ are defined as

$$s_1 = \frac{4\kappa^2 - (c_3 - c_4)\lambda}{2(3\lambda + 2\mu)(6\kappa^2 + (c_3 - c_4)\mu)}, \quad (54)$$

$$s_2 = \frac{c_3 - c_4}{4(6\kappa^2 + (c_3 - c_4)\mu)}, \quad (55)$$

$$s_5 = \frac{2\kappa^2 + c_3\mu}{4(2c_3 + c_4)(6\kappa^2 + (c_3 - c_4)\mu)}, \quad (56)$$

$$s_6 = \frac{2\kappa^2 - c_3\mu - c_4\mu}{4(2c_3 + c_4)(6\kappa^2 + (c_3 - c_4)\mu)}, \quad (57)$$

$$s_8 = \frac{\kappa}{4(6\kappa^2 + (c_3 - c_4)\mu)}. \quad (58)$$

The parameters $s_{3,4,7}$ have more complicated form,

$$s_3 = \frac{\kappa^2(4c_1^2 - 12c_1c_3 - 8c_2c_3 - 10c_1c_4 - 8c_2c_4 - 4c_2c_5 - 4c_3c_5 - 3c_4c_5 - 2(2c_3 + c_4)^2)}{4(2c_3 + c_4)(10c_1^2 + 12c_1c_3 + 4c_2c_3 + 4c_3^2 - 4c_1c_4 - 8c_2c_4 - 2c_3c_4 - 2c_4^2 - 10c_2c_5 - c_3c_5 - 3c_4c_5)(6\kappa^2 + (c_3 - c_4)\mu)}$$

$$\frac{\mu(4c_1^2c_3 + 2c_1c_3^2 + c_1^2c_4 + c_1c_3c_4 - 2c_2c_3c_4 + c_1c_4^2 - 4c_2c_3c_5 - c_3^2c_5 - c_2c_4c_5 - c_3c_4c_5)}{4(2c_3 + c_4)(10c_1^2 + 12c_1c_3 + 4c_2c_3 + 4c_3^2 - 4c_1c_4 - 8c_2c_4 - 2c_3c_4 - 2c_4^2 - 10c_2c_5 - c_3c_5 - 3c_4c_5)(6\kappa^2 + (c_3 - c_4)\mu)}, \quad (59)$$

$$s_4 = \frac{\kappa^2(8c_1^2 + 24c_1c_3 + 8c_2c_3 + 4c_1c_4 - 4c_2c_4 - 8c_2c_5 + 10c_3c_5 + 3c_4c_5 + 2(2c_3 + c_4)^2)}{4(2c_3 + c_4)(10c_1^2 + 12c_1c_3 + 4c_2c_3 + 4c_3^2 - 4c_1c_4 - 8c_2c_4 - 2c_3c_4 - 2c_4^2 - 10c_2c_5 - c_3c_5 - 3c_4c_5)(6\kappa^2 + (c_3 - c_4)\mu)}$$

$$\frac{\mu(-2c_1^2c_3 - 3c_1^2c_4 - 4c_1c_3c_4 + 2c_2c_4^2 + 2c_2c_3c_5 + 2c_3^2c_5 + 3c_2c_4c_5)}{4(2c_3 + c_4)(10c_1^2 + 12c_1c_3 + 4c_2c_3 + 4c_3^2 - 4c_1c_4 - 8c_2c_4 - 2c_3c_4 - 2c_4^2 - 10c_2c_5 - c_3c_5 - 3c_4c_5)(6\kappa^2 + (c_3 - c_4)\mu)}, \quad (60)$$

$$s_7 = \frac{\kappa^2(4c_1^2 + 24c_1c_3 + 40c_2c_3 + 8c_1c_4 + 16c_2c_4 - 4c_2c_5 + 2c_3c_5 + 4(2c_3 + c_4)^2)}{4(2c_3 + c_4)(10c_1^2 + 12c_1c_3 + 4c_2c_3 + 4c_3^2 - 4c_1c_4 - 8c_2c_4 - 2c_3c_4 - 2c_4^2 - 10c_2c_5 - c_3c_5 - 3c_4c_5)(6\kappa^2 + (c_3 - c_4)\mu)}$$

$$\frac{\mu(-6c_1^2c_3 - 4c_1c_3^2 + 4c_2c_3^2 - 4c_1^2c_4 - 4c_1c_3c_4 + 6c_2c_3c_5 + c_3^2c_5 + 4c_2c_4c_5 + 2c_3c_4c_5 + c_4^2c_5)}{4(2c_3 + c_4)(10c_1^2 + 12c_1c_3 + 4c_2c_3 + 4c_3^2 - 4c_1c_4 - 8c_2c_4 - 2c_3c_4 - 2c_4^2 - 10c_2c_5 - c_3c_5 - 3c_4c_5)(6\kappa^2 + (c_3 - c_4)\mu)}. \quad (61)$$

4.3 | Comparison to solving the defining linear system for the compliances

For a verification of the above expressions of the compliances $s_{1...8}$, we set up linear equations for these parameters by inserting \mathbf{H}_2 and \mathbb{H}_3 in Hooke's law, inserting the obtained stresses in the inverted Hookean law and subtracting \mathbf{H}_2 and \mathbb{H}_3 from the output, concisely written as

$$\mathbb{S}[\mathbb{C}[\mathbf{H}]] = \mathbf{H}, \quad (62)$$

where $\mathbb{S}_{4,5,6}$, $\mathbb{C}_{4,5,6}$ and $\mathbf{H}_2, \mathbb{H}_3$ are abbreviated as \mathbb{S} , \mathbb{C} and \mathbf{H} and the squared bracket is the linear mapping according to Equations (5)–(8). This needs to hold for arbitrary \mathbf{H} , that is, the above equation needs to hold component-wise, hence we can vary each of the components of \mathbf{H}_2 and \mathbb{H}_3 independently, for example by setting $H_{11} = 1$ and leaving all other

components of \mathbf{H} equal to zero. Together, \mathbf{H}_2 and \mathbb{H}_3 have 24 independent components. This gives an inhomogeneous system of $24 \times 24 = 576$ linear equations for the components of \mathbb{S} . Because of universal index symmetries and material symmetries, we have much less independent equations. We already know that only eight independent equations can exist in the hemitropic case. We obtained closed form expressions for $s_{1\dots 8}$ by the aid of a computer algebra system. A comparison of the obtained compliances with the ones calculated on the basis of Equations (23)–(25) and Equations (30)–(32) shows the identity of all three variants. The solution was obtained by using the scientific computing system “Mathematica”, the script is provided as supplementary material.

5 | DISCUSSION

Tensorial relations for determining the compliance tensors within the linear theory of coupled gradient elasticity are obtained for arbitrary material symmetry classes. The presence of the coupling term \mathbb{C}_5 in the equation for the potential energy density Equation (1) complicates the problem. Explicit expression for the compliances in terms of the stiffnesses are obtained by substituting a modified strain or a modified strain gradient as an auxiliary variable that decouples the strains and the strain gradient (Equation (1)). This decoupling, or diagonalization, makes it possible to invert Hooke’s law even in the coupled case, since it leaves only invertible tensors on the main diagonal, essentially as quadratic component matrices w.r.t. a suitable basis. In summary, one can easily obtain the coupled compliance tensors $\mathbb{S}_4, \mathbb{S}_5, \mathbb{S}_6$ by forward modification $\{\mathbb{C}_4, \mathbb{C}_5, \mathbb{C}_6\} \rightarrow \{\mathbb{C}_4, \mathbb{C}_6^m\}$ or $\{\mathbb{C}_4, \mathbb{C}_5, \mathbb{C}_6\} \rightarrow \{\mathbb{C}_4^m, \mathbb{C}_6\}$, inversion $\{\mathbb{C}_4, \mathbb{C}_6^m\} \rightarrow \{\mathbb{S}_4, \mathbb{S}_6^m\}$ or $\{\mathbb{C}_4^m, \mathbb{C}_6\} \rightarrow \{\mathbb{S}_4^m, \mathbb{S}_6\}$, and then finally backward modification $\{\mathbb{S}_4, \mathbb{S}_6^m\} \rightarrow \{\mathbb{S}_4, \mathbb{S}_5, \mathbb{S}_6\}$ or $\{\mathbb{S}_4^m, \mathbb{S}_6\} \rightarrow \{\mathbb{S}_4, \mathbb{S}_5, \mathbb{S}_6\}$. It is demonstrated that the compliance tensors obtained from the both variants of modified energy density are identical.

In the case of hemitropic materials, the compliance tensors have the same symmetry and the same form as the stiffness tensors and are characterized by eight independent constants. Explicit expressions for these eight parameters are obtained from the tensorial relations for the compliance tensors Equations (23)–(25) and Equations (30)–(32) and are compared with the direct solution of a linear system for the compliances. All three solutions are identical.

The symbolic solutions given in the mentioned equations imply matrix inversions and matrix multiplications when the constitutive tensors are represented with respect to suitable bases. As in classical elasticity where only $\mathbb{S}_4 = \mathbb{C}_4^{-1}$, these calculations are easy to do with numerical values. For highly symmetric materials, a symbolic evaluation is also doable, but for anisotropic materials this is likely to give unmanageable expressions. As in classical elasticity, there is not much that can be done about that.

To summarize, for the coupled gradient elasticity, the block diagonalization allows to obtain the inequality constraints for the positive definiteness of the stiffness tensors, to invert Hooke’s law to obtain the relations for compliance tensors, which allows to write down the complementary strain energy as a closed form expression even in terms of the stiffnesses. This, in turn, opens the door to extend different theorems from classical elasticity to the coupled strain gradient case, like the variational principle of the maximum of the complementary potential, from which in turn bounds for effective elasticities like the Reuss average as well as uniqueness theorems can be generalized.

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REFERENCES

- [1] Cosserat, F., Cosserat, E.: *Théorie des corps déformables*. A. Herman et Fils, Paris (1909)
- [2] Toupin, R.A.: Elastic materials with couple-stresses. *Arch. Ration. Mech. Anal.* 11(1), 385–414 (1962)
- [3] Mindlin, R.D.: Micro-structure in linear elasticity. *Arch. Ration. Mech. Anal.* 16(1), 51–78 (1964)
- [4] Germain, P.: The method of virtual power in continuum mechanics. Part 2: Microstructure. *SIAM J. Appl. Math.* 25(3), 556–575 (1973)
- [5] Cordero, N.M., Forest, N.M., Busso, E.P.: Second strain gradient elasticity of nano-objects. *J. Mech. Phys. Solids* 97, 92–124 (2016), SI: Pierre Suquet Symposium. <https://doi.org/10.1016/j.jmps.2015.07.012>
- [6] Forest, S., Cordero, N.M., Busso, E.P.: First vs. second gradient of strain theory for capillarity effects in an elastic fluid at small length scales. *Comput. Mater. Sci.* 50(4), 1299–1304 (2011), Proceedings of the 19th International Workshop on Computational Mechanics of Materials

- [7] Ferretti, M., Madeo, A., dell'Isola, F., Boisse, P.: Modeling the onset of shear boundary layers in fibrous composite reinforcements by second-gradient theory. *Z. Angew. Math. Phys.* 65(3), 587–612 (2014)
- [8] Abdoul-Anziz, H., Seppecher, P.: Strain gradient and generalized continua obtained by homogenizing frame lattices. *Math Mech Complex Syst* 6(3), 213–250 (2018)
- [9] Eugster, S.R., dell'Isola, F., Steigmann, D.J.: Continuum theory for mechanical metamaterials with a cubic lattice substructure. *Math Mech Complex Syst* 7(1), 75–98 (2019)
- [10] Placidi, L., Barchiesi, E., Turco, E., Rizzi, N.L.: A review on 2d models for the description of pantographic fabrics. *Z. Angew. Math. Phys.* 67(5), 121 (2016)
- [11] Rahali, Y., Giorgio, I., Ganghoffer, J.F., dell'Isola, F.: Homogenization à la piola produces second gradient continuum models for linear pantographic lattices. *Int. J. Eng. Sci.* 97, 148–172 (2015)
- [12] Volkov-Bogorodsky, D.B., Evtushenko, Y.G., Zubov, V.I., Lurie, S.A.: 'Calculation of deformations in nanocomposites using the block multipole method with the analytical-numerical account of the scale effects. *Comput. Math. Math. Phys.* 46(7), 1234–1253 (2006)
- [13] Askes, H., Aifantis, E.C.: Gradient elasticity in statics and dynamics: An overview of formulations, length scale identification procedures, finite element implementations and new results. *Int. J. Solids Struct.* 48(13), 1962–1990 (2011)
- [14] Eremeyev, V.A., Lurie, S.A., Solyaev, Y.O., dell'Isola, F.: On the well posedness of static boundary value problem within the linear dilatational strain gradient elasticity. *Z. Angew. Math. Phys.* 71(182), 16 (2020)
- [15] Javili, A., dell'Isola, F., Steinmann, P.: Geometrically nonlinear higher-gradient elasticity with energetic boundaries. *J. Mech. Phys. Solids* 61(12), 2381–2401 (2013)
- [16] Askes, H., Suiker, A.S.J., Sluys, L.J.: A classification of higher-order strain-gradient models - linear analysis. *Arch. Appl. Mech.* 72(2), 171–188 (2002)
- [17] Georgiadis, H.G., Anagnostou, D.S.: Problems of the Flamant–Boussinesq and Kelvin type in dipolar gradient elasticity. *J. Elast.* 90(1), 71–98 (2008)
- [18] Reiher, J.C., Giorgio, I., Bertram, A.: Finite-element analysis of polyhedra under point and line forces in second-strain gradient elasticity. *J. Eng. Mech.* 143(2), 04016112 (2017)
- [19] Sinclair, G.B.: Stress singularities in classical elasticity – I: Removal, interpretation, and analysis. *Appl. Mech. Rev.* 57(4), 251–298 (2004)
- [20] Peerlings, R.H.J., Geers, M.G.D., de Borst, R., Brekelmans, W.A.M.: A critical comparison of nonlocal and gradient-enhanced softening continua. *Int. J. Solids Struct.* 38(44–45), 7723–7746 (2001)
- [21] Lazar, M., Maugin, G.A., Aifantis, E.C.: On a theory of nonlocal elasticity of bi-Helmholtz type and some applications. *Int. J. Solids Struct.* 43(6), 1404–1421 (2006)
- [22] Forest, S., Bertram, A.: Formulations of strain gradient plasticity. In: Altenbach, H., Maugin, G.A., Eremeyev, V.A. (eds.) *Mechanics of generalized continua*. vol. 7 of *Advanced Structured Materials*, pp. 137–149. Springer, Berlin Heidelberg (2011), Chapter Formulations of strain gradient plasticity
- [23] Lim, C.W., Zhang, G., Reddy, J.N.: A higher-order nonlocal elasticity and strain gradient theory and its applications in wave propagation. *J. Mech. Phys. Solids* 78, 298–313 (2015)
- [24] Abali, B.E., Barchiesi, E.: Additive manufacturing introduced substructure and computational determination of metamaterials parameters by means of the asymptotic homogenization. *Contin. Mech. Thermodyn.* (2020). <https://doi.org/10.1007/s00161-020-00941-w>
- [25] Abali, B.E., Yang, H., Papadopoulos, P.: A Computational Approach for Determination of Parameters in Generalized Mechanics. In: Altenbach, H., Müller, W.H., Abali, B.E. (eds.) *Higher gradient materials and related generalized continua*, vol. 120 of *Advanced Structured Materials*, pp. 1–18. Springer International Publishing, Cham (2019), chapter A Computational Approach for Determination of Parameters in Generalized Mechanics
- [26] Forte, S., Vianello, M.: Symmetry classes for elasticity tensors. *J. Elast.* 43(2), 81–108 (1996)
- [27] Olive, M., Auffray, N.: Symmetry classes for even-order tensors. *Math Mech Complex Syst* 1(2), 177–210 (2013)
- [28] Olive, M., Auffray, N.: Symmetry classes for odd-order tensors. *ZAMM - J. Appl. Math. Mech. / Z. Angew. Math. Mech.* 94(5), 421–447 (2014)
- [29] Lazar, M.: Irreducible decomposition of strain gradient tensor in isotropic strain gradient elasticity. *ZAMM - J. Appl. Math. Mech. / Z. Angew. Math. Mech.* 96(11), 1291–1305 (2016)
- [30] Mindlin, R.D., Eshel, N.N.: On first strain-gradient theories in linear elasticity. *Int. J. Solids Struct.* 4(1), 109–124 (1968)
- [31] Glüge, R., Kalisch, J., Bertram, A.: The eigenmodes in isotropic strain gradient elasticity. In: Altenbach, H., Forest, S. (eds.) *Generalized continua as models for classical and advanced materials*, vol. 42 of *Advanced Structured Materials*, pp. 157–172. Springer International Publishing, Cham (2016), chapter The eigenmodes in isotropic strain gradient elasticity
- [32] Auffray, N., Le-Quang, H., He, Q.C.: Matrix representations for 3D strain-gradient elasticity. *J. Mech. Phys. Solids* 61(5), 1202–1223 (2013)
- [33] Kholkin, A.L., Pertsev, N.A., Goltsev, A.V.: Piezoelectricity and Crystal Symmetry. In: Safari, A., Akdoğan, E.K. (eds.) *Piezoelectric and acoustic materials for transducer applications*, pp. 17–38. Springer, Boston (2008), chapter Piezoelectricity and Crystal Symmetry
- [34] Papanicolopulos, S.A.: Chirality in isotropic linear gradient elasticity. *Int. J. Solids Struct.* 48(5), 745–752 (2011)
- [35] Boutin, C.: Microstructural effects in elastic composites. *Int. J. Solids Struct.* 33(7), 1023–1051 (1996)
- [36] Lebée, A., Sab, K.: The Bending-Gradient Theory for Laminates and In-Plane Periodic Plates. In: Altenbach, H., Eremeyev, V. (eds.) *Shell-like structures: Advanced theories and applications*, vol. 572 of *CISM International Centre for Mechanical Sciences*, pp. 113–148. Springer, Vienna (2016), chapter The Bending-Gradient Theory for Laminates and In-Plane Periodic Plates

- [37] Smyshlyaev, V.P., Cherednichenko, K.D.: On rigorous derivation of strain gradient effects in the overall behaviour of periodic heterogeneous media. *J. Mech. Phys. Solids* 48(6), 1325–1357 (2000)
- [38] Nazarenko, L., Glüge, R., Altenbach, H.: Positive definiteness in coupled strain gradient elasticity. *Contin. Mech. Thermodyn.* (2020). <https://doi.org/10.1007/s00161-020-00949-2>
- [39] dell'Isola, F., Sciarra, G., Vidoli, S.: Generalized hooke's law for isotropic second gradient materials. *Proc. R. Soc. Lond. A: Math. Phys. Eng. Sci.* 465(2107), 2177–2196 (2009)
- [40] Gusev, A.A., Lurie, S.A.: Symmetry conditions in strain gradient elasticity. *Math Mech Solids* 22(4), 683–691 (2017)

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